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SIMILARITY METHOD FOR THE JACOBIAN PROBLEM

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Abstract. This is an expository article giving a modified version of my talk at the October 2006 Conference in Hanoi.

1. Introduction

Two given bivariate polynomials are said to form a jacobian pair if their jacobian equals a nonzero constant, and they are said to form an automorphic pair if the variables can be expressed as polynomials in the given polynomials. By the chain rule we see that every automorphic pair is a jacobian pair. The jacobian problem asks if conversely every jacobian pair is an automorphic pair. It turns out that a useful method for attacking this problem is to study the similarity of polynomials. Two bivariate polynomials are similar means their degree forms, i.e., highest degree terms, are powers of each other when they are multiplied by suitable nonzero constants. Geometrically this amounts to saying that the corresponding plane curves have the same points at infinity counting multiplicities. At any rate, the points at infinity correspond to the distinct irreducible factors of the degree form.

2. History

Before getting into technicalities, I shall first give a short history of the problem or rather the history of my acquaintance with the problem. For that we have to go back to 1965 when a German mathematician, Karl Stein who created Stein Manifolds, wrote me a letter asking a question. He said that there was an interesting 1955 paper in the Mathematische Annalen by Engel [Eng]. In this paper Engel claims to prove the jacobian theorem or what is now known as the jacobian problem or the jacobian conjecture or whatever. Karl Stein said to me that it is an interesting theorem but he cannot understand the proof. Can I help him? He also reduced it, or generalized it, to a conjecture about complex spaces. I wrote back to Stein giving a counterexample to his complex space conjecture. But I did not look at the Engel paper. Then in 1968, Max Rosenlicht of Berkeley asked me the same question and still I did not look at the Engel paper. Finally in 1970, my own guru (= venerable teacher) Oscar Zariski asked me the same

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question. Then, following the precept that one must obey one's guru, I looked up the Engel paper and found it full of mistakes and gaps.

The primary mistake in the Engel paper, which was repeated in a large number of published and unpublished wrong proofs of the jacobian problem in the last thirty-five years, is the presumed "obvious fact" that the order of the derivative of a univariate function is exactly one less than the order of the function. Being a prime characteristic person I never made this mistake. Indeed, the "fact" is correct only if the order of the function in nondivisible by the characteristic. Of course you could say that the jacobian problem is a characteristic zero problem, and zero does not divide anybody. But zero does divide zero. So the "fact" is incorrect if the order of the function is zero, i.e., if the value of the function is nonzero. Usually this mistake is well hidden inside a long argument, because you may start with a function which has a zero or pole at a given point and your calculation may lead to a function having a nonzero value at a resulting point.

A gap is a spot where you are not sure of the argument because of imprecise definitions or what have you. The gap in the Engel paper seems to be the uncritical use of the Zeuthen-Segre invariant. For this invariant of algebraic surfaces see the precious 1935 book of Zariski [Zar]. Over the years I have made several attempts to understand the somewhat mysterious theory of this invariant, and I still continue to do so.

In 1970-1977 I discussed the matter in my courses at Purdue and also in India and Japan. Mostly I was suggesting to the students to fix the proof and, to get them started, I proved a few small results. Notes of my lectures were taken down by Heinzer, van der Put, Sathaye, and Singh. These appeared in [Ab1] and [Ab4]. Then I put the matter aside for thirty years. Seeing that the problem has remained unsolved inspite of a continuous stream of wrong proofs announced practically every six months, I decided to write up my old results, together with some enhancements obtained recently, in the form of a series of three long papers [Ab5], [Ab8], [Ab9], in the Journal of Algebra, dedicated to the fond memory of my good friend Walter Feit. The Hanoi Conference has given me a welcome opportunity of introducing these papers to the young students with an invitation to further investigate the problem.

Now one of my old results says that the jacobian conjecture is equivalent to the implication that each member of a jacobian pair can have only one point at infinity. Another says that each member of any jacobian pair has at most two points at infinity. Note that the first result is a funny statement; it only says that to prove the jacobian conjecture, it suffices to show that each member of any jacobian pair has only one point at infinity. The second result is of a more definitive nature, and it remains true even when we give weights to the variables which are different from the normal weights. Very recently I noticed that, and this is one of the enhancements, the weighted two point theorem yields a very short new proof of Jung's 1942 automorphism theorem [Jun]. This automorphism theorem says that every automorphism of a bivariate polynomial ring is composed of a finite number of linear automorphisms and elementary automorphisms. In a linear automorphsim both variables are sent to linear expressions in them. In an elementary automorphism, one variable is unchanged and a polynomial in it is added to the second variable. In his 1972 lecture notes [Nag], Nagata declared the automorphism theorem to be very profound and so it did come as a pleasant surprise to me that the weighted two point theorem yields a five line proof of the automorphism theorem. For other recent enhancements let me refer to my Feit memorial papers cited above.

The present short note is only meant to whet the student's appetite. At any rate the material of this paper is based on my 2006 Hanoi Conference Talk and on a very short course I gave at Purdue in 1997.

3. Similarity

Let $f = f(X,Y)$ and $g = g(X,Y)$ be two nonzero polynomials in indeterminates X and Y with coefficients in a field k of characteristic zero. Let deg $f = N$ and deg $g = M$. Let f^+ and g^+ be the degree forms of f and g, i.e., $f = f^+ +$ terms of degree $< N$, and $g = g^+ +$ terms of degree $< M$. Let $J(f,g) = J_{(X,Y)}(f,g) = f_X g_Y - f_Y g_X$ be the jacobian of f and g with respect to X and Y, where f_X is the X-partial of f and so on. Let $\not\theta$ be Abhyankar's nonzero, i.e., let it stand for a generic nonzero element of k; note that $u = \not\theta$ and $v = \not\theta$ does not imply $u = v$. The JC (= Jacobian Conjecture) says that: $J(f,g) = \not\theta \Rightarrow k[f,g] = k[X,Y]$. We say that f is similar to g, and we write $f \sim g$, to mean that $(f^+)^M = \not\theta\,(g^+)^N$.

LEMMA (L3.1). *If* $M + N > 2$ *and* $J(f,g) = \not\theta \Rightarrow f \sim g$.

LEMMA (L3.2). *JC is true for* $GCD(M,N) \le 8$.

In my 1971 Purdue Lectures [Ab4], the above two lemmas are proved by using Euler's Theorem on homogeneous polynomials. The same proofs are reproduced in my recent papers [Ab8] and [Ab9]. In Section 4 we give a "power series" proof reproduced in my recent paper [Ab5].

4. Jacobian Problem

We shall now provide a "power series" proof of Lemma (L3.1). So given $0 \neq F = F(X,Y) \in k[X,Y]$ and $0 \neq G = G(X,Y) \in k[X,Y]$ let

$$\deg_X F = a \ \text{ and } \ \mathrm{deco}_X F = P = P(Y) \ \text{ with } \ u = \deg_Y P \in \mathbb{N}$$

and

$$\deg_X G = b \ \text{ and } \ \mathrm{deco}_X G = Q = Q(Y) \ \text{ with } \ v = \deg_Y Q \in \mathbb{N}$$

where deco stands for highest degree coefficient and $\mathbb{N}$ for nonnegative integers, i.e.,

$$F = X^a P(Y) + \text{ terms of } \deg_X < a$$

and

$$G = X^b Q(Y) + \text{ terms of } \deg_X < b.$$

For

$$J = J(F,G) = J_{(X,Y)}(F,G) = F_X G_Y - F_Y G_X$$

we have

$$F_X = aX^{a-1}P + \text{ terms of } \deg_X < a-1$$

and

$$G_Y = X^b Q' + \text{ terms of } \deg_X < b$$

where prime denotes Y-derivative, and hence

$$F_X G_Y = aX^{a+b-1}PQ' + \text{ terms of } \deg_X < a+b-1$$

and similarly

$$F_Y G_X = bX^{a+b-1}P'Q + \text{ terms of } \deg_X < a+b-1$$

and therefore

$$J = X^{a+b-1}(aPQ' - bP'Q) + \text{ terms of } \deg_X < a+b-1. \tag{4.1}$$

By (4.1) we see that

$$\deg_X J = a+b-1 \Rightarrow aPQ' - bP'Q = \operatorname{deco}_X J \tag{4.2}$$

and

$$\deg_X J \neq a+b-1 \Leftrightarrow \deg_X J < a+b-1 \Leftrightarrow aPQ' - bP'Q = 0. \tag{4.3}$$

If $aPQ' - bP'Q = 0$ with $u+v \geq 2$ and $ab \neq 0$ then by looking at terms of degree $u+v-1$ we see that $av = bu$ and hence $uPQ' - vP'Q = 0$ with $uv \neq 0$; thus

(4.4)

$$aPQ' - bP'Q = 0 \text{ with } u+v \geq 2 \text{ and } ab \neq 0 \Rightarrow \begin{cases} uPQ' - vP'Q = 0 \text{ with} \\ uv \neq 0 \text{ and } av = bu. \end{cases}$$

By the product rule we have

$$(P^v Q^{-u})' = -P^{v-1}Q^{-u-1}(uPQ' - vP'Q)$$

and hence upon multiplying both sides of the above equation by $-P^{v-1}Q^{-u-1}$ we get $(P^v Q^{-u})' = 0$ and hence $P^v Q^{-u} = \theta$ and therefore $P^v = \theta Q^u$; thus

$$uPQ' - vP'Q = 0 \text{ with } uv \neq 0 \Rightarrow P^v = \theta Q^u. \tag{4.5}$$

Let

$$\operatorname{GCD}(u,v) = d.$$

We claim that then

$$P^v = \theta Q^u \text{ with } uv \neq 0 \Rightarrow \begin{cases} P = \theta R^{u/d} \text{ and } Q = \theta R^{v/d} \text{ where} \\ 0 \neq R \in k[Y] \text{ with } \deg_Y R = d. \end{cases} \tag{4.6}$$

To see this, by factoring we can write

$$P = \theta\, P_1^{u_1} \dots P_\sigma^{u_\sigma} \quad \text{and} \quad Q = \theta\, Q_1^{v_1} \dots Q_\tau^{v_\tau}$$

where $u_1, \dots, u_\sigma, v_1, \dots, v_\tau$ are positive integers, $P_1, \dots, P_\sigma$ are pairwise distinct irreducible nonconstant polynomials in $k[Y]$, and $Q_1, \dots, Q_\tau$ are pairwise distinct irreducible nonconstant polynomials in $k[Y]$. Substituting these factorizations in the equation $P^v = \theta\, Q^u$, by the uniqueness property we conclude that $\sigma = \tau$ and after suitable relabelling we have $P_i = Q_i$ and $vu_i = uv_i$ for $1 \le i \le \sigma$. Now

$$\frac{u/d}{v/d} = \frac{u_i/d_i}{v_i/d_i} \quad \text{where } \operatorname{GCD}(u_i, v_i) = d_i \quad \text{for } 1 \le i \le \sigma$$

and by the uniqueness of fractional representation we get

$$u/d = u_i/d_i \quad \text{and} \quad v/d = v_i/d_i \quad \text{for } 1 \le i \le \sigma$$

and hence upon letting

$$R = P_1^{d_1} \dots P_\sigma^{d_\sigma}$$

we conclude that $0 \ne R \in k[Y]$ with $P = \theta\, R^{u/d}$ and $R = \theta\, R^{v/d}$. It follows that $\deg_Y R = (d/u)\deg_Y P = (d/u)u = d$. Combining (4.3) to (4.6) we get

LEMMA (L4.1). *We have*

$deg_X J \ne a + b - 1$ *with* $u + v \ge 2$ *and* $ab \ne 0$

$\Rightarrow uPQ' - vP'Q = 0$ *with* $uv \ne 0$

$\Rightarrow P^v = \theta\, Q^u$ *with* $uv \ne 0$

$\Rightarrow P = \theta\, R^{u/d}$ *and* $Q = \theta\, R^{v/d}$ *where* $0 \ne R \in k[Y]$ *with* $deg_Y R = d$.

We shall now deduce (L3.1) from (L4.1). By rotating coordinates, i.e., by making a homogeneous linear transformation, we may assume that $\deg_Y f = \deg f = N$ and $\deg_Y g = \deg g = M$; this will only multiply the jacobian

$$h = h(X, Y) = J(f, g)$$

by a nonzero constant. Now we make the transformation $(X, Y) \mapsto (X, XY)$, i.e., we put

$$F(X, Y) = f(X, XY) \quad \text{and} \quad G(X, Y) = g(X, XY) \quad \text{and} \quad H(X, Y) = h(X, XY)$$

and we note that then

$$(a, b) = (N, M) = (u, v) \quad \text{and} \quad (P(Y), Q(Y)) = (f^+(1, Y), g^+(1, Y)) \tag{4.7}$$

and by the chain rule we have

$$J(F, G) = XH(X, Y) \tag{4.8}$$

and hence by (L4.1) we get the following stronger version of (L3.1).

LEMMA (L4.2). *If* $MN \ne 0$ *and* $deg\ J(f, g) \ne M + N - 2$ *then* $(f^+)^M = \theta\, (g^+)^N$ *and moreover* $f^+ = \theta\, r^{N/d}$ *and* $g^+ = \theta\, r^{M/d}$ *where* $GCD(M, N) = d$ *and* r *is a nonzero homogeneous polynomial of degree* d *in* $k[X, Y]$.

5. Homework and Research Problems

Here are some HPs = Homework Problems = known results, and some RPs = Research Problems = expected results which are not yet "known." A GOOD method of studying is to get a Research Problem ASAP and read around it. Problem 1 below is a jacobian related, i.e, a zero characteristic problem, while Problem 2 is a fundamental group related, i.e., a nonzero characteristic problem.

PROBLEM 1. Let $F = X + X^2Y$.

HP1: Show that F is not a member of an automorphic pair, or a jacobian pair, or an automorphic triple.

RP1: Show that F is not a member of a jacobian triple.

PROBLEM 2. Let $f(Y) = Y^n + a_1Y^{n-1} + \cdots + a_n$ in $k[[X_1, \ldots, X_t]]$ where k is an algebraically closed field of characteristic p which may or may not be 0. Let $G = \mathrm{Gal}(f, K)$ where $K = k((X_1, \ldots, X_t))$.

HP2.1: Show that, for $t = 1$ and $p = 0$, Newton's Theorem says that G is cyclic.

HP2.2: Show that, for $t = 1$ and $p \neq 0$, Generalized Newton's Theorem says that G has a unique p-Sylow subgroup H and G/H is cyclic.

HP2.3: Show that, for $t > 1$ and $p = 0$, if the Y-discriminant of f has a normal crossing then G is abelian.

HP2.4: Show that, for $t > 1$ and $p \neq 0$, if the Y-discriminant of f has a normal crossing then $G/p(G)$ is abelian where $p(G)$ is the subgroup of G generated by all its p-Sylow subgroups. By examples show that $p(G)$ may be unsolvable.

RP2.1: Show that, for $t = 2$, if the Y-discriminant of f is radicalwise analytically irreducible and has only one characteristic pair (m, n), with m and n nondivisible by p, then $G/p(G)$ is generated by two generators P and Q with the relation $P^m = Q^n$. Also show that any such group G can occur.

6. Detail

For more details see my books [Ab1], [Ab3], [Ab6], [Ab7], and my papers [Ab2], [Ab4], [Ab5], [Ab8], [Ab9]. Also see Zariski's book [Zar].

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