THE THEORY OF THE FLUCTUATIONS IN BRIGHTNESS OF THE MILKY WAY. I

S. Chandrasekhar and G. Münch
Yerkes Observatory
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ABSTRACT

In this paper the following statistical problem is considered. Let stars and interstellar clouds occur with a uniform distribution. Let the system extend to a linear distance L in the direction of a line of sight. Let a cloud reduce the intensity of the light of the stars immediately behind it by a factor q. Let the occurrence of clouds with a transparency factor q be governed by a frequency function $\psi(q)$. Given all this, it is required to find the probability distribution, $g(I, L)$, of the observed brightness, I. From a consideration of this problem it is shown that the following integral equation governs the distribution of brightness:

$$g(u, \xi)+\frac{\partial g}{\partial u}+\frac{\partial g}{\partial \xi}=\int_0^1 g\left(\frac{u}{q}, \xi\right)\psi(q)\frac{dq}{q},$$

where u is the observed brightness measured in suitable units and ξ is the average number of clouds in the direction of the line of sight. It is further shown that the foregoing integral equation enables us to obtain explicit formulae for all the moments of g as functions of ξ and the moments of $\psi(q)$. As an illustration of the use of these general formulae for the moments, an example investigated by Markarian has been reconsidered in an attempt to derive the mean and mean-square deviation of the optical thicknesses of the interstellar clouds.

1. *Introduction.*—The fact, now generally recognized, that interstellar matter occurs in the form of clouds and that the average number of such clouds intersected by a line of sight is of the order of 5 per kiloparsec requires a reorientation of the problems and objectives of stellar statistics. That such a reorientation is needed is brought out most clearly by the definiteness and the precision of the conclusions reached by Ambarzumian and his collaborators in three relatively brief investigations,[1] in which the cloud structure of interstellar matter is explicitly introduced as an essential element of the problem. Thus, in the investigation by Ambarzumian and Gordeladse, in which the observed association of emission and reflection nebulae is quantitatively accounted for on the hypothesis of a random distribution of stars and interstellar clouds by considering the volumes of space illuminated by stars of various spectral types, estimates are obtained for the average number of clouds per unit volume ($\sim 1.2 \times 10^{-4}$ per cubic parsec) and the average number of clouds intersected by a straight line (5–7 per kiloparsec). Similarly, from a simple analysis of the statistics of extragalactic nebulae, Ambarzumian deduced that the photographic absorption per cloud is of the order of 0.2 mag.; this, combined with the earlier estimate of the number of clouds in a line of sight, leads to a photographic extinction coefficient of 1.0–1.5 mag. per kiloparsec, which is in general agreement with other determinations of this quantity. The far-reaching nature of these conclusions—they were revolutionary at the time that they were drawn—should convince one that stellar statistics will gain enormously by making the distribution and the properties of interstellar clouds more immediate objectives of the investigation than has been customary so far. For example, the known fluctuations in the brightness of the Milky Way can be interpreted most readily in terms of the fluctuations in the numbers of the absorbing clouds in the line of sight; for, while other factors doubtless contribute to the observed fluctuations, these must be secondary to the effect of the fluctuations in the numbers of

[1] V. A. Ambarzumian and S. G. Gordeladse, *Bull. Abastumani Obs.*, No. 2, p. 37, 1938; V. A. Ambarzumian, *Bull. Abastumani Obs.*, No. 4, p. 17, 1940; and B. E. Markarian, *Contr. Burakan Obs. Acad. Sci. Armenian S.S.R.*, No. 1, 1946.

clouds, since so few of these are generally involved. Indeed, in a short note published in 1944, Ambarzumian[2] formulated the following problem which he considered basic for such an analysis:

Let stars and absorbing clouds occur with a uniform distribution in a plane of infinite extent. Further, let a cloud reduce the intensity of the light of the stars immediately behind it by a factor q. Let the occurrence of clouds with a "transparency" q be governed by a frequency function $\psi(q)$. What is, then, the probability distribution of the observed brightness?

Ambarzumian derived an integral equation for the required probability distribution and showed how its first and second moments can be expressed quite simply in terms of $\bar{q}$ and $\overline{q^2}$. However, when Markarian[1] came to applying this theory to observations, he found that Ambarzumian's assumption of the infinite extent of the system in the direction of the line of sight was too restrictive and that the problem must be considered for the case in which the average number of clouds in the line of sight is finite. The need for this generalization is apparent when we remember that the average number of clouds in the direction of galactic latitude β is n cosec β, where n is the corresponding number in the direction of the galactic pole; thus in our own galaxy $n \sim 3$ and n cosec $\beta \sim 10$ for $\beta = 20°$. Moreover, this dependence of the average number of clouds on the galactic latitude will provide a valuable check on the analysis.

Markarian did not derive the integral equation governing the distribution of brightness for the more general problem; but he did obtain explicit expressions for the first and the second moments for the case in which all the clouds are equally transparent. In this paper we shall derive the general integral equation governing the distribution of brightness and show how all its moments can be found. And we shall illustrate the use of these general relations for the moments by an example.

2. *The basic integral equation.*—Let I denote the observed brightness and L the distance of the observer to the limits of the system in the direction of the line of sight. Then

$$I = \int_0^L \prod_{i=1}^{n(s)} q_i \eta \, d s \,, \tag{1}$$

where η is the emission per unit volume by the stars assumed to be uniformly distributed, $n(s)$ is the number of clouds in the distance interval $(0, s)$ in the line of sight and is a chance variable, and $q_i\,[i = 1, 2, \ldots, n(s)]$ is the factor by which the ith cloud cuts down the intensity of the light from the stars immediately behind it. As we have already stated in § 1, we shall assume that the q's occur with a known frequency, $\psi(q)$.

If ν is the average number of clouds per unit distance, then $n(s)$ will be governed by the Poisson distribution,

$$e^{-\nu s} \frac{(\nu s)^n}{n!} \qquad (n = 0, 1, \ldots) \,, \tag{2}$$

having the variance νs.

The problem is to determine the probability distribution of I. It is convenient to reformulate this problem in dimensionless variables. For this purpose we shall let

$$u = I \frac{\nu}{\eta} \qquad \text{and} \qquad r = \nu s \,. \tag{3}$$

Then

$$u = \int_0^\xi \prod_{i=1}^{n(r)} q_i \, d r \,, \tag{4}$$

where

$$\xi = \nu L \tag{5}$$

[2] *C.R.* (*Doklady*) *Acad. Sci. URSS,* **14,** 223, 1944.

is the average number of clouds to be expected in the distance L. Also, according to formula (2), the occurrence of a particular number of clouds, n, in the interval $(0, r)$ will be governed by the Poisson distribution,

$$e^{-r}\frac{r^n}{n!} \qquad (n=0, 1, \ldots) . \tag{6}$$

Let $g(u, \xi)$ denote the frequency distribution of u for a given ξ. Since the q_i's are all less than, or equal to, 1, it follows from the definition of u as the integral (4) that u *can never exceed* ξ. Hence

$$g(u, \xi)=0 \qquad \text{for} \qquad u>\xi . \tag{7}$$

In addition to $g(u, \xi)$, it is convenient to define the probability that u exceeds a specified value. Let

$$f(u, \xi)=\int_u^\xi g(u, \xi)\, du \tag{8}$$

denote this probability. An integral equation governing $f(u, \xi)$ can be derived in the following manner:

By definition

$$f(u, \xi)=\text{Probability that} \int_0^\xi \prod_{i=1}^{n(r)} q_i d r \geqslant u . \tag{9}$$

Equivalently, we may also write

$$f(u, \xi)=\text{Probability that}\left\{\int_0^a \prod_{i=1}^{n(r)} q_i d r+\prod_{j=1}^{n(a)} q_j \int_a^\xi \prod_{i=1}^{n(r)-n(a)} q_i d r\right\} \geqslant u , \tag{10}$$

where a is an arbitrary positive constant $\leqslant \xi$. Replacing $r-a$ by r in the second integral on the right-hand side, we have

$$f(u, \xi)=\text{Probability that}\left\{\int_0^a \prod_{i=1}^{n(r)} q_i d r+\prod_{j=1}^{n(a)} q_j \int_0^{\xi-a} \prod_{i=1}^{n(r)} q_i d r\right\} \geqslant u . \tag{11}$$

Now let $a \ll 1$. Then up to $O(a^2)$, we have only two possibilities: either there is no cloud in the interval $(0, a)$, or there is just exactly one cloud. The probabilities of these two occurrences—again, up to $O(a^2)$—are $1-a$ and a, respectively. Hence

$$\begin{aligned}\prod_{i=1}^{n(a)} q_i = 1 &\text{ with probability } 1-a ,\\ \geqslant q \text{ and } \leqslant q+dq &\text{ with probability } a\psi(q)\, dq .\end{aligned} \tag{12}$$

We may, therefore, rewrite equation (11) in the form

$$\begin{aligned} f(u, \xi) = {} & a\int_0^1 dq\psi(q) \times \text{probability that}\left\{\theta a+q\int_0^{\xi-a}\prod_{i=1}^{n(r)} q_i d r\right\} \geqslant u \\ & +(1-a) \times \text{probability that}\left\{a+\int_0^{\xi-a}\prod_{i=1}^{n(r)} q_i d r\right\} \geqslant u+O(a^2) ,\end{aligned} \tag{13}$$

where $\theta \leqslant 1$ is some positive constant. Since we are neglecting all quantities of $O(a^2)$ and higher, it is clearly sufficient to evaluate the integral in equation (13) which occurs with the factor a to zero order in a. Thus

$$f(u, \xi) = a\int_0^1 dq\psi(q) \times \text{probability that}\left\{\int_0^{\xi}\prod_{i=1}^{n(r)} q_i dr\right\} \geqslant \frac{u}{q}$$
$$+ (1-a) \times \text{probability that}\left\{\int_0^{\xi-a}\prod_{i=1}^{n(r)} q_i dr\right\} \geqslant u-a+O(a^2); \quad (14)$$

or, remembering the definition of $f(u, \xi)$, we have

$$f(u, \xi) = a\int_0^1 dq\psi(q)\, f\left(\frac{u}{q}, \xi\right) + (1-a)\, f(u-a, \xi-a) + O(a^2). \quad (15)$$

Hence

$$f(u, \xi) = a\int_0^1 dq\psi(q)\, f\left(\frac{u}{q}, \xi\right) + f(u, \xi) - af(u, \xi) - a\frac{\partial f}{\partial u} - a\frac{\partial f}{\partial \xi} + O(a^2). \quad (16)$$

The function $f(u, \xi)$ must therefore satisfy the integral equation,

$$f + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \xi} = \int_0^1 \psi(q)\, f\left(\frac{u}{q}, \xi\right) dq. \quad (17)$$

Now, differentiating this equation with respect to u, we obtain the integral equation governing $g(u, \xi)$,

$$g(u, \xi) + \frac{\partial g}{\partial u} + \frac{\partial g}{\partial \xi} = \int_0^1 \psi(q)\, g\left(\frac{u}{q}, \xi\right)\frac{dq}{q}. \quad (18)$$

We have already pointed out that u cannot take values *exceeding* ξ. But it can take the value ξ itself with exactly the probability $e^{-\xi}$, since this is the probability that no cloud will occur in the interval $(0, \xi)$. The distribution of u has therefore a delta function, $\delta(u-\xi)$, at $u=\xi$ with an "amplitude" $e^{-\xi}$. Therefore, writing

$$g(u, \xi) = \phi(u, \xi) + e^{-\xi}\delta(u-\xi),$$

we find that the differential equation for ϕ is

$$\phi(u, \xi) + \frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial \xi} = \psi\left(\frac{u}{\xi}\right)\frac{e^{-\xi}}{\xi} + \int_{u/\xi}^1 \psi(q)\,\phi\left(\frac{u}{q}, \xi\right)\frac{dq}{q}.$$

In deriving this equation the assumption has been made that since

$$\left(1 + \frac{\partial}{\partial u} + \frac{\partial}{\partial \xi}\right) e^{-\xi} \times \text{a function of } (u-\xi) \equiv 0,$$

the same is also true of $e^{-\xi}\delta(u-\xi)$.

For the case in which the system extends to infinity in the direction of the line of sight, equation (18) reduces to the one given by Ambarzumian,[2] namely,

$$g(u) + \frac{dg}{du} = \int_0^1 \psi(q)\, g\left(\frac{u}{q}\right)\frac{dq}{q}. \quad (19)$$

Equation (18) is the basic equation of the problem.

3. *The moments of the distribution function $g(u, \xi)$.*—We shall now show how the integral equation (18) enables us to determine all the moments,

$$\mu_n = \int_0^{\xi} g(u, \xi)\, u^n du \qquad (n = 0, 1, \ldots)\,, \tag{20}$$

of the distribution function $g(u, \xi)$. But first we may note that, by definition,

$$\mu_0 = \int_0^{\xi} g(u, \xi)\, du \equiv 1\,; \tag{21}$$

all the other moments will, however, be functions of ξ.

Now, multiplying equation (18) by u^n and integrating over the range of u, we obtain

$$\mu_n + \int_0^{\xi} \frac{\partial g}{\partial u} u^n du + \frac{d\mu_n}{d\xi} = \int_0^1 dq\, q^n \psi(q) \int_0^{\xi} \frac{du}{q} \left(\frac{u}{q}\right)^n g\left(\frac{u}{q}, \xi\right). \tag{22}$$

An integration by parts reduces the integral on the left-hand side to $-n\mu_{n-1}$ if use is made of equation (7). Also, the integral on the right-hand side can be reduced in the following manner:

$$\int_0^1 dq\, q^n \psi(q) \int_0^{\xi} \frac{du}{q} \left(\frac{u}{q}\right)^n g\left(\frac{u}{q}, \xi\right)$$

$$= \int_0^1 dq\, q^n \psi(q) \int_0^{\xi/q} dx\, x^n g(x, \xi) \tag{23}$$

$$= \int_0^1 dq\, q^n \psi(q) \int_0^{\xi} dx\, x^n g(x, \xi) = \mu_n \int_0^1 dq\, q^n \psi(q)\,.$$

In the foregoing reductions use has again been made of the fact that $g(u, \xi) = 0$ for $u > \xi$. Thus equation (22) becomes

$$\frac{d\mu_n}{d\xi} + (1 - q_n)\, \mu_n = n\mu_{n-1} \qquad (n = 0, 1, \ldots)\,, \tag{24}$$

where, for the sake of brevity, we have written

$$q_n = \overline{q^n} = \int_0^1 dq\, q^n \psi(q)\,. \tag{25}$$

It may be noticed here that, by writing

$$\mu_n = \int_0^{\xi} \phi(u, \xi)\, u^n du + \xi^n e^{-\xi} = u_n + \xi^n e^{-\xi}$$

in equation (24), we find that the equation satisfied by u_n is

$$\frac{du_n}{d\xi} + (1 - q_n)\, u_n = q_n \xi^n e^{-\xi} + n u_{n-1}\,.$$

But this same differential equation also follows directly from the equation satisfied by ϕ.

It is evident that all the moments μ_n must vanish at $\xi = 0$. On the other hand, from equation (22) for $n = 2$, namely,

$$\frac{d\mu_2}{d\xi} + (1 - q_2)\, \mu_2 = 2\mu_1\,, \tag{26}$$

we conclude that $d\mu_2/d\xi$ also vanishes at $\xi = 0$. And by induction it follows quite generally from equation (24) that μ_n *and its first* $(n - 1)$ *derivatives must vanish at* $\xi = 0$. Also, the μ_n's must be bounded for $\xi \to \infty$. As we shall see presently, these boundary conditions suffice to solve the system of equations (24) uniquely.

By successive applications of equation (24) for $n = 1, 2$, etc., we obtain

$$\left[\prod_{j=1}^{n}\left(\frac{d}{d\xi}+a_j\right)\right]\mu_n = n! , \tag{27}$$

where

$$a_j = 1 - q_j . \tag{28}$$

The solution of equation (27) which satisfies the boundary condition at $\xi = \infty$ is

$$\mu_n = \sum_{k=1}^{n} A_k e^{-a_k\xi} + \frac{n!}{\prod_{j=1}^{n} a_j} , \tag{29}$$

where the A_k's$(k = 1, \ldots, n)$ are n constants of integration.

The boundary conditions,

$$\mu_n = 0 \quad \text{and} \quad \frac{d^j\mu_n}{d\xi^j} = 0 \qquad (j = 1, \ldots, n-1) , \tag{30}$$

now require that

$$\sum_{k=1}^{n} A_k = -\frac{n!}{\prod_{j=1}^{n} a_j}$$

and

$$\sum_{k=1}^{n} A_k a_k^j = 0 \qquad (j = 1, \ldots, n-1) . \tag{31}$$

The matrix of the system of equations (30) is seen to be the Vandermondie matrix;[3] its reciprocal is the matrix[4]

$$\left\| \frac{S_{n-l-1,\, r}}{\prod_{j\neq r}^{(1,\, n)} (a_r - a_j)} \right\| , \tag{32}$$

in which the $S_{n-l-1,\, r}$'s $(l = 0, 1, \ldots, n-1;\ r = 1, \ldots, n)$ are the independent symmetric functions in the $(n - 1)$ variables $(a_1, \ldots, a_{r-1};\ a_{r+1}, \ldots, a_n)$,

$$S_{0,\, r} = 1 ; \qquad S_{1,\, r} = -\sum_{j\neq r}^{(1,\, n)} a_r ; \ldots ; \quad S_{n-1,\, r} = (-1)^n \prod_{j\neq r}^{(1,\, n)} a_j . \tag{33}$$

The constants A_k are therefore given by

$$A_k = -\frac{S_{n-1,\, k}}{\prod_{j\neq k}^{(1,\, n)} (a_k - a_j)} \frac{n!}{\prod_{j=1}^{n} a_j} = \frac{(-1)^n n!}{a_k \prod_{j\neq k}^{(1,\, n)} (a_k - a_j)} \qquad (k = 1, \ldots, n) . \tag{34}$$

[3] Cf. O. Perron, *Algebra* (Leipzig: De Gruyter, 1932), **1**, No. 22, 92–94.

[4] Cf. S. Chandrasekhar, *Ap. J.*, **101**, 328, 1945, esp. eqs. (75) and (81).

Reverting to the variables $1 - q_j$ (eq. [28]), we have

$$A_k = \frac{n!}{\prod\limits_{j \neq k}^{(0,\,n)} (q_k - q_j)} \qquad (k = 1, \ldots, n) \,. \tag{35}$$

Extending definition (35) also to $k = 0$, we can write the solution for μ_n in the form

$$\mu_n = n! \sum_{k=0}^{n} \frac{e^{-(1-q_k)\xi}}{\prod\limits_{j \neq k}^{(0,\,n)} (q_k - q_j)} \qquad (n = 1, \ldots) \,. \tag{36}$$

In particular, for $n = 1$ and 2 we have

$$\mu_1 = \frac{1}{1 - q_1} [1 - e^{-(1-q_1)\xi}]$$

and

$$\mu_2 = \frac{2}{(1 - q_1)(1 - q_2)} + \frac{2 e^{-(1-q_1)\xi}}{(q_1 - 1)(q_1 - q_2)} + \frac{2 e^{-(1-q_2)\xi}}{(q_2 - 1)(q_2 - q_1)} \,. \tag{37}$$

If all the clouds are characterized by the same value of q, then

$$q_j = q^j \,, \tag{38}$$

and equations (37) reduce to the ones given by Markarian.[5]

4. *A direct proof of the relations satisfied by the moments.*—It is of interest to verify that the relations between the moments of g obtained in § 3 and, in particular, the differential equation (eq. [24]) governing them are deducible directly from the definition of u as the integral (4). For this purpose we first establish the following lemma:

Lemma.—Let $f(x_1, \ldots, x_n)$ be a symmetrical function in the n variables $x_1, \ldots, x_n$. Then

$$\begin{aligned} I_n = \int_b^a dx_n \int_b^a dx_{n-1} \int_b^a dx_{n-2} \ldots \int_b^a dx_1 f(x_1, \ldots, x_n) \\ = n! \int_b^a dx_n \int_{x_n}^a dx_{n-1} \int_{x_{n-1}}^a dx_{n-2} \ldots \int_{x_2}^a dx_1 f(x_1, \ldots, x_n) \,. \end{aligned} \tag{39}$$

Proof.—First, we verify that the lemma is true for two variables; this is very readily done. Next we show that, on the assumption that the lemma is true for all multiple integrals of $(n - 1)$ and lower folds, the truth of the lemma for n-fold integrals can be deduced. The general truth of the lemma would then follow by induction.

Considering, then, the n-fold integral I_n and transforming the $(n - 1)$-fold integral over $x_{n-1}, x_{n-2}, \ldots, x_1$ in accordance with the lemma, we have

$$I_n = (n-1)! \int_b^a dx_n \int_b^a dx_{n-1} \int_{x_{n-1}}^a dx_{n-2} \ldots \int_{x_2}^a dx_1 f(x_1, \ldots, x_n) \,. \tag{40}$$

Splitting the range of integration over x_{n-1} from a to x_n and x_n to b, we have

$$I_n = (n-1)! \int_b^a dx_n \int_{x_n}^a dx_{n-1} \int_{x_{n-1}}^a dx_{n-2} \ldots \int_{x_2}^a dx_1 f(x_1, \ldots, x_n) + J \,, \tag{41}$$

[5] *Op. cit.*, eqs. (10) and (13).

where

$$J=(n-1)!\int_b^a dx_n\int_b^{x_n} dx_{n-1}\int_{x_{n-1}}^a dx_{n-2}\dots\int_{x_2}^a dx_1 f(x_1,\dots,x_n)\,. \tag{42}$$

Now, inverting the order of the integration over x_n and x_{n-1} in J and using the symmetry of $f(x_1, \dots, x_n)$ in x_{n-1} and x_n, we have

$$\begin{aligned} J&=(n-1)!\int_b^a dx_{n-1}\int_{x_{n-1}}^a dx_n\int_{x_{n-1}}^a dx_{n-2}\dots\int_{x_2}^a dx_1 f(x_1,\dots,x_n)\\ &=(n-1)!\int_b^a dx_n\int_{x_n}^a dx_{n-1}\int_{x_n}^a dx_{n-2}\int_{x_{n-2}}^a dx_{n-3}\dots\int_{x_2}^a dx_1 f(x_1,\dots,x_n)\,. \end{aligned} \tag{43}$$

The $(n-2)$-fold integral over $x_{n-2}, x_{n-3}, \dots, x_1$ in this last expression for J can be transformed in accordance with the converse form of the lemma for $(n-2)$ variables and leads to

$$J=(n-1)\int_b^a dx_n\int_{x_n}^a dx_{n-1}\int_{x_n}^a dx_{n-2}\int_{x_n}^a dx_{n-3}\dots\int_{x_n}^a dx_1 f(x_1,\dots,x_n)\,. \tag{44}$$

By a further application of the lemma for the $(n-1)$ variables $x_{n-1}, x_{n-2}, \dots, x_1$, we obtain

$$J=(n-1)(n-1)!\int_b^a dx_n\int_{x_n}^a dx_{n-1}\int_{x_{n-1}}^a dx_{n-2}\dots\int_{x_2}^a dx_1 f(x_1,\dots,x_n)\,. \tag{45}$$

With J given by equation (45), equation (41) becomes

$$I_n=n!\int_b^a dx_n\int_{x_n}^a dx_{n-1}\int_{x_{n-1}}^a dx_{n-2}\dots\int_{x_2}^a dx_1 f(x_1,\dots,x_n)\,; \tag{46}$$

and this verifies the lemma for n variables. The general truth of the lemma therefore follows by induction.

An immediate corollary of the lemma is

$$I_n=n\int_b^a dx_n\int_{x_n}^a dx_{n-1}\int_{x_n}^a dx_{n-2}\dots\int_{x_n}^a dx_1 f(x_1,\dots,x_n)\,. \tag{47}$$

This alternative expression for I_n follows from an application of the lemma in the converse form to the $(n-1)$-fold integral over $x_{n-1}, x_{n-2}, \dots, x_1$ in equation (46).

Returning to the problem of deriving the differential equation (24) connecting the moments of g, we first observe that, by definition,

$$\mu_m=\left[\int_0^\xi \prod_{i=1}^{n(r)} q_i\,dr\right]^m_{\text{average}}\,. \tag{48}$$

Alternatively, we can write

$$\mu_m=\int_0^\xi dr_m\int_0^\xi dr_{m-1}\int_0^\xi dr_{m-2}\dots\int_0^\xi dr_1\left\{\prod_{j=1}^{m}\prod_{i=1}^{n(r_j)} q_i\right\}_{\text{average}}\,. \tag{49}$$

The integrand of this m-fold integral is clearly a symmetrical function of the variables.

Accordingly, using the lemma in the form (47), we can rewrite the foregoing expression for μ_m in the form

$$\mu_m = m\int_0^{\xi} d\,r_m \int_{r_m}^{\xi} d\,r_{m-1} \int_{r_m}^{\xi} d\,r_{m-2} \ldots \int_{r_m}^{\xi} d\,r_1 \left\{ \prod_{j=1}^{m} \prod_{i=1}^{n(r_j)} q_i \right\}_{\text{average}} . \tag{50}$$

With the integration over the variables carried out in this fashion,

$$r_j \geqslant r_m \qquad \text{for all} \qquad j \leqslant m-1 \,. \tag{51}$$

Under these circumstances

$$n\,(r_j) \;-n\,(r_m) \;=n\,(r_j - r_m)\,, \tag{52}$$

and

$$\prod_{j=1}^{m} \prod_{i=1}^{n(r_j)} q_i = \left\{ \prod_{i=1}^{n(r_m)} q_i^m \right\} \left\{ \prod_{j=1}^{m-1} \prod_{i=1}^{n(r_j - r_m)} q_i \right\} . \tag{53}$$

In view of the inequality (51), it is evident that the occurrence of clouds in the interval $(0, r_m)$ is uncorrelated with the occurrence of clouds in any of the intervals $r_j - r_m$ $(j = m-1, \ldots, 1)$. Hence

$$\left\{ \prod_{j=1}^{m} \prod_{i=1}^{n(r_j)} q_i \right\}_{\text{average}} = \left\{ \prod_{i=1}^{n(r_m)} q_i^m \right\}_{\text{average}} \times \left\{ \prod_{j=1}^{m-1} \prod_{i=1}^{n(r_j - r_m)} q_i \right\}_{\text{average}} . \tag{54}$$

Using this result in equation (50), we have

$$\mu_m = m\int_0^{\xi} d\,r_m \left\{ \prod_{i=1}^{n(r_m)} q_i^m \right\}_{\text{average}} \times \int_{r_m}^{\xi} d\,r_{m-1} \int_{r_m}^{\xi} d\,r_{m-2} \ldots \int_{r_m}^{\xi} d\,r_1 \times \left\{ \prod_{j=1}^{m-1} \prod_{i=1}^{n(r_j - r_m)} q_i \right\}_{\text{average}} . \tag{55}$$

Now, writing r_j in place of $r_j - r_m (j = m-1, \ldots, 1)$ in equation (55), we have

$$\mu_m = m\int_0^{\xi} d\,r \left\{ \prod_{i=1}^{n(r)} q_i^m \right\}_{\text{average}} \times \int_0^{\xi - r} d\,r_{m-1} \int_0^{\xi - r} d\,r_{m-2} \ldots \int_0^{\xi - r} d\,r_1 \times \left\{ \prod_{j=1}^{m-1} \prod_{i=1}^{n(r_j)} q_i \right\}_{\text{average}} , \tag{56}$$

where, for brevity, we have suppressed the subscript m in r_m. The $(m-1)$-fold integral in equation (56) is clearly $\mu_{m-1}(\xi - r)$. Accordingly, we may rewrite equation (56) in the form

$$\mu_m = m\int_0^{\xi} d\,r \left\{ \prod_{i=1}^{n(r)} q_i^m \right\}_{\text{average}} \mu_{m-1}\,(\xi - r)\,. \tag{57}$$

On the other hand (cf. eqs. [6] and [25]),

$$\left\{\prod_{i=1}^{n(r)} q_i^m\right\}_{\text{average}} = \sum_{n=0}^{\infty} \frac{e^{-r} r^n}{n!} \prod_{i=1}^{n} \int_0^1 d q_i q_i^m \psi(q_i)$$
$$= \sum_{n=0}^{\infty} e^{-r} \frac{(r q_m)^n}{n!} = e^{-r(1-q_m)} . \quad (58)$$

Hence

$$\mu_m = m \int_0^{\xi} d r e^{-r(1-q_m)} \mu_{m-1}(\xi - r) . \quad (59)$$

But this is the integrated form of equation (24) when the boundary condition $\mu_m = 0$ at $\xi = 0$ is also satisfied. The equations and boundary conditions from which solution (36) for the moments was derived in § 3 have now been obtained directly from the definition of u.

5. *An application of the formulae for the moments μ_n to derive certain statistical properties of the interstellar clouds.*—As an illustration of the application of the formulae for the moments derived in § 3, we shall reconsider an example investigated by Markarian,[1] on the assumption that all the interstellar clouds are equally transparent.

Markarian's example is based principally on van Rhijn's tabulation of the counts[6] of the number of stars $N_m(\beta, \lambda)$, per square degree, brighter than a given apparent magnitude m, and in a region of the sky centered at galactic latitude β and galactic longitude λ. In terms of these numbers[7] Markarian evaluated the quantity

$$I(\beta, \lambda) = \sum_m \{N_{m+1}(\beta, \lambda) - N_m(\beta, \lambda)\} \times 10^{-0.4m} , \quad (60)$$

as a function of β and λ. This quantity, $I(\beta, \lambda)$, is clearly a measure of the brightness of the Milky Way in the region considered. It is, therefore, comparable, apart from a constant of proportionality, to I as we have defined it in equation (1).

In a detailed investigation it will be necessary to compare the observed fluctuations of the quantity (60) from the mean with the theoretical distributions derived on the basis of the integral equation (18). However, in a first attempt, it may suffice to restrict ourselves to the dispersion of the brightness about the mean.

Since we may expect the average number of absorbing clouds in the direction of galactic latitude β to vary with β as cosec β, it is clear that, in determining the dispersion of $I(\beta, \lambda)$ about the mean, we must treat the regions with different β's, separately. Thus, denoting the dispersion in the brightness of the Milky Way at galactic latitude β as $\delta^2(\beta)$, we have

$$\delta^2(\beta) = \frac{\text{Mean}\{I^2(\beta, \lambda)\}}{[\text{Mean}\{I(\beta, \lambda)\}]^2} - 1 , \quad (61)$$

where, in taking the means, β is kept constant.

In the investigation we have quoted, Markarian has derived values for the dispersion $\delta^2(\beta)$, according to equation (61) for those values of β for which van Rhijn's (Groups I–IV) and Baker and Kiefer's (Groups V–VII) tables permit a determination. His final results are summarized in Table 1.

Now, on the model of the distribution of stars and clouds adopted in this paper (§ 1), the value of $\delta^2(\beta)$ should be compared with the theoretical quantity,

$$\delta^2(\xi) = \frac{\mu_2}{\mu_1^2} - 1 . \quad (62)$$

[6] *Groningen Pub.*, No. 43, 1924. Markarian has also used the data given in R. H. Baker and L. Kiefer, *Ap. J.*, **94**, 482, 1941.

[7] Since the areas actually used in van Rhijn's tabulation extend over an appreciable range of β, Markarian had to reduce the observed numbers to a mean latitude β by applying a correction based on the observed mean variation of $N_m(\beta, \lambda)$ with β.

With μ_1 and μ_2 given by equations (37), we have

$$\delta^2(\xi) = \frac{2(1-q_1)}{(1-q_2)(q_1-q_2)} \frac{(1-q_2)[1-e^{-(1-q_1)\xi}] - (1-q_1)[1-e^{-(1-q_2)\xi}]}{[1-e^{-(1-q_1)\xi}]^2} - 1 , \quad (63)$$

where it may be recalled that ξ is the average number of clouds in the direction of the line of sight and q_1 and q_2 are the mean and the mean square of the transparency factor q of the clouds:

$$q_1 = \int_0^1 q\psi(q)\,dq \quad \text{and} \quad q_2 = \int_0^1 q^2\psi(q)\,dq . \quad (64)$$

While q_1 and q_2 occur as two independent parameters in equation (63), it is evident that,

TABLE 1

RESULTS OF MARKARIAN'S ANALYSIS OF $N_m(\beta, \lambda)$

	GROUP						
	I	II	III	IV	V	VI	VII
β	0°	±10°	±30°	±40°	−10°	±10°	0°
λ	100°	100°	100°	100°	160°	130°	130°
No. of regions used	11	12	10	11	10	9	11
$\delta^2(\beta)$	0.092	0.075	0.030	0.020	0.082	0.100	0.126

by virtue of definitions (64), our freedom in assigning values to q_2 for a given q_1 is strictly limited; for the inequality

$$q_1^2 \leqslant q_2 < q_1 \quad (65)$$

is a consequence of the definitions of these quantities only. And, moreover, the equality between q_1^2 and q_2 can occur only when all the clouds are equally transparent with a factor q_1. It may also be noted in this connection that, according to equation (63),

$$\delta^2(\xi) \to \frac{(1-q_1)(q_1-q_2)}{1-q_2} \quad \text{as} \quad \xi \to \infty$$

and

$$\delta^2(\xi) \to 0 \quad \text{as} \quad \xi \to 0 . \quad (66)$$

For a comparison of the observed values of $\delta^2(\beta)$ with the theoretical predictions based on equation (63), we require a knowledge of the three parameters, ξ, q_1, and q_2, which the theoretical expression for $\delta^2(\xi)$ involves. However, of the three parameters, ξ and q_1 are not independent if we make use of the results of the counts of extragalactic nebulae.[8] According to these latter counts, the mean photographic absorption, $\Delta m(\beta)$, in the direction of galactic latitude β is given by

$$\Delta m(\beta) = 0^m\!.25 \operatorname{cosec} \beta . \quad (67)$$

If τ is the mean absorption, in magnitudes, per cloud, the number of clouds to be expected, on the average, in the direction β is

$$\xi = \frac{0^m\!.25\,|\operatorname{cosec} \beta|}{\tau} . \quad (68)$$

[8] E. P. Hubble, *Ap. J.*, 79, 8, 1934.

But, by definition, τ is related to $\bar{q} = q_1$ by the equation

$$\tau = -2.5 \log q_1 . \tag{69}$$

Hence

$$\xi(\beta) = -0.1 \frac{|\operatorname{cosec} \beta|}{\log q_1} . \tag{70}$$

Consequently, for any assigned value of q_1, we can determine the average number of clouds in the direction β. Values of ξ derived in this fashion for three assigned values of q_1 (0.75, 0.80, and 0.85) are listed in Table 2.

TABLE 2

AVERAGE NUMBER OF CLOUDS IN DIRECTION OF GALACTIC LATITUDE β FOR THREE ASSIGNED VALUES OF q_1

q_1	τ (MAG.)	ξ			
		$\beta=0°$	$\beta=\pm 10°$	$\beta=\pm 30°$	$\beta=\pm 40°$
0.75	0.31	∞	4.61	1.60	1.24
.80	.24	∞	5.94	2.06	1.61
0.85	0.18	∞	8.15	2.83	2.20

In interpreting his deduced values of $\delta^2(\beta)$, Markarian made the assumption that all the clouds are equally transparent. On this assumption, $q_2 = q_1^2$, and $\delta^2(\beta)$ becomes determinate when q_1 is given. However, the expression for $\delta^2(\xi)$, which allows for an arbitrary distribution of q, involves the additional parameter q_2. Accordingly, using equation (63), we have computed $\delta^2(\xi)$ for various values of q_2 and for $q_1 = 0.75$, 0.80, and 0.85, respectively. The results of the calculation are illustrated in the form of curves in Figure 1, *a* ($q_1 = 0.75$), *b*($q_1 = 0.80$), and *c*($q_1 = 0.85$).

To appreciate what latitude we have for changing q_2 for a given q_1 and what a difference in q_2 from q_1^2 means in terms of the distribution of q, we may note that, for the frequency function,

$$\psi(q) = (n+1) q^n \qquad (0 \leqslant q \leqslant 1) , \tag{71}$$

$$q_1 = \frac{n+1}{n+2} , \qquad q_2 = \frac{n+1}{n+3} , \qquad \text{and} \qquad \frac{q_2}{q_1^2} - 1 = \frac{1}{(n+3)(n+1)} . \tag{72}$$

In particular, for $n = 5$,

$$q_1 = 0.857 , \qquad q_1^2 = 0.735 , \qquad q_2 = 0.75 , \qquad \text{and} \qquad \frac{q_2}{q_1^2} - 1 = 0.021 . \tag{73}$$

Accordingly, for $q_1 = 0.85$, a change of q_2 from $(0.85)^2 = 0.7225$ to 0.7325 is by no means an unduly large change. Bearing this in mind, we conclude from an examination of the curves in Figure 1 that the predicted variation of $\delta^2(\xi)$ for a given q_1 depends rather sensitively on q_2; indeed, it would appear that relatively slight changes in q_2 (keeping q_1 fixed) affect $\delta^2(\xi)$ almost as much as quite appreciable changes in q_1 (keeping $q_2 = q_1^2$). This is a somewhat unexpected result disclosed by the present analysis.

In Figure 1, *a*, *b*, and *c* we have plotted the observed values of $\delta^2(\beta)$ against the ξ's appropriate for the values of q_1 to which each of the figures refers (cf. Table 2). It is seen that, with the present data, we cannot distinguish in a unique manner the different effects of q_1 and q_2. However, it does appear that

$$q_1 = 0.85 \qquad \text{and} \qquad q_2 = 0.7325 \tag{74}$$

give the best fit with the observations. It is of interest to recall in this connection that, on the balance of all the evidence available, Markarian favored[9] the acceptance of a value of $q_1 = 0.85$, though the agreement of his observed points with the curve $q_1 = 0.75$ and $q_2 = q_1^2 = 0.5625$ is definitely better than with the curve $q_1 = 0.85$ and $q_2 = q_1^2 = 0.7225$ (cf. Fig. 1, *a* and *c*).

While the values derived for q_1 and q_2 (eq. [74]) are uncertain, it is nevertheless of some interest to observe that these values correspond to a root-mean-square deviation of 0.1 in q. A variation in q of this amount (i.e., ± 0.1) about the mean value $q_1 = 0.85$ corresponds to a variation in the true optical thicknesses of interstellar clouds in the range

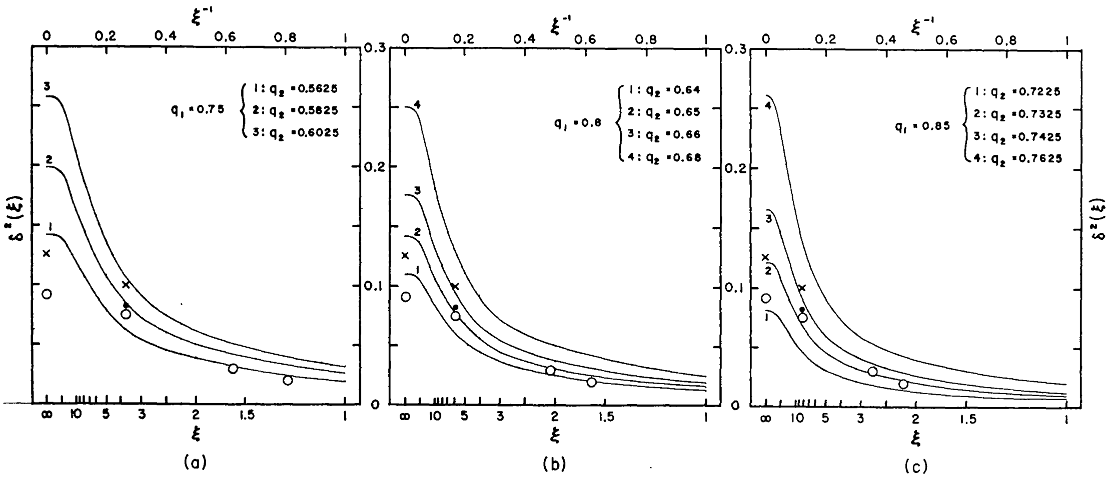

FIG. 1.—The transparency factor q of the interstellar clouds as derived from the observed dispersion in the brightness of the Milky Way at various galactic latitudes. The curves represent the predicted variation of the dispersion $\delta^2(\xi)$ with the average number of absorbing clouds, ξ, in the line of sight for various values of the mean (q_1) and mean square (q_2) of the transparency factor of the clouds. The different sets of curves are for different values of q_1 (0.75 in *a;* 0.80 in *b;* and 0.85 in *c*), the parameter distinguishing the curves in each set being q_2. The lowest curve in each is for the case in which all the clouds are equally transparent and $q_2 = q_1^2$.

The dispersion of the observed brightness of the Milky Way, $\delta^2(\beta)$, at various galactic latitudes can be represented as a variation with ξ if a value of q_1 is assumed (cf. eq. [70]). The values of $\delta^2(\beta)$ deduced by Markarian are plotted in the figure against the ξ's appropriate for each figure (cf. Table 2). The crosses, the open circles, and the solid dots refer to the regions centered at $\lambda = 130°$, $\lambda = 100°$, and $\lambda = 160°$, respectively.

0.29 and 0.05; a variation in the absorptive power of clouds of this amount is entirely reasonable.

Again, if we assume that the average photographic extinction coefficient is 2 mag. per kiloparsec,[10] then we shall need an average of 10–11 clouds per kiloparsec. This estimate is not necessarily in discord with the usual estimate[11] of 7 clouds per kiloparsec; for it may be estimated that a dispersion of 0.1 in q means that about three-fourths of all the clouds (i.e., 7 or 8 in the present instance) will have values in the range 0.75–0.95 and it is possible that the few dense clouds, recognized as such, are not included in the general survey. In any case the present rediscussion of Markarian's example would seem to indicate that a great deal of information concerning the interstellar clouds can be derived by an extension of the basic observational material and their discussion along the lines of this paper.

[9] This value of q_1 is also favored by L. Spitzer, *Ap. J.*, **108**, 276, 1948, esp. p. 278.

[10] H. van de Hulst, *Rech. Astr. Obs. Utrecht*, Vol. **11**, Part II, 1949.

[11] Cf. Spitzer, *op. cit.*; and B. Strömgren, *Ap. J.*, **108**, 242, 1948.