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Proc. R. Soc. A 2007 **463**, 369-383
doi: 10.1098/rspa.2006.1768

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doi:10.1098/rspa.2006.1768
Published online 20 September 2006

Theory of free surface flow over rough seeping beds

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A new theory is developed for the steady free surface flow over a horizontal rough bed with uniform upward seepage normal to the bed. The theory is based on the Reynolds averaged Navier–Stokes (RANS) equations applied to the flow domain that is divided into a fully turbulent outer layer and an inner layer (viscous sublayer plus buffer layer), which is a transition zone from viscous to turbulent regime. In the outer layer, the Reynolds stress far exceeds viscous shear stress, varying gradually with vertical distance. Near the free surface, the velocity gradient in vertical direction becomes lesser giving rise to wake flow. On the other hand, in the composite inner layer close to the bed, the viscous shear stress exists together with the turbulent stress. Thus, for the outer layer, a logarithmic law having modified coefficients from the traditional logarithmic law is obtained for the streamwise velocity, whereas for the inner layer, a fifth-degree polynomial including effective height of protrusions holds. The exact velocity expressions for inner and outer layer, which contain principal terms in addition to infinitesimally small terms, are in agreement with the experimental data obtained from laboratory measurements through an acoustic Doppler velocimeter. The experiments were run on two conditions of no seepage and a low upward seepage. Expressions for the Reynolds stress are also derived and computed for validation by the experimental data.

Keywords: open channel flow; turbulent flow; steady flow; non-uniform flow; seepage; hydraulics

1. Introduction

The theory of turbulent shear flow of an incompressible fluid over smooth or rough boundaries, such as free surface flow and closed conduit flow, is a problem of primary importance in turbulence. The Reynolds averaged Navier–Stokes (RANS) equations for the simple type of turbulent shear flow contain time-averaged flow velocity, piezometric pressure and turbulent fluctuations, but the system is short of one equation for its solution. This is usually supplemented by an additional physical theory like *Prandtl's mixing-length theory*. It is concluded that in the fully developed turbulent flow, the *law of the wall* for streamwise flow velocity is a logarithmic law in depth. There are several competing theories for

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Received 19 March 2006
Accepted 3 August 2006

the layer adjacent to the wall (Stanišić 1984). A particular simple theory, by Tien & Wasan (1963) (see Stanišić 1984), assumes differentiability of the streamwise flow velocity at any point to a few orders with respect to the depth, so that the result matches with the logarithmic law.

Free surface flow over a permeable sedimentary bed is important in hydraulics. In general, the streamwise flow coupled with the seepage from (injection) or into (suction) the bed is a common occurrence. The seepage can be normal or tangential to the bed. Normal seepage has the potential to influence the streamwise velocity distribution even in the outer region of flow. Hence, it is questionable whether the law of the wall for velocity distribution remains valid under seepage condition. Based on Prandtl's mixing-length theory, Clarke *et al.* (1955) and Stevenson (1963) put forward a modified law of the wall to describe the velocity distribution with injection from the boundary. Willetts & Drossos (1975) proposed an exponential law for the velocity distribution over a bed with suction. Maclean (1991*a*) observed through a series of experiments that the velocity distribution over a bed with suction consists of a suction boundary layer. To evaluate the bed shear stress, Maclean (1991*b*) studied the threshold motion of indicator grains with predetermined threshold shear stress. For downward (suction) seepage, Oldenziel & Brink (1974) and Maclean (1991*b*) experimentally found that the streamwise velocity decreases in the outer region and increases near the bed. Prinos (1995) studied the effect of bed suction on turbulent (free surface) flow field by numerically solving the RANS equations. On the other hand, the modified logarithmic laws for the velocity distributions subjected to upward and downward seepages were proposed by Cheng & Chiew (1998) and Chen & Chiew (2004), respectively. Recently, Dey & Cheng (2005) derived the Reynolds stress profile in non-uniform unsteady flow over a bed having upward seepage.

In presenting a new theory of free surface flow with uniform upward seepage, the present study deviates from the traditional method of assuming additional physical theory like Prandtl's mixing-length theory, which is not applicable in the presence of seepage (Tennekes & Lumley 1972). It is hypothesized that in the fully turbulent outer layer, the Reynolds stress varying gradually with the vertical distance dominates the viscous shear stress. Using this concept in the RANS and adopting the methodology of Tien & Wasan (1963), the equation of streamwise velocity is determined in the inner layer (viscous sublayer plus buffer layer). The Reynolds stress is then derived from the developed theory of two layers. Experiments were conducted to measure the velocity and turbulence by the acoustic Doppler velocimeter (ADV) for validation of the theory.

2. The RANS equations

Under consideration is a unidirectional streamwise flow of nearly constant flow depth H with low uniform upward seepage v_0 from the horizontal permeable bed into the main flow. To model the flow field, the horizontal bed is assumed to be rough consisting of sediment particles, as shown schematically in figure 1*a*. Any convenient point on the bed is taken as the origin O. The streamwise flow in the direction of the x-axis, assumed to be fully developed, becomes two dimensional owing to injection from the bed. According to the Reynolds decomposition, the (*a*) (*b*)

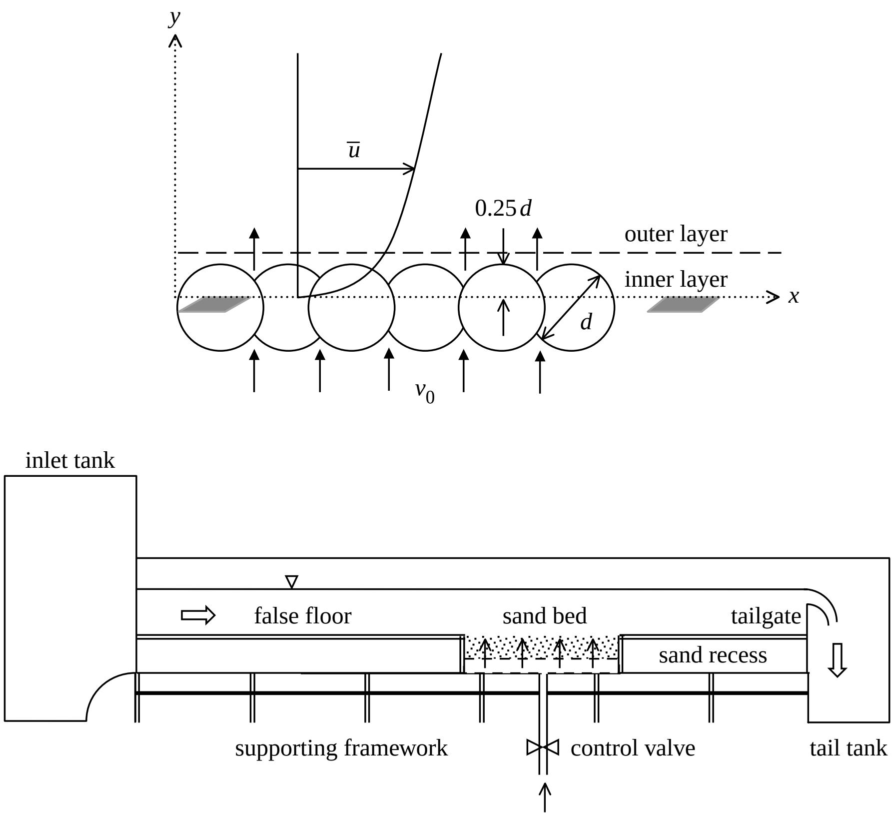

Figure 1. (*a*) Definition sketch of steady flow over a sedimentary bed with constant upward seepage and (*b*) schematic of laboratory experimental set-up.

instantaneous velocity components (u, v) are split into time-averaged part $(\bar{u}, \bar{v})$ and fluctuation part (u', v') as

$$u = \bar{u}(x, y) + u'(x, y, t), \qquad v = \bar{v}(x, y) + v'(x, y, t), \tag{2.1}$$

where x and y are the streamwise and normal distances, respectively, and t is the time. For steady flow, the continuity equations of time-averaged velocity and fluctuation components are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \qquad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \tag{2.2}$$

and the RANS equations are

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial \tau}{\partial y} + \upsilon\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial\left(\overline{u'^2}\right)}{\partial x}, \tag{2.3a}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = -\frac{\partial \bar{p}}{\partial y} + \frac{\partial \tau}{\partial x} + \upsilon\frac{\partial^2 \bar{v}}{\partial x^2} - \frac{\partial\left(\overline{v'^2}\right)}{\partial y}, \tag{2.3b}$$

where $\bar{p}(x, y)$ is the time-averaged piezometric pressure relative to the mass density of fluid ρ, $\tau(x, y)$ is the Reynolds stress relative to ρ, i.e. $-\overline{u'v'}$ and υ is the kinematic viscosity.

Equations (2.2)–(2.3*b*) form an undetermined system, since there are six dependent parameters $\bar{u}$, $\bar{v}$, u', v', $\bar{p}$ and τ. Even if one assumes that flow takes place under a given pressure gradient, the system remains undetermined. Focusing on the stresses, it is pertinent to note that since kinematic viscosity υ is small, the viscous stress becomes negligible in the outer layer compared with the Reynolds stress, while near the bed both the stresses may exist for the low flow velocity. Thus, the flow zone can be considered to be consisting of (i) fully turbulent outer layer with negligible viscous stress and (ii) inner layer (viscous sublayer plus buffer layer), where both the Reynolds and viscous stresses prevail subject to appropriate boundary conditions. Under suitable mathematical representation, the two-zone flow solutions are constructed satisfying the continuity of flow at the interface, as was done by Tien & Wasan (1963) (see Stanišić 1984).

The mathematical analysis of this study is characterized by four local scales (Tennekes & Lumley 1972). They are (i) horizontal length-scale conservatively taken as flow depth H (assuming fully developed flow), (ii) velocity scale as shear velocity u_τ, (iii) vertical length-scale as υ/u_τ (used for transitional flow regime), and (iv) mass-transfer (through the wall) velocity scale as seepage velocity v_0. Importantly, in the smooth flow regime, the shear Reynolds number Re_* $(=\lambda u_\tau/\upsilon$, where λ is the equivalent roughness or protrusion height) being less than 4, the protrusions are well submerged by the viscous sublayer and the corresponding length-scale is υ/u_τ. On the other hand, in the rough flow regime ($Re_*>70$), the protrusions are fully exposed (beyond the viscous sublayer) to the main flow and the corresponding length-scale is λ. However, in the transitional flow regime ($4<Re_*\leq 70$), protrusions are neither fully exposed to the main flow nor fully submerged by the viscous sublayer. Though the effects of both viscosity and roughness prevail on the main flow, its convenient length-scale is υ/u_τ (Reichardt 1951). The present study corresponds to the transitional flow regime (see table 2) and, therefore, length-scale is considered as υ/u_τ.

3. Flow in fully turbulent outer layer

It is assumed that the turbulent flow is fully developed in the outer layer governed by equations (2.1)–(2.3*b*). The flow parameters vary over x and y owing to seepage velocity v_0 and kinematic viscosity υ, respectively. Thus, the appropriate non-dimensional space variables are

$$x^* = \frac{x}{H}, \qquad y^* = \frac{u_\tau y}{\upsilon}, \tag{3.1}$$

where $u_\tau(v_0)$ is the shear velocity, i.e. $(\tau_0/\rho)^{0.5}$ at $(x^*, 0)$, $\tau_0(v_0)$ is the bed shear stress at the same point and v_0 is the seepage velocity. Apart from v_0, as the seepage velocity is assumed to be low in comparison to the main flow velocity, u_τ (obviously τ_0) may also vary slowly with x^*. Using equation (3.1), equations

(2.2)–(2.3*b*) can thus be written as

$$\frac{\partial \bar{u}}{\partial x^*} + Re\frac{\partial \bar{v}}{\partial y^*} = 0, \tag{3.2}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x^*} + Re\bar{v}\frac{\partial \bar{u}}{\partial y^*} = -\frac{\partial \bar{p}}{\partial x^*} + Re\frac{\partial \tau}{\partial y^*} + u_\tau Re\frac{\partial^2 \bar{u}}{\partial y^{*2}} - \frac{\partial\left(\overline{u'^2}\right)}{\partial x^*}, \tag{3.3a}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x^*} + Re\bar{v}\frac{\partial \bar{v}}{\partial y^*} = -Re\frac{\partial \bar{p}}{\partial y^*} + \frac{\partial \tau}{\partial x^*} + \frac{v}{H}\cdot\frac{\partial^2 \bar{v}}{\partial x^{*2}} - Re\frac{\partial\left(\overline{v'^2}\right)}{\partial y^*}, \tag{3.3b}$$

where Re is the Reynolds number, i.e. $u_\tau H/v$. As mentioned in the introduction, the set of equations (3.2)–(3.3*b*) are undetermined. For completion, the following model for turbulence closure is adopted.

(*a*) *Turbulence closure assumption*

In fully turbulent outer layer, the viscous stress contribution in the streamwise direction is largely dominated by the Reynolds stress contribution, varying slowly with y^*. The nature of the variation with y^* is concluded from the fact that in inviscid flow, such a variation must be absent. Consequently, comparing the two contributions in equation (3.3*a*), one gets

$$\left|\frac{\partial^2 \bar{u}}{\partial y^{*2}}\right| \ll \frac{1}{u_\tau Re}\left|Re\frac{\partial \overline{u'v'}}{\partial y^*} - \frac{\partial\left(\overline{u'^2}\right)}{\partial x^*}\right|, \tag{3.4}$$

in terms of some slowly varying variable instead of y^*. Introducing $\bar{u}^* = \bar{u}/u_\tau$ and the slow variable $\eta = \ln y^*$, so that $\mathrm{d}y^* = \exp(\eta)\mathrm{d}\eta$, the above condition implies that

$$\left|\frac{\partial^2 \bar{u}^*}{\partial \eta^2}\right| \ll \frac{\exp(2\eta)}{u_\tau^2 Re}\left|Re\frac{\partial \overline{u'v'}}{\partial \eta} - \frac{\partial\left(\overline{u'^2}\right)}{\partial x^*}\right|. \tag{3.5}$$

Hence, one can infer that

$$\left|\frac{\partial^2 \bar{u}^*}{\partial \eta^2}\right| \le \varepsilon \ll \min\left\{\frac{1}{u_\tau^2 Re}\left[\left|Re\frac{\partial \overline{u'v'}}{\partial \eta}\right| + \left|\frac{\partial\left(\overline{u'^2}\right)}{\partial x^*}\right|\right]\right\} \ge 0, \tag{3.6}$$

where ε is a small non-dimensional constant ($\varepsilon \ge 0$), such that

$$-\varepsilon \le \frac{\partial^2 \bar{u}^*}{\partial \eta^2} \le \varepsilon. \tag{3.7}$$

Integrating equation (3.7), the expression for $\bar{u}^*$ in terms of y^* is obtained as

$$\bar{u}^* = A(x^*) + B(x^*)\ln y^* + \theta\varepsilon(\ln y^*)^2, \tag{3.8}$$

where θ is an uncertain function of x^* lying between -0.5 and 0.5. As the first two terms of the right-hand side of equation (3.8) are the principal contributions, the term containing θ is the uncertain residual of the exact solution. The variability of the functions of integration A and B with x^* may be owing to added local flux by seepage. Experiments by Cheng & Chiew (1998) and those cited here indicate that in the outer layer, the velocity profile changes with y^* in the presence of seepage.

(b) Seepage assumption

When a low upward seepage v_0 through the bed is present, it modifies the term A from that in traditional logarithmic law. It now depends on the seepage velocity v_0, which is practically independent of location x^*. However, the term B is assumed to remain independent of v_0, as it is the reciprocal of the von Karman constant in logarithmic law. Thus, the expressions for A and B are assumed to be

$$A(x^*) = A_0(\hat{v}_0), \tag{3.9a}$$

$$B(x^*) = B_0, \tag{3.9b}$$

where A_0 and B_0 are the non-dimensional constants and $\hat{v}_0$ is the non-dimensional seepage velocity, i.e. v_0/u_τ. Thus, the expression for $\bar{u}^*$ in fully turbulent outer layer is assumed to be

$$\bar{u}^* = A_0(\hat{v}_0) + B_0 \ln y^* + \theta\varepsilon(\ln y^*)^2. \tag{3.10}$$

The first two terms of the above equation constitute the principal contribution to the streamwise velocity distribution, and the last term represents the uncertain residual present in its estimation. Besides the last uncertain term, equation (3.10) constitutes the generalized logarithmic law. Numerous experiments on flow over rough boundaries with different kinds of roughness yield the value of $B_0=2.44$ (reciprocal of the von Karman constant) as in the case of smooth boundaries. Here, the same value should hold, since the layer under consideration is away from the slowly seeping bed.

The low transverse velocity, induced by the seepage, can be estimated from the continuity equation. Substituting equation (3.10) in equation (3.2) yields

$$\frac{\partial \bar{v}^*}{\partial y^*} = -\frac{1}{Re}\cdot\frac{\partial \bar{u}^*}{\partial x^*} - \frac{\bar{u}^*}{Re u_\tau}\cdot\frac{\partial u_\tau}{\partial x^*}, \tag{3.11}$$

where $\bar{v}^* = \bar{v}/u_\tau$. The first term of the right-hand side of equation (3.11) is of the order of ε. The second term arising from the seepage from permeable bed essentially depends on the viscous sublayer and must also be small. With these considerations, one obtains on integration of equation (3.11)

$$\bar{v}^* = \varepsilon\hat{v}_0\theta_1(x^*, y^*), \tag{3.12}$$

where θ_1, like θ, is an uncertain term.

The Reynolds stress in non-dimensional form $\tau^* = \tau/u_\tau^2$ can be estimated from equations (3.3a) and (3.3b). Inserting $\bar{u}$ and $\bar{v}$ from equations (3.10) and (3.12) into equations (3.3a) and (3.3b), respectively, and eliminating $\bar{p}$ by appropriate differentiations yield the following differential equation:

$$\frac{\partial^2(Re\tau^*)}{\partial y^{*2}} - \frac{\partial^2(\tau^*/Re)}{\partial x^{*2}} = -\frac{2ReB_0}{y^{*3}} + O(\varepsilon) + \frac{1}{u_\tau^2}\cdot\frac{\partial^2}{\partial x^*\partial y^*}\left(\overline{u'^2} - \overline{v'^2}\right). \tag{3.13}$$

If Re were a constant (as in the case of no seepage), then the general solution of the linear partial differential equation would consist of the complementary solution $f_1(y^* + x^* Re^{-1}) + f_2(y^* - x^* Re^{-1})$, where the functions f_1 and f_2 are linear in x^* for low seepage velocity. But the solution for the inner layer indicates that it does not explicitly depend on x^* (see equation (4.23) in the succeeding section). Hence, the complementary solution may be considered a constant, say C_0, even for variable Re and the general solution of equation (3.13) at any section at a distance x^* from origin O takes the form

$$\tau^* = C_0 - \frac{B_0}{y^*} + O(\varepsilon) + \text{fluctuation term containing } \overline{u'^2} \text{ and } \overline{v'^2}. \tag{3.14}$$

It is intuitive that the fluctuation term must be small, in as much as the streamwise variation of flow is feeble, owing to the low seepage velocity. The first two terms mainly contribute towards the Reynolds stress τ^*. Eventually, $\tau^* \to C_0$ as $y^* \to \infty$. The constant C_0 depends implicitly on $\hat{v}_0$.

(c) *Wake zone*

The scale υ/u_τ of measurement of y in y^* is very small (of the order of 10^{-2} mm) and so y^* can be very large even for small flow depth. High streamwise time-averaged velocity given by equation (3.10) may not be sustainable above some level, $y^* = \hat{y}_2$, by the pressure gradient responsible for the flow. The motion in $y^* > \hat{y}_2$ is thus wake flow. Numerous experiments including the present one estimate that approximately $\hat{y}_2 = Re/5$. A model of wake flow was given by Coles (1956) and Nezu *et al.* (1997) in the absence of seepage. In the present study, it is noted that the turbulence assumption (3.4) continues to hold in the wake region and so the representations (3.10), (3.12) and (3.14) remain valid with new constants A_0' and B_0' in place of A_0 and B_0, respectively. As the rate of increase with ln y^* in $\bar{u}$ is lower in the wake, $B_0' < B_0$ or 2.44. The constant A_0' depends implicitly on $\hat{v}_0$, owing to the continuity of flow across $y^* = \hat{y}_2$ of the turbulent outer layer.

4. Flow in inner layer

The flow is usually considered to be viscous in a sublayer close to the bed, followed by an intermediate layer (buffer layer), where transition to turbulent flow takes place (Schlichting 1968). Tien & Wasan (1963) gave an elegant treatment of the composite two layers, termed inner layer in the present study, based on the power series expansions in y measured on the scale υ/u_τ. The only underlying assumption in the method is differentiability of the velocity components up to a few desired orders, implying smooth matching with the solution for the outer layer. The method adopted here is based on (2.2) and (2.3) as governing equations. The boundary conditions are crucial, as the bed is rough with three-dimensional corrugation. Miksis & Davis (1994) examined it in the context of the passage of a two-dimensional wave in a two-layer fluid over a rough boundary. For analysis, the corrugations are modelled by the equivalent two-dimensional roughness model. Noting that the scale of roughness is much smaller than the scale of interest in the low viscous flow close to the boundary, it may be possible to replace the actual boundary by a mean smooth boundary on which an effective *Navier slip condition* holds. After detailed analysis following the method of matched asymptotic expansions, they conclude that if λ represents the effective mean height of protrusions (that is equivalent roughness height) over $y=0$, then the streamwise and vertical velocity components must satisfy for $y=0$, the Navier slip conditions

$$\bar{u}+\lambda\frac{\partial \bar{u}}{\partial y}=0, \qquad \bar{v}=0. \tag{4.1}$$

In the present case, the seepage velocity is present in the vertical direction. Hence, at $y=0$, one can take the Navier slip conditions as

$$\bar{u}+\lambda\frac{\partial \bar{u}}{\partial y}=0, \qquad \bar{v}=v_0, \qquad u'=v'=0. \tag{4.2}$$

The last condition indicates that turbulence does not exist in the viscous sublayer. Assuming admissibility of power series expansions satisfying the boundary conditions (4.2), one obtains

$$\bar{u}=U_1(y-\lambda)+U_2y^2+U_3y^3+U_4y^4+U_5y^5+\cdots \tag{4.3}$$

where U_1, U_2, U_3, ... are functions of x by virtue of the low seepage velocity. Substituting equation (4.3) in equation (2.2) (continuity equation of time-averaged velocity components) and integrating the resulting equation yields

$$\bar{v}=v_0+\lambda U_1'y-\frac{1}{2}U_1'y^2-\frac{1}{3}U_2'y^3-\frac{1}{4}U_3'y^4-\frac{1}{5}U_4'y^5-\cdots \tag{4.4}$$

where U_1', U_2', U_3', ... denote differentiation with respect to x. Similarly, turbulence fluctuation components are represented by

$$u'=u_1y+u_2y^2+u_3y^3+\cdots \tag{4.5a}$$

$$v'=v_1y+v_2y^2+v_3y^3+\cdots \tag{4.5b}$$

where u_1, u_2, u_3, ... and v_1, v_2, v_3, ... are functions of x and t. Inserting equations (4.5a) and (4.5b) in equation (2.2) (continuity of turbulence fluctuations) yields

$$\frac{\partial u_1}{\partial x}y+\frac{\partial u_2}{\partial x}y^2+\frac{\partial u_3}{\partial x}y^3+\cdots+v_1+2v_2y+3v_3y^2+\cdots=0. \tag{4.6}$$

Equating coefficients of different powers of y to 0, one gets

$$v_1=0, \qquad v_2=-\frac{1}{2}\cdot\frac{\partial u_1}{\partial x}, \qquad v_3=-\frac{1}{3}\cdot\frac{\partial u_2}{\partial x},\ldots. \tag{4.7}$$

The first condition is significant to determine the expression of τ from equations (4.5a) and (4.5b) as

$$\tau=-\overline{u'v'}=-[\overline{u_1v_2}y^3+(\overline{u_2v_2}+\overline{u_1v_3})y^4+\cdots]. \tag{4.8}$$

Using equations (4.3)–(4.5b) and (4.8) in equation (2.3b) with $v_1=0$ and differentiating with respect to x, one obtains

$$\frac{\partial^2\bar{p}}{\partial x\partial y}=-\frac{\partial^2}{\partial x^2}[\overline{u_1v_2}y^3+(\overline{u_2v_2}+\overline{u_1v_3})y^4+\cdots]-\left(4\overline{v_2^2}y^3+10\overline{v_2v_3}y^4+\cdots\right). \tag{4.9}$$

Therefore, integrating equation (4.9) with respect to y, one has

$$\frac{\partial \bar{p}}{\partial x} = \frac{\partial \bar{p}}{\partial x}\bigg|_{y=0} - \left[\frac{1}{4}\frac{\partial^2}{\partial x^2}(\overline{u_1 v_2}) + \overline{v_2^2}\right] y^4 - \cdots. \tag{4.10}$$

Inserting equations (4.3)–(4.5b) and (4.10) in equation (2.3a) with $v_1 = 0$ and following an integration with respect to y, one obtains another expression for τ as

$$\begin{aligned}\tau =& \left[\frac{\partial \bar{p}}{\partial x}\bigg|_{y=0} + \lambda^2 U_1 U_1' + v_0 U_1 - 2\upsilon U_2\right] y + \left[-\lambda U_1 U_1' + 2 v_0 U_2 - 6\upsilon U_3\right]\frac{y^2}{2} \\ &+ \left[\frac{1}{2} U_1 U_1' + \lambda(U_2 U_1' - U_1 U_2') + 3 v_0 U_3 - 12\upsilon U_4 + \frac{\partial}{\partial x}\left(\overline{u_1^2}\right)\right]\frac{y^3}{3} \\ &+ \left[\frac{2}{3} U_1 U_2' + \lambda(2 U_3 U_1' - U_1 U_3') + 4 v_0 U_4 - 20\upsilon U_5 + 2\frac{\partial}{\partial x}(\overline{u_1 u_2})\right]\frac{y^4}{4} + \cdots.\end{aligned} \tag{4.11}$$

Comparing equations (4.8) and (4.11) and neglecting the $O(\lambda^2)$ term, one gets

$$2\upsilon U_2 = \frac{\partial \bar{p}}{\partial x}\bigg|_{y=0} + v_0 U_1, \qquad 6\upsilon U_3 = 2 v_0 U_2 - \lambda U_1 U_1'. \tag{4.12}$$

To get more insight, at the bed ($y=0$), $\bar{v} \approx v_0$ and turbulence (u' and v') disappears. Thus, equations (2.2)–(2.3b) reduce to

$$\frac{\partial \bar{u}}{\partial x} \approx 0, \qquad v_0 \frac{\partial \bar{u}}{\partial y} \approx -\frac{\partial \bar{p}}{\partial x} + \upsilon \frac{\partial^2 \bar{u}}{\partial y^2}, \qquad \frac{\partial \bar{p}}{\partial y} \approx 0. \tag{4.13}$$

From the above set of equations, it is inferred that $\bar{p}$ is almost independent of y and

$$\upsilon \frac{\partial^2 \bar{u}}{\partial y^2} - v_0 \frac{\partial \bar{u}}{\partial y} \approx \frac{\partial \bar{p}}{\partial x} = K(x), \tag{4.14}$$

where $K(x)$ is a function of x, since $\bar{p}$ is independent of y. The general solution of equation (4.14) is given by

$$\bar{u} \approx C_1(x) + C_2(x)\exp\left(\frac{v_0}{\upsilon}\right) y - \frac{K(x)}{v_0} y. \tag{4.15}$$

It is intuitive that since $\bar{v}$ cannot increase exponentially, $C_2(x)$ equals 0. The boundary condition in equation (4.2) at $y=0$ yields $C_1(x) = \lambda K(x)/v_0$. Hence, the solution is

$$\bar{u} \approx -\frac{K(x)}{v_0}(y - \lambda). \tag{4.16}$$

Comparing equation (4.16) with equation (4.3), it is possible to assume that $U_1 = -K(x)/v_0$ and equation (4.12) yields

$$U_2 = 0, \qquad U_3 = -\frac{\lambda}{6\upsilon} U_1 U_1'. \tag{4.17}$$

The bed shear stress τ_0 and the shear velocity u_τ are given by

$$\tau_0 = \rho\upsilon \frac{\partial \bar{u}}{\partial y}\bigg|_{y=0} = \rho\upsilon U_1(x), \qquad u_\tau = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\upsilon U_1(x)}. \tag{4.18}$$

In the absence of seepage ($v_0 = 0$), U_1, U_4, U_5, … are constants, and hence equation (4.17) yields $U_2 = U_3 = 0$. Thus, in this case, $\bar{u}$ is a power series in y given by equation (4.3), whereas in equation (4.4), all the derivative terms drop out and

$\bar{v} = v_0$. Similar is the case for τ, given by equation (4.11), in which the fluctuation terms $\partial(\cdot)/\partial x$ also drop out. On the other hand, in the presence of seepage ($v_0 \neq 0$), the expression (4.3) would match with that in equation (3.8) under seepage assumption if

$$U_1', U_3', U_4', U_5', \ldots = \hat{v}_0 \times \text{terms of the order of } \varepsilon. \tag{4.19}$$

Thus, using the expression for u_τ in equation (4.18), $\bar{u}^*$ in the inner layer is given by

$$\bar{u}^* = y^* - \frac{u_\tau \lambda}{\upsilon} + \hat{U}_3 y^{*3} + \hat{U}_4 y^{*4} + \hat{U}_5 y^{*5} + \cdots, \tag{4.20}$$

$$\bar{v}^* = \hat{v}_0 + \frac{\upsilon \lambda U_1'}{u_\tau} y^* - \frac{1}{2} \cdot \frac{\upsilon^2 U_1'}{u_\tau^2} y^{*2} - \cdots, \tag{4.21}$$

where

$$\hat{U}_3 = -\frac{\upsilon \lambda U_1'}{6u_\tau^2}, \qquad \hat{U}_4 = \left(\frac{\upsilon}{U_1}\right)^{3/2} \frac{U_4}{U_1}, \qquad \hat{U}_5 = \left(\frac{\upsilon}{U_1}\right)^2 \frac{U_5}{U_1}. \tag{4.22}$$

The set of expressions in equation (4.22) is obtained using equation (4.17). The non-dimensional Reynolds stress τ^* ($=\tau/u_\tau^2$), given in equation (4.11), becomes

$$\tau^* = \left[\frac{1}{6}\left(1 - \frac{v_0 \lambda}{\upsilon}\right) \frac{\upsilon^2 U_1'}{u_\tau^3} - 4\hat{U}_4 + \frac{1}{3U_1^2} \sqrt{\frac{\upsilon}{U_1}} \cdot \frac{\partial\left(\overline{u_1^2}\right)}{\partial x}\right] y^{*3} + \left[\frac{v_0}{u_\tau} \hat{U}_4 - 5\hat{U}_5 + \frac{\upsilon}{2U_1^2} \cdot \frac{\partial(\overline{u_1 u_2})}{\partial x}\right] y^{*4} + \cdots. \tag{4.23}$$

In the series representations of equations (4.20) and (4.23), it is sufficient to consider the terms up to fifth and fourth power, respectively. In equation (4.20), the term containing $\hat{U}_3$ may be dropped owing to its smallness. In equation (4.12), fluctuating terms in both the coefficients of y^{*3} and y^{*4} are proportional to υ^3 according to equation (4.18). Therefore, the terms are negligible. Moreover, the first term of the coefficient of y^{*3} is negligible owing to the smallness of $\upsilon^2 U_1'$.

5. Experimentation

The experiments were conducted in a horizontal glass-walled flume of 0.6 m wide, 0.71 m deep and 10 m long, as shown in figure 1*b*. The seepage test zone consisted of a sand recess 0.3 m deep and 2 m long having a width of the flume located 6 m downstream of the flume inlet. An arrangement, similar to Chen & Chiew (2004), was made to apply uniform upward seepage from the bottom of the recess through the sand bed. A false floor at an elevation of 0.3 m from the flume bottom was constructed along the length of the flume to maintain the same level of the sand bed in the sand recess. The uniform sand having same size used for the test was glued over the false floor to simulate the turbulent flow over a rough planar sand bed. The flow discharge, controlled by an inlet valve, was measured using a calibrated V-notch weir fitted at the inlet of the flume. An adjustable tailgate in the downstream of the flume controlled the flow depth in the flume. Two uniform sands of median diameters $d=0.81$ and 1.86 mm were used in the experiments. The degree of uniformity of the particle size distribution of a sand sample is defined by the value of geometric standard deviation, given by $(d_{84}/d_{16})^{0.5}$, which is less than 1.4 for uniform sand (Dey *et al.* 1999). The flow conditions were set in such a way that sediment transport in the sand recess was absent.

The instantaneous velocity components were detected by a SonTek-made 5 cm downlooking ADV. The ADV functioned on a pulse-to-pulse coherent Doppler shift to provide instantaneous velocity components at a rate of 50 Hz. Output data from the ADV was filtered using a spike removal algorithm.

6. Determination of constants

The constants appearing in the representation of flow, equations (3.10), (3.14), (4.20) and (4.23), are u_τ, B_0, A_0, C_0, $\hat{U}_4$, $\hat{U}_5$ and A_0' and B_0' for the wake. In the experiments, the vertical distribution of flow characteristics was measured by the ADV along the vertical central section $x=50$ cm of the test bed, when vertical seepage was not allowed. The origin of coordinates O is assumed to be located at the upstream end of the test section. When upward seepage was allowed, flow measurements were carried out along three sections at $x=33.3$, 50 (central section) and 66 cm downstream. By these choices, any disturbance at the entry ($x=0$) and exit ($x=100$ cm) sections was avoided.

The normalization of $\bar{u}$ as well as that of Reynolds stress τ depends on u_τ. The shear velocity u_τ is essentially a constant for a given bed sediment, seepage velocity and flow condition. Its direct estimation is difficult in the ambient motion close to the bed. This difficulty is circumvented by noting that for the outer layer below the wake, $B_0=2.44$ as argued earlier. The least-square fit of the generalized logarithmic law for $\bar{u}^*$, namely equation (3.10), to the experimental dataset for a particular flow condition and section of measurement was carried out for that value of u_τ which yields $B_0=2.44$. The least-square fit also yields the estimate of A_0. The computed value of u_τ was used to estimate the constants A_0' and B_0' in the wake region $y^*>\hat{y}_2=Re/5$. This was done by least-square fitting of the experimental value of $\bar{u}^*$ in the region of the theoretical curve of $A_0'+B_0'\ln y^*$.

The two unknowns $\hat{U}_4$ and $\hat{U}_5$, as in Tien & Wasan (1963), can be numerically determined by matching $\bar{u}^*$, $\partial\bar{u}^*/\partial y^*$ and $\partial^2\bar{u}^*/\partial y^{*2}$ at the interface $y^*=\hat{y}_1$ with the inner layer. The following equations are obtained from the last three conditions:

$$\hat{U}_4\hat{y}_1^4 = 1.25B_0 - \hat{y}_1 - 1.5\hat{U}_3\hat{y}_1^3, \tag{6.1a}$$

$$\hat{U}_5\hat{y}_1^5 = -0.8B_0 + 0.6\hat{y}_1 + 0.6\hat{U}_3\hat{y}_1^3, \tag{6.1b}$$

$$B_0\ln\hat{y}_1 - 0.6\hat{y}_1 - 0.1\hat{U}_3\hat{y}_1^3 - 0.45B_0 + \frac{u_\tau\lambda}{\upsilon} + A_0 = 0, \tag{6.1c}$$

where $\hat{y}_1$ is the real-valued solution of equation (6.1c) and can only be solved for a particular experiment and estimate of λ.

The sand bed can be regarded as a dense packing of spherical particles, assuming spheres to be of equal diameter d. If the protrusion height is considered

Table 1. Expressions for $\bar{u}^*$ for $d=0.81$ mm.

v_0 (cm s^{-1})	u_τ (cm s^{-1})	expression for $\bar{u}^*$	range of y^*
0	3.484	$y^*-21.92-1.72\times10^{-5}\ y^{*4}+2.73\times10^{-7}\ y^{*5}$	$27.44\le y^*\le 38$
		$-7.08+2.44\ \ln y^*$	$38<y^*\le 1040$
		$2.62+1.08\ \ln y^*$	$1040<y^*$
0.0153	3.505	$y^*-22.06-1.68\times10^{-5}\ y^{*4}+2.64\times10^{-7}\ y^{*5}$	$27.55\le y^*\le 38$
		$-7.05+2.44\ \ln y^*$	$38<y^*\le 1000$
		$4.85+0.76\ \ln y^*$	$1000<y^*$

Table 2. Expressions for $\bar{u}^*$ for $d=1.86$ mm.

v_0 (cm s^{-1})	u_τ (cm s^{-1})	expression for $\bar{u}^*$	range of y^*
0	3.777	$y^*-54.58-1.05\times10^{-6}\ y^{*4}+6.48\times10^{-9}\ y^{*5}$	$67.11\le y^*\le 97$
		$-6.29+2.44\ \ln y^*$	$97<y^*\le 1170$
		$-1.39+1.75\ \ln y^*$	$1170<y^*$
0.0153	4.157	$y^*-60.07-8.03\times10^{-7}\ y^{*4}+4.52\times10^{-9}\ y^{*5}$	$74.37\le y^*\le 110$
		$-6.44+2.44\ \ln y^*$	$110<y^*\le 1640$
		$-1.31+1.74\ \ln y^*$	$1640<y^*$

to be at the height above the point of contact with the spheres at the base, on average, it can be proved that $\lambda=0.78d$. Using such an estimate of λ, u_τ and A_0, equation (6.1*c*) can be numerically solved to determine $\hat{y}_1$, letting us to estimate $\hat{U}_4$ and $\hat{U}_5$ from equations (6.1*a*) and (6.1*b*), respectively. This completes the method of estimation of all constants u_τ, B_0, A_0, $\hat{U}_4$, $\hat{U}_5$, A_0' and B_0' for any cross-section, which appear in the principal terms of the expression of $\bar{u}^*$, given by equations (3.10) and (4.20). The same is the case for τ^* given by equations (3.14) and (4.23). The resulting expressions for $\bar{u}^*$ (in the case of no seepage and upward seepage) along with the ranges (ranges of y^*) of their applicability are given in tables 1 and 2. Here, it is pertinent to point out that the lowest value of y^* indicates the zero-velocity level. For example, the zero-velocity level of the first expression in table 1 is at $y^*=27.44$.

The constant C_0 appearing in Reynolds stress is estimated by equating (3.14) and (4.23) at $y^*=y_1$. It turns out to be unity in the case of zero seepage. This is in accordance with the solution for the corresponding Couette flow problem (Schlichting 1968). For $\hat{v}_0>0$, the computed value of C_0 is found to be less than unity.

7. Results and discussion

The theoretical curves of non-dimensional streamwise velocity $\bar{u}^*$ (equations (3.10) and (4.21)) and the corresponding experimental data are presented in figure 2*a*–*d*. The theoretical curves of non-dimensional Reynolds stress τ^* (equations (3.14) and (4.23)) are presented in figure 3*a*–*d*. The experimental values of the Reynolds shear

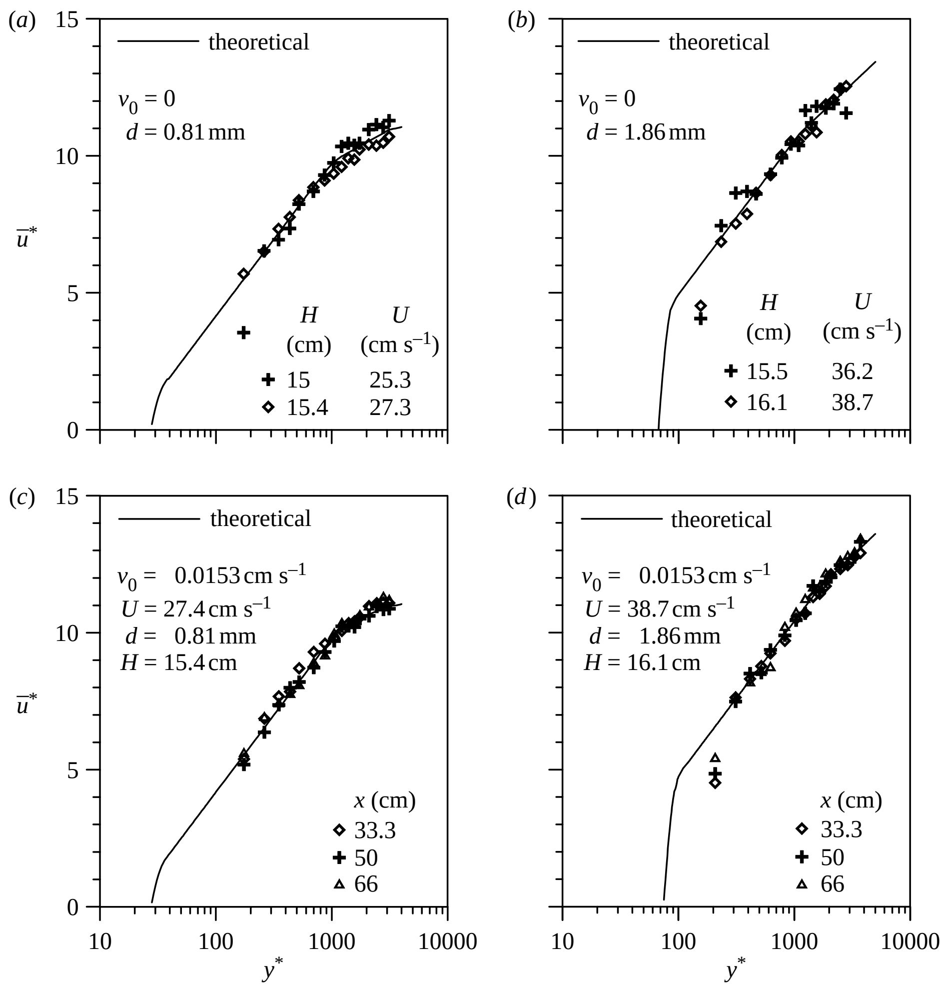

Figure 2. Distribution of time-averaged velocity $\bar{u}^*(y^*)$ with (a) no seepage for sediment size $d=0.81$ mm, (b) no seepage for sediment size $d=1.86$ mm, (c) upward seepage ($v_0=0.0153$ cm s^{-1}) for sediment size $d=0.81$ mm at different distances, and (d) upward seepage ($v_0=0.0153$ cm s^{-1}) for sediment size $d=1.86$ mm at different distances.

stress τ estimated from its definition $-\overline{u'v'}$ based on the velocity fluctuations were relatively small, compared with the data of streamwise time-averaged velocity $\bar{u}$, by a factor of at least 0.1. Moreover, the ADV has a measuring volume of 0.09 cm^3. This makes values of fluctuating velocity and hence the Reynolds stress subject to uncertain attenuation and error. In spite of this fact, the data show trends similar to the theoretical curves.

For the distribution of streamwise velocity $\bar{u}^*(y^*)$, figure 2a,b, under no seepage condition ($\hat{v}_0=0$) with sediment sizes $d=0.81$ and 1.86 mm, show that the experimental data points collapse on the theoretical curves. Similar observations are made in the case of upward seepage velocity $v_0=0.0153$ cm s^{-1} at different streamwise distances x (figure 2c,d). For the distribution of the Reynolds stress $\tau^*(y^*)$, figure 3a–d shows a satisfactory agreement between the theory and the experimental data without and with seepage velocity.

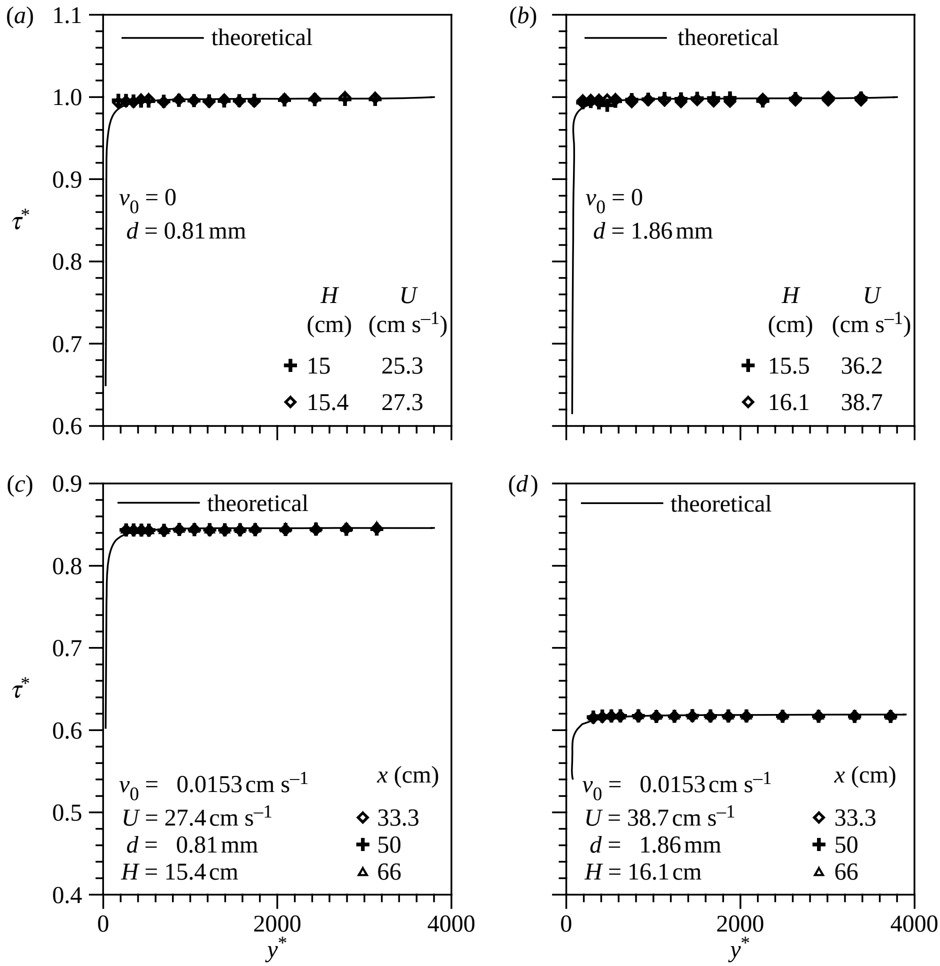

Figure 3. Distribution of Reynolds stress $\tau^*(y^*)$ with (*a*) no seepage for sediment size $d=0.81$ mm, (*b*) no seepage for sediment size $d=1.86$ mm, (*c*) upward seepage ($v_0=0.0153$ cm s^{-1}) for sediment size $d=0.81$ mm at different distances, and (*d*) upward seepage ($v_0=0.0153$ cm s^{-1}) for sediment size $d=1.86$ mm at different distances.

8. Conclusions

A theory has been developed for the steady free surface flow over a horizontal rough bed with uniform upward seepage normal to the bed. It is based on the concepts that in the fully turbulent outer layer, the Reynolds stress, varying gradually with vertical distance, is much greater than the viscous shear stress. Using this concept in the RANS equations and adopting the methodology of Tien & Wasan (1963) for inner layer, exact expressions for the streamwise and normal components of velocity have been constructed containing some bounding and infinitesimal terms. The principal parts of the expressions contain some unknown constants, which were estimated using the experimental data. In this way, for the streamwise velocity distribution, a modified logarithmic law and a fifth-degree polynomial law including effective height of protrusions have been obtained in the outer layer and inner layer, respectively. The expressions for the Reynolds stress have also been derived from the developed theory of two layers. The exact expressions for all the quantities contain some small bounding or infinitesimal terms, which are ignored in the computations. The velocity and Reynolds stress distributions estimated from the present theory are in agreement with the experimental data for no seepage and upward seepage conditions.

The first author is deeply thankful to the Centre for Theoretical Studies at the Indian Institute of Technology, Kharagpur for providing fellowship to visit the Institute during the course of this study.

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