Proc. Indian Acad. Sci. (Math. Sci.) Vol. 114, No. 2, May 2004, pp. 107–122.

Picard groups of the moduli spaces of semistable sheaves I

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MS received 9 January 2003; revised 12 March 2004

Abstract. We compute the Picard group of the moduli space U' of semistable vector bundles of rank n and degree d on an irreducible nodal curve Y and show that U' is locally factorial. We determine the canonical line bundles of U' and U'_L, the subvariety consisting of vector bundles with a fixed determinant. For rank 2, we compute the Picard group of other strata in the compactification of U'.

Keywords. Picard groups; semistable sheaves; nodal curve.

1. Introduction

In our previous paper [3] we proved that the Picard group of the moduli space $U'_L(n, d)$ of semistable vector bundles of rank n with fixed determinant L (L being a line bundle of degree d) on an irreducible projective nodal curve Y of geometric genus $g \geq 2$ is isomorphic to $\mathbb{Z}$ (except possibly in the case $g = 2, n = 2, d$ even). We used this to show that $U'_L(n, d)$ is locally factorial. Interestingly, the results for irreducible nodal curves are very similar to those for smooth curves. However, the proofs are different and much more difficult. Unlike in the smooth case, the moduli space of vector bundles on a nodal curve is not projective. Moreover its complement in the compactification U (moduli of torsion-free sheaves) has codimension 1. The computation of Picard group needs codimension of the non-semistable and non-stable strata (see [6,11] for smooth case). Since HN-filtrations of vector bundles contain non-locally free sheaves and tensor products of stable bundles are not semistable (on Y), in general it is impossible to determine this codimension directly on Y. We did it by using parabolic bundles on the normalization X of Y and hence had to assume $g \geq 2$ and exclude the case $g = n = d = 2$.

In this paper, we do a detailed analysis for rank 2 and extend these results to nodal curves of arithmetic genus $g_Y \geq 0$ (rank 2). Combining this with results of [3], we have the following theorem.

Theorem 1. *Let Y be an irreducible reduced curve with only ordinary nodes as singularities. Assume that for $n \geq 3$, the geometric genus $g \geq 2$. Then*

(1) Pic $U'_L(n, d) \approx$ Pic $U'^s_L(n, d) \approx \mathbb{Z}$,
(2) U'_L *is locally factorial.*

We also show that the dualising sheaf ω_L of $U'_L(n, d)$ is isomorphic to the line bundle $-2\delta\mathbb{L}$, where $\delta = gcd(n, d)$ and $\mathbb{L}$ is the ample generator of Pic $U'_L(n, d)$ (Theorem 4).

We then compute the Picard group of the moduli space $U'(n, d)$ (resp. $U'^s(n, d)$) of semistable (resp. stable) vector bundles of rank n and degree d on Y. Let J denote the generalised Jacobian of degree d on Y.

Theorem (Theorem 3(A)). *Let the assumptions be above.*

(a) Pic $U'^s \approx$ Pic $J \oplus \mathbb{Z}$,
(b) Pic $U' \approx$ Pic $J \oplus \mathbb{Z}$,
(c) *U' is locally factorial.*

This completes the extension of results of [6] *to nodal curves.*

Let $U = U(n, d)$ denote the moduli space of torsion-free sheaves of rank n and degree d on Y. If Y has only a single ordinary node as singularity, then the variety $U(2, d)$ has a stratification, $U = U' \cup U_1 \cup U_0$, a disjoint union. Points of U_1 correspond to torsion-free sheaves F of rank 2 with $F_y \approx \mathcal{O}_y \oplus m_y$. Let L be a rank 1 torsion-free sheaf which is not locally free. Let $U_{1,L}(2, d)$ be the subscheme of U_1 corresponding to torsion-free sheaves of rank 2 with determinant isomorphic to L.

Theorem (Theorem 2, Theorem 3(B)). *Let $g_Y \geq 2$; if $g_Y = 2$, assume that d is odd for* (b), (c), (d). *Then*

(a) Pic $U_{1,L}(2, d) \approx \mathbb{Z}$,
(b) Pic $U_1^s(2, d) \approx$ Pic $J_X \oplus \mathbb{Z}$,
(c) Pic $U_1(2, d) \approx$ Pic $J_X \oplus \mathbb{Z}$,
(d) *$U_1(2, d)$ is locally factorial.*

In a subsequent paper, we study the Picard group of a seminormal variety. As an application we compute the Picard groups of the compactified Jacobian and some subvarieties of $U(2, d)$.

Notation. Let Y denote an irreducible reduced projective curve with ordinary nodes y_j, $j = 1, \ldots, m$ as only singularities. Let g be the geometric genus and g_Y the arithmetic genus of Y. For $y \in Y$, let $(\mathcal{O}_y, m_y)$ be the local ring at y. A torsion-free sheaf N on Y is locally free on the subset U of non-singular points of Y. The rank $r(N)$ of N is the rank of the locally free sheaf $N|_U$. The degree $d(N)$ of N is defined by $d(N) = \chi(N) + r(N)(g - 1)$, where χ denotes the Euler characteristic. Let N^* denote the torsion-free sheaf $\mathrm{Hom}(N, \mathcal{O})$.

Let J and $\overline{J}$ be respectively the generalised Jacobian and the compactified Jacobian of Y (of a fixed degree) and $\mathcal{P}$ the Poincaré bundle. Let p_J denote the projection to $\overline{J}$. Let $U = U(n, d)$ be the moduli space of semistable torsion-free sheaves of rank n and degree d on Y. Let $\delta = gcd(n, d)$. Let $U' \subset U$ be the open subvariety corresponding to vector bundles (i.e. S-equivalence classes of E such that gr$E$ is a vector bundle). Fix a rank 1 torsion-free sheaf L of degree d on Y. Let U'_L (resp. $U_{1,L}$) be the subscheme of U corresponding to vector bundles (resp. torsion-free sheaves) with determinant isomorphic to L and U_L its closure in U. Let $U'^s \subset U'$, $U'^s_L \subset U'_L$ etc. be the open subvarieties corresponding to stable torsion-free sheaves. The variety U is seminormal ([13], Theorem 4.2), U' and U'_L are normal being GIT-quotients of non-singular varieties [10]. For $m = 1$, U has a filtration $U \supset W_{n-1} \supset \cdots \supset W_0$, with W_i seminormal closed subvarieties [13]. W_{i-1} is the non-normal locus of W_i, $i = 1, \ldots, n$ and W_0 is normal. Let $U' = U - W_1$, $U_i = W_i - W_{i-1} (i = 1, \ldots, n-1)$, $U_0 = W_0$.

2. Torsion-free sheaves of rank 2

In this section we study $U_L(2, d)$ and $U(2, d)$. Throughout the section E will denote a torsion-free sheaf of rank 2 and degree d on Y.

Lemma 2.1. *Let E be a torsion-free sheaf with $\wedge^2 E = L$ torsion-free. Let N_1 be a rank* 1 *subsheaf of E such that the quotient $N_2 = E/N_1$ is torsion-free.*

(1) *If N_1 or L is locally free, then $N_2 \approx N_1^* \otimes L$,*
(2) *If N_2 is locally free, then $N_1 \otimes N_2 \approx L$.*

Proof. The canonical alternating form $E \times E \to L$ induces an alternating form $N_1 \times N_1 \to L$. We claim that this form is zero. This is clear at $y \in Y$ such that the stalk $(N_1)_y$ is free. If $(N_1)_y \not\approx \mathcal{O}_y$, then $(N_1)_y = m_y$, also $L_y = \mathcal{O}_y$ or m_y ([12], Prop. 2, p. 164). Let u, v be the two generators of $(N_1)_y$. Since any $\mathcal{O}_y$-linear map from m_y to m_y (or $\mathcal{O}_y$) is given by the multiplication by $a \in \overline{\mathcal{O}}_y$ (= normalisation of $\mathcal{O}_y$) ([12], p. 169), the map $(N_1)_y \to L_y$ defined by $w \mapsto w \wedge u$ is given by $w \wedge u = wa, a \in \overline{\mathcal{O}}_y$. In particular, $0 = u \wedge u = ua$. Since $\overline{\mathcal{O}}_y$ is a domain, this implies $a = 0$. Thus $v \wedge u = 0$ and hence $(N_1)_y \wedge (N_1)_y = 0$.

Define an $\mathcal{O}$-bilinear map $b\colon N_1 \times N_2 \to L$ by $b(n_1, n_2) = n_1 \wedge n_3$, where n_3 is a lift of n_2 in E. This is well-defined as any two lifts n_3, n_3' differ by an element of N_1 and $N_1 \wedge N_1 = 0$ as seen above. The bilinear map b induces an injective sheaf homomorphism $N_2 \to \mathrm{Hom}(N_1, L)$ which is an isomorphism outside the singular set of Y. If N_1 or L is locally free, then $d(\mathrm{Hom}(N_1, L)) = d(L) - d(N_1)$ ([4], Lemma 2.5(B)) and hence $d(\mathrm{Hom}(N_1, L)) = d(N_2)$. It follows that $N_2 \approx \mathrm{Hom}(N_1, L)$.

If N_2 is locally free, the bilinear map b gives an injective homomorphism of torsion-free sheaves $N_1 \otimes N_2 \to L$. Since $d(N_1 \otimes N_2) = d(N_1) + d(N_2) = d(L)$, this is an isomorphism. This proves the lemma.

We remark that if both N_1, N_2 are not locally free then $N_1 \otimes N_2$ has a torsion and b gives a homomorphism $N_1 \otimes N_2/\text{torsion} \to L$ which is not an isomorphism.

Lemma 2.2. *Assume that Y has only one node y. Let $\pi\colon X \to Y$ be the normalisation map and $\pi^{-1}y = \{x, z\}$. Let N_1, N_2 be line bundles of degree -1 on X.*

(a) *Given a line bundle L on Y with $\pi^*L = N_1 \otimes N_2(x + z)$, there exists a vector bundle E of rank* 2 *and determinant L on Y such that E is S-equivalent to $\pi_*N_1 \oplus \pi_*N_2$.*
(b) *There exists a torsion-free sheaf E of rank* 2 *on Y such that* (1) $E_y \approx \mathcal{O}_y \oplus m_y$, (2) *determinant of E is isomorphic to $\pi_*(N_1 \otimes N_2(z))$ and* (3) *E is S-equivalent to $\pi_*N_1 \oplus \pi_*N_2$.*

Proof.

(a) We shall construct a generalised parabolic bundle $(E', F_1(E'))$ on X which gives the required vector bundle E on Y. Take $E' = L_1 \oplus L_2, L_1 = N_1(x + z), L_2 = N_2$. Let e_1, e_2 be basis elements of $(L_1)_x, (L_1)_z$ respectively. Let f_1, f_2 be basis elements of $(L_2)_x, (L_2)_z$ respectively. Define $F_1(E') = (e_2 - f_1, ce_1 + f_2)$, c being a non-zero scalar. Since the projections p_1, p_2 from $F_1(E')$ to E'_x, E'_z are both isomorphisms, E is a vector bundle [1]. Choose c such that L corresponds to the generalised parabolic line bundle $(\pi^*L, (c, 1)), (c, 1) \in \mathbf{P}^1$ [1]. One has $\det(E', F_1(E')) = (\det E', (c, 1)) =$

$(\pi^* L, (c, 1))$. Hence $\det E = L$. Since $F_1(L_1) = 0$, $\pi_* L_1(-x-z)$ is a sub-bundle of E. The quotient is $\pi_* L_2$ as the projection from $F_1(E')$ to $(L_2)_x \oplus (L_2)_z$ is onto. Thus E is S-equivalent to $\pi_*(N_1 \oplus N_2)$.

(b) Take E' as in the above proof, define $F_1(E') = (e_1 + f_2, f_1)$. Since p_1 is an isomorphism and p_2 has rank 1, $E_y \approx \mathcal{O}_y \oplus m_y$. Since $(e_1+f_2) \wedge f_1 = 0 e_1 \wedge e_2 + f_1 \wedge f_2 + \cdots$, one has $\det(E', F_1(E')) = (L_1 \otimes L_2, (0,1))$. Hence $\det(E) = \pi_*(L_1 \otimes L_2(-x)) = \pi_*(N_1 \otimes N_2(z))$. The final assertion follows as in the above proof.

PROPOSITION 2.3

Let $g_Y = 1$. *Then one has the following*:

(1) $U_L(2,1) = \{a\ point\}$ *for* $L \in \overline{J}$,
$U(2,1) \approx \overline{J} \approx Y$, $U'(2,1) \approx J \approx Y - \{node\}$.

(2) $U_{\mathcal{O}}(2,0) \approx \overline{J}/i \approx \mathbb{P}^1$, *where* $i: \overline{J} \to \overline{J}$ *is defined by* $N \mapsto N^*$,
$U_L(2,0) \approx \mathbb{P}^1$ *and* $U'_L(2,0) \approx \mathbb{A}^1$, *for* $L \in J$.

Proof.

(1) For $y \in Y$, let I_y denote the ideal sheaf of y. The dual I_y^* is a rank 1 torsion-free sheaf of degree 1 [5]. It is well-known that $y \mapsto I_y^*$ gives an isomorphism $Y \to \overline{J}^1$, where $\overline{J}^1$ is the compactified Jacobian of degree 1 torsion-free sheaves.

Let E be a stable rank 2 torsion-free sheaf of degree 1 on Y. Then $h^1(E) = 0$ as E is stable and hence $h^0(E) = 1$. Any non-zero section $s \in H^0(E)$ must be everywhere non-vanishing, otherwise it will generate a rank 1 torsion-free subsheaf of degree ≥ 1 contradicting the stability of E. Hence $s \in H^0(E)$ generates a unique trivial line sub-bundle $\mathcal{O}$ of E. The quotient $E/\mathcal{O}$ must be torsion-free, if not then the kernel of $E \to (E/\mathcal{O})/\text{torsion}$ will contradict the stability of E. Thus we have a morphism $h: U(2,1) \to \overline{J}^1$ given by $E \mapsto E/\mathcal{O}$. Conversely, given $L \in \overline{J}^1$, $\mathrm{Ext}^1(L, \mathcal{O}) = H^1(L^*)$ ([4], Proof of Lemma 2.5(B)). Since $h^0(L^*) = 0$, $h^1(L^*) = 1$, any non-zero element in $\mathrm{Ext}^1(L, \mathcal{O})$ determines a unique (up to isomorphism) torsion-free rank 2 sheaf E of degree 1. It is easy to check that E is stable. This gives the inverse of h. Note that h is in fact the determinant map.

(2) We first prove that W_0 consists of a single point. Any element in W_0 has stalk at the node y isomorphic to $m_y \oplus m_y$. By [12], Proposition 10, p. 174, such an element is the direct image of a vector bundle E_0 on the desingularisation $\mathbb{P}^1$. Since $\pi_* E_0$ is semistable, so is E_0. Hence $E_0 = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. By Lemma 2.2(a), for every line bundle L there exists a vector bundle E with determinant L such that E is S-equivalent to $\pi_*(\mathcal{O}(-1)) \oplus \pi_*(\mathcal{O}(-1))$. Thus for any $L \in J$, $U_L(2,0)$ contains the point $\pi_* E_0$. One has $U_L \cap W_1 = W_0$ [1]. Thus every element of $U_L(2,0)$ is S-equivalent to a vector bundle with determinant L. It follows that $U_L \approx U_{\mathcal{O}}$.

We now prove that $U_{\mathcal{O}}(2,0) \approx \overline{J}/i \approx \mathbb{P}^1$. Note first that the involution i keeps the unique element $\pi_* \mathcal{O}(-1)$ of $\overline{J} - J$ invariant and under the isomorphism $Y \approx \overline{J}$, the map $\overline{J} \to \overline{J}/i$ is the double cover $Y \to \mathbb{P}^1$ ramified at the image of the node. Let E be a semistable vector bundle of rank 2 with trivial determinant. Let E_1 be the vector bundle of degree 2 obtained by tensoring E with a line bundle of degree 1. Since E_1

is semistable, with slope > 0, $h^1(E_1) = 0$, $h^0(E_1) = 2$. Since the evaluation map $Y \times H^0(E_1) \to E_1$ cannot be an isomorphism, there is a section of E_1 vanishing at a point and hence generating a (torsion-free) subsheaf N_1 of rank 1, degree ≥ 1. Since E_1 is semistable, one must have $d(N_1) = 1$. Hence E has a rank 1 subsheaf N of degree 0. The quotient E/N is torsion-free in view of the semistability of E. By Lemma 2.1(1), $E/N \approx N^*$. Thus E is S-equivalent to $N \oplus N^*$. Using the Poincaré bundle and the properties of moduli spaces, one sees that this proves the proposition.

Lemma 2.4. *For $g_Y \geq 2$, d even and $L \in J$, one has*

$$\operatorname{codim}_{U'_L}(U'_L - U'^s_L) = 2g_Y - 3.$$

Proof. A rank 2 vector bundle E which is semistable but not stable contains a torsion-free subsheaf N_1 with a torsion-free quotient $N_2 \approx \operatorname{Hom}(N_1, L) = N_1^* \otimes L$, where L is determinant of E (Lemma 2.1(1)). Thus E is S-equivalent to $N_1 \oplus (N_1^* \otimes L)$, hence dim $U'_L - U'^s_L = \dim J = g_Y$ and $\operatorname{codim}_{U'_L} U'_L - U'^s_L = 2g_Y - 3 \geq 3$ if $g_Y \geq 3$.

Lemma 2.5.

(1) $\operatorname{Codim}_{U_L}(U_L - U'_L) \geq 3$ *for* $g_Y \geq 3$.
(2) *For* $g_Y = 2$, $\operatorname{codim}_{U_L}(U_L - U'_L) = 3$ *if d is odd*, $U_L = U'_L = \mathbb{P}^3$ *if d is even.*

Proof.

(1) The points of $U_L - U'_L$ correspond to torsion-free sheaves which are direct images of semistable vector bundles with fixed determinant on partial normalisations of Y. Hence $U_L - U'_L$ is a finite union of irreducible components each of dimension $3(g_Y - 1) - 3 = 3g_Y - 6$ for $g_Y \geq 3$. Thus $\operatorname{codim}_{U'_L}(U_L - U'_L) \geq 3$.
(2) For $g_Y = 2$ the partial normalisations are of arithmetic genus 1. It follows from Proposition 2.3(1) that for d odd, $U_L - U'_L$ consists of one or two points according as $g = 1$ or $g = 0$. For d even, $U_L = U'_L \approx \mathbb{P}^3$ ([2], Lemmas 3.3, 3.4, Corollary 3.5). We remark that Proposition 2.3(2) implies that the subset $U_{0,L}$ of non-locally free sheaves in U_L is isomorphic to $\mathbb{P}^1$ if $g = 1$ and it consists of two smooth rational curves intersecting in a point if $g = 0$. The intersection point is the direct image of the unique semistable bundle of degree $d - 2$ on the desingularisation $\mathbb{P}^1$. Note also that $U_{0,L} = U_L - U^s_L$ in this case.

Lemma 2.6. $\operatorname{Codim}_{U'} U' - U'^s \geq 3$ *for* $g_Y \geq 3$ (d *even*).

Proof. The surjective determinant map $U' \to J$ is a fibration with fibres isomorphic to U'_L, L a fixed line bundle of degree d. Hence the lemma follows from Lemma 2.4.

Remark 2.7.

(1) Let $g_Y = 1$. Then Pic $U(2,1) \approx G_m \oplus \mathbb{Z}$. For $L \in J$, Pic $U_L(2,0) \approx \mathbb{Z}$, and Pic $U'_L(2,0)$, Pic $U'(2,0)$, Pic $U'(2,1)$ are trivial.
(2) If $g_Y = 2$, then Pic $U'_L(2,d) \approx \mathbb{Z} \approx$ Pic $U'^s_L(2,d)$ for all d.

Proof. Part (1) follows from Proposition 2.3. Part (2) is proved in [3], §2.4.

PROPOSITION 2.8

For $g_Y \geq 3$, *one has*:

(1) $U_L'^s(2,d) \approx \mathbb{Z}$,
(2) $U_L'(2,d) \approx \mathbb{Z}$.

Proof. Let $p\colon \tilde{U}_L \to U_L$ be a (finite) normalisation. Since U_L' is normal, p is an isomorphism over U_L' and p gives a finite map $\tilde{U}_L - p^{-1}U_L' \to U_L - U_L'$. Therefore codim $\tilde{U}_L - p^{-1}U_L' =$ codim $U_L - U_L' \geq 3$ by Lemma 2.5. Since $\tilde{U}_L$ is normal, this implies that Pic $\tilde{U}_L \hookrightarrow$ Pic$(p^{-1}U_L') \approx$ Pic U_L'. Since U_L is projective, so is $\tilde{U}_L$ and hence rank(Pic $\tilde{U}_L) \geq 1$. It follows that rank(Pic $U_L') \geq 1$. Since U_L' is normal and by Lemma 2.4, codim$(U_L' - U_L'^s) \geq 3$ we have Pic $U_L' \hookrightarrow$ Pic $U_L'^s$. Thus rank(Pic $U_L'^s) \geq 1$. By [3], Proposition 2.3, one has Pic $U_L'^s \approx \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, $m \in \mathbb{Z}$. It follows that Pic $U_L'^s \approx \mathbb{Z}$ and hence Pic $U_L' \approx \mathbb{Z}$.

Remark 2.9. Putting together the results of [3] and Proposition 2.8, we have Theorem 1.

2.10 *Varieties* U_1 *and* $U_{1,L}$

Henceforth we assume that there is only one node y. We first remark that if E is a rank 2 vector bundle then E cannot be S-equivalent to a direct sum of a line bundle and a non-locally free torsion-free rank 1 sheaf. For, then, one has an exact sequence $0 \to L_1 \to E \to L_2 \to 0$ with one of the $(L_1)_y$ or $(L_2)_y$ isomorphic to $\mathcal{O}_y$ and the other isomorphic to m_y. Since $\mathrm{Ext}^1(m_y, \mathcal{O}_y) = 0 = \mathrm{Ext}^1(\mathcal{O}_y, m_y)$, this means $E_y \approx \mathcal{O}_y \oplus m_y$, i.e., E is not locally free. Similarly one sees that if $E_y \approx \mathcal{O}_y \oplus m_y$, then E cannot be S-equivalent to a direct sum of two locally free sheaves. In particular E with $E_y \approx \mathcal{O}_y \oplus \mathcal{O}_y$ cannot be S-equivalent to E' with E'_y not free unless $[E] = [E'] \in W_0$. Hence taking determinant gives a well-defined morphism det: $U' \cup U_1 \to \overline{J}_Y$ with $\det(U') = J_Y$, $\det(U_1) = \overline{J}_Y - J_Y \approx J_X$. This morphism induces a morphism of normalisations det: $P' \cup P_1 \to \tilde{J}_Y$, $\tilde{J}_Y$ being the desingularisation of $\overline{J}_Y$ and P', P_1 are respectively the pull backs of U', U_1 in the normalisation.

Lemma 2.11. *Let* $L \in \overline{J}_Y - J_Y$ *with degree of* L *even.*

(1) $\dim(U_{1,L} - U_{1,L}^s) = g_Y$, *for all* L,
(2) codim $U_{1,L} - U_{1,L}^s \geq 3$ *for* $g \geq 3$.

Proof.

(1) From §2.10, one sees that $E \in U_{1,L} - U_{1,L}^s$ is S-equivalent to $N_1 \oplus N_2$ with one of N_1, N_2 locally free and the other torsion-free but not locally free. Also, one of them is a subsheaf and the other is a quotient sheaf. By Lemma 2.1, $E \sim M \oplus (M^* \otimes L)$, $M \in J_Y$. It follows that $\dim(U_{1,L} - U_{1,L}^s) = g_Y$. In fact, one has $U_{1,L} - U_{1,L}^s \approx J_Y$.
(2) One has dim $U_{1,L} = 3g_Y - 3$. Hence codim$(U_{1,L} - U_{1,L}^s) = 2g_Y - 3 \geq 3$ for $g_Y \geq 3$.

Lemma 2.12. *For $L \in \overline{J}_Y - J_Y$ and $g_Y \geq 2$, one has* $\mathrm{codim}_{U_L}(U_L - U_{1,L}) \geq 2$.

Proof. The subset $U_L - U_{1,L}$ consists of torsion-free (semistable) rank 2 sheaves $E \approx \pi_* E_0$, E_0 semistable vector bundle of rank 2 on X with $\det E_0 \approx (\pi^* L/\text{torsion})(-x)$ or $(\pi^* L/\text{torsion})(-z)$ [1]. Hence $\dim(U_L - U_{1,L}) = 3g_X - 3$ if $g_X \geq 2$, $\dim(U_L - U_{1,L}) = 0$ if $g_X = 1$ and d is odd, $\dim(U_L - U_{1,L}) = 1$ if $g_X = 1$ and d is even. Therefore, one has for $g_Y \geq 3$, $\dim U_L - U_{1,L} = 3g_Y - 6$ and $\mathrm{codim}_{U_L}(U_L - U_{1,L}) = (3g_Y - 3) - (3g_Y - 6) = 3$. For $g_Y = 2$, $\mathrm{codim}_{U_L}(U_L - U_{1,L}) = 3$ if d is odd and $\mathrm{codim}_{U_L}(U_L - U_{1,L}) = 2$ if d is even.

Lemma 2.13.

(1) U_1^s *is non-singular*, U_1 *is normal.*
(2) $U_{1,L}$ *is normal*, $U_{1,L}^s$ *is non-singular.*
(3) W_0^s *is non-singular*, W_0 *is normal.*

Proof. The moduli space U is the geometric invariant theoretic quotient of R^{ss} by a projective linear group. Let $\mathcal{E}$ be the universal quotient sheaf on $R^{ss} \times Y$. Let $R_1 = \{t \in R^{ss} | (\mathcal{E}_t)_y \approx \mathcal{O}_y \oplus m_y\}$, $R_0 = \{t \in R^{ss} | (\mathcal{E}_t)_y \approx m_y \oplus m_y\}$, $R_{1,L} = \{t \in R_1 | \det \mathcal{E}_t = L\}$. At any point $p \in R^{ss}$, the analytic local model for $R_1 \hookrightarrow R^{ss}$ at p is Spec $A/(u, v) \hookrightarrow$ Spec A where $A = \mathbb{C}[u, v]/(uv)$ ([9], Theorem 2(2), p. 576). Since the spectrum of a point is a regular scheme, R_1 is regular. Since U_1^s is a geometric quotient of R_1^s, it follows that U_1^s is a regular scheme. Since R_1, $\overline{J}_Y - J_Y$ are regular and $R_{1,L}$ are all isomorphic, $R_{1,L}$ is regular. Hence the assertion (2) follows. We remark here that R_1, $R_{1,L}$ are not saturated for S-equivalence; U_1 and $U_{1,L}$ are G.I.T. quotients of open subsets of R_1 and $R_{1,L}$ consisting of sheaves not S-equivalent to elements in R_0 and hence are normal. The assertion (3) follows as (2) using [9], Theorem 2(3).

PROPOSITION 2.14

Let Y be an irreducible projective curve (*with one ordinary node*), $g_Y \geq 2$ *and* $n = 2$. *Then*

$$\mathrm{Pic}\, U_{1,L}^s \approx \mathbb{Z} \quad \text{or} \quad \mathbb{Z}/m\mathbb{Z}, m \in \mathbb{Z}.$$

Proof. The idea of the proof is the same as that of [6] or [3], Proposition 2.3. Hence we only indicate the necessary modifications. We may assume $d \gg 0$. Then $R^1 p_{J*}\mathcal{P}^*$ is a vector bundle on $\overline{J}_Y$. Let $\mathbb{P} = \mathbb{P}(R^1 p_{J_*}(\mathbb{P}^*))$, $\mathbb{P}_L$ = fibre of $\mathbb{P}$ over $L \in \overline{J}_Y$. One has a universal family $\mathcal{E}$ of rank 2 torsion-free sheaves E of degree d on $\mathbb{P} \times Y$. Let $\mathbb{P}^s$, $\mathbb{P}_L^s$ be the subvarieties corresponding to stable sheaves. Since $\mathrm{Ext}^1(\mathcal{O}_y, \mathcal{O}_y) = 0 = \mathrm{Ext}^1(m_y, \mathcal{O}_y)$, one has $E_y \approx \mathcal{O}_y \oplus \mathcal{O}_y$ or $\mathcal{O}_y \oplus m_y$. Hence by the universal property of moduli spaces, one has morphisms $f_\epsilon \colon \mathbb{P}^s \to (U - W_o)^s$ and $f_{\epsilon,L} \colon \mathbb{P}_L^s \to U_L'^s$ (or $U_{1,L}^s$) if $L \in J_Y$ (or $L \in \overline{J}_Y - J_Y$). By [10], Chapter 7, Lemma 5.2′, any semistable torsion-free sheaf E of $d \gg 0$ is generated by global sections. If $E_y \approx \mathcal{O}_y \oplus \mathcal{O}_y$ or $\mathcal{O}_y \oplus m_y$, then by [1], Lemma 2.7, one has an exact sequence $0 \to \mathcal{O}_Y \to E \to G \to 0$ with G torsion-free. Also $G \approx \det E$ by Lemma 2.1(1). Hence f_ϵ and $f_{\epsilon,L}$ are surjective. One shows that the induced map $f_{\epsilon,L}^*$ on Picard groups is injective. This was checked in [3] for $L \in J_Y$, the same proof goes through for $L \in \overline{J}_y - J_Y$ as $R_{1,L}^s$ and $U_{1,L}^s$ are non-singular (Lemma 2.13(2)). Let

$\mathbb{P}_{\overline{J}-J} = \mathbb{P}(R^1 p_{J_*}(\mathcal{P}^*|_{\overline{J}-J}))$ and $f_1 \colon \mathbb{P}^s_{\overline{J}-J} \to U^s_1$. The same argument gives that f_1^* is injective and one has exact sequences

$$0 \to \operatorname{Pic} U_1^s \to \operatorname{Pic} \mathbb{P}^s_{\overline{J}-J} \to \mathbb{Z}/((n-1)d/a)\mathbb{Z} \to 0, a = gcd(n, d),$$
$$0 \to \operatorname{Pic} U^s_{1,L} \to \operatorname{Pic} \mathbb{P}^s_L \to \mathbb{Z}/((n-1)d/a)\mathbb{Z} \to 0.$$

Since $\mathbb{P}^s_L$ is an open subset of a projective space, Pic $\mathbb{P}^s_L$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$ and the same as true for Pic $U^s_{1,L}$.

We remark that the injectivity of f_ϵ^* does not seem to follow similarly. In the notations of [6], Corollary 7.4, one certainly gets a codimension one subvariety $\Gamma_0 - \Gamma_0'$ of Γ_0. Since $(U - W_0)^s$ is not necessarily non-singular it is not clear that $\Gamma_0 - \Gamma_0'$ is a Cartier divisor, i.e., its ideal sheaf is locally free. $U - W_0$ is seminormal, but not normal in general, in particular it is not locally factorial.

PROPOSITION 2.15

Let the notations be as in Proposition 2.14. *Then for* $g_Y \geq 3$, $n = 2$ *and* $g_Y = 2$, $n = 2$, d *odd, one has*

$$\operatorname{Pic} U_{1,L} \approx \operatorname{Pic} U^s_{1,L} \approx \mathbb{Z}.$$

Proof. For d odd, $U_{1,L} = U^s_{1,L}$. Since $U_{1,L}$ is normal and $\operatorname{codim}(U_{1,L} - U^s_{1,L}) \geq 3$ (Lemma 2.11), Pic $U_{1,L} \hookrightarrow$ Pic $U^s_{1,L}$ for d even, $g_Y \geq 3$ as in the proof of Proposition 2.8. Going to a finite normalisation we see that rank (Pic $U_{1,L}) \geq 1$. We need Lemma 2.12 for this. The result now follows from Proposition 2.14.

2.16

Assume that $g_Y = 2, g_X = 1, n = 2, d = 0$. Let M be the moduli space of α-semistable GPBs $(E, F_1(E))$ of rank 2, degree 0 on a smooth elliptic curve $X, 0 < \alpha < 1, \alpha$ being close to 1 [1]. Let M_L be the closed subscheme of M corresponding to E with determinant $L, L \in J_X$. Let $p_1 \colon F_1(E) \to E_x, p_2 \colon F_1(E) \to E_z$ be the projections. Define $D_L = \{(E, F_1(E)) \in M_L | p_2$ has rank $\leq 1\}$ and $D_{1,L} = \{(E, F_1(E)) \in D_L |$rank $p_2 = 1$, p_1 isomorphism$\}$. $D_{1,L}$ is an open subscheme of D_L and D_L is a closed subscheme of codimension 1 in D. There is a surjective birational morphism $f \colon M \to U$ such that D_L maps onto $U_{L'}$ inducing an isomorphism $D_{1,L} \approx U_{1,L'}$ where $L' = \pi_*(L(-z))$. We shall determine D_L, $D_{1,L}$ explicitly and use the explicit description to compute Pic $U_{1,L'}$. Note that $D_L \approx D_{\mathcal{O}}$ for all L.

PROPOSITION 2.17

D_L *is isomorphic to a* $\mathbb{P}^2$*-bundle over* $\mathbb{P}^1$. *Outside* $\mathbb{P}^1 - \{4$ *points*$\}$, *this bundle is of the form* $\mathbb{P}(\mathcal{O} \oplus \epsilon)$, ϵ *being a rank* 2 *vector bundle.*

Proof. It is not difficult to check that $(E, F_1(E))$ of degree 0, rank 2 is α-semistable if and only if E is a semistable vector bundle and for any line sub-bundle L of E of degree 0, $F_1(E) \neq L_x \oplus L_z$. Moreover, $(E, F_1(E))$ is α-stable if and only if E is semistable and $F_1(E) \cap (L_x \oplus L_z) = 0$ for any sub-bundle of degree 0.

Let e_1, e_2 and e_3, e_4 be the bases of E_x and E_z respectively. The subspace $F_1(E)$ defines a point in the Grassmannian Gr of two-dimensional subspaces of $V = E_x \oplus E_z$. Let $\mathrm{Gr} \subset \mathbb{P}(\wedge^2 V)$ be the Plücker embedding, let $(X_1, Y_1, X_2, Y_2, X_3, Y_3)$ be the Plücker coordinates. Any element in $\wedge^2 V$ is of the form $X_1 e_1 \wedge e_2 + Y_1 e_3 \wedge e_4 + X_2 e_1 \wedge e_4 + Y_2 e_2 \wedge e_3 + X_3 e_3 \wedge e_1 + Y_3 e_2 \wedge e_4$. The Grassmannian quadric is given by $X_1 Y_1 + X_2 Y_2 + X_3 Y_3 = 0$. Since E is semistable, one has either (a) $E = M \oplus M^*$, $M \in J_X$ or (b) there is a non-trivial extension $0 \to M_1 \xrightarrow{g} E \xrightarrow{h} M_2 \to 0$ with $M_1 \approx M_2 \approx M \in J_X$, $M^2 = \mathcal{O}$. In either case E is an extension of M_2 by M_1; $M_1, M_2 \in J_X$. Choose e_1, e_2, e_3, e_4 to be basis elements of $(M_1)_x, (M_2)_x, (M_1)_z, (M_2)_z$ respectively. Let $D_V \subset \mathrm{Gr}$ be defined by $Y_1 = 0$.

Case (a). Assume that $E = M_1 \oplus M_2, M_1^* = M_2, M_1 \neq M_2$. The group $\mathbb{P}$ (Aut $E) = \mathbb{P}(G_m \times G_m) \approx G_m$ acts on $D_V \subset \mathbb{P}(\wedge^2 V)$ by $t(X_1, X_2, Y_2, X_3, Y_3) = (X_1, X_2, Y_2, tX_3, t^{-1} Y_3)$. It is easy to see that $D_V // G_m \approx \mathbb{P}^2$, the quotient map $D_V \to \mathbb{P}^2$ being given by $(X_1, X_2, Y_2, X_3, Y_3) \mapsto (X_1, X_2, Y_2)$. Let $D_{1,V} = D_V - \{(X_1 = 0) \cup (1, 0, 0, 0, 0)\}$. The image of $D_{1,V}$ in $\mathbb{P}^2$ is given by $\mathbb{P}^2 - \{(X_1 = 0) \cup (1, 0, 0)\}$.

Let $\mathcal{P}_X \to J_X \times X$ be the Poincaré bundle, $\mathcal{P}_x = \mathcal{P}|_{J'_X \times x}$, $\mathcal{P}_z = \mathcal{P}|_{J'_X \times z}$, $J'_X = J_X - J_2$, J_2 being the group of 2-torsion points of J_X. The group $G_m \times G_m$ acts on the bundles $\mathbb{V} = (\mathcal{P}_x \oplus \mathcal{P}_x^*) \oplus (\mathcal{P}_z \oplus \mathcal{P}_z^*)$, and $\wedge^2 \mathbb{V}$ as above, giving G_m-action on $\mathbb{P}(\wedge^2 \mathbb{V})$ and $D_{\mathbb{V}} // G_m \approx \mathbb{P}^2$-bundle over J'_X. This $\mathbb{P}^2$-bundle is in fact the bundle $\mathbb{P}(\mathcal{O} \oplus (\mathcal{P}_x \otimes \mathcal{P}_z^*) \oplus (\mathcal{P}_z \otimes \mathcal{P}_x^*))$. The involution on J_X given by $i(M) = M^*$ lifts to an action on this bundle (switching second and third factors), hence it descends to a bundle on $J'_X / i = \mathbb{P}^1 - \{4$ points$\}$, of the form $\mathbb{P}(\mathcal{O} \oplus \epsilon)$, ϵ a vector bundle of rank 2 on J'_X / i.

Case (b). There are, up to isomorphism, exactly four bundles E given by extension of type (b). Since any automorphism of E is of the form $\lambda Id + \mu g \circ h$, one has $\mathbb{P}$ (Aut $E) \approx G_a$ under the isomorphism $(\lambda, \mu) \mapsto t = \mu \lambda^{-1} \in G_a$. The action of G_a on V is given by $te_1 = e_1, te_3 = e_3, te_2 = e_2 + te_1, te_4 = e_4 + te_3$ and that on D_V is given by $t(X_1, X_2, Y_2, X_3, Y_3) = (X_1, Y_2 + tY_3, X_2 + tY_3, X_3 - t(X_2 + Y_2) - t^2 Y_3, Y_3)$. It is not difficult to see that the ring of invariants for G_a-action on D_V (resp. on the hyperplane $Y_1 = 0$ of $\mathbb{P}(\wedge^2 V)$) is generated by $X_1, X_2 - Y_2, Y_3$ (resp. $X_1, X_2 - Y_2, Y_3, X_2 Y_2 + X_3 Y_3$). The non-semistable points for the G_a-action are $\{X_1 = Y_3 = X_2 - Y_2 = 0\}$. It follows that $D_V // G_a \approx \mathbb{P}^2$, the quotient map $D_V \to \mathbb{P}^2$ being given by $(X_1, X_2, Y_2, X_3, Y_3) \to (X_1, X_2 - Y_2, Y_3)$. Clearly, $D_{1,V} // G_a \approx \mathbb{P}^2 - (\{X_1 = 0\} \cup (1, 0, 0))$. We remark that non-stable GPBs correspond to the line $Y_3 = 0$ in $\mathbb{P}^2$. In case $E = M_1 \oplus M_2$, $M_1 = M_2$ with $M_1^2 = \mathcal{O}$, one sees that corresponding quotient $D_V // G_a$ is $\mathbb{P}^1$ which is identified to the line $Y_3 = 0$ in the above $\mathbb{P}^2$. Note that there are no stable GPBs in the last case.

It follows that there is a $\mathbb{P}^2$-fibration $\phi\colon D_L \to \mathbb{P}^1$ which is locally trivial outside the set of four points in $\mathbb{P}^1$. By Tsen's theorem ([8], p. 108, Case (d)), ϕ is a locally trivial fibration. This completes the proof.

COROLLARY 2.18

Let $g_X = 1$, $g_Y = 2$, d even, $n = 2$.

(1) *$U_{1,L'}$ is non-singular.*

(2) Pic $U_{1,L'} \approx \mathbb{Z}$.

Proof.

(1) It follows immediately from the proof of Proposition 2.18 that $D_{1,L}$ is a (locally trivial) fibration over $\mathbb{P}^1$ with non-singular fibres isomorphic to $\mathbb{P}^2 - \{(X_1 = 0) \cup (1, 0, 0)\}$. Hence $D_{1,L}$ and $U_{1,L'}$ are non-singular.

(2) $D_L - D_{1,L} \cong$ (hyperplane H) $\cup$ {a line ℓ}, $H \cap \ell = \Phi$, Pic $D_L \approx$ Pic $\mathbb{P}^1 \oplus$ Pic $\mathbb{P}^2$. Since D_L is non-singular, $0 \to \mathbb{Z}H \to \text{Pic } D_L \to \text{Pic}(D_L - H) \to 0$ is exact. It follows that Pic $D_L - H \cong$ Pic $\mathbb{P}^1 = \mathbb{Z}$. Since ℓ is of codimension 2, $\text{Pic}(D_{1,L}) \approx \text{Pic}(D_L - H) \cong \mathbb{Z}$. Thus Pic $U_{1,L'} \approx$ Pic $D_{1,L} \approx \mathbb{Z}$.

Remark 2.19. Note that $H \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle. The fibres of this bundle are given by $X_1 = 0$ in D_V, the restriction of this bundle to $\mathbb{P}^1 - \{4 \text{ points}\}$ is $\mathbb{P}(\epsilon)$. Under the map $D_L \to U_{L'}$, this $\mathbb{P}^1$-bundle maps onto one component in $U_{L'} - U_{1,L'}$ isomorphic to J_X/i ($\approx \mathbb{P}^1$). This component corresponds to sheaves of the form $\pi_* E_0$, det $E_0 \approx L(-x-z)$. The line ℓ maps isomorphically onto the other component isomorphic to $\mathbb{P}^1$, it corresponds to $\pi_* E_0$, det $E_0 \approx L(-2z)$. Since $g_X = 1$, E_0 are semistable but not stable. Thus unlike in the case when L' is a line bundle (Y smooth or nodal) $U_{L'} - U^s_{L'}$ is not the Kummer variety. It has an open subset isomorphic to J_Y (Proof of Lemma 2.11(1)) whose complement is the union of two disjoint smooth rational curves.

Putting together Proposition 2.15 and Corollary 2.18, we have proved the following.

Theorem 2. *Let Y be an irreducible projective curve of arithmetic genus ≥ 2 with only a single ordinary node as singularity. Let L be a rank 1 torsion-free sheaf which is not locally free. Then*

$$\text{Pic } U_{1,L} \approx \mathbb{Z}.$$

3. Pic and local factoriality of $U'(n,d)$, $U_{1,L}(2,d)$

3.1

In this section we prove Theorems 3A and 3B. Throughout the section, we assume that $n \geq 2$ and if $n \geq 3$ then $g \geq 2$. One has a map $U'_L \times J \to U'$ given by tensorisation. We first remark that Pic U' cannot be computed easily using this map. The map induces a map of Picard groups Pic $U' \approx$ Pic $U'_L \oplus$ Pic $J \to$ Pic $U'_L \oplus$ Pic J. The induced map Pic $J \to$ Pic J is not identity, it is multiplication by n. The right map to consider is the determinant morphism which does induce identity on Pic J as we show below:

Theorem 3A. *One has the following*:

(a) Pic $U'^s \approx$ Pic $J \oplus \mathbb{Z}$,

(b) Pic $U' \approx$ Pic $J \oplus \mathbb{Z}$,

(c) *U' is locally factorial.*

Proof.

(a) Without loss of generality, we may assume that $d \gg 0$. Then a semistable vector bundle E of degree d is globally generated ([10], Lemma 5.2) and contains a trivial sub-bundle of rank $n-1$. Let $\mathbb{P} = \mathbb{P}(R^1_{p_{j*}}(\mathcal{P}^* \otimes \mathbb{C}^{n-1}))$, it is a projective bundle over J. Let $\mathbb{P}_L$ denote its fibre over $L \in J$, $\mathbb{P}_L$ is a projective space. $\mathbb{P}$ parametrises a family $\mathcal{E}$ of vector bundles on Y of rank n, degree d and containing a trivial sub-bundle of rank $n-1$. Let $\mathbb{P}^s = \{p \in \mathbb{P} | \mathcal{E}_p \text{ stable}\}$, $\mathbb{P}^s_L = \mathbb{P}^s \cap \mathbb{P}_L$. One has canonical surjective morphisms $f\colon \mathbb{P}^s \to U'^s(n,d)$, $f_L\colon \mathbb{P}^s_L \to U'^s_L(n,d)$ such that the induced maps $f^*\colon \text{Pic } U'^s \to \text{Pic } \mathbb{P}^s$, $f^*_L\colon \text{Pic } U'^s_L \to \text{Pic } \mathbb{P}^s_L$ are injective ([3], Proposition 2.3; [6], Propositions 7.6, 7.8, 7.9). Clearly, Pic $\mathbb{P} \approx$ Pic $J \times$ Pic $\mathbb{P}_L \approx$ Pic $J \times \mathbb{Z}$. Under the conditions of the theorem we know that ([3], Theorem I) Pic $U'^s_L \approx \mathbb{Z}$ and hence Pic $\mathbb{P}^s_L \approx \mathbb{Z}$. Hence the surjective restriction map Pic $\mathbb{P}_L \to$ Pic $\mathbb{P}^s_L$ is an isomorphism for all $L \in J$. Hence $\text{codim}_{\mathbb{P}_L}(\mathbb{P}_L - \mathbb{P}^s_L) \neq 1$ and therefore $\text{codim}_{\mathbb{P}}(\mathbb{P} - \mathbb{P}^s) \geq 2$. Thus Pic $\mathbb{P}^s \approx$ Pic $\mathbb{P} \approx$ Pic $J \oplus \mathbb{Z}$ and hence

$$\text{Pic } U'^s \hookrightarrow \text{Pic } J \oplus \mathbb{Z}.$$

The natural map $p\colon \mathbb{P}^s \to J$ factors as $p = \det \circ f$, where det is the determinant map $E \mapsto \overset{n}{\wedge} E$. Since both f and det are surjections, so is p. Note that $f^* \circ \det^* = p^*$: Pic $J \to$ Pic $\mathbb{P}^s$ is injective. It follows that $\det^*$ is injective.

One has the following diagram with the last column exact.

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\text{Pic } J & = & \text{Pic } J & = & \text{Pic } J \\
\downarrow & & \downarrow & & \downarrow \\
\text{Pic } U' & \to & \text{Pic } U'^s & \hookrightarrow & \text{Pic } J \oplus \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Pic } U'_L & \overset{\approx}{\to} & \text{Pic } U'^s_L & \overset{\approx}{\to} & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Here $\mathbb{Z}$ denotes the image of Pic U'^s_L in Pic $\mathbb{P}^s_L$. The map Pic $U'^s \to$ Pic U'^s_L is the restriction map and is surjective ([3], Proposition 3.2 and 3.5). It now follows from the diagram that the injection Pic $U'^s \to$ Pic $J \oplus \mathbb{Z}$ is an isomorphism and the second column is exact.

(b) and (c). Since $\text{codim}_{U'}(U' - U'^s) \geq 2$ under the conditions of the theorem and U' is normal ([3], Proposition 3.4(i)), it follows that the restriction map Pic $U' \to$ Pic U'^s is injective. The restriction morphism Pic $U' \to$ Pic U'_L is surjective ([3], Propositions 3.2, 3.5). The restriction map Pic $U'_L \to$ Pic U'^s_L is an isomorphism [3]. It now follows from the commutative diagram that Pic $U' \approx$ Pic U'^s under the restriction map. By arguments similar to those in the proof of [3], Proposition 3.6, this implies that U' is locally factorial.

Theorem 3B. *Let Y be an irreducible projective curve of arithmetic genus $g_Y \geq 2$ with only a single ordinary node as singularity. If $g_Y = 2$, then assume that d is odd. Let L be a* *rank* 1 *torsion-free sheaf of degree* d *which is not locally free. Let* $U_{1,L}$ *be the subscheme of* U *corresponding to torsion-free sheaves of rank* 2 *with determinant isomorphic to* L.

(a) Pic $U_1^s \approx$ Pic $J_X \oplus \mathbb{Z}$,
(b) Pic $U_1 \approx$ Pic $J_X \oplus \mathbb{Z}$,
(c) U_1 *is locally factorial.*

Proof. The proof is more or less identical with that of Theorem 3A. One has only to replace f, f_L by the maps f_1, $f_{\epsilon,L}$ of Proposition 2.14 and use Theorem 2 instead of Theorem 1.

4. The dualising sheaves of U' and U'_L

4.1

Let $K(Y)$ denote the Grothendiéck group of vector bundles on Y. Then $K(Y) \approx \mathbb{Z} \oplus$ Pic Y under the map $[E] \mapsto$ (rank E, det E), $[E]$ being the class of a vector bundle E in $K(Y)$. The inverse map is given by $n \mapsto [n \cdot \mathcal{O}_Y]$ for $n \in \mathbb{Z}$ and $L \mapsto [L] - [\mathcal{O}_X]$ for $L \in$ Pic Y.

Let $\chi = d + n(1-g)$, $P(m) = \chi + rm$, fix $m \gg 0$. Let $Q = \mathrm{Quot}(\mathbb{C}^{P(m)} \otimes \mathcal{O}_Y(-m), P)$ be the Hilbert scheme ('the Quot scheme') of quotients of $\mathbb{C}^{P(m)} \otimes \mathcal{O}_Y(-m)$ with Hilbert polynomial P. Let $\mathcal{F} \to Q \times Y$ be the universal family. Let $R_m \subset Q$ be the open subset consisting of $q \in Q$ such that $H^1(\mathcal{F}_q(m)) = 0$, $H^0(\sum(m)) \simeq H^0(\mathcal{F}(m))$ under the canonical map, $\sum = \mathbb{C}^{P(m)} \otimes \mathcal{O}_Y(-m)$. The open subvariety R^{ss} of Q consisting of $q \in Q$ such that $\mathcal{F}_q$ is a semistable torsion-free sheaf is contained in R_m. The subset R'^{ss} of R^{ss} corresponding to semistable vector bundles is a smooth variety, so is the closed subset $R_L'^{ss} \subset R'^{ss}$ consisting of semistable vector bundles with fixed determinant L ([10], Remark, p. 167).

The moduli space U' (resp. U'_L) is a geometric invariant theoretic good quotient of the smooth irreducible scheme R'^{ss} (resp. $R_L'^{ss}$) by the group $G = P(\mathrm{Aut} \sum) \approx PGL(N)$, $N \gg 0$ [10,12]. The restriction of the universal family on $Q \times Y$ gives a universal family $\mathcal{F} \to R_L'^{ss} \times Y$ of vector bundles on Y of rank n, degree d. Let Pic$^G(R_L'^{ss})$ denote the group of line bundles on $R_L'^{ss}$ with G-action (compatible with the G-action on $R_L'^{ss}$). For a vector bundle E on Y, one defines an element $\lambda_{\mathcal{F}}(E) \in \mathrm{Pic}^G(R_L'^{ss})$ by

$$\lambda_{\mathcal{F}}(E) := \otimes_i (\det R^i_{p_{1*}}(\mathcal{F} \otimes p_2^* E))^{(-1)^{i+1}},$$

where p_1 and p_2 are projections to $R_L'^{ss}$ and Y respectively. $\lambda_{\mathcal{F}}(E)$ depends only on the class of E and $\lambda_{\mathcal{F}}\colon K(Y) \to \mathrm{Pic}^G(R_L'^{ss})$ is a group homomorphism.

PROPOSITION 4.2

Let E be a vector bundle on Y with $\mathrm{rank}(E) = n/\delta$, $\det(E) = \mathcal{O}_Y(-\frac{\chi}{\delta})$, $\chi = d + n(1-g)$, $\delta = gcd(n,d)$. *Then $\lambda_{\mathcal{F}}(E)$ descends to $U'_L(n,d)$ as the generator $\mathbb{L}$ of* Pic $U'_L(n,d)$.

Proof. By [3], Propositions 3.2, 3.5, the generator $\mathbb{L}$ is obtained by the descent of the line bundle $\mathbb{L}'$ on $R_L'^{ss}$ given by

$$\mathbb{L}' = (\det Rp_{1_*}\mathcal{F})^{\frac{n}{\delta}} \otimes (\overset{n}{\wedge} (\mathcal{F}|_{R_L'^{ss} \times y_0}))^{\chi/\delta},$$

y_0 being a non-singular point of Y. Here det $Rp_{1_*}\mathcal{F}$ denotes the determinant of cohomology ([7], Ch.VI, pp. 135–136). However, our definition is different from the standard one, it is the inverse of the line bundle defined in [7] as det $Rp_{1_*}\mathcal{F}$. One has det $R_{p_{1*}}(\mathcal{F}) = \lambda_{\mathcal{F}}(1)$, $1 =$ class of $\mathcal{O}_Y$. If h denotes the class of the structure sheaf of the point y_0, $h = [\mathcal{O}_Y(y_0)] - [\mathcal{O}_Y]$, then we claim that

$$\overset{n}{\wedge}\mathcal{F}\,|_{R_L'^{ss}\times y_0} = -\lambda_{\mathcal{F}}(h).$$

Proof of the Claim. For $m \gg 0$ one has the exact sequence

$$0 \to \mathcal{F}(m) \to \mathcal{F}(m+1) \to \mathcal{F}(m)\,|_{R_L'^{ss}\times y_0} \to 0,$$

$\mathcal{F}(m) = \mathcal{F}\otimes\mathcal{O}_Y(m)$, $\mathcal{O}_Y(1)$ being a line bundle of degree 1 on Y. Since $R^1_{p_{1*}}(\mathcal{F}(m')) = 0$ $\forall m' \geq m$, $R^1 p_{1_*}(\mathcal{F}(m)|_{R_L'^{ss}\times y_0}) = 0$, the direct image sequence gives

$$0 \to R^0 p_{1_*}(\mathcal{F}(m)) \to R^0 p_{1_*}(\mathcal{F}(m+1)) \to R^0 p_{1_*}(\mathcal{F}(m)|_{R'^{ss}_{R_L}\times y_0}) \to 0.$$

Since det $p_{1_*}(\mathcal{F}(m')) = -\lambda_{\mathcal{F}}(1 + m'h)$, $m' \geq m$, and

$$\det(p_{1_*}\mathcal{F}(m)|_{R_L'^{ss}\times y_0}) \approx \det(p_{1_*}\mathcal{F}|_{R_L'^{ss}\times y_0}) = \overset{n}{\wedge}\mathcal{F}\,|_{R_L'^{ss}\times y_0},$$

one has

$$\begin{aligned}\overset{n}{\wedge}\mathcal{F}|_{R_L'^{ss}\times y_0} &= -\lambda_{\mathcal{F}}((m+1)h) + \lambda_{\mathcal{F}}(1+mh)\\ &= -\lambda_{\mathcal{F}}(h).\end{aligned}$$

This proves the claim.

Thus we have

$$\begin{aligned}\mathbb{L}' &= \frac{n}{\delta}\lambda_{\mathcal{F}}(1) - \frac{\chi}{\delta}\lambda_{\mathcal{F}}(h)\\ &= \lambda_{\mathcal{F}}\left(\frac{n}{\delta} - \frac{\chi h}{\delta}\right) = \lambda_{\mathcal{F}}(E).\end{aligned}$$

Remark 4.3. Note that the line bundle $\mathbb{L}'$ exists on R^{ss} and descends to U' ([3], Proposition 3.5). Also $\lambda_{\mathcal{F}}(E)$ makes sense for $\mathcal{F} \to R^{ss} \times Y$, the universal family on $R^{ss} \times Y$. The above relation between $\lambda_{\mathcal{F}}(E)$ and $\mathbb{L} \in$ Pic $U'_L(n,d)$ holds for $\lambda_{\mathcal{F}}(E)$ and $\mathbb{L} \in$ Pic $U'(n,d) \approx$ Pic $U'_L \oplus$ Pic J.

4.4 *Computation of the dualising sheaves*

Both U' and U'_L are normal and Cohen–Macaulay as they are quotients of smooth varieties by $PGL(N)$. They are also locally factorial ([3], Theorem 2; Theorem 1). A locally factorial Cohen–Macaulay variety is Gorenstein, i.e., its dualising sheaf ω is locally free. The tangent sheaf $T_{U'}$ of U' is locally free on the smooth open subscheme U'^s of codimension ≥ 2. Hence the determinant of $T_{U'}$ defines a line bundle det $T_{U'}$ on U'. Since it coincides with ω^{-1} on U'^s, it follows that $\omega^{-1} =$ det $T_{U'}$. Similarly one has a locally free dualising sheaf ω_L on U'_L with $\omega_L^{-1} =$ det $T_{U'_L}$.

Theorem 4. *Let the assumptions be as in Theorem* 1. *Then one has the following*:

(a) $\omega \approx -2\delta\mathbb{L}$, $\mathbb{L} =$ *generator of* Pic $U'_L(n,d)$,
(b) *Let* F_0 *be a vector bundle on* Y *of rank* $2r$ *and degree* $2(-d + r(g-1))$. *Then* $\omega \approx \lambda_{\mathcal{F}}(F_\circ) \otimes \det \wedge$, *where* $\wedge$ *is a line bundle on* J *given by*

$$\wedge = \det(p_{J!}[\mathcal{P}] \otimes \det p_{J!}[\mathcal{P}^*])^{r-1} \otimes \det p_{J!}([\mathcal{P} \otimes p_2^* F_0])^{-1}.$$

Proof. In view of the injective morphism f_L^*: Pic $U'_L \to$ Pic P_L^s mapping $\mathbb{L}$ to $\mathcal{O}_{p_L^s}(\frac{d}{\delta}(r-1))$, it suffices to prove that

$$\det f_L^* T_{U'_L} \approx \mathcal{O}_{\mathbb{P}_L^s}(2d(r-1)).$$

One has $f^* T_{U'} \approx R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E}^* \otimes \mathcal{E})$, $f_L^* T_{U'_L} \approx R^1_{p_{\mathbb{P}^s_{L*}}}(Ad\mathcal{E}) \approx R^1_{p_{\mathbb{P}^s_*}}(Ad\mathcal{E})\,|_{\mathbb{P}^s_L}$. Also, det $R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E}^* \otimes \mathcal{E}) \approx \det R^1_{p_{\mathbb{P}^s_*}}(Ad\mathcal{E})$, so that det $f_L^* T_{U'_L} \approx \det R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E} \otimes \mathcal{E}^*)|\mathbb{P}^s_L$.

Computation of det $R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E} \otimes \mathcal{E}^*)$

There is a universal exact sequence on $\mathbb{P}^s \times Y$.

$$0 \to \mathcal{O}_{\mathbb{P}^s \times Y} \otimes \mathbb{C}^{r-1} \to \mathcal{E} \to (1 \times p)^* \mathcal{P} \otimes p^*_{\mathbb{P}^s} \mathcal{O}_{\mathbb{P}^s}(-1) \to 0. \tag{1}$$

For $d \gg 0$, $H^0(\mathcal{E}_t^*) = 0 \forall t \in \mathbb{P}^s$, $H^0(\mathcal{E}_t \otimes \mathcal{E}_t^*)$ consists of scalars as $\mathcal{E}_t$ is stable. Hence by tensoring (1) with $\mathcal{E}^*$ and taking direct images, one gets (for $d \gg 0$ and $(1 \times p)^* = p^\#$)

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^s} &\to \mathcal{O}_{\mathbb{P}^s}(-1) \otimes p_{\mathbb{P}^s_*}(p^\# \mathcal{P} \otimes \mathcal{E}^*) \to R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E}^* \otimes \mathbb{C}^{r-1}) \\ &\to R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E}^* \otimes \mathcal{E}) \to 0. \end{aligned}$$

Hence,

$$\begin{aligned} \det R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E} \otimes \mathcal{E}^*) &\approx \det(R^1_{p_{\mathbb{P}^s_*}} \mathcal{E}^*)^{r-1} \\ &\otimes \det(\mathcal{O}_{\mathbb{P}^s}(-1) \otimes p_{\mathbb{P}^s_*} p^\# \mathcal{P} \otimes \mathcal{E}^*)^{-1}. \end{aligned} \tag{2}$$

$R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E}^*)$ is computed by taking dual of (1) and direct images as follows:

$$0 \to p^\# \mathcal{P}^* \otimes p^*_{\mathbb{P}^s} \mathcal{O}_{\mathbb{P}^s}(1) \to \mathcal{E}^* \to \mathcal{O}_{\mathbb{P}^s \times Y} \otimes \mathbb{C}^{r-1} \to 0. \tag{1*}$$

Since $p_{\mathbb{P}^s_*} p^\# \mathcal{P}^* = 0 = p_{\mathbb{P}^s *}(\mathcal{E}^*)$ for $d \gg 0$, one has the direct image sequence

$$\begin{aligned} 0 &\to \mathcal{O}_{\mathbb{P}^s} \otimes \mathbb{C}^{r-1} \to \mathcal{O}_{\mathbb{P}^s}(1) \otimes R^1_{p_{\mathbb{P}^s_*}}(p^\# \mathcal{P}^*) \to R^1_{p_{\mathbb{P}^s_*}} \mathcal{E}^* \\ &\to \mathcal{O}_{\mathbb{P}^s} \otimes \mathbb{C}^{(r-1)g} \to 0 \end{aligned}$$

and hence

$$\det R^1_{p_{\mathbb{P}^s_*}} \mathcal{E}^* \approx \det(\mathcal{O}_{\mathbb{P}^s}(1) \otimes R^1_{p_{\mathbb{P}^s_*}}(p^\# \mathcal{P}^*)).$$

Since $h^1(\mathcal{P}_t^*) = -\chi(\mathcal{P}_t^*) = d + g - 1$ for $t \in J$, one gets

$$\det R^1_{p_{\mathbb{P}^s_*}} \mathcal{E}^* \approx \mathcal{O}_{\mathbb{P}^s}(d + g - 1) \otimes \det R^1_{p_{\mathbb{P}^s_*}}(p^{\#}\mathcal{P}^*). \tag{3}$$

Tensoring $(1)^*$ with $p^{\#}\mathcal{P}$ gives

$$0 \to p^*_{\mathbb{P}^s}\mathcal{O}_{\mathbb{P}^s}(1) \to \mathcal{E}^* \otimes p^{\#}\mathcal{P} \to \mathcal{O}_{\mathbb{P}^s \times Y} \otimes \mathbb{C}^{r-1} \otimes p^{\#}\mathcal{P} \to 0,$$

and hence the direct image sequence

$$0 \to \mathcal{O}_{\mathbb{P}^s}(1) \to p_{\mathbb{P}^s_*}(\mathcal{E}^* \otimes p^{\#}\mathcal{P}) \to p_{\mathbb{P}^s_*}(\mathbb{C}^{r-1} \otimes p^{\#}\mathcal{P}) \to 0.$$

By tensoring with $\mathcal{O}_{\mathbb{P}^s}(-1)$ and taking det, one has

$$\det(p_{\mathbb{P}^s_*}(p^{\#}\mathcal{P} \otimes \mathcal{E}^*) \otimes \mathcal{O}_{\mathbb{P}^s}(-1)) \approx \det(p_{\mathbb{P}^s_*}(p^{\#}\mathcal{P} \otimes \mathbb{C}^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^s}(-1)).$$

Since $h^0(\mathcal{P}_t) = d + 1 - g$ for $t \in J$, the latter is isomorphic to $\det p_{\mathbb{P}^s_*}(p^{\#}\mathcal{P} \otimes \mathbb{C}^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^s}((g - d - 1)(r - 1))$. Thus we have

$$\begin{aligned} &\det(p_{\mathbb{P}^s_*}(p^{\#}\mathcal{P} \otimes \mathcal{E}^*) \otimes \mathcal{O}_{\mathbb{P}^s}(-1)) \\ &\quad \approx \det p_{\mathbb{P}^s_*}(p^{\#}\mathcal{P} \otimes \mathbb{C}^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^s}((r - 1)(g - d - 1)). \end{aligned} \tag{4}$$

Substituting in (2) from (3) and (4) gives

$$\det R^1_{p_{\mathbb{P}^s_*}}(\mathcal{E}^* \otimes \mathcal{E}) \approx \mathcal{O}_{\mathbb{P}^s}(2(r - 1)d) \otimes \Delta^{r-1}, \tag{5}$$

where $\Delta^{-1} = \det(R^1_{p_{\mathbb{P}^s_*}} p^{\#}\mathcal{P}^*) \otimes \det(p_{\mathbb{P}^s_*} p^{\#}\mathcal{P})$.

Since $\Delta|_{\mathbb{P}^s_L}$ is trivial, from (5) one has

$$\det f_L^* T_{U'_L} \approx \mathcal{O}_{\mathbb{P}^s_L}(2(r - 1)d),$$

this proves (a).

If F_0 is a vector bundle of rank $2r$ and degree $2(-d + r(g - 1))$, then from sequence (1), one sees that

$$\lambda_{\mathcal{E}}([F_0]) \approx \mathcal{O}_{\mathbb{P}^s}(-2d(r - 1)) \otimes \det{}^*(p_{J_*}\mathcal{P} \otimes p_Y^* F_0),$$

so that (5) becomes

$$\det(R^1 p_{\mathbb{P}^s_*}(\mathcal{E} \otimes \mathcal{E}^*)) \approx \lambda_{\mathcal{E}}([F_0])^{-1} \otimes \det{}^*(p_{J_*}(\mathcal{P} \otimes p_Y^* F_0)) \otimes \Delta^{r-1}.$$

Since $p = \det \circ f$, $p^* = f^* \circ \det^*$ and f^* is injective, (b) also follows.

Acknowledgement

We would like to thank C S Seshadri for useful discussions and correspondence.

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