Astron. Astrophys. 239, 335–339 (1990)

ASTRONOMY AND ASTROPHYSICS

A correction to Spruit's equation for the dynamics of thin flux tubes

A.R. Choudhuri

Department of Physics, Indian Institute of Science, Bangalore-560012, India

Received November 3, 1989; accepted April 20, 1990

Abstract. It is pointed out that a term was overlooked in the derivation of the equation of motion for a thin flux tube by Spruit (1981). The correction to be applied in an inertial frame and in a rotating frame are discussed. This correction makes the formulation self-consistent, though it does not invalidate the qualitative results obtained by various investigators who have used Spruit's equation.

Key words: solar magnetic fields – flux tubes – hydromagnetics

1. Introduction

The equation of motion for a thin flux tube moving in an ambient atmosphere was derived in a paper by Spruit (1981). Since Spruit's equation is generally believed to be the basic equation for studying the dynamics of magnetic flux tubes in the solar convection zone, it has been used by several investigators (Spruit and van Ballegooijen, 1982; van Ballegooijen, 1983; Moreno-Insertis, 1986; Choudhuri and Gilman, 1987; van Ballegooijen and Choudhuri 1988; Chou and Fisher, 1989; Choudhuri, 1989). In this paper, we wish to point out a small correction that should be applied to this equation. This correction seems to have been overlooked so far. Since the calculations done by the above authors were mainly aimed at elucidating broad qualitative features of the dynamics of flux tubes, their conclusions still remain valid in spite of this small correction being overlooked. However, application of this small correction makes the equation completely satisfactory at a conceptual level. We shall point out that the equation may lead to some inconsistencies if this correction is not applied.

It is well-known that an object of cylindrical shape accelerating through a fluid imparts kinetic energy to the surrounding fluid, and hence the transverse component of motion involves an effective mass instead of just the mass of the cylinder (Lamb, 1945, p. 77). In the case of a right circular cylinder, the effective mass turns out to be its own mass plus the mass of fluid displaced by it. This effect is introduced through Eq. (12) of Spruit (1981), which is

$$(\rho+\rho_e)\left(\frac{d\boldsymbol{v}}{dt}\right)_\perp=\boldsymbol{F}_\perp \tag{1}$$

where ρ is the density inside the flux tube and ρ_e the density of the surrounding fluid, and the subscript $\perp$ used throughout this paper refers to components of vectors transverse to the local tangent at any point of the flux tube. This equation is certainly true for a cylinder with a straight axis. However, in the case of a flux tube with a curved axis, even when the fluid inside the flux tube just moves parallel to the axis of the flux tube without causing any displacement of the surrounding fluid, there is a transverse acceleration arising out of the fact that the fluid inside the flux tube is constrained to move in a curved path. Hence some correction has to be applied to (1) such that the part of transverse acceleration merely arising out of curvature is not associated with any extra effective mass. Failure to do so may lead to inconsistensies, as can be seen in some of the equations of Choudhuri and Gilman (1987) where this correction was overlooked. They obtained the equation of motion in spherical coordinates for a flux ring symmetric around the rotation axis. The r-component of the equation of motion for the flux ring is given in Eq. (4) of Choudhuri and Gilman (1987), which is

$$(m_i+m_e)\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2-r\left(\frac{d\varphi}{dt}\right)^2\sin^2\theta-2r\Omega\left(\frac{d\varphi}{dt}\right)\sin^2\theta\right]=f_r. \tag{2}$$

Since the effective mass m_i+m_e approximately equals $2m_i$ for a flux tube in the convection zone, Choudhuri and Gilman (1987) actually write $2m_i$ instead of m_i+m_e. The r-component of the total force due to magnetic buoyancy, magnetic tension and drag is denoted above by f_r. We are, however, going to neglect the drag term throughout this paper, since it is not relevant for our discussions and can be put easily in the equations if the need arises. It is possible to have a situation where the forces balance in such a way that there is no motion in the (r, θ) plane, though there may be a flow in the φ-direction. Such possibilities are discussed in detail in van Ballegooijen and Choudhuri (1988). It seems that (2) should reduce in such situations to

$$-(m_i+m_e)\left[r\left(\frac{d\varphi}{dt}\right)^2\sin^2\theta+2r\Omega\left(\frac{d\varphi}{dt}\right)\sin^2\theta\right]=f_r.$$

Though there is flow only in the φ-direction and the surrounding fluid is not disturbed at all, we seem to be still stuck with the effective mass when we would expect only the actual mass of the flux ring to appear.

We derive the correct equation in an inertial frame in the next Section. Then Sect. 3 is devoted to transforming the equation to a rotating frame of reference. The conclusions are summarized in the last Section.

2. Correct equation in an inertial frame

Let us use the same notation as Spruit (1981). If $\hat{\boldsymbol{l}}$ be the tangent vector at a point of the flux tube where internal density is ρ and magnetic field B, then the curvature vector is

$$\boldsymbol{k}=\partial_l \hat{\boldsymbol{l}}$$

where $\partial_l=\hat{\boldsymbol{l}}\cdot\boldsymbol{V}$. The fluid velocity parallel to the flux tube axis is $\hat{\boldsymbol{l}}\cdot\boldsymbol{v}$, and one would naively think that the transverse acceleration due to curvature is $(\hat{\boldsymbol{l}}\cdot\boldsymbol{v})^2\boldsymbol{k}$. However, this is not so, since $\hat{\boldsymbol{l}}\cdot\boldsymbol{v}$ is not in general the velocity with which the fluid turns around the curvature. This becomes clear if we consider a bent part of the flux tube to undergo uniform translation through the surrounding fluid. Such translation certainly does not involve any turning around of the fluid inside the flux tube, though $\hat{\boldsymbol{l}}\cdot\boldsymbol{v}$ will not in general be zero at an arbitrary point of the flux tube. This forces us to conclude that $\hat{\boldsymbol{l}}\cdot\boldsymbol{v}$ may correspond to translation also, and we have to isolate that part of fluid velocity parallel to the local axis which corresponds to the fluid turning around the curvature there. This is done in the following way.

In Fig. 1, let s be the distance of the point A measured along the flux tube from some reference point, and let the corresponding distance for a neighbouring point B be $s+ds$. Suppose the arc AB has a curvature $\boldsymbol{k}$ with the centre of curvature at the point O, i.e.

$$OA\simeq OB\simeq 1/k.$$

If $\hat{\boldsymbol{l}}(s)$ is the tangent vector at A and $\boldsymbol{v}(s+ds)$ the velocity at B, then the component of velocity at point B in the direction of the tangent at the neighbouring point A is

$$\hat{\boldsymbol{l}}(s)\cdot\boldsymbol{v}(s+ds)=\hat{\boldsymbol{l}}\cdot\boldsymbol{v}+ds\hat{\boldsymbol{l}}\cdot\frac{d\boldsymbol{v}}{ds}+\frac{ds^2}{2}\hat{\boldsymbol{l}}\cdot\frac{d^2\boldsymbol{v}}{ds^2}+O(ds^3)$$

$$=\hat{\boldsymbol{l}}\cdot\boldsymbol{v}+ds\hat{\boldsymbol{l}}\cdot\frac{d\boldsymbol{v}}{ds}-k^{-2}\hat{\boldsymbol{l}}\cdot\frac{d^2\boldsymbol{v}}{ds^2}(\cos kds-1)+O(ds^3) \quad (3)$$

(Here it is to be noted that d/ds is the same thing as ∂_l.) There is a term in this Taylor series expansion which is proportional to cos kds. Note that kds is just the angle subtended by the arc AB at the centre of curvature O. If v_c be the part of the axial velocity which corresponds to the fluid turning around this arc, then the component of this velocity parallel to $\hat{\boldsymbol{l}}(s)$ (i.e. the tangent vector at A) at a distance ds from A is v_c cos kds, since kds is also the angle between $\hat{\boldsymbol{l}}(s)$ and the tangent vector at $s+ds$. Hence we identify the coefficient of cos kds in (3) as the velocity of the fluid turning around the arc, i.e.

$$v_c=-k^{-2}\hat{\boldsymbol{l}}\cdot\partial_l^2\boldsymbol{v}. \quad (4)$$

Since the transverse acceleration corresponding to this velocity of turning around is $v_c^2\boldsymbol{k}$, we have to subtract $\rho_e v_c^2\boldsymbol{k}$ from the left side of (1), which then becomes

$$\rho\left(\frac{d\boldsymbol{v}}{dt}\right)_\perp+\rho_e\left[\left(\frac{d\boldsymbol{v}}{dt}\right)_\perp-v_c^2\boldsymbol{k}\right]=\boldsymbol{F}_\perp.$$

The expression for the transverse force $\boldsymbol{F}_\perp$ was given in (10) of Spruit (1981). On substituting this expression,

$$\rho\left(\frac{d\boldsymbol{v}}{dt}\right)_\perp+\rho_e\left[\left(\frac{d\boldsymbol{v}}{dt}\right)_\perp-v_c^2\boldsymbol{k}\right]=\frac{B^2}{4\pi}\boldsymbol{k}+(\rho-\rho_e)(\hat{\boldsymbol{l}}\times\boldsymbol{g})\times\hat{\boldsymbol{l}}. \quad (5)$$

The equation for acceleration parallel to $\hat{\boldsymbol{l}}$ remains the same as (5) of Spruit (1981):

$$\rho\left(\frac{d\boldsymbol{v}}{dt}\right)_\parallel=-\partial_l p+\rho\boldsymbol{g}\cdot\hat{\boldsymbol{l}}. \quad (6)$$

On adding (5) and (6), and using the fact that

$$\nabla\left(p+\frac{B^2}{8\pi}\right)=\nabla p_e=\rho_e\boldsymbol{g},$$

we find

$$\rho\frac{d\boldsymbol{v}}{dt}+\rho_e\left[\left(\frac{d\boldsymbol{v}}{dt}\right)_\perp-v_c^2\boldsymbol{k}\right]$$

$$=\frac{B^2}{4\pi}\boldsymbol{k}+\partial_l\left(\frac{B^2}{8\pi}\right)\hat{\boldsymbol{l}}+(\rho-\rho_e)\boldsymbol{g}$$

$$=-\nabla\left(p+\frac{B^2}{8\pi}\right)+\frac{1}{4\pi}\left(\boldsymbol{B}\cdot\nabla\right)\boldsymbol{B}+\rho\boldsymbol{g} \quad (7)$$

which can be regarded as the basic equation. One can also divide (5) by $(\rho+\rho_e)$ and (6) by ρ, and then add them to obtain

$$\frac{d\boldsymbol{v}}{dt}=-\frac{1}{\rho}\partial_l p\hat{\boldsymbol{l}}+\boldsymbol{g}\cdot\hat{\boldsymbol{l}}\hat{\boldsymbol{l}}+\frac{\rho v_A^2+\rho_e v_c^2}{\rho+\rho_e}\boldsymbol{k}+\frac{\rho-\rho_e}{\rho+\rho_e}(\hat{\boldsymbol{l}}\times\boldsymbol{g})\times\hat{\boldsymbol{l}} \quad (8)$$

where $v_A=B/(4\pi\rho)^{\frac{1}{2}}$ is the Alfvén speed inside the tube. This equation should be compared with (14) of Spruit (1981). Since most calculations on the dynamics of thin flux tubes begin with (14) of Spruit, we suggest that the above equation (6) should be used instead.

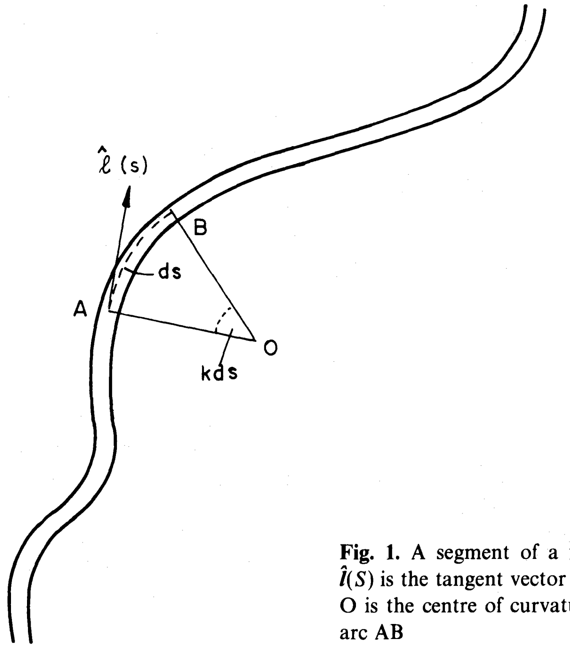

Fig. 1. A segment of a flux tube: $\hat{\boldsymbol{l}}(S)$ is the tangent vector at A, and O is the centre of curvature of the arc AB

3. Correct equation in a rotating frame

3.1. General considerations

In order to study the evolution of flux tubes in the solar convection zone, it is necessary to write down our equations in a rotating frame of reference. In order to go from an inertial to a rotating frame, we have to replace $\boldsymbol{v}$ and $d\boldsymbol{v}/dt$ in the equations in the inertial frame by the following expressions

$$\boldsymbol{v} \rightarrow \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{r} \tag{9}$$

$$\frac{d\boldsymbol{v}}{dt} \rightarrow \frac{d\boldsymbol{v}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \tag{10}$$

To find out the replacement for v_c, we substitute from (9) in (4), i.e.

$$v_c \rightarrow -k^{-2}\hat{\boldsymbol{l}}.[\partial_l^2 \boldsymbol{v} + \boldsymbol{\Omega} \times \partial_l^2 \boldsymbol{r}].$$

Since $-k^{-2}\hat{\boldsymbol{l}}.\partial_l^2 \boldsymbol{v}$ is v_c in the rotating frame and $\partial_l^2 \boldsymbol{r} = \boldsymbol{k}$, we write

$$v_c \rightarrow v_c - k^{-2}\hat{\boldsymbol{l}}.(\boldsymbol{\Omega} \times \boldsymbol{k}). \tag{11}$$

Substituting (9), (10) and (11) in (7), and using the fact that

$$\nabla\left(p + \frac{B^2}{8\pi}\right) = \nabla p_e = \rho_e[\boldsymbol{g} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})]$$

in a rotating frame, we find

$$\rho\left[\frac{d\boldsymbol{v}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right] + \rho_e\left[\frac{d\boldsymbol{v}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})\right]_\perp - \rho_e\left[v_c - k^{-2}\hat{\boldsymbol{l}}.(\boldsymbol{\Omega} \times \boldsymbol{k})\right]^2 \boldsymbol{k}$$

$$= (\rho - \rho_e)[\boldsymbol{g} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})] + \frac{1}{4\pi}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}. \tag{12}$$

This is the general equation. Since the rotation period in the sun is slow compared to the typical dynamical period for the evolution of a flux tube, we can neglect terms of the order $\boldsymbol{\Omega}^2$ in our equations. Neglecting such terms in (12), we write down the parallel part of the equation of motion

$$\rho\left(\frac{d\boldsymbol{v}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\parallel} = (\rho - \rho_e)\boldsymbol{g}.\hat{\boldsymbol{l}} + \partial_l\left(\frac{B^2}{8\pi}\right) = -\partial_l p + \rho \boldsymbol{g}.\hat{\boldsymbol{l}}, \tag{13}$$

whereas the transverse part turns out to be

$$(\rho + \rho_e)\left(\frac{d\boldsymbol{v}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_\perp - \rho_e[v_c^2 - 2v_c k^{-2}\hat{\boldsymbol{l}}.(\boldsymbol{\Omega} \times \boldsymbol{k})]\boldsymbol{k} = \frac{B^2}{8\pi}\boldsymbol{k} + (\rho - \rho_e)(\hat{\boldsymbol{l}} \times \boldsymbol{g}) \times \hat{\boldsymbol{l}}. \tag{14}$$

These two parts can again be combined to give the equation

$$\frac{d\boldsymbol{v}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{1}{\rho}\partial_l p + \boldsymbol{g} \cdot \boldsymbol{l} + \frac{\rho - \rho_e}{\rho + \rho_e}(\hat{\boldsymbol{l}} \times \boldsymbol{g}) \times \hat{\boldsymbol{l}} + \frac{\rho v_A^2 + \rho_e[v_c^2 - 2v_c k^{-2}\hat{\boldsymbol{l}}.(\boldsymbol{\Omega} \times \boldsymbol{k})]}{\rho + \rho_e}\boldsymbol{k}. \tag{15}$$

3.2. *Application to a symmetric ring*

In order to understand the implications of these equations, let us consider an application to a flux ring symmetric around the rotation axis. The equation of motion for such a flux ring was derived in Choudhuri and Gilman (1987) by associating the effective mass with the whole of transverse acceleration. To find out the departures from their equations, we have to evaluate the correction term

$$-\rho_e[v_c^2 - 2v_c k^{-2}\hat{\boldsymbol{l}}.(\boldsymbol{\Omega} \times \boldsymbol{k})]\boldsymbol{k}$$

in (14). The velocity at a point of the symmetric flux ring is

$$\boldsymbol{v} = \frac{dr}{dt}\hat{\boldsymbol{e}}_r + r\frac{d\theta}{dt}\hat{\boldsymbol{e}}_\theta + r\frac{d\varphi}{dt}\sin\theta\,\hat{\boldsymbol{e}}_\varphi \tag{16}$$

whereas the curvature is given by

$$\boldsymbol{k} = \frac{1}{r\sin\theta}(-\sin\theta\,\hat{\boldsymbol{e}}_r - \cos\theta\,\hat{\boldsymbol{e}}_\theta). \tag{17}$$

Since $\partial_l = (r\sin\theta)^{-1}\partial/\partial\varphi$, (4) reduces to

$$v_c = -\hat{\boldsymbol{e}}_\varphi \cdot \frac{\partial^2 \boldsymbol{v}}{\partial\varphi^2}. \tag{18}$$

Using the relations

$$\frac{\partial\hat{\boldsymbol{e}}_r}{\partial\varphi} = \sin\theta\,\hat{\boldsymbol{e}}_\varphi, \qquad \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial\varphi} = \cos\theta\,\hat{\boldsymbol{e}}_\varphi, \qquad \frac{\partial\hat{\boldsymbol{e}}_\varphi}{\partial\varphi} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$$

it trivially follows from (16) and (18) that

$$v_c = r\frac{d\varphi}{dt}\sin\theta. \tag{19}$$

Since

$$k^{-2}\hat{\boldsymbol{l}} \cdot (\boldsymbol{\Omega} \times \boldsymbol{k}) = -\Omega r \sin\theta,$$

using (17) and (19), the correction term becomes

$$-\rho_e[v_c^2 - 2v_c k^{-2}\hat{\boldsymbol{l}}.(\boldsymbol{\Omega} \times \boldsymbol{k})]\boldsymbol{k} = \rho_e\left[r\left(\frac{d\varphi}{dt}\right)^2 \sin^2\theta + 2r\Omega\left(\frac{d\varphi}{dt}\right)\sin^2\theta\right]\hat{e}_r + \rho_e\left[r\left(\frac{d\varphi}{dt}\right)^2 \sin\theta\cos\theta + 2r\Omega\left(\frac{d\varphi}{dt}\right)\sin\theta\cos\theta\right]\hat{e}_\varphi. \tag{20}$$

If we now integrate over the whole ring as was done in Choudhuri and Gilman (1987) and apply the correction term, then the 3 components of our equation turn out to be

$$(m_i + m_e)\left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] + m_i\left[-r\left(\frac{d\varphi}{dt}\right)^2 \sin^2\theta - 2r\Omega\left(\frac{d\varphi}{dt}\right)\sin^2\theta\right] = (m_i - m_e)g - \frac{\Psi^2}{2\pi\sigma^2}\sin\theta, \tag{21}$$

$$(m_i + m_e)\left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right] + m_i\left[-r\left(\frac{d\varphi}{dt}\right)^2 \sin\theta\cos\theta - 2r\Omega\left(\frac{d\varphi}{dt}\right)\sin\theta\cos\theta\right] = -\frac{\Psi^2}{2\pi\sigma^2}\cos\theta, \tag{22}$$

$$m_i\left[r\frac{d^2\varphi}{dt^2}\sin\theta + 2\frac{dr}{dt}\frac{d\varphi}{dt}\sin\theta + 2r\frac{d\theta}{dt}\frac{d\varphi}{dt}\cos\theta + 2\Omega\left(r\frac{d\theta}{dt}\cos\theta + \frac{dr}{dt}\sin\theta\right)\right] = 0, \tag{23}$$

where σ is the radius of cross-section of the flux tube and Ψ is the flux through it. Comparing these equations with (4), (5) and (6) of Choudhuri and Gilman (1987) shows that the terms involving $(d\varphi/dt)$ in the r and θ components of the equation are now associated with just the mass of the flux ring rather the effective mass $m_i + m_e$. This ensures that we are not led into the sorts of inconsistencies that we found on putting

$$\frac{dr}{dt} = \frac{d\theta}{dt} = 0$$

in (2).

In addition to giving correct results when the flux ring is held in a fixed position in the (r, θ) plane, our equations should also give the correct dynamics when the flux ring moves through the surrounding fluid. To demonstrate that this is so, we have considered a hypothetical problem in the Appendix. All forces acting on a flux ring (including magnetic bouyancy and magnetic tension) are assumed to be suddenly switched off at an instant until which the ring was made to rotate uniformly at a fixed position by the application of various forces. We have shown that it is necessary to use corrected equations in order to conserve energy.

4. Conclusion

We thus find that our corrected equations are free from the conceptual difficulties mentioned in the Introduction. However, the general mathematical character of the equations has remained the same, and hence we expect the qualitative results obtained by the previous investigators to remain valid. In fact, we repeated some calculations presented in Choudhuri and Gilman (1987) by using the corrected Eqs. (21), (22) and (23). Comparing these equations with (4), (5) and (6) of Choudhuri and Gilman (1987), we find that the Coriolis force terms in the r and θ components have now effectively become weaker by a factor of 2 with respect to the other force terms. Consequently, a slightly smaller value of magnetic buoyancy is now needed to overcome the Coriolis force. But when we make runs for values of magnetic buoyancy either substantially smaller or larger than the critical value needed to suppress the Coriolis force, then the results appear essentially the same.

Perhaps it would not be out of place to mention another limitation of Spruit's equation. It has been assumed by most authors using this equation that it holds for flux tubes moving through the convection zone. In fact, the very title of Spruit's paper (1981) is "Motion of Magnetic Flux Tubes in the Solar Convection Zone and Chromosphere". However, the ambient medium is assumed to be in static equilibrium in the derivation, as seen from Eq. (9) of Spruit (1981). Not only the convection zone is not in static equilibrium, but the turbulence present there may interact with the moving flux tubes in complicated ways. This is obviously a very complex process. A phenomenological study of how the turbulence may interact with the flux tubes is presented in a forthcoming paper by Choudhuri and D'Silva (1989). If this interaction is not taken into account, then it seems that the magnetic flux starting from the bottom of the convection zone appears at rather high latitudes on the solar surface instead of appearing where sunspots are seen (Choudhuri and Gilman, 1987; Choudhuri, 1989). This unphysical result is probably due to the fact that the equation of motion for the thin flux tubes was used for motions in the convection zone without taking any account of the turbulence present there (Choudhuri and D'Silva, 1989). Hence we conclude that Spruit's equation is strictly valid for motions of thin flux tubes in ambient atmospheres in stable equilibrium. It is only a rough approximation when applied to flux tubes in the convection zone, though one may try to make it more realistic by incorporating the influence of turbulence in the phenomenological way proposed by Choudhuri and D'Silva (1989).

Acknowledgements. I wish to thank the referee of this paper, Henk Spruit, who waived his right to anonymity and sent an insightful referee's report. The Appendix, which is a belated addition, resulted from certain points raised in this report.

Appendix: evolution of a rotating ring

Let us consider a flux ring of radius r_0 which is made to rotate uniformly in an inertial frame due to the application of external forces, and let the linear velocity of rotation at any point of the ring be v_0. The external forces are supposed to suddenly 'switch off' at time $t=0$ and we want to find out the subsequent evolution of the ring. Equation (1) would imply

$$\left(\frac{d\boldsymbol{v}}{dt}\right)_{\perp} = 0, \tag{A1}$$

whereas it follows from the corrected Eq. (5) that

$$\left(\frac{d\boldsymbol{v}}{dt}\right)_{\perp} = \frac{\rho_e}{\rho+\rho_e} v_{\parallel}^2 \boldsymbol{k} \tag{A2}$$

where we have put $v_c = v_{\parallel}$, since v_c for a circular ring turns out to be just the parallel component of velocity (see Eq. [18]). We want to show that (A2) is the correct result.

If each fluid particle of the ring moves with a tangential velocity v_0, then the ring as a whole will expand in radius. Though $v_{\perp}=0$ at $t=0$, if (A1) is obeyed, the value of $v_{\perp}$ at the moment when the ring attains a radius r starting from r_0 is

$$v_{\perp} = v_0 \sqrt{1 - \frac{r_0^2}{r^2}}. \tag{A3}$$

As $v_{\perp}$ increases from zero, the ring will have to impart kinetic energy to the surrounding fluid. If (A1) were true, then there would have been no source from which this kinetic energy could come. Hence (A1) cannot be true and the outward expansion of the ring must slow down in order to compensate for the kinetic energy gained by the surrounding fluid. Equation (A2) represents a deceleration of magnitude

$$\frac{\rho_e}{\rho+\rho_e} \frac{v_{\parallel}^2}{r}$$

directed radially inwards. We now proceed to show that the presence of this deceleration would conserve the total kinetic energy.

As the ring expands, the ring will in general have both transverse velocity $v_{\perp,1}$ and longitudinal velocity $v_{\parallel,1}$ at time $t=t_1$ when the radius of the ring is r_1. If the velocity v_1 is inclined to the radial direction at angle θ_1, as shown in Fig. 2, then

$$\left.\begin{aligned} v_{\perp,1} &= v_1 \cos\theta_1 \\ v_{\parallel,1} &= v_1 \sin\theta_1 \end{aligned}\right\} \tag{A4}$$

Let us first consider how the ring would evolve if (A1) were true. Afterwards we shall include the effect of deceleration as given by (A2). If each fluid particle of the ring moves with uniform velocity, as implied by (A1), then the magnitude of velocity still

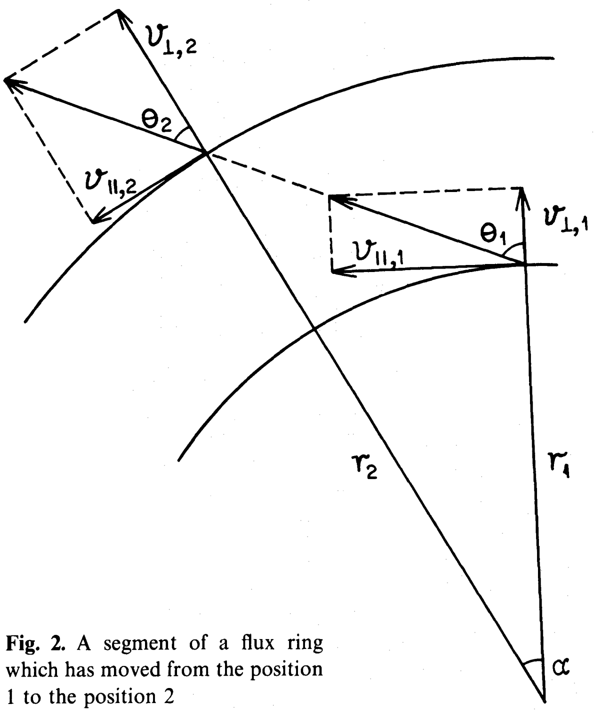

Fig. 2. A segment of a flux ring which has moved from the position 1 to the position 2

remains v_1 when the ring expands to radius r_2, whereas the radius vector of the fluid particle sweeps through an angle α (see Fig. 2). If Δt is the time taken by the ring radius to expand from r_1 to r_2, then

$$\alpha = \frac{v_{\parallel,1}}{r_1}\Delta t + O(\Delta t^2) \tag{A5}$$

It is also easy to see that

$$\left.\begin{aligned} v_{\perp,2} &= v_1 \cos\theta_2 \\ v_{\parallel,2} &= v_1 \sin\theta_2 \end{aligned}\right\} \tag{A6}$$

Noting the fact that $\theta_2 = \theta_1 - \alpha$ and keeping terms up to the first order in Δt, we find from (A4), (A5) and (A6) that

$$\left.\begin{aligned} v_{\perp,2} &= v_{\perp,1} + \frac{v_{\parallel,1}^2}{r_1}\Delta t + O(\Delta t^2) \\ v_{\parallel,2} &= v_{\parallel,1} - \frac{v_{\perp,1} v_{\parallel,1}}{r_1}\Delta t + O(\Delta t^2) \end{aligned}\right\} \tag{A7}$$

Instead of each fluid particle moving uniformly, if a deceleration took place, as given by (A2), then we have to subtract

$$\frac{\rho_e}{\rho+\rho_e}\frac{v_{\parallel,1}^2}{r_1}\Delta t$$

from the expression of $v_{\perp,2}$ in (A7). Hence, if (A2) were the governing equation instead of (A1), then the expressions for $v_{\perp,2}$ and $v_{\parallel,2}$ would have been

$$\left.\begin{aligned} v_{\perp,2} &= v_{\perp,1} + \frac{\rho}{\rho+\rho_e}\frac{v_{\parallel,1}^2}{r_1}\Delta t + O(\Delta t^2) \\ v_{\parallel,2} &= v_{\parallel,1} - \frac{v_{\perp,1} v_{\parallel,1}}{r_1}\Delta t + O(\Delta t^2) \end{aligned}\right\} \tag{A8}$$

The kinetic energy density within the ring would be $\frac{1}{2}\rho(v_\perp^2 + v_\parallel^2)$. When the motion of the surrounding fluid is also taken into account, then the effective kinetic energy density becomes

$$\tfrac{1}{2}(\rho+\rho_e)v_\perp^2 + \tfrac{1}{2}\rho v_\parallel^2.$$

When the ring radius expands from r_1 to r_2 without any external force acting, the effective kinetic energy density is expected to be conserved, i.e.

$$\tfrac{1}{2}(\rho+\rho_e)v_{\perp,2}^2 + \tfrac{1}{2}\rho v_{\parallel,2}^2 = \tfrac{1}{2}(\rho+\rho_e)v_{\perp,1}^2 + \tfrac{1}{2}\rho v_{\parallel,1}^2. \tag{A9}$$

It is easy to check that (A7) would not satisfy (A9), whereas (A8) satisfies (A9) to order Δt as we want. Hence (A2), from which (A8) follows, gives the correct expression of deceleration caused by imparting kinetic energy to the surrounding fluid.

References

Chou, D.-Y., Fisher, G.H.: 1989, *Astrophys. J.* **341**, 533
Choudhuri, A.R.: 1989, *Solar Phys.* **123**, 217
Choudhuri, A.R., D'Silva, S.: 1990a, *Astron. Astrophys.* **239**, 326
Choudhuri, A.R., Gilman, P.A.: 1987, *Astrophys. J.* **316**, 788
Lamb, H.: 1945, *Hydrodynamics*, Dover, New York
Moreno-Insertis, F.: 1986, *Astron. Astrophys.* **166**, 291
Spruit, H.C.: 1981, *Astron. Astrophys.* **98**, 155
Spruit, H.C., van Ballegooijen, A.A.: 1982, *Astron. Astrophys.* **106**, 58
van Ballegooijen, A.A.: 1983, *Astron. Astrophys.* **118**, 275
van Ballegooijen, A.A., Choudhuri, A.R.: 1988, *Astrophys. J.* **333**, 965