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Very smooth points of spaces of operators

T S S R K RAO

Indian Statistical Institute, R.V. College Post, Bangalore 560 059, India
E-mail: tss@isibang.ac.in

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Abstract. In this paper we study very smooth points of Banach spaces with special emphasis on spaces of operators. We show that when the space of compact operators is an M-ideal in the space of bounded operators, a very smooth operator T attains its norm at a unique vector x (up to a constant multiple) and $T(x)$ is a very smooth point of the range space. We show that if for every equivalent norm on a Banach space, the dual unit ball has a very smooth point then the space has the Radon–Nikodým property. We give an example of a smooth Banach space without any very smooth points.

Keywords. Very smooth points; spaces of operators; M-ideals.

1. Introduction

A Banach space X is said to be *very smooth* if every unit vector has a unique norming element in X^{***} (here X is being considered as a subspace of X^{**} under the canonical embedding, see [S]). In this paper we study a local version of the notion of 'very smooth space' by calling a unit vector of X a very smooth point if it is also a smooth point of X^{**} (recall that a unit vector is a smooth point if it has a unique norming functional in the dual). These notions are related to differentiability of the norm at these points, see [S]. In particular for the space of compact operators we will be considering differentiability in the direction of every bounded operator.

Identification of smooth points of spaces of operators , C^*-algebras and their generalizations has received a lot of attention in the literature. See [H, GY1, GY2, KY, MR, TW, W1, W2, HWW] Chapter VI. Motivated by these results, in this paper we study very smooth points of spaces of operators and JB^*-triples.

To do this we first prove a proposition involving the notion of an M-ideal that allows us to 'lift' very smooth points from subspaces. We use this to show that depending on the 'position' of an M-ideal, all smooth points of a Banach space can be very smooth. We give several examples from spaces of operators where these ideas apply.

In order to study very smooth points of the space of compact operators $\mathcal{K}(X, Y)$ we make use of a characterization due to Heinrich [H] of smooth points of this space (see §2). Let X_1 denote the closed unit ball of X. We recall from [S] and [GI] that a unit vector $x \in X_1$ is a very smooth point if and only if its unique norming functional x^* is a point of continuity for the identity map $i : (X_1^*, w^*) \to (X_1^*, weak)$. Our approach involves studying these points of $w^* - w$ continuity. For $\mathcal{K}(X, Y)$ we show that a very smooth point attains its norm and the image vector is a very smooth point. When $Y = C(K)$ for a compact space K we could give a complete description of very smooth points. More generally we show that $f \in C(K, X)$ (the space of X-valued continuous functions on K, equipped with the supremum norm) is a very smooth point iff there is a unique isolated point k such that $f(k)$ is a very smooth point of X. We also show that any very smooth point of a JB^*-triple is a Fréchet smooth point. We give a description of very smooth points of $\mathcal{L}(X, Y)$ when $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$. See Chapter VI of [HWW] and [KW] for several examples of Banach spaces X and Y for which $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$.

In the third section of this paper we prove a result analogous to a result of Ruess and Stegall ([RS1, RS2]) by showing that when $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$, then the dual unit balls of both the spaces have the same points of $w^* - w$ continuity.

In the final section of the paper we consider spaces that fail to have very smooth points. Considering the inclusion $X \subset Y \subset X^{**}$ under the canonical embedding, we show that if X is a proper L-ideal in Y then X_1 has no very smooth points. This allows us to give an example of a smooth space in which no unit vector is very smooth. We complement our earlier work from [R2] and [R3] by showing that for an infinite compact set K and for a reflexive Banach space X, if $\mathcal{L}(X, C(K))$ is a dual space then there is no point of $w^* - w$ continuity in the unit ball.

Our notation and terminology is standard and can be found in [HWW]. For a Banach space X by $\partial_e X_1$ we denote the set of extreme points.

2. Very smooth points

Let $M \subset X$ be a closed subspace. It was observed in [MR] that if $x \in M$ is a smooth point of X then it is a smooth point of M. It is easy to see that if every continuous linear functional on M has a unique norm preserving extension to X then every smooth point of M is also a smooth point of X. Our first result addresses this question for very smooth points and involves the notion of an M-ideal. We recall from Chapter I of [HWW] that a closed subspace $M \subset X$ is an M-ideal, if there is a projection P in $\mathcal{L}(X^*)$ such that $\|P(x^*)\| + \|x^* - P(x^*)\| = \|x^*\|$ for all $x^* \in X^*$ and ker $P = M^\perp$. We note from Proposition 1.1.12 in [HWW] that if M is an M-ideal in X then continuous linear functionals on M have unique norm preserving extensions to X. We also note that in C^*-algebras M-ideals are precisely closed two sided ideals. We first prove a lemma on smooth points in the ℓ^∞-direct sum of two spaces.

Lemma 1. *Suppose $X = M \oplus_\infty N$. Let $m \in M$, $\|m\| = 1$ and $n \in N$, $\|n\| < 1$. $x = m + n$ is a smooth point of X if and only if m is a smooth point of M.*

Proof. Suppose x is smooth in X. Let $\|m_i^*\| = 1 = m_i^*(m)$, $m_i^* \in M^*$ for $i = 1, 2$. Then $(m_1^*, 0)$, $(m_2^*, 0)$ are two linear functionals in $X^* = M^* \oplus_1 N^*$ (ℓ^1-direct sum) that attain their norm at x. Hence $m_1^* = m_2^*$ and therefore m is smooth in M.

Conversely suppose m is smooth in M. Suppose $\|x_i^*\| = 1 = x_i^*(x)$ for $i = 1, 2$. Since $X^* = M^\perp \oplus_1 N^\perp = M^* \oplus_1 N^*$, $x_i^* = m_i^* + n_i^*$ and $1 = \|m_i^*\| + \|n_i^*\| = m_i^*(m) + n_i^*(n)$. Suppose $n_i^* \neq 0$. Thus $1 = \|m_i^*\| m_i^*(m)/\|m_i^*\| + \|n_i^*\| n_i^*(n)/\|n_i^*\|$ implies $\|n\| = 1$. Hence $n_i^* = 0$ for $i = 1, 2$ and $m_1^* = m_2^*$. Therefore x is smooth in X. □

PROPOSITION 2

Let $m \in M$ be a very smooth point of X then it is a very smooth point of M. If $M \subset X$ is an M-ideal then a very smooth point of M is also a very smooth point of X.

Proof. Let $m \in M$ be a very smooth point of X . Since $M^{**} = M^{\perp\perp} \subset X^{**}$ under the canonical embedding and since m is a smooth point of X^{**}, by applying Lemma 2.1 of [MR] we get that m is a smooth point of M^{**}. Thus m is a very smooth point of M.

Assume further that M is an M-ideal. We have from the definition that $X^{**} = M^{**} \oplus_\infty (M^*)^\perp$. Now if $m \in M$ is a very smooth point then it is a smooth point of M^{**} and hence it follows from our lemma that it is a smooth point of X^{**} and hence a very smooth point of X. □

Remark 3. Part of Proposition 2 is an abstract version of Proposition 3.2 in [GI], where the authors showed by different methods that if X is an M-ideal in its bidual (under the canonical embedding), then very smooth points of X are also very smooth in X^{**}. The M-ideal condition in the above proposition cannot be replaced by 'M is an ideal in X' (we recall from [GKS] that $M \subset X$ is an ideal if there is a projection P of norm one in the dual such that $\ker P = M^\perp$). In [GI] the authors have constructed an example of a Banach space X and a very smooth point of X that is not very smooth in X^{**} (X is canonically embedded in X^{**} and hence an ideal).

However, for $M \subset X$ if one assumes that there is a norm-one projection $P : X^* \to X^*$ such that $\ker P = M^\perp$ and $P(X^*)_1$ is w^*-dense in X_1^* then since under these conditions $M \subset X \subset M^{**}$ (under the canonical embedding, see [R6]), we have that a very smooth point of M is a smooth point of X.

If X^* or Y has the compact metric approximation property (CMAP), then it is easy to see that $K(X, Y) \subset \mathcal{L}(X, Y)$ satisfies the above condition (see [R6]). Thus for a compact operator that is a very smooth point, directional derivatives exist in the direction of all bounded operators.

Smooth points of operator spaces has been extensively studied in the literature. We now undertake to study very smooth points of these spaces. As noted before one of the motivations is that under the assumption of the CMAP, a very smooth point of $K(X, Y)$ is a smooth point of $\mathcal{L}(X, Y)$.

Before doing this we indicate one more proposition that exhibits the presence of very smooth points depending on the position of an M-ideal and involves the notion of a Hahn–Banach smooth space considered in [S].

To study very smooth points in this set up we use a characterization of very smooth points obtained in [GI], as smooth points for which the unique norming linear functional in X_1^* is also a point of continuity for the identity map on X_1^* equipped with the weak* and weak topologies on the domain and the range respectively.

PROPOSITION 4

Let J be a Hahn–Banach smooth space. Suppose $J \subset X$ is an M-ideal and X/J does not have any smooth points. Then every smooth point of X is a very smooth point.

Proof. Let $x \in X$ be a smooth point. Since X/J does not have any smooth points, and since J is an M-ideal, there exists a unique $j^* \in J_1^*$ such that $j^*(x) = 1$. Since J is Hahn–Banach smooth, j^* is a point of $w^* - w$ continuity of J_1^* (see [HWW], Lemma III.2.14). We now claim that j^* is a point of $w^* - w$ continuity of X_1^*. Let $\{x_\alpha^*\}_{\alpha \in I} \subset X_1^*$ be a net converging to j^* in the weak*-topology of X^*. Clearly the net $\{P(x_\alpha^*)\}_{\alpha \in I}$ converges to j^* in the weak*-topology of J^*. In view of our assumption on J, this convergence is thus in the weak topology. Also $1 = \lim \|P(x_\alpha^*)\| = \lim \|x_\alpha^*\|$. By the defining property of P we get that the net $\{\|x_\alpha^* - P(x_\alpha^*)\|\}_{\alpha \in I}$ converges to 0. Now it is easy to see that the net $\{x_\alpha^*\}_{\alpha \in I}$ converges to j^* in the weak topology. Hence x is a very smooth point of X. □

Remark 5. We have used the assumption X/J has no smooth points to get a $j^* \in J_1^*$. Thus this hypothesis can also be replaced by $d(x, J) < 1$.

Remark 6. If J has property $(**)$ of [S] then arguments similar to the ones given above can be used to show that under the same hypothesis every smooth point of X is a Fréchet smooth point.

Example 7. Let $1 < p < \infty$ and E be a Banach space such that $K(\ell^p, E)$ is an M-ideal in $\mathcal{L}(\ell^p, E)$ (see Corollary 6.4 of [KW] for a necessary and sufficient condition for this to happen). If $T \in \mathcal{L}(\ell^p, E)$ is a smooth point, it follows from Theorem 1 of [GY2] that $\mathrm{d}(T, K(\ell^p, E)) < 1$. Thus if E further satisfies the condition $K(\ell^p, E)$ is Hahn–Banach smooth, then any smooth point of $\mathcal{L}(\ell^p, E)$ is very smooth.

Example 8. Let X be an M-ideal in its bidual. See Chapters III and VI of [HWW] for several examples of such spaces from among function spaces and spaces of operators. It follows from our Proposition 2 and Theorem 2 of [R7] where we have proved that X is an M-ideal under appropriate canonical embeddings in all the duals of even order, that every smooth point of X is very smooth and continues to be a very smooth point of all the duals of even order of X. Assume further X^{**}/X has no smooth points then since X is Hahn–Banach smooth (see [HWW], Corollary III.2.15), we get that any smooth point of X^{**} is very smooth. $c_0 \subset \ell^\infty$ and $\mathcal{K}(\ell^2) \subset \mathcal{L}(\ell^2)$ are well-known examples of this phenomenon (see [KY]). It may be worth recalling here that any such X (see [HWW], Theorem III.4.6) can be renormed to have a strictly convex dual (and hence every unit vector is a smooth point) and continues to be an M-ideal in the bidual with respect to the new norm. Thus in this renorming, all the unit vectors are very smooth and continue to be very smooth in all the duals of X of even order.

The next example is once again from operator theory.

Example 9. Let X, Y be separable Banach spaces in the classes (M_p) and (M_q) for some $1 < p, q < \infty$ (see [HWW], Chapter VI and [W1]). Since $\mathcal{K}(X, Y)$ is an M-ideal in its bidual, $\mathcal{L}(X, Y)$ and since the quotient space has no smooth points (see [HWW], Lemma VI.5.18), it follows that every smooth point of $\mathcal{L}(X, Y)$ is very smooth. In particular in $\mathcal{L}(\ell^p)$ every smooth point is very smooth. It follows from Theorem 6 of [R7] that there are no very smooth points in ℓ^1. It can thus be deduced from Proposition 17 in this paper that $\mathcal{L}(\ell^1)$ does not have very smooth points. The question of very smooth points of $\mathcal{L}(\ell^\infty)$ will also be considered later in this paper.

Our next example shows that unlike the situation described in the previous examples, the fourth dual of X (denoted by $X^{(IV)}$) quotiented by X can have smooth points.

Example 10. Let $X = c_0$. It follows from Theorem 2 of [R7] that it is an M-ideal in $X^{(IV)}$. It is well known that $X^{(IV)}$ can be identified with a $C(K)$ space and isolated points of K correspond precisely to one-dimensional L-summands (or atoms) of X^{***}. Now $X = \{f \in C(K) : f(K') = 0\}$, where $K' = \{k \in K : f(k) = 0$ for all $f \in c_0\}$. See Chapter I of [HWW]. Thus $X^{(IV)}/X$ can be identified as $C(K')$. To exhibit smooth points it is therefore enough to exhibit isolated points in K'. Now for any $\tau \in \partial_e(c_0^\perp)_1$ since $\mathrm{span}\{\tau\}$ is an L-summand, it is an isolated point of $K \cap K'$.

We do not know if a similar result is true in the non-commutative situation of $\mathcal{K}(\ell^2) \subset \mathcal{L}(\ell^2)$.

In the next proposition we again impose a condition on the quotient space to preserve very smooth points and this result is modeled on Proposition 3.4 of [GI]. Here we will be assuming that the subspace J is only an ideal (in the sense of [GKS]) but make up for the loss in the geometry of the projection in the dual, by making the topological assumption that the quotient space is a Grothendieck space, that is, weak* and weak sequential convergence coincide in the dual of the quotient space. It is known that any von Neumann algebra is a Grothendieck space, by [P]. We recall from [S] and [GI] that for a smooth point to be very smooth one only needs to verify weak*-weak sequential continuity of the norm attaining functional. We also recall that any smooth Grothendieck space is reflexive.

PROPOSITION 11

Let X be a smooth Banach space and let $J \subset X$ be an ideal such that X/J is a Grothendieck space. Then a very smooth point of J is also a very smooth point of X.

Proof. Let P be a projection of norm one in X^* such that ker $P = J^\perp$. Let j_0 be a very smooth point of J and let x_0^* be the unique functional with $x_0^*(j_0) = 1$. We shall show that for any sequence $\{x_n^*\}_{n\geq 1} \subset X_1^*$ such that $x_n^* \to x_0^*$ in the weak*-topology also converges in the weak topology.

Since for any x^*, $P(x^*)$ is the norm preserving extension of x^*/J, we have that $P(x_0^*) = x_0^*$ and also $P(X^*)$ is isometric to J^* via the map $P(x^*) \to x^*/J$. Thus $P(x_n^*) \to x_0^*$ in the weak*-topology of J^* and hence by hypothesis it would converge in the weak topology. Therefore $\{P(x_n^*) - x_n^*\}_{n\geq 1} \subset J^\perp$ converges to zero in the weak*-topology and as X/J is a Grothendieck space, this convergence is also in the weak topology. Hence $x_n^* \to x_0^*$ in the weak topology. Therefore, j_0 is a very smooth point of X. □

We now recall a well-known characterization of smooth points of $K(X, Y)$ due to Heinrich [H].

Theorem [**H**]. *$T \in K(X, Y)$ is a smooth point if and only if T^* attains its norm at a unique (up to a constant multiple) $y_0^* \in \partial_e Y_1^*$ and $T^* y_0^*$ is a smooth point of X^*.*

We next consider very smooth points of $K(X, Y)$. Let $T \in K(X, Y)$ be a smooth point. Note that since $T^*(y_0^*)$ is a smooth point, there exists a unique $x_0^{**} \in \partial_e X_1^{**}$ such that $x_0^{**}(T^*(y_0^*)) = 1$. Thus the linear functional $x_0^{**} \otimes y_0^*$ defined by $(x_0^{**} \otimes y_0^*)(S) = x_0^{**}(S^*(y_0^*))$ is an extreme point of $K(X, Y)_1^*$ and is the unique functional attaining its norm at T. Keeping the criterion from [GI] mentioned earlier in view, we first look at points of $w^* - w$ continuity. As a further geometric motivation for studying this concept, we recall from [HWW, p. 125] that $x^* \in X_1^*$ is a point of $w^* - w$ continuity if and only if under the canonical embeddings, x^* has a unique norm preserving extension to X^{**}.

Theorem 12. *Let $x_0^{**} \in \partial_e X_1^{**}$ and let $y_0^* \in \partial_e Y_1^*$. Suppose $x_0^{**} \otimes y_0^* \in \partial_e K(X, Y)_1^*$ is a point of $w^* - w$ continuity. Then x_0^{**} and y_0^* are points of $w^* - w$ continuity. In particular $x_0^{**} = x_0 \in \partial_e X_1$.*

Proof. Let $\{x_\alpha^{**}\}_{\alpha\in I}$ be a net in X_1^{**} such that $x_\alpha^{**} \xrightarrow{w^*} x_0^{**}$. For any $T \in K(X, Y)$, $(x_\alpha^{**} \otimes y_0^*)(T) = x_\alpha^{**}(T^*(y_0^*)) \to x_0^{**}(T^*(y_0^*))$. Thus $x_\alpha^{**} \otimes y_0^* \xrightarrow{w^*} x_0^{**} \otimes y_0^*$. Hence by the hypothesis, $x_\alpha^{**} \otimes y_0^* \xrightarrow{w} x_0^{**} \otimes y_0^*$.

Now let $\tau \in X_1^{***}$ and let $y_0 \in Y$ be such that $y_0^*(y_0) = 1$. Consider the functional, $(\tau \otimes y_0)(x^{**} \otimes y^*) = \tau(x^{**}) y^*(y_0)$. It follows from the well-known inclusions, $X^{***} \otimes_\epsilon Y \subset (X^* \otimes_\epsilon Y)^{**} \subset \mathcal{K}(X, Y)^{**}$, that $\tau \otimes y_0 \in K(X, Y)^{**}$. Hence $(\tau \otimes y_0)(x_\alpha^{**} \otimes y_0^*) = \tau(x_\alpha^{**}) \to \tau(x_0^{**})$. Hence $x_\alpha^{**} \xrightarrow{w} x_0^{**}$. Since X_1 is w^*-dense in X_1^{**}, we get that $x_0^{**} = x_0 \in \partial_e X_1$. Similar arguments work for y_0^*. □

Remark 13. It is clear from the proof that the arguments also go through for the space $\mathcal{L}(X, Y)$. We note that for $x \in \partial_e X_1^{**}$ and $y^* \in \partial_e Y_1^*$, $x \otimes y^*$ in general need not be an extreme point of $\mathcal{L}(X, Y)_1^*$ (see [HWW], p 267 for an example).

COROLLARY 14

Every very smooth point T of $K(X, Y)$ attains its norm at a unique unit vector (*up to constants*) x_0 *and* $T(x_0)$ *is a very smooth point of* Y.

Proof. Let T be a very smooth point of $K(X, Y)$. As noted before, there exists a $x_0^{**} \in \partial_e X_1^{**}$ and a $y_0^* \in \partial_e Y_1^*$ such that $x_0^{**} \otimes y_0^*$ is the unique element of $\partial_e K(X, Y)_1^*$ that attains its norm at T. Since T is very smooth, $x_0^{**} \otimes y_0^*$ is a point of $w^* - w$ continuity. Therefore, $x_0^{**} = x_0 \in X$. Hence $1 = y_0^*(T(x_0)) \leq \|T(x_0)\| \leq \|T\| = 1$. Thus $\|T\| = 1 = \|T(x_0)\|$. That $T(x_0)$ is very smooth again follows from the above theorem. □

We do not know if the converse of the above theorem is always true. However when $Y = C(K)$, for a compact set K we have the following complete description of very smooth points of $\mathcal{K}(X, C(K))$. It is well known that $\mathcal{K}(X, C(K))$ can be idetified with the space of vector-valued functions $C(K, X^*)$. Since in this case the arguments given during the proof of Theorem 12 are much simpler we present the complete proof.

COROLLARY 15

Let K be a compact Hausdorff space and let X be any Banach space. $f \in C(K, X)$ is very smooth if and only if there exists a unique point $k \in K$ which is an isolated point such that $f(k)$ is a very smooth point of X.

Proof. We begin by noting that the dual of $C(K, X)$ can be identified with $M(K, X^*)$, the space of X^*-valued regular Borel measures equipped with the total variation norm.

Suppose $f \in C(K, X)$ is very smooth. Let $k \in K$ and $x^* \in \partial_e X_1^*$ be such that $1 = x^*(f(k))$.

Let $\{x_\alpha^*\}_{\alpha \in I}$ be a net in X_1^* such that $x_\alpha^* \xrightarrow{w^*} x^*$. Consider for any $\tau \in X^{**}$, $F \in M(K, X^*)$, $(\delta(k) \otimes \tau)(F) = \tau(F(\{k\}))$. Then $\delta(k) \otimes \tau \in C(K, X)^{**}$. Also $\delta(k) \otimes x_\alpha^* \xrightarrow{w^*} \delta(k) \otimes x^*$ and hence $\delta(k) \otimes x_\alpha^* \xrightarrow{w} \delta(k) \otimes x^*$. Therefore $(\delta(k) \otimes \tau)(\delta(k) \otimes x_\alpha^*) = \tau(x_\alpha^*) \to (\delta(k) \otimes \tau)(\delta(k) \otimes x^*) = \tau(x^*)$. Hence $x_\alpha^* \xrightarrow{w} x^*$.

Also if $\{k_\alpha\}_{\alpha \in I}$ is a net in K such that $k_\alpha \to k$, then similar arguments show that $\delta(k_\alpha) \xrightarrow{w} \delta(k)$ and hence $\{k\}$ is an isolated point of K. Similar arguments show that $f(k)$ is a very smooth point.

Conversely suppose that $k \in K$ is an isolated point such that $f(k)$ is a very smooth point. Since $M = \{f \in C(K, X) : f(k) = 0\}$ is an M-summand in $C(K, X)$, with X (under the canonical identification) as the complementary summand, it follows from Proposition 2 that f is a very smooth point. □

COROLLARY 16

Any very smooth point of $\mathcal{K}(X, C(K))$ is a very smooth point of $\mathcal{L}(X, C(K))$.

Proof. Since $C(K)$ has the MAP we have from our earlier remark that a very smooth point of $\mathcal{K}(X, C(K))$ is a smooth point of $\mathcal{L}(X, C(K))$. We now use the identification of $\mathcal{L}(X, C(K))$ as the space $W^*C(K, X^*)$ of functions on K, that are continuous when X^* is equipped with the w^*-topology, equipped with the supremum norm. Suppose $f \in$

$C(K, X^*)$ is very smooth. There exists an isolated point $k_0 \in K$ and an $x_0 \in \partial_e X_1$ that is a point of $w^* - w$ continuity of X_1^{**} such that $f(k_0)(x_0) = 1$. Since $f \to f\chi_{\{k_0\}}$ is an M-projection in $W^*C(K, X^*)$, once more using arguments similar to the ones given during the proof of the previous corollary, we get that f is a very smooth point of $W^*C(K, X^*)$. $\square$

It follows from the arguments given above that if a very smooth point $f \in W^*C(K, X^*)$ attains its norm at a $k \in K$ and $f(k)$ attains its norm, then $\{k\}$ is an isolated point and $f(k)$ is a very smooth point of X^*. We use similar ideas in the next proposition to describe very smooth points in ℓ^∞-direct sums of Banach spaces. For any set I, if $\beta(I)$ denotes the Stone–Cech compactification of the discrete space I, the space $W^*C(K, X^*)$ can be identified as the ℓ^∞ direct sum of $|I|$-many copies of X^*. Our arguments run parallel to those given during the proof of Theorem 1 in [GY1].

PROPOSITION 17

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces. Let $X = \oplus_\infty X_i$. $x \in X$ is a very smooth point iff there exists a unique i_0 such that $\|x(i_0)\| = 1 > \text{Sup}\ \{\|x(i)\| : i \neq i_0\}$ and $x(i_0)$ is a very smooth point of X_{i_0}.

Proof. Suppose there is no i_0 such that $\|x(i_0)\| = 1$ or there is an i_0 such that $\|x(i_0)\| = 1 = \text{Sup}\ \{\|x(i)\| : i \neq i_0\}$. In either case arguing as in the proof of Theorem 1 in [GY1], we get a disjoint partition $\{A, B\}$ of I, such that the supremum of $\|x(i)\|$ over both sets is 1. Now using the canonical projection into the summands of X formed out of X_i's taken from A and B, we get a decomposition $x = y + z$ with $\|y\| = 1 = \|z\|$ and such that $\|y - z\| \leq 1$. Thus it follows from Lemma 2 of [GY1] that x is not a smooth point. This contradiction establishes that there is a unique i_0 such that $\|x(i_0)\| = 1$. Since X_{i_0} is an M-summand and hence an M-ideal of X, the conclusions follow from Proposition 2. $\square$

Inherent in the discussion of this section is that a very smooth point of $C(K)$ is the indicator function of an isolated point and hence is a Fréchet smooth point. We use this in the next proposition to show that for a JB^*-triple (see [MR] for the definitions) very smooth points and Fréchet smooth points coincide.

PROPOSITION 18

Let X be a JB^*-triple. Any very smooth point of X is a Fréchet smooth point.

Proof. Let $x \in X$ be a very smooth point. Let $C(x)$ be the JB^*-subtriple generated by odd powers of x. By Proposition 2 we get that x is a very smooth point of $C(x)$. Since $C(x)$ is a commutative C^*-algebra, we get that x is a Fréchet smooth point. Since the sequence of odd powers of x converge in the weak*-topology of the bidual to a tripotent $u(x)$ and as the norm closed faces $\{x\}$ and $\{u(x)\}$ coincide (see [MR], Lemma 2.4) we conclude as in the proof of Theorem 3.1 in [MR] that x is a Fréchet smooth point of X. $\square$

We conclude this section with a renorming result for very smooth points. We have from the results in §3 of [S] that if X is very smooth then X^* has the Radon–Nikodým property and if X^* is very smooth then X is reflexive. Also if X has the Radon–Nikodým property then X^* has a point of Fréchet differentiability and hence a very smooth point (see [B], Theorem 5.7.4). It is therefore natural to ask for renormings that admit very smooth points.

PROPOSITION 19

Suppose for every renorming of X the dual unit ball has a very smooth point. Then X has the Radon–Nikodým property.

Proof. Suppose $x^* \in X_1^*$ is a very smooth point. As noted before, there exists a unit vector x such that $x^*(x) = 1 = \|x^*\|$. We claim that x is an extreme point of the unit ball of the fourth dual $X^{(IV)}$ of X. Suppose $x = \frac{1}{2}\{\lambda_1 + \lambda_2\}$ where $\lambda_i \in X_1^{(IV)}$. Thus $1 = x^*(x) = \lambda_1(x^*) = \lambda_2(x^*)$. Since x^* is a smooth point of X^{***} we get that $x = \lambda_1 = \lambda_2$. Therefore $x \in \partial_e X_1^{(IV)}$. Thus for every equivalent norm on X, the unit ball has an extreme point of the unit ball (w. r. t the equivalent norm) of $X^{(IV)}$. Hence it follows from Corollary 6 of [Hu] that X has the Radon–Nikodým property. □

3. Connection with the work of Ruess and Stegall

In a series of papers in the 80's Ruess and Stegall ([RS1], [RS2]) have showed that w^*-denting points of $\mathcal{K}(X, Y)_1^*$ and $\mathcal{L}(X, Y)_1^*$ coincide and are precisely points of the form $x^{**} \otimes y^*$, where x^{**} and y^* are w^*-denting points of X_1^{**} and Y_1^* respectively. Using a result of Lin *et al* [LLT], a denting (w^*-denting) point is precisely an extreme point that is a point of weak-norm (w^*-norm) continuity of the identity map in the dual unit ball.

While investigating the question of very smooth points in the unit ball of spaces of operators one encounters extreme points of $\mathcal{L}(X, Y)_1^*$ (or $K(X, Y)_1^*$) that are points of $w^* - w$ continuity. Our theorem in the previous section is an attempt to prove a Ruess–Stegall type result in one direction (see Remark 13). We could describe them fully only for the space of compact operators when $Y = C(K)$. Next proposition gives another partial answer and should again be compared with Remark 13.

PROPOSITION 20

Suppose X^* or Y has the CMAP. *Let $x \otimes y^* \in \partial_e \mathcal{K}(X, Y)_1^*$ be a $w^* - w$ point of continuity. Then $x \otimes y^* \in \partial_e \mathcal{L}(X, Y)_1^*$.*

Proof. As noted before the hypothesis implies that $\mathcal{K}(X, Y) \subset \mathcal{L}(X, Y) \subset \mathcal{K}(X, Y)^{**}$. Applying Lemma III.2.14 from [HWW] once more we have that $x \otimes y^*$ has a unique norm preserving extension to a functional on $\mathcal{K}(X, Y)^{**}$. Since $x \otimes y^* \in \mathcal{L}(X, Y)^*$ is one such extension and as $x \otimes y^* \in \partial_e \mathcal{K}(X, Y)_1^*$, we get that $x \otimes y^* \in \partial_e \mathcal{L}(X, Y)_1^*$. □

In the next proposition we show that under the M-ideal assumption the dual unit balls of the space of both compact and bounded operators have the same $w^* - w$ points of continuity.

PROPOSITION 21

Let $\mathcal{K}(X, Y) \subset \mathcal{L}(X, Y)$ be an M-ideal. Any $\tau \in \partial_e \mathcal{L}(X, Y)_1^*$ that is a $w^* - w$ point of continuity is of the form $x \otimes y^*$, where $x \in \partial_e X_1^{**}$ is a $w^* - w$ point of continuity and $y^* \in \partial_e Y_1^*$ is also a $w^* - w$ point of continuity.

Proof. Let $\tau \in \partial_e \mathcal{L}(X, Y)_1^*$ be a $w^* - w$ point of continuity. We shall show that $\tau \in \partial_e \mathcal{K}(X, Y)_1^*$. It would then follow from the arguments given during the proof of Lemma 1 in [R5] that it is a $w^* - w$ point of continuity of $\mathcal{K}(X, Y)_1^*$. Hence the conclusion follows from Theorem 12.

It follows from the arguments given during the proof of Proposition 1 in [R4] that a net of convex combinations of functionals of the form $\{x \otimes y^* : \|x\| = 1 = \|y^*\|\}$ (which are in $\mathcal{K}(X, Y)_1^* \subset \mathcal{L}(X, Y)_1^*$) converges to τ in the weak* topology of $\mathcal{L}(X, Y)^*$. Hence by our assumption this convergence occurs in the weak topology. Therefore $\tau \in \partial_e \mathcal{K}(X, Y)_1^*$. □

The following corollary is now easy to see and should be compared with the description of smooth points where one needed extra assumptions on the quotient space, see [KY], [GY2] and [HWW] Chapter VI.

COROLLARY 22

Let $\mathcal{K}(X, Y) \subset \mathcal{L}(X, Y)$ be an M-ideal. If $T \in \mathcal{L}(X, Y)$ is a very smooth point then it attains its norm at a unique vector x and $T(x)$ is a very smooth point of Y.

4. Spaces without points of $w^* - w$ continuity

In the concluding part of this paper we consider situations where X_1^* fails to have points of $w^* - w$ continuity. For such spaces, X_1 fails to have very smooth points. We recall from [HWW] that $Y \subset Z$ is said to be an L-ideal if there is an onto projection $P : Z \to Y$ such that $\|z\| = \|P(z)\| + \|z - P(z)\|$ for all $z \in Z$. Also if a Banach space X, when considered under the canonical embedding, as a subspace of X^{**}, is an L-ideal then X is said to be an L-embedded space. See [HWW], Chapter IV and VI for several examples of L-embedded spaces from among function spaces and spaces of operators. In what follows we only consider non-reflexive spaces. Our first result leads to an example of a smooth space that does not have any very smooth points.

PROPOSITION 23

Let $X \subset Y \subset X^{**}$ (under the canonical embedding) and suppose X is a proper L-ideal in Y. Then X_1^* has no $w^* - w$ points of continuity. In particular if X is an L-embedded space then X_1 has no very smooth points.

Proof. Let P be the L-projection in Y whose range is X. Let $x^* \in X_1^*$ be a $w^* - wPC$. We get the necessary contradiction by showing that x^* has no unique norm preserving extension to Y^* and hence to X^{**}. Choose a $\tau \in X^{\perp}$, i.e., such that $P^*(\tau) = 0$ and $0 < \|\tau\| \leq \|P^*(x^*)\| \leq \|x^*\|$. We note that since $X \subset Y \subset X^{**}$ under the canonical embedding, x^* has a natural extension as the evaluation functional to Y. We continue to denote this extension by x^*. Now for any $x \in X$, $(P^*(x^*) + \tau)(x) = x^*(x)$ and as P^* is a M-projection, $\|P^*(x^*) + \tau\| = \max\{\|P^*(x^*)\|, \|\tau\|\} \leq \|x^*\|$. Hence there is no unique extension. □

Remark 24. It is well known that the Hardy space on the unit circle, H_0^1, is a smooth L-embedded space (see [HWW], p. 167). However, the unit ball has no very smooth points.

In view of this remark the following question can be asked: 'does there exist a very smooth space X so that no unit vector is a smooth point of the fourth dual of X?' This however is not the case. To see this note that since X is very smooth, X^* has the Radon–Nikodým property (see [S], §3), thus X has points of Fréchet differentiability (see [B], Proposition 5.6.13). It is well known that a point of Fréchet differentiability continues to be a point of Fréchet differentiability of all the duals of even order of X.

COROLLARY 25

Let X be an L-embedded space with the metric approximation property and let Y be any Banach space. There are no points of $w^* - w$ continuity in the dual unit ball of the projective tensor product space $X \otimes_\pi Y$.

Proof. We use the known inclusions (under canonical embeddings)

$$X \otimes_\pi Y \subset X^{**} \otimes_\pi Y \subset (X \otimes_\pi Y)^{**}.$$

See [R6]. Since X is an L-ideal in X^{**} it follows from Theorem VI.6.8 of [HWW] that $X \otimes_\pi Y$ is an L-ideal in $X^{**} \otimes_\pi Y$. It thus follows from our above proposition that there are no points of $w^* - w$ continuity in the dual unit ball. □

Remark 26. For any positive measure μ, $L^1(\mu)$ is an L-embedded space with the metric approximation property. For any Banach space Y, let $L^1(\mu, Y)$ denote the space of Y-valued Bochner integrable functions. Since $L^1(\mu) \otimes_\pi Y = L^1(\mu, Y)$ we get that there are no $w^* - w$ points of continuity in the dual unit ball of $L^1(\mu, Y)$. We recall that $\mathcal{L}(X, Y^*)$ can be identified with $(X \otimes_\pi Y)^*$. Thus when X is an L-embedded space with the metric approximation property there are no points of $w^* - w$ continuity in $\mathcal{L}(X, Y^*)_1$.

Let X be a Banach space. Suppose there is a $u \in X^{**} \backslash X$ such that $\|u + x\| = \|u\| + \|x\|$ for every $x \in X$. Then $Y = \text{span}\,\{X, u\}$ satisfies the hypothesis of the above proposition. This geometric condition has been well studied in the literature, see [DGZ] . We use these ideas in the next corollary.

Turning once more to spaces of operators we recall that a Banach space X is said to have the Daugavet property if $\|I + T\| = 1 + \|T\|$ for all compact operators T (see [W], Definition 2.1 and Theorem 2.7). Any space with the Daugavet property contains an isomorphic copy of ℓ^1 (see [W], Theorem 2.6).

COROLLARY 27

Let X be a Banach space having the Daugavet property and the metric approximation property. Then $\mathcal{K}(X)_1^*$ has no points of $w^* - w$ continuity.

Proof. Since X has the metric approximation property it follows from Example 1 in [R6] that $\mathcal{K}(X) \subset \mathcal{L}(X) \subset \mathcal{K}(X)^{**}$ under the canonical embedding. Since X satisfies the Daugavet property the conclusion follows. □

This author has proved recently ([R2], [R3]) that when K is infinite, for the space $\mathcal{L}(X, C(K))$ there are no point of w-norm continuity in the unit ball. When $\mathcal{L}(X, C(K))$ is a dual space it would be interesting to know if the unit ball can have points of $w^* - w$ continuity (with respect to a predual!) . We have some partial results.

PROPOSITION 28

Let X be a reflexive Banach space. If $\mathcal{L}(X, C(K))$ is a dual space then the unit ball has no points of $w^* - w$ continuity. In particular the predual has no very smooth points.

Proof. It follows from [CG] that when X is reflexive the assumption $\mathcal{L}(X, C(K))$ is a dual space implies that K is hyperstonean. Hence the (unique) predual is of the form $L^1(\vartheta, X)$ for a category measure ϑ on K. Since X is reflexive, it follows from p. 200 of [HWW], that $L^1(\vartheta, X)$ is an L-embedded space. Hence $\mathcal{L}(X, C(K))_1$ has no points of $w^* - w$ continuity. □

Remark 29. Similar arguments work when X is the predual of a von Neumann algebra. In this case if $\mathcal{L}(X, C(K))$ is a dual space, then K is extremally disconnected and hence $\mathcal{L}(X, C(K))$ is a von Neumann algebra and hence is the dual of an L-embedded space.

Remark 30. Turning to the case of weakly compact operators valued in a $C(K)$ space or more generally if one considers the space $WC(K, X)$, (functions continuous when X has the weak topology), this author has proved in [R1] that if $WC(K, X)$ is a dual space then X is reflexive and K is hyperstonean. Therefore, it follows from the above proposition that when $WC(K, X)$ is a dual space the unit ball has no $w^* - w$ PC's.

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