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RENORMINGS AND EXTREMAL STRUCTURES

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Abstract. In this paper we use renorming techniques to settle several questions on extremal structures of Banach spaces. We construct a unitary vector in a dual space which is not a weak*-unitary. We construct an exposed point in the unit ball of a Banach space X that remains exposed in the unit ball of $X^{(4)}$ but is not extreme in the unit ball of $X^{(6)}$.

1. Introduction

This work is motivated by our attempt to understand several new extremal structures of Banach spaces that have been recently studied in [2], [7] and [5]. For a Banach space X let X_1 denote the closed unit ball. The authors of [2] have introduced a new class of extreme points by calling a unit vector $u \in X$ whose state space $S = \{x^* \in X_1^* : x^*(u) = 1\}$ spans X^*, a unitary. When u belongs to a dual space X^*, if $S' = \{x \in X_1 : u(x) = 1\}$ spans X then u is called a weak*-unitary. These notions are the abstract analogues of the corresponding notion of a unitary in a unital C^*-algebra. It follows from Theorem 9.5.16 of [10] (see also[1]) that a vector u in a C^* algebra is a unitary in this sense if and only if it is unitary in the (usual) algebraic sense. A unitary is in particular a strongly extreme point (see [2]).

It was shown in [2] that any weak*-unitary of X^* is a unitary. The converse holds in several natural situations: it follows from Theorem 3 in [1] that when X^* is a von Neumann algebra every unitary in X^* is a weak*-unitary. More generally, this is true under an upper semi-continuity assumption on the duality map (Proposition 2.2). However, we show below (Theorem 2.4) that this converse fails for general Banach spaces: indeed, if X is non-reflexive and X^* is separable, there is a renorming of X such that the dual space X^*

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contains a unitary u which is not a weak*-unitary. In fact, u does not even attain its norm: hence, while the subset K of the bidual unit ball where u attains its norm is so large that it spans X^{**}, this set K does not meet X.

Any unitary u is clearly a weak*-unitary (under the canonical embedding) of the bidual. Thus a natural question is the following: When is $\{x^* \in X_1^* : x^*(u) = 1\}$ weak*-dense in $\{\tau \in X_1^{***} : \tau(u) = 1\}$? It turns out that this density condition is frequently satisfied for natural examples of unitaries. For instance, when X is a C^*-algebra, a unitary in X remains a unitary of the enveloping von Neumann algebra; thus it follows from Proposition 3.3 in [2] that the density condition holds in this case. Following the terminology of Theorem 3.1 in [6], when this density condition is satisfied, we call u a point of norm-weak upper semi-continuity (norm-weak usc) for the duality map $x \to \{f \in X_1^* : f(x) = \|x\|\}$. We show (Theorem 2.3) that any non-reflexive Banach space with a unitary u can be renormed so that in the new norm u is still a unitary, but is not a point of norm-weak usc for the duality map.

For a convex set K we denote by $\partial_e K$ its set of extreme points. We always consider a Banach space as canonically embedded in its bidual. For $n > 3$ we denote by $X^{(n)}$ the n-th dual of X. It is easily seen that a unitary $u \in X$ is extreme in all dual unit balls $B_{X^{(2n)}}$.

For a non-reflexive space X, $x \in \partial_e X_1$ is said to be a weak*-extreme point if it is also an extreme point of X_1^{**}. In [5] the authors gave an example of a space with a smooth dual, whose unit vectors are all extreme points of X_1^{**} but none is an extreme point of $X_1^{(4)}$. By a renorming result that is applicable, e.g., to certain separable Asplund spaces, we construct (Theorem 2.5) a point $x \in X \cap \partial_e X_1^{(4)}$ that is not an extreme point of $X_1^{(6)}$. Such counterexamples suggest that it is hopeless to find a condition which ensures that an extreme point of the unit ball of a Banach space remains extreme in all duals of even order, and which boils down to usual extremality in reflexive spaces (see [5]).

The gist of our results is that, although the notion of a unitary element behaves properly in the C^* algebra context or at least when some natural algebraic structure is available (see [2]), everything that can go wrong in the Banach space context does go wrong. Remark 2.6 shows however that one can sometimes be pleasantly surprised and meet positive results.

We denote by $CO(E)$ the convex hull of a set E. When E is a subset of a dual space, the weak*-closure of E is denoted E^-, and $CO^-(E)$ is the weak*-closed convex hull. We denote by $CO^=(E)$ the norm closed convex hull of E.

2. Main results

We first observe that the simplest way to obtain unitaries through renorming techniques actually provides unitaries which are also points of norm-weak usc for the duality map.

PROPOSITION 2.1. *Let X be a Banach space and $0 \neq u \in X$. There is a renorming on X in which u is a unitary and a point of norm-weak usc of the duality map.*

Proof. As remarked in [2], given a non-zero vector u in a Banach space we can renorm the space so that u is a unitary in the new norm. It suffices indeed to consider the unit ball B_0 of some equivalent norm such that $u \notin B_0$, and to consider the equivalent norm whose unit ball is $B_1 = CO(B_0 \cup \{\pm u\})$. So we may and do assume that u is a unitary. Put $B^* = CO(S \cup -S)$. Then B^* is the dual unit ball of an equivalent norm on X whose state space is S and clearly u is still a unitary. It is easy to see that the unit ball of the triple dual is given by $B' = CO(S^- \cup -S^-)$ (where the closure is taken with respect to the weak*-topology). Now if $\tau \in B'$, then $\tau(u) = 1$ if and only if $\tau \in S^-$. Thus u is a point of norm-weak usc for the duality map in this norm. □

In the notation of the above proof, the point u is a (QP) point (in the sense of [13]) of X when this space is equipped with the norm whose unit ball is B_1. Hence it is in particular, for this norm as well, a point where the duality map is even norm-to-norm upper semi-continuous.

The following simple proposition shows that in the presence of norm-weak upper semi-continuity unitaries in dual spaces are weak*-unitaries. We refer to [7] for the definition and basic statements on norm-weak upper semi-continuity of the pre-duality map, $x^* \to \{x \in X_1 : x^*(x) = \|x^*\|\}$.

PROPOSITION 2.2. *Let $x_0^* \in X^*$ be a unitary. If x_0^* is a point of norm-weak usc for the pre-duality map then x_0^* is a weak*-unitary. In particular, if the duality map is norm-weak usc at x_0^*, then x_0^* is a weak*-unitary.*

Proof. By Lemma 2.1 in [7], the functional x_0^* is norm-attaining. Let $S' = \{x \in X_1 : x_0^*(x) = 1\}$. To show that x_0^* is a weak*-unitary we shall show that $B = CO^{=}(S' \cup -S')$ is the unit ball of an equivalent norm on X. By Corollary 3.2 in [2] the conclusion follows. Note that $B^- = CO(S'^- \cup -S'^-)$ (where the closure is taken in the weak*-topology of X^{**}). Thus by our assumption of upper semi-continuity, B^- is also the absolute convex hull of $\{\tau \in X_1^{**} : \tau(x_0^*) = 1\}$. As x_0^* is a unitary, by Theorem 3.1 in [2] we have that B^- is the unit ball of an equivalent dual norm on X^{**}. Thus by the bipolar theorem we see that B is an equivalent norm on X. The last part follows from Theorem 2.3 in [7], since if a norm attaining functional is a point of norm-weak usc for the duality map then it is also a point of norm-weak usc for the pre-duality map. □

We will now use finer renorming techniques for exhibiting unitaries that are *not* points of norm-weak usc for the duality map.

Theorem 2.3. *Let X be a non-reflexive Banach space, and u a non-zero vector in X. Then X can be renormed so that in the new norm u is a unitary but not a point of norm-weak usc for the duality map.*

Proof. As shown in [2], we may and do assume that u is a unitary in the original norm of X. We first consider the case of a separable Banach space X. Let K denote the state space of u. Let $x^{**} \in X^{**} \backslash X$. Let $\alpha = \sup\{|x^{**}(x^*)| : x^* \in K\}$. Since $x^{**}|\ker(u)$ is not weak*-sequentially continuous, we can choose a sequence $\{y_n^*\}$ with $x^{**}(y_n^*) > 3 + \alpha$, $y_n^*(u) = 1$ and $y_n^* \to x^* \in K$ (with respect to the weak*-topology, here and in the rest of the proof).

Let $z_n^* = (1 - 1/n)y_n^*$ and let $B = CO^-(X_1^* \cup \pm\{z_n^*\})$. The convex set B is the dual unit ball for an equivalent norm on X. As before it is easy to see that u is still a unitary with respect to this norm with the same state space K.

We now show that u is not a point of norm-weak usc of the duality map of this norm. We show that the criterion in Theorem 2.1 of [6] is violated. Let $V = \{x^* : |x^{**}(x^*)| < 1\}$. Pick any $\delta > 0$. Since $z_n^* \to x^* \in K$, these functionals are eventually in the set $\{f \in B : f(u) > 1 - \delta\}$. It follows that this set is not contained in $(K + V)$. Indeed, for $k \in K$ and $v \in V$, $x^{**}(k+v) < \alpha+1$, while $x^{**}(z_n^*) > 2+\alpha$ for n large enough. Thus for n large enough we get that $z_n^* \in \{f \in B : f(u) > 1 - \delta\}$ but $z_n^* \notin (K + V)$.

The general case follows easily from the separable one. Let $u \in Y \subset X$ and Y be separable non-reflexive. We construct z_n^* in Y^* as above, and we denote by $z_n'^*$ norm preserving extensions to X of the functionals z_n^* obtained above. Let $B = CO^-(X_1^* \cup \pm\{z_n'^*\})$. It is easily checked that u is still a unitary in the new norm but is not a point of norm-weak usc for the duality map. □

In a slightly more specific situation, we now construct examples of unitaries in a dual space that fail to attain norm, and hence are in particular *not* weak*-unitaries. Note that Proposition 2.2 implies that Theorem 2.4 is an improvement of Theorem 2.3, with a dual norm in the (separable) dual case.

Theorem 2.4. *Let X be a non-reflexive Banach space such that X^* is separable. Then X can be equivalently renormed so that, in the new dual norm, X^* contains a unitary which fails to attain its norm.*

Proof. Since X^* is separable, we may assume (see [4], Theorem II.7.1) that X is equipped with an equivalent norm such that X^{**} is strictly convex. Since X is not reflexive, by James' theorem there exists a unit vector x_0^* that is not norm attaining. Let $\|x_0^{**}\| = x_0^{**}(x_0^*) = 1$. Clearly $x_0^{**} \notin X$. Let $d = d(x_0^{**}, X)$. Let $K = \{x^{**} \in X^{**} : x^{**}(x_0^*) = 1, \|x^{**} - x_0^{**}\| \leq d/2\}$. Clearly $\operatorname{span} K = X^{**}$ and $K \cap X = \emptyset$.

We now renorm X such that K is the state space of x_0^* in this norm. It will clearly follow that x_0^* is a unitary that does not attain its norm.

As X^* is separable and K is bounded, let $K = \{x_n^{**}\}^-$. We can write $K = \cap W_l = \cap W_l^-$ for a sequence $\{W_l\}$ of weak*-open subsets of $(1+d/2)X_1^{**}$. For each n, we can choose a sequence $\{x_{n,k}\} \subset (1+d/2)X_1$ such that $x_{n,k} \to x_n^{**}$ in the weak*-topology and such that $|x_0^*(x_{n,k})| < 1$, $x_{n,k} \in W_l$ for all $n \geq l$ and for all k.

Let $B' = CO^=(X_1 \cup \pm\{x_{n,k}\})$. Let $\|.\|'$ denote the equivalent norm on X whose unit ball is B'. As $|x_0^*(x_{n,k})| \leq 1$, we have that x_0^* is a unit vector with respect to the new norm.

Also in the new norm the bidual unit ball is given by $B'^{**} = CO(X_1^{**} \cup CO^-(\pm\{x_{n,k}\}))$. Now suppose $x^{**} = \lambda x_1^{**} + (1-\lambda)x_2^{**}$ for some $x_1^{**} \in X_1^{**}$, $x_2^{**} \in CO^-(\pm\{x_{n,k}\})$, $\lambda \in [0,1]$ and $x^{**}(x_0^*) = 1$. Then $1 = x_1^{**}(x_0^*) = x_2^{**}(x_0^*)$. Since X^{**} is strictly convex, $x_1^{**} = x_0^{**} \in K$.

Since for any $l \geq 1$ all $x_{n,k}$'s but a finite number are contained in W_l, we have that $(\pm\{x_{n,k}\})^- = \pm(K \cup \{x_{n,k}\})$. We now claim that $x_2^{**} \in K$. To see this we use the description of $CO^-(\pm\{x_{n,k}\})$ in terms of barycenters (Proposition 1.2 in [12]). Thus there is a probability measure μ with $\mu((\pm\{x_{n,k}\})^-) = 1$ and x_2^{**} is the barycenter of μ. Then $1 = x_2^{**}(x_0^*) = \int x_0^* d\mu$. Since $(\pm\{x_{n,k}\})^- = \pm(K \cup \{x_{n,k}\})$ and by the choice of $x_{n,k}$, this implies $\mu(K) = 1$. As K is a weak*-compact convex set by Proposition 1.2 of [12] again, we have $x_2^{**} \in K$. Hence $x^{**} \in K$ and so K is also the state space for this norm.

Since $K \cap X = \emptyset$, it follows that x_0^* does not attain its supremum on B', in other words, that x_0^* fails to attain its norm. $\square$

A unitary in X remains unitary in X^{**}, and it follows through an obvious induction that a unitary is in particular an extreme point in the unit ball of every dual space $X^{(2n)}$. It is well known that such a stability fails for general extreme points. Our last result shows that, even if stability holds to begin with, it may fail afterwards.

Theorem 2.5 is inspired by Proposition 4.1 in [7] and partially answers the questions raised in [5]. We will now use similar but more elaborate renormings to climb the tower of successive duals. The existence of spaces which satisfy the hypothesis of Theorem 2.5 below follows from [9].

THEOREM 2.5. *Let X be a separable space such that X^{***}/X^* is separable and non-reflexive. There is an equivalent norm on X and a vector $f \in X$ of norm one, which is an exposed point of the unit ball of the fourth dual but is not an extreme point of the unit ball of the sixth dual.*

Proof. The proof below is a modification of the proof of Proposition 4.1 in [7] that the reader is invited to consult before dwelling upon this proof. Our strategy is to adjust what has been done for proving [7, Prop. 4.1], in such a way that the renormed space is actually a dual space. In order to keep the notation of the proof of [7, Prop. 4.1], we will denote by x a smooth point of X^* and by $f \in X$ the corresponding differential.

We first note that the original dual norm is Fréchet smooth on a dense set since X^{**} is separable. Indeed, the dual of X^{**}/X is isomorphic to X^{***}/X^*. Let $x \in X^*$ be a unit vector where the norm is Fréchet differentiable. It is an easy consequence of Smulyan's lemma that when a dual norm is Fréchet differentiable at a given point x, the differential f at this point belongs to the predual. Hence, let $f \in X$ be such that $x(f) = \|f\| = 1$. Let $\{\phi_j\}_{j\geq 1} \subset X^{***}$ be such that $\{\phi_{2j}\}_{j\geq 1}$ is norm dense in X^* and $\{\phi_{2j+1}\}_{j\geq 0}$ is norm dense in $X^{\perp}$.

Since X^{***}/X^* is separable and non-reflexive, we can choose a sequence of unit vectors $\{t_n\}_{n\geq 1} \subset (X^*)^{\perp} \subset X^{(4)}$ such that $t_n \to 0$ in the weak *-topology of $X^{(4)}$ and there exists $0 \neq F \in X^{(6)}$ with $\{\pm F\} \subset \{t_n\}^-$ (where the closure is taken with respect to the weak*-topology of $X^{(6)}$).

Again by separability there exist sequences $\{f_{n,k}\} \subset X_1^{**}$ such that $f_{n,k} \to t_n$ for each n, in the weak*-topology of $X^{(4)}$. Note that this in particular implies that for fixed n and k tending to infinity, $f_{n,k} \to 0$ in the weak*-topology of X^{**} as $\{t_n\} \subset (X^*)^{\perp}$. Without loss of generality we may assume that:

(1) $|t_n(\phi_j)| < 1/2^n$ *for* $n > j$;
(2) $|\phi_j(f_{n,k})| < 1/2^n$ *for* $n > j$ *and for all* k;
(3) $|f_{n,k}(x)| < 1/2^n$ *for all* n, k.

We finally choose a sequence $\{z_{n,k,l}\} \subset X_1$ such that $z_{n,k,l} \to f_{n,k}$ in the weak*-topology of X^{**} and assume again without loss of generality:

(1) $|\phi_{2j}(z_{n,k,l})| < 1/2^n$ *for* $n > 2j$, *for all* k, l;
(2) $|\phi_{2j}(z_{n,k,l} - f_{n,k})| < 1/2^k$ *for* $k > j$, *for all* n, l;
(3) $|x(z_{n,k,l})| < 1/2^n$ *for all* n, k, l.

As in the previous renormings, we now let $z'_{n,k,l} = z_{n,k,l} + (1-1/n)f$ and take $B' = CO^{=}(X_1 \cup \pm\{z'_{n,k,l}\})$ as the new unit ball.

As before, the unit ball $B'_{X^{**}}$ of the bidual in the new norm is given by $B'_{X^{**}} = CO^{-}(X_1^{**} \cup L^{-})$ with $L = \pm\{z'_{n,k,l}\}$), where the closures are taken in the weak*-topology of X^{**}. By our choice of these sequences, we have:

(1) $z'_{n,k,l} \to f$ *in* (X, w) *as* $n \to \infty$, *for any* $k = k(n)$ *and* $l = l(n)$;
(2) for any n_0, $z'_{n_0,k,l} \to (1-1/n_0)f$ *in* (X, w) *for* $k \to \infty$ *and any* $l = l(k)$;
(3) for any n_0, k_0, $z'_{n_0,k_0,l} \to f_{n_0,k_0} + (1-1/n_0)f$ *in* (X^{**}, w^*) *for* $l \to \infty$.

Therefore $L^- = L \cup \pm\{f_{n,k} + (1-1/n)f\}_{n,k\ \geq 1} \cup \pm\{(1-1/n)f\}_{n\geq 1} \cup \pm\{f\}$. Now since X^{**} has the Radon-Nikodým property, we have (see page 327 of [7]) that the weak*-closed convex hull of any weak*-compact subset of X^{**} coincides with its norm-closed convex hull. Therefore $B'_{X^{**}} = CO\{X_1^{**} \cup$

$CO^{=}(L^{-})\}$ with

$$L^{-} = \pm \left\{ z_{n,k,l} + \left(1 - \frac{1}{n}\right) f \right\}_{n,k,l \geq 1} \cup \pm \left\{ f_{n,k} + \left(1 - \frac{1}{n}\right) f \right\}_{n,k \geq 1} \cup \pm \left\{ \left(1 - \frac{1}{n}\right) f \right\}_{n \geq 1} \cup \pm \{f\}.$$

We now show that x is a smooth point of the third dual in the new norm (in other words, a very smooth point, with the notation used in [7], of the dual), and thus f is an exposed point (and so in particular an extreme point) of the fourth dual unit ball.

To achieve this we first note that x is a smooth point of X^* equipped with the new dual norm, with derivative f. Indeed, we clearly have

$$L^{-} \cap x^{-1}(\{1\}) = \{f\},$$

and since x is a smooth point of the original dual norm,

$$X_1^{**} \cap x^{-1}(\{1\}) = \{f\},$$

and our claim easily follows.

Also by the choice of the sequences we have:

(1) $|\phi_j(f_{n,k})| < 1/2^n$ *for* $n > j$, *for all* k;
(2) $|\phi_{2j}(z_{n,k,l})| < 1/2^n$ *for* $n > 2j$, *for all* k , l;
(3) $\phi_{2j+1}(z_{n,k,l}) = 0$ *for all* j, n, k, l.

Therefore when n goes to infinity, the sequences $(f_{n,k})$ and $(z_{n,k,l})$ converge weakly to 0 in X^{**} regardless of k and l. Now, using convex combinations as in the proof of Fact 6 in [7], we can show that f is a point of weak*-weak continuity for the identity map on the unit ball of the bidual. Thus by Remark 3.1 in [7] we get that x is a smooth point of the third dual, and thus f is exposed by x in the new unit ball $B'_{X^{(4)}}$ of $X^{(4)}$.

We finally invoke the choice of $F \in X^{(6)}$ we made at the start of the proof. By our construction we have that

$$f_{n,k} + \left(1 - \frac{1}{n}\right) f \in B'_{X^{**}}$$

for every n and k, and thus

$$t_n + \left(1 - \frac{1}{n}\right) f \in B'_{X^{(4)}}$$

for every n, and finally $f + F \in B'_{X^{(6)}}$ and $f - F \in B'_{X^{(6)}}$. Therefore f is not an extreme point of the sixth dual unit ball.

□

REMARKS 2.6. (1) It is very likely that a suitable modification of the arguments given above provides, when assuming separability and non-reflexivity of quotients of higher order, an equivalent norm with a point of X in $\partial_e X_1^{(2n)}$ but not in $\partial_e X_1^{(2n+2)}$.

(2) Although Theorem 2.5 has been shown under the quite restrictive assumption that X^{***}/X^* is separable and non-reflexive, it is clear that our construction is flexible enough and would apply to many spaces. However, it is interesting to observe that non-reflexivity does not suffice. Indeed, it follows from [7, Prop. 3.4] that if X^{***}/X^* is a Grothendieck space and $f \in X$ is exposed in the unit ball of the fourth dual by $x \in X^*$, it is exposed by $x \in X^*$ in the unit ball of the sixth dual as well. This observation applies in particular to quasi-reflexive spaces, and also by [11] when X^* is a von Neumann algebra. Finally, we refer to [8] and [3] for the investigation of the topological complexity of the set of weak*-exposed points, which is not necessarily a Borel set, even in the separable case.

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