An analogue of the Wiener Tauberian Theorem for the Heisenberg Motion group

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Abstract | We show that the Wiener Tauberian property holds for the Heisenberg Motion group $\mathbb{T}^n \rhd\!< H^n$. This is a special case of the same result for a wider class of groups. However, our exposition is almost self contained and the techniques used in the proof are relatively simple.

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1. Introduction

A locally compact group is said to have Wiener's property if every two sided ideal in the Banach $*$ algebra $L^1(G)$ is contained in the kernel of a nondegenerate Banach $*$ representation of $L^1(G)$ on a Hilbert space. It is well known from the work of Leptin [7] that semidirect product of abelian groups and connected nilpotent groups have the Wiener's property. On the other hand it was established by M. Duflo that no semisimple Lie group has this property [7].

If G is compact extension of a nilpotent group then it it is of polynomial growth [4] and G has a symmetric group algebra. These two properties together imply that G has Wiener's property [8]. For the case of general locally compact motion groups see the work of Gangolli [3].

We consider one such group $G = \mathbb{T}^n \rhd\!< H^n$, the semidirect product of the n-dimensional torus $\mathbb{T}^n$ and $2n+1$ dimensioanl Heisenberg group H^n. The above theory points out that it has Wiener's property. However this far reaching general theory related to the groups of polynomial growth involves heavy machinary and hence not easily accessible.

We offer here a direct and independent proof of the fact for G as above, starting from a result of Hulanicki and Ricci [5]. We interpret the result in [5] on H^n as a Wiener's theorem for $\mathbb{T}^n$-biinvariant functions on G. Using elementary arguments we extend this result to a Wiener's theorem for the full group G. Going towards the proof we find the representations in $\widehat{G}$ and establish the Plancherel theorem. Then we obtain the Wiener's theorem explicitly in terms of the representations. Precisely, we find the sufficient condition that an L^1-function on G generates a dense ideal in $L^1(G)$. This condition is also necessary. We may conjecture that this method of extending the theorem from the biinvariant functions in $L^1(G)$ to the full $L^1(G)$ would work for other compact extensions.

This exposition is almost self contained and techniques are simple. Our group G is a subgroup of the so called Heisenberg Motion group, which is the semidirect product of the Heisenberg group H^n and the unitary group $U(n)$. This latter group acts on H^n as automorphisms and the semidirect product $U(n) \rhd\!< H^n$ turns out to be the natural group of isometries for the Heisenberg geometry; see the works of Koranyi [6] and Strichartz [10]. In [9] R. Rawat has studied a Wiener's theorem for the action of $U(n) \rhd\!< H^n$ on H^n.

2. Notation and Preliminaries

Let $H^n = \mathbb{C}^n \times \mathbb{R}$, with group law

$$(z,t)\circ(z',t') = \left(z+z', t+t' - \frac{1}{2} Im\, z'.\bar{z}\right),$$

denote the $(2n+1)$-dimensional Heisenberg group. Let $\mathbb{T}^n = S^1 \times \cdots \times S^1$ (n times) be the n-dimensioanl Torus. The group $\mathbb{T}^n$ acts naturally on H^n by automorphisms and $(H^n, \mathbb{T}^n)$ forms a Gelfand pair. Let $G = \mathbb{T}^n \rhd\!< H^n$ be the semidirect product of H^n and $\mathbb{T}^n$. Let us denote the elements of G by (σ, z, t) where $\sigma = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{T}^n$ and $(z, t) \in H^n$. The group law of G is given by:

$$(\sigma, z, t).(\tau, w, s) = \left(\sigma\tau, z + \sigma w, t + s - \frac{1}{2} Im\, \sigma w.\bar{z}\right).$$

As $\mathbb{T}^n$ is a subgroup of G through the identification of σ and $(\sigma, 0, 0)$, it acts on G from left and right through the group law as:

$$(\sigma, 0, 0).(\tau, w, s) = (\sigma\tau, \sigma w, s)$$

and

$$(\tau, w, s).(\sigma, 0, 0) = (\sigma\tau, w, s).$$

The Heisenberg group H^n can also be identified naturally both as a subgroup of G and as the quotient $G/\mathbb{T}^n$. For an element $(\sigma, z, t) \in G$, $(\sigma, z, t)^{-1} = (\sigma^{-1}, -\sigma^{-1}z, -t) = (\sigma^{-1}, 0, 0)(1, -z, -t)$.

We use $\mathbb{R}_+^n$ for $\mathbb{R}_+ \times \cdots \times \mathbb{R}_+$ (n times) where $\mathbb{R}_+$ is the set of positive reals. The n-dimesional Euclidean space is denoted by $\mathbb{R}^n$. For this article '.' is the usual bilinear inner product. For $\Theta = (\theta_1, \ldots \theta_n) \in [0, 2\pi)^n$ we denote $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{T}^n$ by $e^{i\Theta}$. A function f on H^n is called *polyradial* if $f(z, t) = f(e^{i\Theta}z, t)$ for all $e^{i\Theta} \in \mathbb{T}^n$, where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $e^{i\Theta}z = (e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)$. We consider $C_c^\infty(G)$ to be the space of compactly supported C^∞-functions. For a function $f \in C_c^\infty(G)$ we define a function f_0 on H^n by $f_0(z, t) = f(1, z, t)$ and a polyradial function f_{00} on H^n by $f_{00}(r, t) = f(1, r, t)$, where $r = (r_1, \ldots, r_n)$, $r_j \geq 0$. For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ the character χ_μ of $\mathbb{T}^n$ is defined by $\chi_\mu(e^{i\Theta}) = e^{\mu.\Theta} = e^{i\mu_1\theta_1} \cdot \cdots \cdot e^{i\mu_n\theta_n}$. Suppose $\mu, \mu' \in \mathbb{Z}^n$. A complex valued function f on G is called spherical of right (resp. left) type μ (resp. μ') if

$$\begin{aligned} f(xk_1) &= f(x)\chi_\mu(k_1) \ \text{(resp. } f(k_2x) \\ &= \chi_{\mu'}(k_2)f(x)) \end{aligned}$$

for all $x \in G$ and $k_1, k_2 \in \mathbb{T}^n$. A function of left type μ' and right type μ is called spherical of type (μ', μ). For a function $f \in C_c^\infty(G)$

$$\int_{\mathbb{T}^n} \overline{\chi_\mu}(k_1)f(xk_1)dk_1 \ \text{ and } \int_{\mathbb{T}^n} \overline{\chi_{\mu'}}(k_2)f(k_2x)dk_2$$

respectively are its projections on the space of right μ and left μ' type functions in $C_c^\infty(G)$. For a suitable function space S , by $S_{\mu',\mu}$ we denote the projection of S on the subspace of left μ' and right μ type functions of S. The polyradial functions in $L^1(H^n)$ (denoted by $\mathcal{A}$ in [5]) is under obvious identification the same as $L^1(G)_{0,0}$, the bi-$\mathbb{T}^n$ invariant functions in $L^1(G)$. The right and left G-translates of f for $x \in G$, are denoted respectively by f^x and ${}^xf$. Precisely, $f^x(y) = f(yx^{-1})$ and ${}^xf(y) = f(x^{-1}y)$. By $*$ we mean convolution in G while $*_{H^n}$ denotes the convolution in H^n. For two elements $m, n \in \mathbb{R}^n$, $m - n$ is the componentwise subtraction. By $m > n$ we mean $m_j > n_j$ for every $j = 1, \ldots, n$.

We conclude this section with the following proposition.

Proposition 2.1. *Let $f, g \in C_c^\infty(G)$. Then,*

(i) left type of $f * g$ is the left type of f and the right type of $f * g$ is right type of g.

(ii) if moreover f and g are of right m-type and left n-type respectively, then, $f * g \equiv 0$ if $m \neq n$ and if $m = n$ then $f * g(1, z, t) = f_0 *_{H^n} g_0(z, t)$

Proof of this proposition follows easily considering the fact that every element $g \in G$ can be decomposed as $g = xk = k_1x_1$ where $x, x_1 \in H^n$ and $k, k_1 \in \mathbb{T}^n$. By (i) above $L^1(G)_{\mu,\mu}$ is a subalgebra of $L^1(G)$ under convolution $*$.

3. Representations of G

We shall construct the representations of G from that of H^n and the Euclidean motion group $M(2)$. For details of the representations of these two groups we refer to [12] and [11] respectively.

For $k = 0, 1, 2, \ldots,$ and $t \in \mathbb{R}$ the Hermite polynomials are defined by

$$H_k(t) = (-1)^k \left(\frac{d^k}{dt^k}\{e^{-t^2}\}e^{t^2}\right).$$

The normalized Hermite functions are defined in terms of the Hermite polynomials as

$$h_k(t) = (2^k\sqrt{\pi}k!)^{-\frac{1}{2}} H_k(t) e^{-\frac{1}{2}t^2}.$$

These Hermite functions $\{h_k : k = 0, 1, 2, \ldots\}$ form an orthonormal basis of $L^2(\mathbb{R})$. For any multiindex α and $x \in \mathbb{R}^n$ we define the higher dimensional Hermite functions Φ_α by taking tensor product:

$$\Phi_\alpha(x) = \Pi_{j=0}^n h_{\alpha_j}(x_j).$$

Then the family $\{\Phi_\alpha\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. For $\lambda \neq 0$ we define the *scaled* Hermite functions

$$\Phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}} \Phi_\alpha\left(|\lambda|^{\frac{1}{2}} x\right).$$

We also consider

$$\Phi_{\alpha\beta}^\lambda(x) = (2\pi)^{-\frac{n}{2}} |\lambda|^{\frac{n}{2}} \langle \pi_\lambda(z,0)\Phi_\alpha^\lambda, \Phi_\beta^\lambda \rangle,$$

which is essentially the matrix coefficient of the Schrödinger representation π_λ at $(z,0)$ of H^n. They are the so called special Hermite functions and $\{\Phi_{\alpha\beta}^\lambda : \alpha, \beta \in \mathbb{N}^n\}$ is a complete orthonormal system in $L^2(\mathbb{C}^n)$. For each $\sigma \in \mathbb{T}^n$, $(z,t) \mapsto (\sigma z, t)$ is an automorphism of H^n, because $\mathbb{T}^n$ preserves the symplectic form $Im(z.\overline{w})$. If ρ is a representation of H^n, then using this automorphism we can define another representation ρ_σ by $\rho_\sigma(z,t) = \rho(\sigma z, t)$ which coincides with ρ at the center. Therefore by Stone-von Neumann theorem ρ_σ is unitarily equivalent to ρ. If we take ρ to be the Schrödinger representation π_λ, then we have the unitary intertwining operator $\mu_\lambda(\sigma)$, i.e.

$$\pi_\lambda(\sigma z, t) = \mu_\lambda(\sigma)\pi_\lambda(z,t)\mu_\lambda(\sigma)^*.$$

The operator valued function μ_λ can be chosen so that it becomes a unitary representation of the double cover of the symplectic group and is called metaplectic representation. For a detailed description of these representations we refer to [2].

For each $\lambda \neq 0, m \in \mathbb{Z}^n$ we consider the representations $(\rho_m^\lambda, L^2(\mathbb{R}^n))$ defined by

$$\rho_m^\lambda(e^{i\Phi}, z, t) = e^{-im.\Phi}\pi_\lambda(z,t)\mu_\lambda(e^{i\Phi}),$$

where $\mu_\lambda(e^{i\Phi})$ is the metaplectic representation and π_λ is the Schrödinger representation of H^n. The action of $\mu_\lambda(e^{i\Phi})$ on the Hermite basis $\{\Phi_\alpha^\lambda : \alpha \in \mathbb{N}^n\}$ is given by

$$\mu_\lambda(e^{i\Phi})\Phi_\alpha^\lambda = e^{i\alpha.\Phi}\Phi_\alpha^\lambda.$$

Since $\rho_m^\lambda(1,z,t) = \pi_\lambda(z,t)$, these representations are irreducible.

Theorem 3.1. *Let π be any unitary representation of G such that $\pi(1,z,t)$ is irreducible as a representation of H^n. If $\pi(1,0,t) = e^{i\lambda t}I$ with $\lambda \neq 0$ then π is unitarily equivalent to ρ_m^λ for some $m \in \mathbb{Z}^n$.*

Proof. Since $\pi(1,z,t)$ is irreducible and $\pi(1,0,t) = e^{i\lambda t}I$ by Stone-von Neumann theorem π is unitarily equivalent to $\pi_\lambda(z,t)$ on $L^2(\mathbb{R}^n)$. If $\mathcal{H}$ is the Hilbert space on which π is realised, we have a unitary operator $U_\lambda : \mathcal{H} \longrightarrow L^2(\mathbb{R}^n)$ such that

$$U_\lambda^*\pi_\lambda(z,t)U_\lambda = \pi(1,z,t).$$

Now

$$\pi(e^{i\Phi}, z, t) = \pi(1,z,t)\pi(e^{i\Phi}, 0, 0)$$

and also

$$\begin{aligned} \pi(e^{i\Phi}, z, t) &= \pi((e^{i\Phi},0,0)(1, e^{-i\Phi}z, t)) \\ &= \pi(e^{i\Phi},0,0)\pi(1, e^{-i\Phi}z, t). \end{aligned}$$

Therefore,

$$\begin{aligned} &U_\lambda\pi(1,z,t)\pi(e^{i\Phi},0,0)U_\lambda^* \\ &\quad = U_\lambda\pi(e^{i\Phi},0,0)\pi(1,e^{-i\Phi}z,t)U_\lambda^* \end{aligned}$$

and

$$\begin{aligned} &\pi_\lambda(z,t)U_\lambda\pi(e^{i\Phi},0,0)U_\lambda^* \\ &\quad = U_\lambda\pi(e^{i\Phi},0,0)U_\lambda^*\pi_\lambda(e^{-i\Phi}z,t). \end{aligned}$$

But $\pi_\lambda(e^{-i\Phi}z,t)$ is unitarily equivalent to $\pi_\lambda(z,t)$ via the metaplectic representation, i.e.

$$\pi_\lambda(e^{-i\Phi}z,t) = \mu_\lambda(e^{-i\Phi})\pi_\lambda(z,t)\mu(e^{-i\Phi})^*.$$

Defining $\rho(e^{i\Phi}) = U_\lambda\pi(e^{i\Phi},0,0)U_\lambda^*$ we have

$$\begin{aligned} &\pi_\lambda(z,t)\rho(e^{i\Phi}) \\ &\quad = \rho(e^{i\Phi})\mu(e^{-i\Phi})\pi(z,t)\mu_\lambda(e^{-i\Phi})^*. \end{aligned}$$

Thus $\rho(e^{i\Phi})\mu(e^{-i\Phi})$ commutes with $\pi_\lambda(z,t)$ for all (z,t) and hence $\rho(e^{i\Phi})\mu_\lambda(e^{-i\Phi}) = \chi(\Phi)I$. That is $\rho(e^{i\Phi}) = \chi(\Phi)\mu_\lambda(e^{i\Phi})$. Therefore $\chi(\Phi).\chi(\Phi') = \chi(\Phi + \Phi')$. Thus χ defines a character of the group $\mathbb{T}^n$. Hence, $\rho(e^{i\Phi}) = e^{-im.\Phi}\mu(e^{i\Phi})$ for some $m \in \mathbb{Z}^n$. Finally

$$U_\lambda\pi(e^{i\Phi},z,t)U_\lambda^* = \pi_\lambda(z,t)e^{-im.\Phi}\mu(e^{i\Phi})$$

which proves the theorem.

We now consider the case when $\lambda = 0$. That is $\pi(e^{i\Phi}, z, t) = \pi(e^{i\Phi}, z, 0)$.

Defining $\rho(z, e^{i\Phi}) = \pi(e^{i\Phi}, z, 0)$ we see that

$$\begin{aligned} \rho(z,e^{i\Phi})\rho(w,e^{i\Theta}) &= \pi(e^{i\Phi},z,0)\pi(e^{i\Theta},w,0) \\ &= \pi(e^{i(\Theta+\Phi)}, z + e^{i\Phi}w, 0) \end{aligned}$$

and hence ρ is a representation of the motion group $\mathbb{C}^n \times \mathbb{T}^n = M(2) \times \cdots \times M(2)$ where $M(2) = \mathbb{C} \times U(1)$.

It is well known (see [11]) that all the irreducible unitary representations of $M(2)$ are given by the following two families

(i) for $a>0$, ρ_a realised on $L^2(\mathbb{T})$ and defined by

$$\rho_a(z,e^{i\phi})g(\theta)=e^{iRe(aze^{-i\theta})}g(\theta-\phi).$$

(ii) for $m\in\mathbb{Z}$, the one-dimensional representations χ_m realised on $\mathbb{C}$ and defined by

$$\chi_m(e^{i\phi})z=e^{im\phi}z.$$

From this we build the representations of $M(2)\times\cdots\times M(2)$ as:

(I) for $a=(a_1,\dots a_n)\in\mathbb{R}^n_+$, ρ_a realised on $L^2(\mathbb{T}^n)$ is given by

$$\rho_a(z,e^{i\Phi})g(e^{i\Theta}) = \Pi_{j=1}^n e^{iRe(a_jz_je^{-i\theta_j})} \times g(e^{i(\Theta-\Phi)}),$$

where $z=(z_1,\dots z_n)\in\mathbb{C}^n$, $e^{i\Theta}\in\mathbb{T}^n$.

(II) for $m\in\mathbb{Z}^n$, χ_m realised on $\mathbb{C}$ is given by

$$\chi_m(e^{i\Phi})w=e^{im.\Phi}w.$$

Hence we have,

Theorem 3.2. *If π is a unitary representation of G such that $\pi(1,0,t)=I$ and $\pi(1,z,t)$ is irreducible then π is unitarily equivalent to either ρ_a for some $a\in\mathbb{R}^n_+$ or χ_m for some $m\in\mathbb{Z}^n$.*

We now show that the representations ρ^λ_m are enough for the Plancherel theorem. Given $f\in L^1\cap L^2(G)$ consider the group Fourier transform

$$\begin{aligned}\widehat{f}(\lambda,m) &= \int f(e^{i\Phi},z,t)\rho^\lambda_m(e^{i\Phi},z,t)\,d\Phi dz dt\\ &= \int f^\lambda(e^{i\Phi},z)e^{-im.\Phi}\pi_\lambda(z,0)\mu_\lambda(e^{i\Phi})\\ &\quad\times d\Phi dz\end{aligned}$$

where $f^\lambda(e^{i\Phi},z)=\int e^{i\lambda t}f(e^{i\Phi},z,t)dt$. We can calculate the Hilbert–Schmidt operator norm of $\widehat{f}(\lambda,m)$ by using the Hermite basis $\{\Phi^\lambda_\alpha\}$:

$$\begin{aligned}\widehat{f}(\lambda,m)\Phi^\lambda_\alpha &= \int f^\lambda(e^{i\Phi},z)e^{-im.\Phi}e^{i\alpha.\Phi}\pi_\lambda\\ &\quad\times(z,0)\Phi^\lambda_\alpha\, d\Phi dz\\ &= \int \tilde{f}^\lambda(m-\alpha,z)\pi_\lambda(z,0)\Phi^\lambda_\alpha dz\end{aligned}$$

where

$$\tilde{f}^\lambda(z,m)=\int f^\lambda(z,e^{i\Phi})e^{-im.\Phi}d\Phi.$$

Thus

$$\begin{aligned}&(\widehat{f}(\lambda,m)\Phi^\lambda_\alpha,\Phi^\lambda_\beta)\\ &\quad=(2\pi)^{\frac{n}{2}}|\lambda|^{-\frac{n}{2}}\int \tilde{f}^\lambda(z,m-\alpha)\Phi^\lambda_{\alpha\beta}(z)dz\end{aligned}$$

so that

$$\begin{aligned}&\|\widehat{f}(\lambda,m)\Phi^\lambda_\alpha\|_2^2\\ &\quad=(2\pi)^n|\lambda|^{-n}\sum_\beta\left|\int\tilde{f}^\lambda(z,m-\alpha)\Phi^\lambda_{\alpha\beta}(z)dz\right|^2\end{aligned}$$

and

$$\begin{aligned}\|\widehat{f}(\lambda,m)\|^2_{HS} &= (2\pi)^n|\lambda|^{-n}\\ &\quad\times\sum_\alpha\sum_\beta\left|\int\tilde{f}^\lambda\times(z,m-\alpha)\Phi^\lambda_{\alpha\beta}(z)dz\right|^2.\end{aligned}$$

This shows that

$$\begin{aligned}\sum_m\|\widehat{f}(\lambda,m)\|^2_{HS} &=(2\pi)^n|\lambda|^{-n}\\ &\quad\times\sum_m\sum_\alpha\sum_\beta\left|\int\tilde{f}^\lambda(z,m-\alpha)\Phi^\lambda_{\alpha\beta}(z)dz\right|^2.\end{aligned}$$

Making a change of variable in the summation over m and noting that $\{\Phi^\lambda_{\alpha\beta}:\alpha,\beta\in\mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{C}^n)$ we obtain

$$\begin{aligned}&\sum_m\|\widehat{f}(\lambda,m)\|^2_{HS}\\ &\quad=(2\pi)^n|\lambda|^{-n}\sum_m\int|\tilde{f}^\lambda(z,m)|^2dz\\ &\quad=(2\pi)^{2n}|\lambda|^{-n}\int\int|f^\lambda(e^{i\Phi},z)|^2d\Phi dz.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int\left(\sum_m\|\widehat{f}(\lambda,m)\|^2_{HS}\right)|\lambda|^n d\lambda\\ &\quad=(2\pi)^{2n}\int\int\int|f^\lambda(e^{i\Phi},z)|^2d\Phi dz\\ &\quad=(2\pi)^{2n+1}\int|f(e^{i\Phi},z,t)|^2d\Phi dz dt.\end{aligned}$$

Theorem 3.3. (Plancherel) *For $f\in L^1\cap L^2(G)$*

$$\begin{aligned}&\int|f(e^{i\Phi},z,t)|^2d\Phi dz dt\\ &\quad=(2\pi)^{-2n-1}\int_{-\infty}^{\infty}\left(\sum_m\|\widehat{f}(\lambda,m)\|^2\right)|\lambda|^n d\lambda.\end{aligned}$$

Remark 3.4. Note that if $f(e^{i\Phi}z,t)$ is right $\mathbb{T}^n$-invariant then $(\widehat{f}(\lambda,m)\Phi_\alpha^\lambda,\Phi_\beta^\lambda)=0$ unless $m=\alpha\geq 0$. Thus $\widehat{f}(\lambda,m)=0$ for $m_j<0$ for some j and for $m\geq 0$

$$\|\widehat{f}(\lambda,m)\|_{HS}^2 = (2\pi)^n|\lambda|^{-n}\sum_{\beta\in\mathbb{Z}^n}\left|\left(\widehat{f}(\lambda,m)\Phi_m^\lambda,\Phi_\beta^\lambda\right)\right|^2.$$

We also use the notation $\rho_m^\lambda(f)$ for $\widehat{f}(\lambda,m)$.

4. Wiener's Theorem

We begin with a theorem of Hulanicki and Ricci [5]

Theorem 4.1. *Let J be a proper closed ideal of $L^1(H^n/\mathbb{T}^n)$. Suppose for every non-zero multiplicative functional Λ_ψ of $L^1(H^n/\mathbb{T}^n)$, given by the bounded spherical function ψ, there is a function $f\in J$ such that*

$$\Lambda_\psi(f)=\int f(z,t)\psi(z,t)\,dz\,dt\neq 0.$$

Then $J=L^1(H^n/\mathbb{T}^n)$.

The actual Theorem of Hulanicki and Ricci is little stronger than this as it says that $L^1(H^n)*_{H^n}J$ is dense in $L^1(H^n)$. We have quoted the part which we are going to use. A detailed proof of this can be found in [1]. Note that in our notation $L^1(H^n/\mathbb{T}^n)$ is $L^1(G)_{0,0}$. In the language of representations this theorem can be restated as: for J as above, suppose

(i) for every $(\lambda,m)\in\mathbb{R}\setminus\{0\}\times\mathbb{Z}^n$ there is a function $f_{\lambda,m}\in J$ such that $\rho_m^\lambda(f_{\lambda,m})\neq 0$
(ii) for every $a=(a_1,\ldots,a_n)$ with $a_i>0$ there is a $f_a\in J$ such that $\rho_a(f_a)\neq 0$
(iii) there is $f\in J$ such that $\int_{H^n}f\,dz\,dt\neq 0$.

Then $J=L^1(G)_{0,0}$. For $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ we consider its *multi-polar* decomposition

$$z=e^{i\Theta}\|z\|=(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n}),$$

where $e^{i\Theta}\in\mathbb{T}^n$, $\|z\|=(r_1,\ldots,r_n)$ and $z_j=r_je^{i\theta_j}$ for $j=1,\ldots,n$. Notice that condition (ii) is then equivalent to this: the Euclidean Fourier transform of f_a at $a\in\mathbb{R}_+^n$ is not equal to zero. That is $\widehat{f_a}(.,t)(a)\neq 0$.

Proposition 4.2. *Let $f\in L^1(G)$. If right (resp. left) type of f is μ (resp. μ') then $\widehat{f}(\lambda,m)\equiv 0$ if $m_j<\mu_j$ (resp. $m_{j'}<\mu_{j'}$) for some j,j'. For $f\in L^1(G)$ of type (μ',μ), $\langle\rho_m^\lambda(f)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\rangle\neq 0$ only when $\mu_j=m_j-\alpha_j$ and $\mu'_j=m_j-\beta_j$ for all j.*

Proof.

$$\begin{aligned}
&\langle\rho_m^\lambda(f)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\rangle\\
&=\int_G f(e^{i\Theta},z,t)\langle\Phi_\alpha^\lambda,\rho_m^\lambda\\
&\quad\times(e^{i\Theta},z,t)^{-1}\Phi_\beta^\lambda\rangle d\Theta\,dz\,dt\\
&=\int_G f(e^{i\Theta},z,t)\langle\Phi_\alpha^\lambda,\rho_m^\lambda\\
&\quad\times(e^{-i\Theta},-e^{-i\Theta}z,-t)\Phi_\beta^\lambda\rangle d\Theta\,dz\,dt\\
&=\int_G f(e^{i\Theta},e^{i\Theta}e^{-i\Theta}z,t)\langle\Phi_\alpha^\lambda,e^{-i(\beta-m)\Theta}\pi_\lambda\\
&\quad\times(-e^{-i\Theta}z,-t)\Phi_\beta^\lambda\rangle d\Theta\,dz\,dt\\
&=\int_G e^{i\mu'.\Theta}f(1,e^{-i\Theta}z,t)e^{i(\beta-m)\Theta}\\
&\quad\times\langle\Phi_\alpha^\lambda,\pi_\lambda(-e^{-i\Theta}z,-t)\Phi_\beta^\lambda\rangle d\Theta\,dz\,dt.
\end{aligned}$$

Therefore, $\langle\rho_m^\lambda(f)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\rangle=0$ unless $\mu'=m-\beta$ in which case,

$$\langle\rho_m^\lambda(f)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\rangle =\int_{H^n}f(1,z,t)\langle\Phi_\alpha^\lambda,\pi_\lambda(-z,-t)\Phi_\beta^\lambda\rangle dz\,dt.$$

The other result will follow similarly.

Lemma 4.3. *Suppose for $f\in L^1(G)$, $\widehat{f}(\lambda,m)\neq 0$ for some $\lambda\in\mathbb{R}\setminus\{0\}$, $m\in\mathbb{Z}^n$ and $\rho_a(f)\neq 0$ for some $a\in\mathbb{R}_+^n$. Assume also that $\mu\in\mathbb{Z}^n$ satisfies $\mu\leq m$. Then there exist functions g and g' of type (μ,μ) in the closure of span of (left and right) G-translates of f such that $\widehat{g}(\lambda,m)\neq 0$ and $\rho_a(g')\neq 0$.*

Proof. Let $\nu=m-\mu$. As the operator $\widehat{f}(\lambda,m)\neq 0$ there exist α and β such that $\langle\widehat{f}(\lambda,m)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\rangle\neq 0$. Since ρ_m^λ is irreducible, for every $\varepsilon>0$ there exist $c_i\in\mathbb{C}$ and $x_i\in G$, $i=1,2,\ldots,s$ for some s such that

$$\Big\|\sum_{i=1}^s c_i\rho_m^\lambda(x_i)\Phi_\nu^\lambda-\Phi_\alpha^\lambda\Big\|<\varepsilon.$$

This implies

$$\Big\|\sum_{i=1}^s c_i\rho_m^\lambda(f)\rho_m^\lambda(x_i)\Phi_\nu^\lambda-\rho_m^\lambda(f)\Phi_\alpha^\lambda\Big\|<\varepsilon\|\rho_m^\lambda(f)\|$$

and hence

$$\big\|\rho_m^\lambda(g_1)\Phi_\nu^\lambda-\rho_m^\lambda(f)\Phi_\alpha^\lambda\big\|<\varepsilon\|\rho_m^\lambda(f)\|$$

where $g_1=\sum_{i=1}^s c_if^{x_i}$.

As $\rho_m^\lambda(f)\Phi_\alpha^\lambda \neq 0$ and $\|\rho_m^\lambda(f)\| < \infty$, $\rho_m^\lambda(g_1)\Phi_\nu^\lambda \neq 0$. By the above proposition this also shows that g_1 has right μ component. Let g_2 be the right μ-th projection of g_1. Then $\rho_m^\lambda(g_1)\Phi_\nu^\lambda = \rho_m^\lambda(g_2)\Phi_\nu^\lambda \neq 0$, because Fourier transforms of other right projections of g_1 will kill Φ_ν^λ. There exists γ such that $\langle \rho_m^\lambda(g_2)\Phi_\nu^\lambda, \Phi_\gamma^\lambda\rangle \neq 0$. For $\varepsilon > 0$, there are $c_j' \in \mathbb{C}$ and $y_j \in G, j = 1, 2, \ldots s'$ for some s' such that

$$\|\sum_{j=1}^{s'} c_j' \rho_m^\lambda(y_j^{-1})\Phi_\nu^\lambda - \Phi_\gamma^\lambda\| < \varepsilon.$$

Applying $\rho_m^\lambda(g_2)^*$ on both sides and using that ρ_m^λ is unitary we get,

$$\|\sum_{j=1}^{s'} c_j' \rho_m^\lambda(g_2)^* \rho_m^\lambda(y_j)^* \Phi_\nu^\lambda - \rho_m^\lambda(g_2)^*\Phi_\gamma^\lambda\| < \varepsilon \|\rho_m^\lambda(g_2)^*\|.$$

That implies

$$\|\sum_{j=1}^{s'} c_j' \rho_m^\lambda({}^{y_j}g_2)^* \Phi_\nu^\lambda - \rho_m^\lambda(g_2)^*\Phi_\gamma^\lambda\| < \varepsilon \|\rho_m^\lambda(g_2)^*\|.$$

Let $g_3 = \sum_{j=1}^{s'} \overline{c_j'}\, {}^{y_j}g_2$. Then we have

$$\|\rho_m^\lambda(g_3)^*\Phi_\nu^\lambda - \rho_m^\lambda(g_2)^*\Phi_\gamma^\lambda\| < \varepsilon\|\rho_m^\lambda(g_2)^*\|.$$

As $\rho_m^\lambda(g_2)^*\Phi_\gamma^\lambda \neq 0$, $\rho_m^\lambda(g_3)^*\Phi_\nu^\lambda \neq 0$. Since g_3 is a finite linear combination of the left G-translates of g_2, right type of g_3 continues to be μ. Hence $\langle \Phi_\nu^\lambda, \rho_m^\lambda(g_3)^*\Phi_\nu^\lambda\rangle = \langle \rho_m^\lambda(g_3)\Phi_\nu^\lambda, \Phi_\nu^\lambda\rangle \neq 0$. Let g be the left μ-th projection of g_3. Then g is of type (μ,μ) and $\langle \rho_m^\lambda(g_3)\Phi_\nu^\lambda, \Phi_\nu^\lambda\rangle \neq 0$.

By similar method we can show that there is a g' of type (μ,μ) in the closure of the span of G-translates of f such that $\rho_a(g') \neq 0$.

Suppose f is a function in $L^1(G)_{\mu,\mu}$. Let $z = e^{i\Theta}\|z\|$ where $\|z\| = (r_1, \ldots, r_n)$ and $e^{i\Theta} \in \mathbb{T}^n$. Then

$$\begin{aligned} f(e^{i\Phi}, z, t) &= f(e^{i\Phi}, e^{i\Theta}\|z\|, t) \\ &= e^{i\mu.\Phi} f(1, e^{i\Theta}\|z\|, t). \end{aligned}$$

Since $(1, e^{i\Theta}\|z\|, t) = (e^{i\Theta}, 0, 0)(e^{-i\Theta}, \|z\|, t)$ we also have

$$\begin{aligned} f(1, e^{i\Theta}\|z\|, t) &= e^{i\mu.\Theta} f(e^{-i\Theta}, \|z\|, t) \\ &= f(1, \|z\|, t). \end{aligned}$$

Thus we have

$$f(e^{i\Phi}, z, t) = e^{i\mu\Phi} f_{00}(\|z\|, t).$$

By the above proposition for $f \in L^1(G)_{\mu,\mu}$, $\rho_m^\lambda(f) = 0$ whenever $m_j < \mu_j$ for some j. If $m \geq \mu$ then

$$\begin{aligned} \rho_m^\lambda(f) &= \int_G f(e^{i\Phi}, z, t)\rho_m^\lambda(e^{i\Phi}, z, t)\, d\Phi\, dz\, dt \\ &= \int_G f_{00}(\|z\|, t) e^{i\mu.\Phi} e^{-im.\Phi} \pi_\lambda(z,t)\mu_\lambda \\ &\quad \times (e^{i\Phi})\, d\Phi\, dz\, dt \\ &= \int_G f_{00}(\|z\|, t)\rho_{m-\mu}^\lambda(e^{i\Phi}, z, t)\, d\Phi\, dz\, dt. \end{aligned}$$

This shows that $\rho_m^\lambda(f) = \rho_{m-\mu}^\lambda(f_{00})$.

Now suppose for some $f \in L^1(G)$

$$\chi_m(f) = \int_G f(e^{i\Phi}, z, t) e^{-im.\Phi}\, dz\, dt\, d\Phi \neq 0.$$

Let f_μ is the right μ-th projection of f. That is

$$\begin{aligned} f_\mu(e^{i\Phi}, z, t) &= \int_G f(e^{i(\Theta+\Phi)}, z, t) e^{-i\mu.\Theta}\, d\Theta \\ &= e^{i\mu.\Phi}\int_G f(e^{i\Theta}, z, t) e^{-i\mu\Theta}\, d\Theta. \end{aligned}$$

Then

$$\begin{aligned} \chi_m(f_\mu) &= \delta_{m,\mu}\int_G f(e^{i\Theta}, z, t) e^{-i\mu.\Theta}\, d\Theta\, dz\, dt \\ &= \delta_{m,\mu}\chi_\mu(f) \end{aligned}$$

where $\delta_{m,\mu}$ is the Kronecker δ and hence if $\mu = m$ then $\chi_\mu(f_\mu) \neq 0$. Similarly we can show that if ${}_\mu f$ is the left μ-th projection of f, then $\chi_m({}_\mu f) = \delta_{m,\mu}\chi_\mu(f)$. Therefore if $\chi_m(f) \neq 0$ for all $m \in \mathbb{Z}^n$, then $\chi_\mu(g) \neq 0$ where g is the (μ,μ)-the projection f. By the above computation that means $\int_{H^n} g_{0,0}(\|z\|, t)\, dz\, dt \neq 0$.

Proposition 4.4. *Let $S \subset L^1(G)_{\mu,\mu}$ Suppose*

(1) for every $\lambda \in \mathbb{R}\setminus\{0\}$ and $m \geq \mu$, there is $f_{\lambda,m} \in S$ such that $\rho_m^\lambda(f_{\lambda,m}) \neq 0$,
(2) for all $a \in \mathbb{R}_+^n$, $\rho_a(f_a) \neq 0$ for some function $f_a \in S$ and
(3) $\chi_\mu(f) \neq 0$ for some function $f \in S$.

Then the span of the ideal (under convolution $*$) generated by elements of S in $L^1(G)_{\mu,\mu}$ is dense in $L^1(G)_{\mu,\mu}$.

Proof. Let $S_{0,0} = \{f_{0,0} : f \in S\}$. If $f \in L^1(G)_{\mu,\mu}$, then $\rho_m^\lambda(f) \neq 0$ for some $m \geq \mu$ implies $\rho_{m-\mu}^\lambda(f_{0,0}) \neq 0$. Also $\rho_a(f) \neq 0$ implies the Euclidean Fourier transform $\widehat{f_{0,0}}(.,t)(a) \neq 0$. Notice also that $\chi_\mu(f) \neq 0$ implies $\int_{H^n} f_{0,0}(\|z\|, t)\, dz\, dt \neq 0$.

Therefore by Theorem 4.1 span of the ideal generated by the elements of $S_{0,0}$ in $L^1(G)_{0,0}$ is dense in $L^1(G)_{0,0}$. Let us take an arbitrary $h \in L^1(G)_{\mu,\mu}$. For $\varepsilon > 0$, there exist $c_j \in \mathbb{C}$, $f^j \in S$ and $g^j \in L^1(G)_{0,0}$, $j = 1, 2, \ldots s$ for some s such that

$$\|\sum_j c_j f^j_{0,0} *_{H^n} g^j - h_{0,0}\|_{L^1(H^n)} < \varepsilon. \tag{4.1}$$

Let us define $\tilde{g^j}(e^{i\Theta}, z, t) = g^j(\|z\|, t)e^{i\mu.\Theta}$. Clearly $\tilde{g^j} \in L^1(G)_{\mu,\mu}$. Then,

$$\begin{aligned}
&\|\sum c_j f^j * \tilde{g^j} - h\|_{L^1(G)} \\
&\quad = \int_G |\sum c_j f^j * \tilde{g^j}(e^{i\Theta}, z, t) - h(e^{i\Theta}, z, t)| \\
&\quad \times d\Theta dz dt.
\end{aligned}$$

Since $f^j * \tilde{g^j}$ and h are functions of type (μ, μ), the above expression equals to

$$\begin{aligned}
&\int_G |\sum c_j (f^j * \tilde{g^j}))_{0,0}(\|z\|, t)e^{i\mu.\Theta} \\
&\quad - h_{0,0}(\|z\|, t)e^{i\mu.\theta}| d\Theta dz dt \\
&\quad = \int_{H^n} |\sum c_j (f^j * \tilde{g^j}))_{0,0}(\|z\|, t) \\
&\quad - h_{0,0}(\|z\|, t)| dz dt.
\end{aligned}$$

Again using the fact that $f^j * \tilde{g^j}$ and h are (μ, μ)-functions we have

$$\begin{aligned}
(f^j * \tilde{g^j})_{0,0}(\|z\|, t) &= f^j * \tilde{g^j}(1, z, t) \\
&= (f^j_0 *_{H^n} \tilde{g^j}_0)(\|z\|, t)
\end{aligned}$$

by Proposition 2.1. Also,

$$\tilde{g^j}_0(z, t) = \tilde{g^j}(1, z, t) = g^j(\|z\|, t) = g^j(z, t)$$

and

$$f^j_0(z, t) = f^j(1, z, t) = f^j_{0,0}(\|z\|, t) = f^j_{0,0}(z, t).$$

Therefore,

$$(f^j * \tilde{g^j})_{0,0}(\|z\|, t) = f^j_{0,0} *_{H^n} g^j(z, t).$$

Hence

$$\begin{aligned}
&\|\sum c_j f^j * \tilde{g^j} - h\|_{L^1(G)} \\
&\quad = \int_{H^n} |\sum c_j (f^j_{0,0} *_{H^n} \tilde{g^j})(\|z\|, t) \\
&\quad - h_{0,0}(\|z\|, t)| dz dt.
\end{aligned}$$

The proposition follows now from (4.1).

We now state and prove a Wiener Tauberian theorem for the action of G on itself.

Theorem 4.5. *If for a function $f \in L^1(G)$, $\widehat{f}(\lambda, m) \neq 0$ for all $\lambda \neq 0$, $m \in \mathbb{Z}^n$, $\widehat{f}(a) \neq 0$ for all $a \in \mathbb{R}^n_+$ and $\chi_m(f) \neq 0$ for all $m \in \mathbb{Z}^n$ then the span of the (left and right) G-translates of f is dense in $L^1(Gx)$.*

Proof. Fix $\mu \in \mathbb{Z}^n$. From lemma 4.3 and the subsequent computations we see that using the given nonvanishing condition on f, we can find functions of type (μ, μ) in the closure of G-translates of f such that they satisfy the hypothesis of proposition 4.4. Hence they can generate $L^1(G)_{\mu,\mu}$. The proof now follows from the fact that any ideal I in $L^1(G)$ which contains $L^1(G)_{\mu,\mu}$ for all $\mu \in \mathbb{Z}^n$ is $L^1(G)$ itself.

Note that instead of a single function f we can also take a subset $S \subset L^1(G)$ such that Fourier transforms of the elements of S have no common zero in the set of above representations parametrized by $\mathbb{R}^* \times \mathbb{Z}^n \cup \mathbb{R}^n_+ \cup \mathbb{Z}^n$, where $\mathbb{R}^*$ is the set of nonzero reals.

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