PHYSICAL REVIEW E VOLUME 54, NUMBER 6 DECEMBER 1996

Nonsimultaneity effects in globally coupled maps

Gabriel Pérez,[1,3,*] Sudeshna Sinha,[2,3,†] and Hilda A. Cerdeira[3‡]

[1]*Departamento de Física Aplicada, CINVESTAV del IPN, Unidad Mérida, Apartado Postal 73 ''Cordemex,'' 97310 Mérida, Yucatán, Mexico*

[2]*Indian Institute of Astrophysics, Sarjaphi Road, Koramazda, Bangalore 560 034, India*

[3]*International Centre for Theoretical Physics, P.O. Box 586, 34100 Trieste, Italy*

(Received 9 July 1996)

We study the behavior of globally coupled maps when the coupling mean field is either delayed or averaged over several time steps. We find that introducing a delay does not reduce, and in some cases increases, the saturation values for the fluctuations of the mean field. The mean field changes its quasiperiodic behavior by introducing more components in its spectrum, and the distance between main components of this spectrum is reduced in a linear way. On the other hand, averaging the mean field reduces the saturation value for fluctuations, but does not fully restore statistical behavior to the system except in the limit of very large averages. As before, quasiperiodicity is changed by the introduction of more beating frequencies, and the distance between the most important among them decreases linearly. As an extra test, we study the effects that a small periodic driving has over this dynamics, and find that although there is some influence, there are not strong resonances to simple sinusoidal driving. [S1063-651X(96)02412-9]

PACS number(s): 05.45.+b, 47.52.+j

There has been in recent years sustained interest in the dynamics of large lattices of coupled chaotic systems [1]. These models give simple approximations to many interesting physical models, such as coupled Josephson junctions, multimode lasers, arrays of coupled nonlinear circuits, etc. A particular area of interest is that of globally coupled maps (GCMs), where the dynamics is discrete and the coupling is made global.

For these systems, a problem that has attracted some attention is how close to statistical is the behavior of the system when the dynamics of the local maps is chaotic and the coupling is weak. In this case, it is tempting to propose a ''simplicity hypothesis,'' which says that since the chaotic behavior would give for independent maps invariant distributions with compact support, and the influence of the average is moderated by a small parameter, it could happen that in the limit of infinite lattices the average would converge to a fixed point, therefore decoupling the maps; and that for large but finite lattices this mean field would have a Gaussian distribution (central limit theorem), with a variance proportional to $1/N$ (law of large numbers). These hypotheses were originally checked by Kaneko [2], who found that the behavior in most cases was clearly nonstatistical, i.e., that the fluctuations of the mean field saturated at a finite value, as shown by their mean square deviation (MSD), and moreover, that this mean field developed some quasiperiodicity. This phenomenon has been confirmed in several types of coupled systems, and has been shown to survive, and in some cases to be enhanced, under the influence of noise or partial coupling [3].

Global coupling can correspond in some cases to the actual physics of the system [4,5], but in many cases is introduced as a simplifying limit (a mean field approach) to local couplings, which are usually diffusive. One aspect of this use of GCMs that is not satisfactory is the fact that, even though under local dynamics information takes time to travel though the lattice, in the globally coupled limit all effects are always instantaneous. As an approximate way of considering the unavoidable delay effects that appear in locally coupled systems, we have studied the behavior of GCMs under both delay and averaging (in time) of the mean field. Two examples are chosen here. One is Kaneko's original model of a globally coupled lattice of logistic maps [2], a system with the following equations:

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\epsilon h_n,$$
$$h_n\equiv\frac{1}{N}\sum_{j=1}^{N} f(x_n(j)),\qquad f(x)=1-ax^2, \tag{1}$$

where $x_n(i)$ is the local variable at location i and iteration n. Here ϵ is the global coupling, and the local map $f(x_n(i))$ is normalized by $1-\epsilon$ so as to avoid getting out of the range $-1\leqslant x\leqslant 1$. The other test case is that of a lattice of nonlinear optical elements susceptible of modeling via the Ikeda equations [4,6]

$$E_{n+1}(i)=A+Bf(E_n(i))+\epsilon h_n,$$
$$h_n\equiv\frac{1}{N}\sum_{j=1}^{N} f(E_n(j)),\qquad f(E)=E\exp(iE^*E), \tag{2}$$

where now E is complex and represents the amplitude of the slowly varying envelope of the electric field in a nonlinear optical element.

I. DELAY OF THE MEAN FIELD

For both systems, we have studied the effects of delaying the mean field h by changing h_n to h_{n-D} in Eqs. (1) and (2), where D denotes how many iterations h is delayed. We have

*Electronic address: gperez@kin.cieamer.conacyt.mx
†Electronic address: sudeshna@iiap.ernet.in
‡Electronic address: cerdeira@ictp.trieste.it

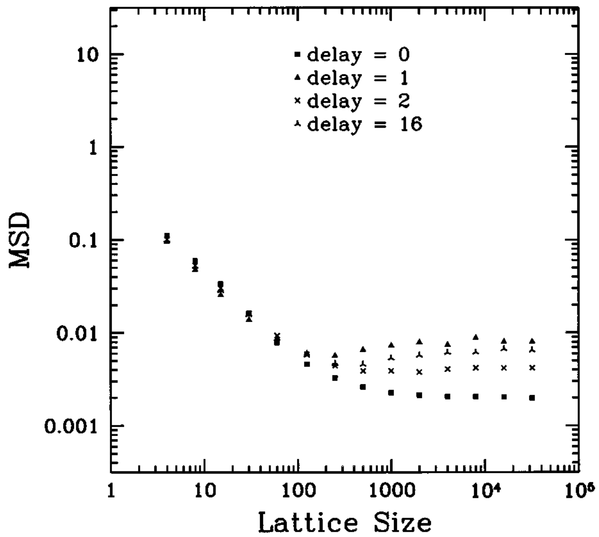

FIG. 1. MSD of the mean field vs lattice size for the delayed logistic GCM. Parameters are given in the text. Here we are averaging over 50 runs of 1024 iterations each, after a transient of 10 000 iterations. The error levels are of the same size or smaller than the markers.

tested values of D from 0 (no delay) up to 17. The parameters used have been $a=1.99$ and $\epsilon=0.1$ for the coupled logistic maps of Eqs. (1), and $A=3$, $B=0.3$, and $\epsilon=0.1$ for the Ikeda mappings Eqs. (2). (Lattice sizes and running times are indicated in the figure captions). The first important finding is that delaying the effects of the mean field does not in any case destroy the subtle coherence responsible for the failure of the simplicity hypothesis, and saturation of the MSD of h as N grows is actually enhanced (see Figs. 1 and 2). This is in itself a nontrivial assertion, since it means that there are memory effects in the dynamics, which allow the sequence of past mean fields $h_{n-D}, h_{n-D-1}, h_{n-D-2}, \ldots,$

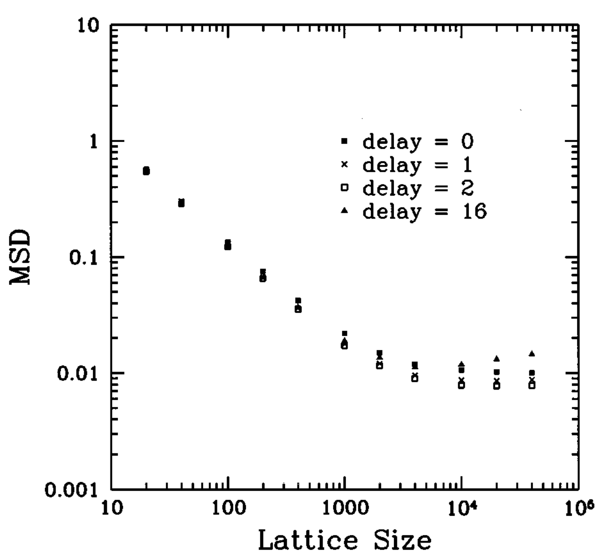

FIG. 2. MSD of the mean field vs lattice size for the delayed Ikeda GCM. Parameters are given in the text, and running times are as in Fig. 1. The error levels are of the same size or smaller than the markers.

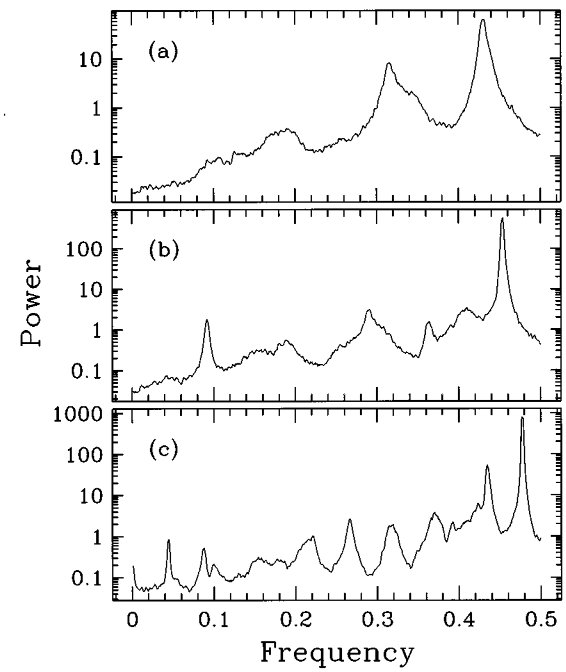

FIG. 3. Power spectra for the mean field of the delayed logistic GCM. The delays are (a) $D=0$ (no delay), (b) $D=4$, and (c) $D=16$. Parameters are given in the text. Here we have performed the Fourier transform over runs of 1024 iterations, and averaged over 50 such runs. The lattice contains 32 000 elements.

to affect the present evolution of the system over a span of D iterations; this for a model where each individual oscillator is losing track of its previous state exponentially fast, due to the existence of positive Lyapunov exponents. In this case, however, the small quasiperiodic fluctuations that affect h for large lattices manage to reproduce themselves even after long delays.

Delaying the mean field does have a clear influence in the form of the quasiperiodicity of the system, as can be seen in the Fourier spectra shown in Figs. 3 and 4. The main effect is an increase in the number of peaks in the spectrum, and a corresponding decrease in their separation. This decrease in separation seems to be linear for the Ikeda mappings [Eqs. (2)], where one can fit the distance between the main peaks in the spectrum (those closest to $\omega=0$) by an approximate formula

$$1/\Delta\omega_D \approx (1/\Delta\omega_0) + \alpha D, \tag{3}$$

where D is the delay, and for the parameters used here, $\Delta\omega_0=0.316\pm0.005$ and $\alpha=1.005\pm0.006$ (i.e., consistent with $\alpha=1$).

For the logistic mappings this particular behavior is not as clear as in the Ikeda system, but estimates made for the largest two peaks in the spectrum (those closest to $\omega=0.5$) show a behavior similar to the one given in Eq. (3), except that we need here to consider odd and even values of D separately. As before, the value found for the constant α is consistent with $\alpha=1$, suggesting an exceedingly simple rule for the accumulation of new peaks in the spectrum: in practice, one gets one new peak for each increase of the delay in one iteration.

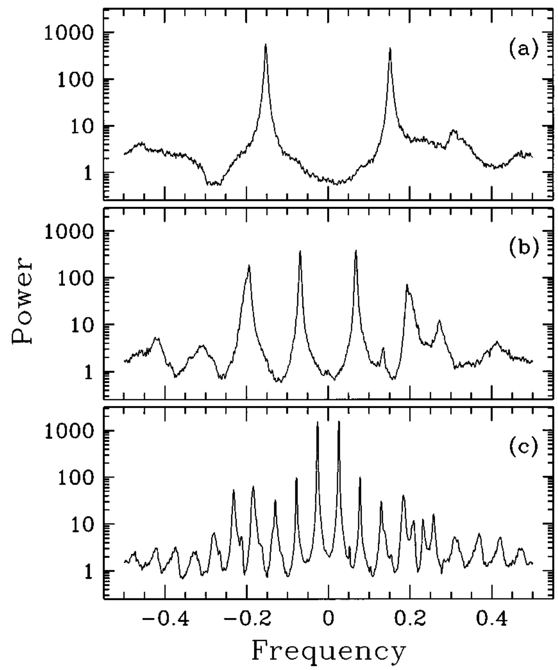

FIG. 4. Power spectra for the mean field of the delayed Ikeda GCM. The delays are (a) $D=0$ (no delay), (b) $D=4$, and (c) $D=16$. Parameters are given in the text. Here we have performed the Fourier transform over runs of 1024 iterations, and averaged over 50 such runs. The lattice contains 40 000 elements.

II. AVERAGING OF THE MEAN FIELD

For both systems, we have studied the averaging of the mean field over a total of P contiguous iterations by changing h_n in Eqs. (1) and (2) to h_n^P where

$$h_n^P \equiv \frac{1}{P}\sum_{j=0}^{P-1} h_{n-j}.$$

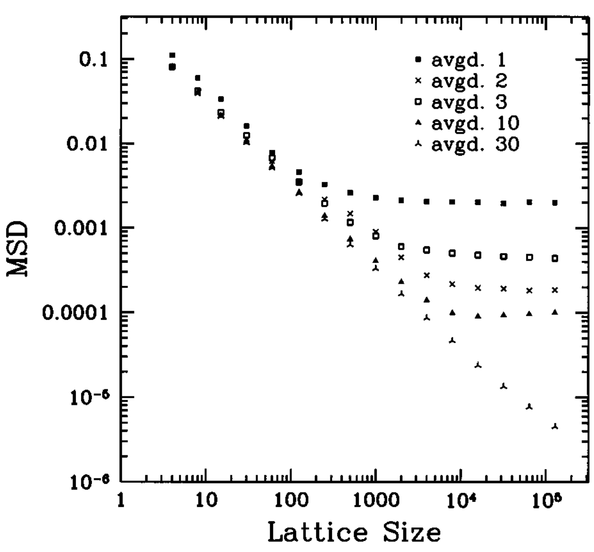

FIG. 5. MSD of the mean field vs lattice size for the averaged logistic GCM. Parameters are given in the text. Here we are averaging over 50 runs of 1024 iterations each, after a transient of 10 000 iterations. The error levels are of the same size or smaller than the markers.

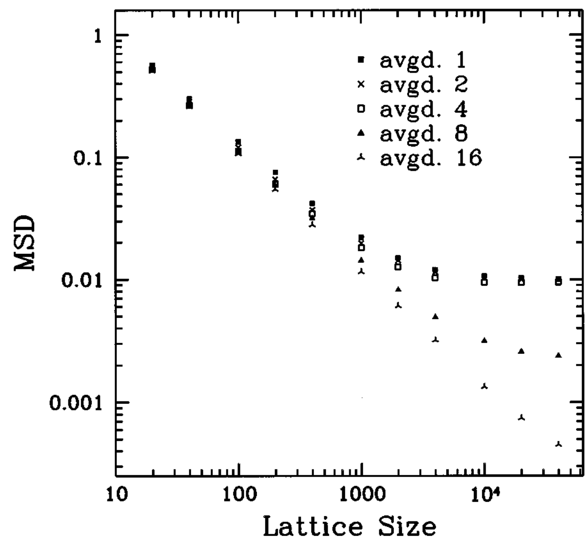

FIG. 6. MSD of the mean field vs lattice size for the averaged Ikeda GCM. Parameters are given in the text, and running times and lattice size are as in Fig. 5. The error levels are of the same size or smaller than the markers.

We have tested values of P up to 30. For this modification of the dynamics one finds, as expected [7], that averaging over several iterations tends to reduce the level of the fluctuations and therefore to render the system closer to statistical. We should notice, however, the following two points: one, the fluctuations in h show robustness, in the sense that, even if it is true that a time average over a few of them reduces the level of saturation of its MSD, it does not restore its statistical behavior. It just increases the value of N where the MSD stops decreasing. Two, the quasiperiodicity of the mean field is still manifest, (see Figs. 5 and 6), and, at least for the case of the Ikeda mappings, show a behavior similar to the one found for delaying: the Fourier spectrum acquires more peaks, and the distance among the two largest of them (the two closest to $\omega=0$) decays linearly, following an approximate rule $1/\Delta\omega_P \approx (1/\Delta\omega_1)+\beta(P-1)$. For the parameters given, we have found $\Delta\omega_1=0.39\pm0.01$ and $\beta=0.554\pm0.008$.

For the logistic mappings, the trend towards multiplicity of peaks and the corresponding decrease in distance between them is visible in the Fourier spectra, but not clearly enough as to be unequivocally quantified.

III. PERIODIC DRIVING

The presence of quasiperiodicity in the mean field for globally coupled maps suggests a test of these systems for possible resonant behavior. For the logistic model, we have done this by changing the action of the mean field h in Eqs. (1) to a mixture $\epsilon h_n \rightarrow \alpha h_n + \beta \sin(n\omega)$, with $\epsilon=\alpha+\beta$ (this in order to keep $|x_{n+1}(i)|\leqslant 1$). We have kept $\epsilon=0.1$, and have checked the behavior of h for two values of α, sweeping over the available ω range. The most important result here is a negative one: we do not find strong resonance in this system, even for the frequencies corresponding to the largest peaks in the nondriven spectrum.

In the first test we have set $\alpha=0.099$ and $\beta=0.001$. Tak-

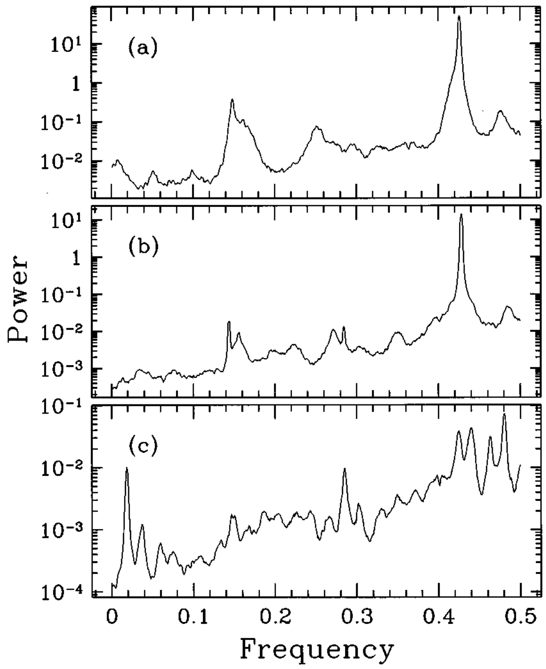

FIG. 7. Power spectra for the mean field of the averaged logistic GCM. Here we are averaging over (a) $P=3$, (b) $P=10$, and (c) $P=30$ iterations. Parameters are given in the text. We have performed the Fourier transform over runs of 1024 iterations, and averaged over 50 such runs. The lattice contains 128 000 elements.

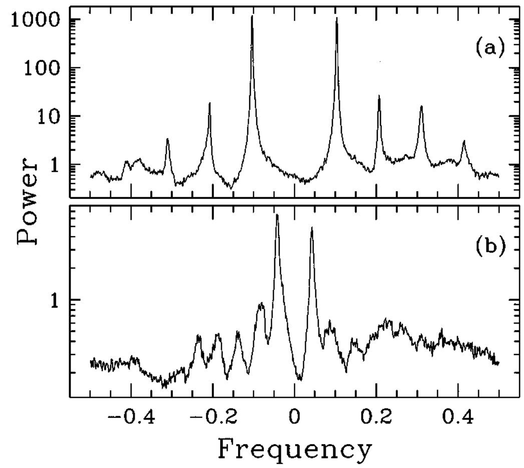

FIG. 8. Power spectra for the mean field of the averaged Ikeda GCM. Here we are averaging over (a) $P=4$ and (b) $P=16$ iterations. Parameters are given in the text. We have performed the Fourier transform over runs of 1024 iterations, and averaged over 50 such runs. The lattice contains 40 000 elements.

ing into account the size of the fluctuations of h for the nondriven system with the same value of ϵ, we have that in this case the autonomous fluctuations should have an amplitude around five times larger than the amplitude of the driving. For this case, there is essentially no response of the system to the periodic driving. The Fourier spectrum obtained in this case is the same as for the nondriven case, with the addition of an isolated δ spike at the frequency of the driving.

In the second test we have set $\alpha=0.09$ and $\beta=0.01$, which gives an approximated ratio of amplitudes of autonomous to driven fluctuations of 1 to 2, so that in this case the external driving is dominant. In this case we have observed very clear effects of the driving, but not the expected strong resonances at the peak frequencies. Typically, the spectrum shows a somewhat deformed version of the quasiperiodic spectrum from the nondriven case, plus several δ spikes at the frequencies of the driving and its harmonics (including aliased harmonics). In no case have we seen a strong enhancement of the main peaks of the spectrum due to the periodic driving.

Results similar to these were obtained for the Ikeda mappings, showing how the introduction of a very small driving just adds a single spike to the spectrum, with no changes in the quasiperiodic background, and that larger drivings do deform the spectrum and introduce harmonics, but does not create strong resonances. These two results indicate that the quasiperiodicity of the mean field cannot be enhanced by an external driving.

In conclusion, we have found that the use of nonsimultaneous mean fields in globally coupled maps affects the character of their quasiperiodic behavior, increasing linearly the number of peaks in the spectra of their mean fields. The nonstatistical character of these mean fields is preserved, except for the obvious case of time averages over very large spans. Finally, we have not found strong resonance effects in these systems.

G. Perez and S. Sinha thank the ICTP (Trieste), where this work was started, for their hospitality. G. Perez acknowledges support from CONACyT (Mexico), through Grant No. 4178-E9405.

[1] See, for instance, J. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by B.-L. Hao (World Scientific, Singapore, 1989); K. Kaneko, Physica D **86**, 158 (1995), and references therein.

[2] K. Kaneko, Phys. Rev. Lett. **65**, 1391 (1990); Physica D **55**, 368 (1992).

[3] G. Perez *et al.*, Physica D **63**, 341 (1993); S. Sinha *et al.*, Phys. Rev. A **46**, 6242 (1992); V. Hakim and J. W. Rappel, Europhys. Lett. **27**, 637 (1994).

[4] G. Perez *et al.*, Phys. Rev. A **45**, 5469 (1992).

[5] D. Dominguez and H. A. Cerdeira, Phys. Rev. Lett. **71**, 3359 (1993).

[6] K. Ikeda, Opt. Commun. **30**, 257 (1979); K. Ikeda *et al.*, Phys. Rev. Lett. **45**, 709 (1980); H. Nakatsuoka *et al.*, *ibid.* **50**, 109 (1983).

[7] A. Pikovsky, Phys. Rev. Lett. **71**, 653 (1993).