THE IDENTICAL EQUATIONS OF THE MULTIPLICATIVE FUNCTION

BY R. VAIDYANATHASWAMY

1. *Introduction.* An arithmetic function $f(N)$ is *multiplicative*, if $f(MN)=f(M)f(N)$, whenever the integers M, N are relatively prime. An analogous definition may be given for functions of more than one argument; thus $\psi(M_1, M_2)$ would be said to be *multiplicative*, if $\psi(M_1N_1, M_2N_2)=\psi(M_1, M_2)\psi(N_1, N_2)$, whenever the two products M_1M_2, N_1N_2 are relatively prime. If $f(N)$ and $\psi(M_1, M_2)$ are multiplicative functions, it is clear that we have

$$f(1) = \psi(1, 1) = 1. \tag{1}$$

If the relation $f(MN)=f(M)f(N)$ is true for *all* integers M, N, we shall say that f is a *linear function.*

The process of *composing* two arithmetic functions $f(N)$, $\phi(N)$ consists in forming their *composite* (represented by $f\cdot\phi$), namely the function ψ defined by the equation

$$\psi(N) = \sum f(\delta)\phi(N/\delta),$$

where the summation is for all divisors δ of N. If f and ϕ are multiplicative, it is easy to see that their composite ψ is also multiplicative. The notion of composition may be obviously extended to functions of several arguments.

Given an arithmetic function $f(N)$, such that $f(1)\neq 0$, it has been shown by E. T. Bell,* that there exists a unique arithmetic function $\psi(N)$, such that the composite $f\cdot\psi$ vanishes for all values of its argument other than 1, and takes the value 1 when the argument is equal to 1. We call ψ the *inverse function* of f, and denote it by f^{-1}; it is easy to see that f^{-1} is multiplicative if f is also.

We shall say that a multiplicative function $F(M_1, M_2)$ is a *cardinal function* of M_1, M_2, if it vanishes whenever either of

* *On a certain inversion in the theory of numbers*, Tôhoku Mathematical Journal, vol. 17 (1920). Also *A ray of numerical functions*, this Bulletin, vol. 32 (1926), p. 341.

M_1, M_2 admits a divisor relatively prime to the other, in other words, whenever the distinct prime factors of M_1 are not identical with those of M_2. We require in what follows, a particular cardinal function $C(M_1, M_2)$ defined as follows:

$C(M_1, M_2)=0$, if the distinct prime factors of M_1, M_2 are not identical,

$C(M_1, M_2)=(-1)^\nu$, if M_1, M_2 have the same ν distinct prime factors.

I have shown elsewhere,* by the method of generating series, that every multiplicative function f (of a single argument) satisfies a certain identical equation. By using the cardinal function C, the identical equation can be put into the form

$$f(MN) = \sum f\left(\frac{M}{\delta_1}\right) f\left(\frac{N}{\delta_2}\right) f^{-1}(\delta_1\delta_2)C(\delta_1, \delta_2),$$

the summation on the right extending over all divisors δ_1 of M, and δ_2 of N. My purpose in the present note is to show that this identical equation can be derived, by straightforward reasoning of a purely arithmetical nature, from certain fundamental properties of the inverse function.

2. *Three Theorems relating to the Inverse Function.*

THEOREM 1. *If f is any multiplicative function of one argument, and f^{-1} its inverse function, then the sum*

$$\sum f(M_1\delta) f^{-1}\left(\frac{M_2N}{\delta}\right),$$

extended over all the divisors δ of N, vanishes unless every prime factor of N divides M_1M_2.

PROOF. Let $N=N_1N_2$, where all the prime factors of N_1 divide M_1M_2, and N_2 is relatively prime to M_1M_2. It is clear that N_1 and N_2 are relatively prime, and therefore any factor δ of N can be expressed uniquely in the form $\delta_1\delta_2$, where δ_1 is a divisor of N_1, and δ_2 of N_2. Hence we have

$$\sum f(M_1\delta) f^{-1}\left(\frac{M_2\,N}{\delta}\right) = \sum f(M_1\delta_1\delta_2) f^{-1}\left(\frac{M_2N_1}{\delta_1}\cdot\frac{N_2}{\delta_2}\right)$$

* In a paper on *The theory of multiplicative arithmetic functions*, to appear shortly in the Transactions of this Society.

$$= \left\{ \sum f(\delta_2) f^{-1}\left(\frac{N_2}{\delta_2}\right) \right\} \left\{ \sum f(M_1\delta_1) f^{-1}\left(\frac{M_2 N_1}{\delta_1}\right) \right\}$$
$$= 0, \text{ if } N_2 \neq 1,$$

which proves the theorem.

COROLLARY. A factor N_1 of N may be called a *block factor*, if it is relatively prime to N/N_1. The sum

$$\sum f\left(\frac{N}{\delta}\right) f^{-1}(\delta),$$

extended over all the divisors δ of a block factor $N_1(\neq 1)$ of N, vanishes. This may be proved directly or as a consequence of the theorem. The defining property of the inverse function is the particular case of this result which arises when the block factor N_1 coincides with N.

Suppose now that N contains ν distinct prime factors, and $N_i^1, N_i^2, \cdots, N_i^k (k = \binom{\nu}{i})$ are the distinct block factors of N which contain exactly i of the prime factors. Consider the sum

$$A = \sum_N f\left(\frac{N}{\delta}\right) f^{-1}(\delta) - \sum_{k=1}^{\nu} \left\{ \sum_{N^k_{\nu-1}} f\left(\frac{N}{\delta}\right) f^{-1}(\delta) \right\}$$
$$+ \sum_{k=1}^{\nu(\nu-1)/2} \left\{ \sum_{N^k_{\nu-1}} f\left(\frac{N}{\delta}\right) f^{-1}(\delta) \right\} - \cdots$$
$$+ (-1)^{\nu-1} \sum_{k=1}^{\nu} \left\{ \sum_{N_1{}^k} f\left(\frac{N}{\delta}\right) f^{-1}(\delta) \right\},$$

where N_i^j below $\sum$ indicates that the sum is extended over all the divisors δ of N_i^j. We shall evaluate the expression A in two ways. In the first place, every partial sum in A, except the first, vanishes by the Corollary to Theorem I. Hence we have

$$A = \sum_N f\left(\frac{N}{\delta}\right) f^{-1}(\delta) = 0, \qquad (N > 1).$$

On the other hand consider a particular divisor d of N, containing i distinct prime factors. The coefficient of $f(N/d)f^{-1}(d)$ in A is

$$1 - \binom{\nu - i}{1} + \binom{\nu - i}{2} - \cdots = 0,$$

if $0<i<\nu$, but $=1$ if $i=\nu$. If $d=1$, the coefficient of

$$f(N/1)f^{-1}(1)=f(N)$$

is

$$1-\binom{\nu}{1}+\binom{\nu}{2}-\cdots+(-1)^{\nu+1}\binom{\nu}{\nu-1}=(-1)^{\nu-1};$$

hence we have

$$A=\sum f\left(\frac{N}{t}\right)f^{-1}(t)+(-1)^{\nu-1}f(N),$$

where the summation is for the divisors t of N which contain *all* its distinct prime factors. Since it has been shown that $A=0$, we have the following theorem.

THEOREM 2. *If N contains ν distinct prime factors, then*

$$\sum f\left(\frac{N}{t}\right)f^{-1}(t)=(-1)^{\nu}f(N),$$

where the summation extends over all the divisors t of N which contain all its ν distinct prime factors.

The next theorem is concerned with two numbers M, N which contain the *same* ν distinct prime factors. We shall denote by M_i^k, N_i^k $(k=1, 2, \cdots, \binom{\nu}{i})$ two block factors of M, N respectively, which contain the *same* i prime factors. We shall also write

$$M=M_i^k\overline{M}_i^k,\qquad N=N_i^k\overline{N}_i^k,$$

so that $\overline{M}_i^k$, $\overline{N}_i^k$ are respectively prime to M_i^k, N_i^k and are block factors of M, N, containing the *same* $\nu-i$ prime factors. Consider the expression

$$\begin{aligned}B=&\sum f\left(\frac{MN}{\delta}\right)f^{-1}(\delta)\\&+\sum_{k=1}^{\nu}\left\{\sum\sum f\left(\frac{M_1{}^k}{t}\cdot\frac{\overline{M}_1{}^k\overline{N}_1{}^k}{\delta}\right)f^{-1}(N_1^k t\delta)\right\}\\&-\sum_{k=1}^{\nu(\nu-1)/2}\left\{\sum\sum f\left(\frac{M_2{}^k}{t}\cdot\frac{\overline{M}_2{}^k\overline{N}_2^k}{\delta}\right)f^{-1}(N_2^k t\delta)\right\}+\cdots\\&-\cdots+(-1)^{\nu-1}\sum f\left(\frac{M}{t}\right)f^{-1}(Nt).\end{aligned}$$

Here the first term of B is a summation over all divisors δ of N. The second (and later) terms contain three summations; the two inner summations relate respectively to *all* divisors δ of $\overline{M}_i^k\overline{N}_i^k$, and to all such divisors t of M_i^k as contain all its distinct prime factors; the outer summation relates to all possible resolutions of M, N into corresponding block factors containing i and $\nu-i$ primes. The signs of the $\nu+1$ terms in B alternate from the second term onwards. In the last term $i=\nu$, and so the outer summation as well as the summation relating to δ, has disappeared, leaving only the summation over all factors t of M containing all its ν prime factors.

As in the proof of Theorem 2, we shall evaluate B in two different ways. In the first place *all terms of B, except the first and the last, vanish.* To see this, take a fixed divisor t of M_i^k (containing all its prime factors), and consider the sum

$$\sum f\left(\frac{M_i^k}{t}\cdot\frac{\overline{M}_i^k\overline{N}_i^k}{\delta}\right)f^{-1}(N_i^k t\delta),$$

extended over all the divisors δ of $\overline{M}_i^k\overline{N}_i^k$. Since $\overline{M}_i^k\overline{N}_i^k$ has no prime factors in common with $M_i^k/t\times N_i^k t=M_i^kN_i^k$, it follows from Theorem 1 that this (and similarly every other sum of the same type) vanishes. It thus follows that

$$B = \sum f\left(\frac{MN}{\delta}\right)f^{-1}(\delta) + (-1)^{\nu-1}\sum f\left(\frac{M}{t}\right)f(Nt),$$

where the first summation is for all divisors δ of N, and the second for all divisors t of M which contain all its ν distinct prime factors.

Secondly, we shall show that B is equal to the sum $\sum f(MN/\delta)f^{-1}(\delta)$, extended over all divisors δ of MN, and is therefore identically zero. A divisor d of MN may divide N; if it does not divide N, then a certain number, say $i(\geqq 1)$, of the ν prime factors of N must occur in d to a higher power than in N. Denote by N_i^k the block factor of N corresponding to these i primes; then we can express d in the form

$$d = N_i^k t\delta,$$

where δ is an arbitrary factor of $\overline{M}_i^k\overline{N}_i^k$, and t is a factor of M_i^k containing all its prime factors (M_i^k, $\overline{M}_i^k$, $\overline{N}_i^k$ having the meanings already explained). But this expression for d is not unique, since there is no restriction on the factor δ of $\overline{M}_i^k\overline{N}_i^k$; in fact, if N_j^l $(1 \leqq j < i)$ is a block factor of N, which divides N_i^k, we can also write

$$d = N_j^l t_1 \delta_1,$$

where t_1 is a divisor of M_j^l containing all its prime factors, and δ_1 is some divisor of $\overline{M}_j^l\overline{N}_j^l$.

Consider now the number of times which $f(MN/d)f^{-1}(d)$ (d being a particular divisor of MN) occurs in (B). In case d divides N, it is clear that it occurs only once, namely in the first term of B. If d does not divide N, let exactly i of the ν prime factors occur in d to a higher power than in N; from the previous explanation, it will be clear then that $f(MN/d)f^{-1}(d)$ will occur $\binom{i}{1}$ times in the second term of B, and generally $\binom{i}{\lambda}$ times in the $(\lambda+1)$th term. Hence its coefficient in B is

$$\binom{i}{1} - \binom{i}{2} + \cdots = 1 - (1-1)^i = 1, \text{ if } i > 0.$$

Thus every term $f(MN/d)f^{-1}(d)$ occurs just once in B, whether d divides N or not. Hence

$$B = \sum_{\delta \mid MN} f\left(\frac{MN}{\delta}\right) f^{-1}(\delta) = 0 \quad (\text{since } MN > 1).$$

Combining this with our previous evaluation of B, we have the following result.

THEOREM 3. *If M and N have the same ν distinct prime factors, then*

$$\sum f\left(\frac{MN}{\delta}\right) f^{-1}(\delta) = (-1)^\nu \sum f\left(\frac{M}{t}\right) f^{-1}(Nt),$$

where on the left the summation is for all divisors δ of N, and on the right for all divisors t of M which contain all its ν distinct prime factors.

COROLLARY 1. *On putting $N=1$ in Theorem 3, we obtain Theorem 2. Hence the theorem must be considered to be true for any M, and $N=1$.*

COROLLARY 2. *Let M_1 be relatively prime to M (and therefore* *also to N). Write $M' = M_1M$, and multiply both sides of the equality in Theorem 3 by $f(M_1)$. We obtain the relation*

$$\sum f\left(\frac{M'N}{\delta}\right)f^{-1}(\delta) = (-1)^{\nu}\sum f\left(\frac{M'}{t}\right)f^{-1}(Nt),$$

where M' is any number containing all the ν distinct prime factors of N, the summation on the left is for all divisors δ of N, and on the right for all divisors t of M' which contain only the ν distinct prime factors of N.

COROLLARY 3. *Theorem 3 is also true for any two numbers M, N, provided* (1) *ν is the number of distinct prime factors in N, and* (2) *the summation on the right is for all divisors t of M which contain the ν distinct primes dividing N, and no others.*

If all the ν prime factors of N occur in M, then Corollary 3 reduces to Corollary 2. If on the other hand, some of the ν prime factors of N do not occur in M, then the left side of the equality of Theorem 3 vanishes by virtue of Theorem I, and the right side also vanishes, since there are now no factors t of M which contain *all* the prime factors of N. Thus, under this interpretation of the summation on the right, Theorem 3 is true for any two numbers M, N.

3. *The Identical Equation of f.* Replacing N by N_1, the equality of Theorem 3 is

$$\sum f\left(\frac{MN_1}{\delta}\right)f^{-1}(\delta) = (-1)^{\nu}\sum f\left(\frac{M}{t}\right)f^{-1}(N_1t),$$

where ν is the number of prime factors in N_1, δ runs over the divisors of N_1, and t over those divisors of M which contain neither more nor less than the ν prime factors of N_1. As has been remarked, this result is true for *any* two numbers M, N_1.

Multiply both sides of the equality by $f(N_2)$ and sum over all values of N_1, N_2, such that $N_1N_2 = N$. On the left side, we carry out the summation in two stages; namely, we first keep N_1/δ fixed, and sum over all values of N_2 and δ such that $N_2\delta = N\delta/N_1$. Thus the left side is

$$\sum\sum f\left(\frac{MN_1}{\delta}\right)f^{-1}(\delta)f(N_2) = \sum f\left(\frac{MN_1}{\delta}\right)\sum f^{-1}(\delta)f(N_2).$$

We have also

$$\sum_{N_2\delta=N\delta/N_1} f^{-1}(\delta)f(N_2) = 0, \text{ if } \frac{N_1}{\delta} \neq N;$$

$$= 1, \text{ if } \frac{N_1}{\delta} = N.$$

Thus the left side reduces to $f(MN)$. The right side is

$$\sum\sum(-1)^\nu f\left(\frac{M}{t}\right) f\left(\frac{N}{\delta}\right) f^{-1}(\delta t),$$

summed for all divisors δ of N, and all such divisors t of M as contain neither more nor less than the ν prime factors of δ. It is clear that this is identical with

$$\sum\sum f\left(\frac{M}{\delta_1}\right) f\left(\frac{N}{\delta_2}\right) f^{-1}(\delta_1\delta_2)C(\delta_1, \delta_2),$$

δ being the special cardinal function defined above, and the summation extending over *all* the divisors δ_1 of M, and δ_2 of N. The identical equation of f, namely,

$$f(MN) = \sum\sum f\left(\frac{M}{\delta_1}\right) f\left(\frac{N}{\delta_2}\right) f^{-1}(\delta_1\delta_2)C(\delta_1, \delta_2),$$

is thus established.

4. *The Identical Equation of the Quadratic Function.* A multiplicative function f may be called *an integral quadratic function*, or simply *a quadratic function*, if it is the composite of two linear functions. More generally, f is an *integral function of the rth degree*, if it is the composite of r linear functions. We shall now show that the identical equation of f assumes a simple form when it is a quadratic function.

A fundamental property of the inverse function is: *the inverse of the composite of any number of multiplicative functions of one argument is the composite of their inverses.* This is easily proved, and is assumed in what follows.

THEOREM 4. *If $f(N)$ is an integral function of the rth degree, its inverse function $f^{-1}(N)$ vanishes whenever N is divisible by an $(r+1)$th power; also, if N is the product of distinct primes,*

$$f^{-1}(N^r) = \{\mu(N)\}^r \lambda_1(N)\lambda_2(N) \cdots \lambda_r(N),$$

where f is the composite of the linear functions $\lambda_1, \lambda_2, \cdots, \lambda_r$, *and* $\mu(N)$ *is Mertens' function.*

For consider the theorem for the case $r=1$, that is, when f is a linear function $\lambda(N)$. If p is an arbitrary prime, we have from the definition of the inverse function

$$\lambda(p)+\lambda^{-1}(p)=0; \; \lambda(p^2)+\lambda(p)\lambda^{-1}(p)+\lambda^{-1}(p^2)=0.$$

Hence $\lambda^{-1}(p)=-\lambda(p)$, and therefore $\lambda^{-1}(N)=\mu(N)\lambda(N)$, when N is a product of distinct primes. Also, remembering that λ is linear, we have $\lambda(p^2)=\lambda(p)\lambda(p)$, and therefore, from the second equation, $\lambda^{-1}(p^2)=0$. Hence $\lambda^{-1}(N)=0$ when N has a square factor. The theorem is thus true for $r=1$.

Assume now the truth of the theorem for the value $r-1$ of r, and let f be the composite of the linear functions $\lambda_1, \lambda_2, \cdots, \lambda_r$. Then $f^{-1}\lambda_r$ is the composite of the $r-1$ linear functions $\lambda_1, \lambda_2, \cdots, \lambda_{r-1}$. Hence, by virtue of our assumption, we may write

$$\begin{aligned} 0 &= (f^{-1}\cdot\lambda_r)(p^r) = f^{-1}(p^r)+f^{-1}(p^{r-1})\lambda_r(p)+f^{-1}(p^{r-2})\lambda_r(p^2)+\cdots, \\ 0 &= (f^{-1}\cdot\lambda_r)(p^{r+1}) = f^{-1}(p^{r+1})+f^{-1}(p^r)\lambda_r(p)+\cdots \\ &= f^{-1}(p^{r+1})+\lambda_r(p)\{f^{-1}(p^r) \\ &\qquad +f^{-1}(p^{r-1})\lambda_r(p)+\cdots\}, \\ &= f^{-1}(p^{r+1}), \end{aligned}$$

since λ_r is linear. Hence $f^{-1}(N)=0$ if N is divisible by an $(r+1)$th power. Again, for the second part of the theorem, we have, by virtue of our assumption,

$$\begin{aligned} (-1)^{r-1}\lambda_1(p)\lambda_2(p)\cdots\lambda_{r-1}(p) &= (f^{-1}\cdot\lambda_r)(p^{r-1}) \\ &= f^{-1}(p^{r-1})+f^{-1}(p^{r-2})\lambda_r(p)+\cdots. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= (f^{-1}\cdot\lambda_r)(p^r) \\ &= f^{-1}(p^r)+\lambda_r(p)\{f^{-1}(p^{r-1})+\lambda_r(p)f^{-1}(p^{r-2})+\cdots\} \\ &= f^{-1}(p^r)+(-1)^{r-1}\lambda_1(p)\lambda_2(p)\cdots\lambda_r(p), \end{aligned}$$

or

$$\begin{aligned} f^{-1}(p^r) &= (-1)^r\lambda_1(p)\lambda_2(p)\cdots\lambda_r(p); \\ f^{-1}(N^r) &= \{\mu(N)\}^r\lambda_1(N)\cdots\lambda_r(N), \end{aligned}$$

N being a product of distinct primes. Thus the theorem is true for the next higher value of r. Since it has been seen to be true for $r=1$, the induction is complete.

THEOREM 5. *If f is the composite of the two linear functions $\lambda_1(N)$, $\lambda_2(N)$, its identical equation takes the form*

$$f(MN) = \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) \lambda_1(\delta)\lambda_2(\delta)\nu(\delta),$$

where the summation is now for all common divisors δ of M, N.

PROOF. The identical equation of f may be written in the form

$$f(MN) = \sum f\left(\frac{M}{\delta_1}\right) f\left(\frac{N}{\delta_2}\right) \cdot (-1)^\nu f^{-1}(\delta_1\delta_2),$$

summed for pairs of divisors δ_1, δ_2 of M, N, which contain the same distinct prime factors, ν in number. Now if f is a quadratic function, $f^{-1}(\delta_1\delta_2)$ vanishes when $\delta_1\delta_2$ contains a cubed factor and therefore (since δ_1, δ_2 contain the same distinct prime factors) when either δ_1 or δ_2 contains a squared factor. It follows that the only non-zero terms in the above sum are those for which

$$\delta_1 = \delta_2 = \delta = \text{a product of distinct primes.}$$

Hence

$$f(MN) = \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) \cdot (-1)^\nu f^{-1}(\delta^2),$$

summed for common divisors δ of M, N. If we use Theorem 4, this becomes

$$f(MN) = \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) f^{-1}(\delta^2)\mu(\delta)$$

$$= \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) \lambda_1(\delta)\lambda_2(\delta)\mu(\delta).$$

As special cases of this theorem, may be mentioned the identical equations of the following functions:

(1) $\sigma_a(N)$ ($=$the sum of the ath powers of the divisors of N). This is a quadratic function, being the composite of the two linear functions $\lambda_1(N)=N^a$, and $\lambda_2(N)=1$. Hence

$$\sigma_a(MN) = \sum \sigma_a\left(\frac{M}{\delta}\right)\sigma_a\left(\frac{N}{\delta}\right)\delta^a\mu(\delta),$$

summed for common divisors δ.

(2) The function $4R(N)$ which is equal to the number of representations of N as a sum of two squares. It is known that $R(N)$ is also equal to the excess of the number of divisors of N of the form $4n+1$ over the number of those of the form $4n-1$. Hence $R(N)$ is a quadratic function, being the composite of the linear functions $\lambda_1(N)$, $\lambda_2(N)$ defined by the quadratic residue symbol

$$\lambda_2(N) = 1;\ \lambda_1(2^n) = 0;\ \lambda_1(N) = \left(\frac{-1}{N}\right),$$

Hence

$$R(MN) = \sum R\left(\frac{M}{\delta}\right)R\left(\frac{N}{\delta}\right)\left(\frac{-1}{\delta}\right)\mu(\delta),$$

summed for common divisors δ of M, N.

MADRAS UNIVERSITY,
INDIA