QUASI-BOOLEAN ALGEBRAS AND MANY-VALUED LOGICS.

By R. Vaidyanathaswamy.

(From the Department of Mathematics, University of Madras, Madras.)

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I. The classical calculus of propositions, found for instance in the *Principia Mathematica*, can be interpreted, as is well-known, as a *truth-value system.* This is done by attributing to each proposition p a truth-value $t(p)$ which is zero or unity according as p is false or true. If now $f(p, q, r, \cdots)$ be any proposition which is formed from the propositions $p, q, r, \cdots$ by the operations of the calculus (that is, $\sim$, $+$, $\cdot$, and $\supset$), it is a condition to be satisfied by any truth-value system that f should be *categorical*, that is, that the truth-value $t(f)$ of f should *not* depend on the actual propositional arguments $p, q, r, \cdots$ but only on their truth-values, $t(p), t(q), t(r), \cdots$. This is easily verified by inspection for the propositional calculus; for:

$$t(p+q) = \text{Max}\,[t(p), t(q)]$$
$$t(p \cdot q) = \text{Min.}\,[t(p), t(q)]$$
$$t(\sim p) = 1 - t(p)$$
$$t(p \supset q) = \text{Max}\,[1 - t(p), t(q)].$$

The laws of the propositional calculus are those propositions $f(p, q, r, \cdots)$, which are true whatever be the truth or falsity of $p, q, r, \cdots$.

This idea has been generalised by Łukasiewicz and Tarski who have constructed a logic of propositions with $n+1$ truth-values denoted for convenience by $0, \frac{1}{n}, \frac{2}{n} \cdots, \frac{n-1}{1,}$, 1. The implication-relation C of the calculus is *defined* by:

$$t(p\,\text{C}\,q) = 1 \text{ if } t(p) \leqslant t(q)$$
$$= 1 - t(p) + t(q), \quad \text{if } t(p) > t(q).$$

Two propositions p, q are logically equivalent when each implies the other; from the truth-value $t(p\,\text{C}\,q)$, this can happen only when p and q have the same truth-value, since

$$[t(p) \leqslant t(q)].\ [t(q) \leqslant t(p)] : \supset .\ t(p) = t(q).$$

Further, *negation* is defined by:

$$t(\sim p) = 1 - t(p).$$

Logical addition ($\vee$) and multiplication ($\wedge$) of propositions are now defined by :

$$p \vee q = p \mathrm{C} q \cdot \mathrm{C} q \ \mathrm{D}f$$

$$p \wedge q = \sim (\sim p \vee \sim q) \ \mathrm{D}f.$$

The *meaning* of the operations C, $\sim$, $\vee$, $\wedge$ thus defined should not be identified with the broad meaning given to these same operations in the two-valued calculus. As a matter of fact, $\vee$, $\wedge$, $\sim$ can not have the ordinary meanings, *or, and,* and *not,* since the law of excluded middle and the law of contradiction do not hold ; for

$$t(p \vee q) = t(p \mathrm{C} q \cdot \mathrm{C} q) \text{ by definition.}$$

If $t(p) \leqslant t(q)$, $t(p \mathrm{C} q) = 1$, and therefore $t(p \mathrm{C} q \cdot \mathrm{C} q) = t(q)$.

If $t(p) > t(q)$, $t(p \mathrm{C} q) = 1 - t(p) + t(q)$ and

$$t(p \mathrm{C} q \cdot \mathrm{C} q) = 1 - [1 - t(p) + t(q)] + t(q) = t(p).$$

Thus $t(p \vee q) = \mathrm{Max}\,[t(p), t(q)]$. Similarly

$$t(p \wedge q) = 1 - t(\sim p \vee \sim q) = \cdot 1 - \mathrm{Max.}\,[1 - t(p), 1 - t(q)$$

$$= \mathrm{Min}\,[t(p), t(p)].$$

In particular :

$$t(p \vee \sim p) = \mathrm{Max}\,[t(p), 1 - t(p)] \neq 1$$

$$t(p \wedge \sim p) = \mathrm{Min}\,[t(p), 1 - t(p)] \neq 0.$$

Thus the laws of excluded middle and contradiction both fail. The actual meaning to be attached to the operations of the many-valued calculus must be discovered from considerations of probability.* For, the limiting form when n becomes infinite, of the Lukasiewicz-Tarski logic is the logic of probability.

II. The purpose of this paper is not the investigation of the meaning of the operations of the many-valued calculus. It is on the other hand to arrive at a view of many-valued logics which is somewhat more general than that of Lukasiewicz and Tarski, and includes their extension as a special case. The view which I wish to advance is : *the truth-values attributed to propositions in any propositional calculus must be elements of, what I shall call, a quasi-boolean algebra.* By a *boolean algebra* is meant an algebra which is constructed on the model of the algebra of all subclasses of a given class. By a *quasi-boolean algebra,* I shall mean an algebra which is constructed on the model of the algebra of all subclasses of a given class *containing groups of like elements.* This requires further explanation, as it is not evident as to what is meant by the algebra of all subclasses of a given class, when the class contains like elements. The explanation is supplied in what follows.

* For these meanings see Lewis I and II.

III. The Simple Quasi-boolean Algebra.

The *simple quasi-boolean algebra* may be defined to be an algebra constructed on the model of the algebra of all subclasses of a class, *all* of whose elements are alike. To study this algebra, consider a class C_n composed of n like elements. Since the elements of C_n are indistinguishable, two subclasses of C_n containing the same number r of elements are indistinguishable from one another. Hence C_n has precisely $n+1$ subclasses, containing respectively $0, 1, 2, \cdots, n$ elements. Thus the subclasses are in (1, 1) correspondence with the integers $\leqslant n$, and are in linear order.

The *sum* and *product* of two subclasses c, c' are defined generally as the classes containing the elements of c, c', and the elements common to c, c', respectively. These definitions would however be ambiguous if applied to two subclasses c_r, c_s containing respectively r, s elements of C_n. To remove the ambiguity we consider the extreme cases of indetermination. We shall say that c_r, c_s are in the position of *maximum incidence*, when they have as many common elements as possible, and in the position of *minimum incidence*, when they have as few common elements as possible. The sum and product of c_r, c_s in the position of maximum incidence are defined to be their quasi-boolean *sum* and *product*. It follows from this definition, that if $r \nless s$,

$$c_r + c_s = c_r$$

$$c_r\, c_s = c_s.$$

Hence also:

$$c_0 + c_k = c_k\,;\; c_0\, c_k = c_0$$

$$c_n + c_k = c_n\,;\; c_n\, c_k = c_k.$$

The associative and commutative laws hold for these two quasi-boolean operations, as well as the existence of zero and the unit. Further, just as in boolean algebra, *each of these quasi-boolean operations distributes the other.*

For

$$c_r\,(c_s + c_t) \;= c_k\,;\; k = \text{Min}\,[r, \max\,(s, t)]$$

$$c_r\, c_s + c_r\, c_t = c_p\,;\; p = \text{Max}\,[\min\,(r, s), \min\,(r, t)].$$

We easily verify:

Min $[r, \max\,(s, t)] = \text{Max}\,[\min\,(r, s), \min\,(r, t)]$, for any three integers r, s, t.

Thus this distributive law and similarly the other distributive law are seen to hold.

The negative $\bar{c}_r$ of the quasi-boolean element c_r is defined to be the subclass which remains when c_r is removed from C_n. It is clear that $\bar{c}_r = c_{n-r}$, and $\bar{\bar{c}}_r = c_r$ as in boolean algebra. Further with this definition of the negative, *the principle of duality holds just as in boolean algebra. For,*

$$\overline{(c_r + c_s)} = \bar{c}_k = c_{n-k}\,;\ k = \max(r, s)$$
$$\bar{c}_r\,\bar{c}_s = c_{n-r}\cdot c_{n-s} = c_{n-k},\ \text{since } n - k = \min(n - r, n - s).$$

Thus $\overline{(c_r + c_s)} = \bar{c}_r\,\bar{c}_s$; similarly $\overline{(c_r \cdot c_s)} = \bar{c}_r + \bar{c}_s$.
However, the two boolean laws $c_r + \bar{c}_r = 1$, $c_r \cdot \bar{c}_r = 0$ no longer hold. We have in fact,

$$c_r + \bar{c}_r = c_r + c_{n-r} = c_k\,;\ k = \max(r, n - r)$$
$$c_r \cdot \bar{c}_r = c_r\, c_{n-r} = c_l\,;\ l = \min(r, n - r).$$

Further, in the simple quasi-boolean algebra we can define the relation 'contained in', ($<$) as follows :

$$c_r < c_s \text{ means } c_r + c_s = c_s.$$

Then, just as in boolean algebra, we can define the two quasi-boolean operations $+$ and $\times$ in terms of the relation $<$ and its converse $>$; namely, $c_r + c_s$ is the element which contains c_r and c_s and which is contained in every element containing c_r and c_s; with a similar definition for $c_r \cdot c_s$. By means of the relation $<$, the simple quasi-boolean algebra is linearly ordered.

IV. The Group-operation of the Simple Quasi-boolean Algebra.

We shall now shew that if we take the subclasses c_r, c_s in their position of minimum incidence, then the formation of their sum and product can be effectively combined into a single operation, which is a group-operation of the simple quasi-boolean algebra, and exhibits it as a cyclic group of order $n + 1$.

For, if $r + s \not> n$, the classes, c_r, c_s have no common element in their position of minimum incidence, hence $c_r + c_s = c_{r+s}$, $c_r \cdot c_s = c_0$; while if $r + s > n$, the classes will have $r + s - n$ common elements in minimum incidence, so that $c_r + c_s = c_n$; $c_r \cdot c_s = c_{r+s-n}$. Discarding the trivial results c_0 and c_n, we see that sum and product in the position of minimum incidence can be combined into a single operation R such that :

$$c_r \,\mathrm{R}\, c_s = c_k\,;\ k = \text{least positive residue of } r + s \bmod (n + 1).$$

Thus $c_r \,\mathrm{R}\, c_s$ is the sum of c_r, c_s in the position of minimum incidence if $r + s < n + 1$, and is otherwise the largest proper subclass of the product of c_r, c_s (in minimum incidence). It is clear that R is a group-operation of period $n + 1$, and that the simple quasi-boolean is a cyclic group with respect to R. Contrary to what happens in boolean algebra, R cannot be expressed in terms of the quasi-boolean operations.

The definition of R is slightly more simple when n is infinite; namely $c_r \mathrm{R}\, c_s = c_r + c_s$ or $c_r\, c_s$ in the position of minimum incidence, according as Measure (c_r) + Measure $(c_s) <$ or $\nless$ Measure (c_n).

V. *The General Quasi-boolean Algebra.*

The general quasi-boolean algebra may be defined as the vector compound of any number of simple quasi-boolean algebras, $\mathrm{S}_1, \mathrm{S}_2, \cdots$. If s_i is an element of S_i, the quasi-boolean operations for the vectors

$$\sigma = (s_1, s_2, \cdots)\,;\ \sigma' = (s_1', s'_2, \cdots)$$

are defined by:

$$\bar{\sigma} = (\bar{s}_1, \bar{s}_2, \cdots)\,;\ \sigma + \sigma' = (s_1 + s_1', s_2 + s_2', \cdots)$$

$$\sigma\sigma' = (s_1\, s_1', s_2 s_2', \cdots).$$

It is clear that $\sigma > \sigma'$ only if each $s_i > s_i'$.

A simple example of the finite quasi-boolean algebra is the algebra (which has been known for a long time) of divisors of a number $\mathrm{N} = p_1^{n_1}\, p_2^{n_2} \cdots p_r^{n_r}$, where the p's are distinct primes. The quasi-boolean sum and product of two divisors d_1, d_2 of N, are respectively their greatest common divisor and their least common multiple. The quasi-boolean negative of any divisor d is the conjugate divisor $\dfrac{\mathrm{N}}{d}$. The divisors of N represent in fact subclasses of a class with n_1 like elements p_1, n_2 like elements $p_2, \cdots, n_r$ like elements p_r. It may be shewn that the number of elements in the general finite quasi-boolean algebra must be of the form $d\,(\mathrm{N})$ (= number of divisors of N) and that the algebra is identical with the algebra of divisors of N.

More generally, it may be shewn that a set of elements with a reflexive transitive relation $<$, and a negation operation $(\bar{a})$, is a quasi-boolean algebra, if the following postulates hold:

(1) $a < b \cdot b < a :\supset\ : a = b$.

(2) $a < b \cdot\!\supset \cdot\ \bar{b} < \bar{a}$.

(3) $\bar{\bar{a}} = a$

(4) For any two elements a, b there exists a unique element x, such that:

$a < x\,;\ b < x\,;\ a < c \cdot b < c :\supset\ \cdot x < c$.

We write $x = a + b$.

(5) There exists two distinct elements 0, 1, such that

$$0 < x < 1 \text{ for every element } x.$$

(6) $a\,(b + c) = ab + ac$, where the product is defined by $xy = \overline{(\bar{x} + \bar{y})}$.

(7) A postulate for ensuring that the elements of a simple quasi-boolean algebra are in linear order.

The quasi-boolean algebra may be split up into its simple quasi-boolean components, by a theory of minimal elements as in the case of the boolean algebra.

VI. The Truth-value System.

Consider a logic of propositions with an implication-operation C. Whatever be the *meaning* of implication, the relation of implication must be reflexive and transitive, and must be related to *denial* in such a way that $p \, C q$ is logically equivalent to $\sim q \; C \sim p$. Assume further two logical operations $\vee$ and $\wedge$ (corresponding to *or*, and *and*), with the properties:

$$p \cdot C \cdot p \vee q \, ; \; q \cdot C \cdot p \vee q \, ;$$

$$p \, C \, r \cdot q \, C \, r : C \cdot p \vee q \, C \, r \, ;$$

(with similar properties of $\wedge$). In the Lukasiewicz-Tarski logics $p \vee q$ is defined in terms of C as $p \; C \; q \cdot C q$.

If we attribute to the propositions p of the calculus, a system of truth-values $[t\,(p)]$, we have to require that the logical relations and operations should be exactly imaged in corresponding relations and operations in the system $[t\,(p)]$ of truth-values. Hence the system $[t\,(p)]$ admits a reflexive transitive relation $<$ corresponding to C, a unary operation of negation, corresponding to *denial*, and standing in such relation to $<$ that $t_1 < t_2$ is equivalent to $\bar{t}_2 < \bar{t}_1$, and further two operations $+$ and $\times$, which can be defined in terms of $<$, by means of:

$$p < p + q \, ; \; q < p + q \, ;$$

$$\text{If } p < r \text{ and } q < r, \text{ then } p + q < r,$$

with similar definitions for $p \cdot q$.

These facts shew that the general features of the structure of the system of truth-values, are such as to render the system a quasi-boolean algebra.

When the quasi-boolean algebra is a *simple* one, we have the truth-value system of Lukasiewicz and Tarski.

When the quasi-boolean algebra reduces to a two-element boolean algebra (which is a particular case of the simple quasi-boolean algebra), we have the ordinary or classical two-valued logic.

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