Environment-induced dynamical chaos

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We examine the interplay of nonlinearity of a dynamical system and thermal fluctuation of its environment in the "physical limit" of small damping and slow diffusion in a semiclassical context and show that the trajectories of c-number variables exhibit dynamical chaos due to the thermal fluctuations of the bath.

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The interplay of nonlinearity of a dynamical system and thermal fluctuations of its environment has been one of the major areas of investigation in the recent past. These studies have enriched our understanding of nonequilibrium processes in several contexts, such as, symmetry between the growth and decay of classical fluctuations in equilibrium [1], interesting topological features of patterns of paths of large fluctuations in nonlinear systems [1], existence of generalized nonequilibrium potential [2], influence of nonlinearity on dissipation in multiphoton processes [3] and higher order diffusion in a nonlinear system [4], etc. While the development in these areas is largely confined to classical domain we examine a related issue in the semiclassical context. Since quantization is likely to add a new dimension to the interplay of nonlinearity and stochasticity in a weakly dissipative system, it is worthwhile to consider the physical limit of small damping and slow diffusion due to thermal fluctuations of the environment and look for the thermal fluctuations-induced features of nonlinearity in the dynamics. In this communication we specifically explore some interesting aspects of dynamical chaos in a driven bistable system whose origin lies at the fluctuations of the environmental degrees of freedom.

To describe the dissipative quantum dynamics of a system we first consider the traditional system-reservoir model developed over the last few decades [3,5,6]. The Hamiltonian of the bare system is coupled to an environment modeled by a reservoir of harmonic oscillator modes characterized by a frequency set $\{\Omega_j\}$. The quantum dynamics is generated by the overall Hamiltonian operator H for the system, environment and their coupling as follows;

$$H = H_0 + \hbar \sum_{j=1}^{\infty} \Omega_j {b_j}^\dagger b_j + \hbar \sum_{j=1}^{\infty} \left[g(\Omega_j) b_j + g^\star(\Omega_j) {b_j}^\dagger \right] x, \qquad (1)$$

where

$$H_0 = \frac{p^2}{2} + V(x) \ , \qquad (2)$$

defines the usual kinetic and potential energy terms corresponding to the system, x and p being the position and momentum operators, respectively. The second and the third terms in (1) specify the reservoir modes and their linear coupling to the system. $g(\Omega)$ denotes the system-reservoir coupling constant.

Systematic elimination of the reservoir modes in the usual way, using Born and Markov approximations leads one to the following standard reduced density matrix equation of the system only [5].

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] + \frac{\gamma}{2}(1+\bar{n})(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) + \frac{\gamma}{2}\bar{n}(2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger) \ . \qquad (3)$$

Here the system operator co-ordinate x is related to the creation and the annihilation operators $a^\dagger, a$ respectively as $x = (\frac{1}{\sqrt{2\omega}})(a + a^\dagger)$. ω is the linearised frequency of the system. Also the spectral density function of the reservoir is replaced by a continuous density $\mathcal{D}(\omega)$. $\gamma > 0$ is the limit of $2\pi |g(\omega)|^2 \frac{\mathcal{D}(\omega)}{\omega}$ as $\omega \to 0_+$ and is assumed to be finite. γ is the relaxation or dissipation rate and $\bar{n}\gamma$ is the diffusion coefficient D. $\bar{n}(= [exp\left(\frac{\hbar\omega}{kT}\right) - 1]^{-1})$ is the average thermal photon number of the reservoir. The terms analogous to Stark and Lamb shifts have been neglected.

The first term in Eq.(3) corresponding to the dynamical motion of the system refers to Liouville flow. The terms containing γ arise due to the interaction of the system with the surroundings. While $\frac{\gamma\bar{n}}{2}$ terms denote the diffusion of fluctuations of the reservoir modes into the system mode, $\frac{\gamma}{2}$ terms refer to the loss of energy from the system into reservoir. In the limit $T \to 0, i.e., \bar{n} \to 0$ the system is influenced by pure quantum noise or vacuum fluctuations [Note that 1 in $(\bar{n}+1)$ in Eq.(3) corresponds to the vacuum].

Our next task is to go over from a full quantum operator problem described by the Eq.(3) to an equivalent c-number problem . To this end we consider the quasi-classical distribution function W $(x, p,$ t) of Wigner. x, p are now c-number variables. Rewriting Eq.(3) in a quasi-classical language we obtain

$$\frac{\partial W}{\partial t} = -p\frac{\partial W}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial W}{\partial p} + \frac{\gamma}{2}\eta\left(\frac{\partial x W}{\partial x} + \frac{\partial p W}{\partial p}\right)$$

$$+\frac{\gamma\eta\hbar}{2\omega}(\bar{n}+1)\frac{\partial^2 W}{\partial x^2}+\frac{\gamma\eta\hbar\omega}{2}(\bar{n}+1)\frac{\partial^2 W}{\partial p^2}$$
$$+\sum_{n\geq 1}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}\frac{\partial^{2n+1}W}{\partial p^{2n+1}} \quad . \qquad (4)$$

η in Eq.7 is a parameter used to identify the environment-induced effect on the dynamics described by Eq.(4) (kept for bookkeeping in the calculation and put $\eta = 1$ at the end).

In the semiclassical limit $\hbar\omega << kT$, we have $\bar{n}+1 \approx \bar{n}$ and $D \approx \gamma kT$ so that Eq.(7) reduces to

$$\frac{\partial W}{\partial t} = -p\frac{\partial W}{\partial x}+\frac{\partial V}{\partial x}\frac{\partial W}{\partial p}+\frac{\gamma}{2}\eta\left(\frac{\partial xW}{\partial x}+\frac{\partial pW}{\partial p}\right)$$
$$+D\eta\left(\frac{1}{\omega^2}\frac{\partial^2 W}{\partial x^2}+\frac{1}{2}\frac{\partial^2 W}{\partial p^2}\right)$$
$$+\sum_{n\geq 1}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}\frac{\partial^{2n+1}W}{\partial p^{2n+1}} \quad . \qquad (5)$$

The overall dynamics described above is a superposition of two contributions, i. e. , the Liouville-Wigner dynamics and the system-reservoir dissipative dynamics. That the two contributions act independently is an assumption. The master equation (3) [or its Wigner function version (4)] is the most popular one in quantum optics. It has been extensively used [7], for the strongly nonlinear processes like three-wave, four-wave mixing and strong coherent light-matter interaction phenomena. The equation has also been applied in the context of chaos, e. g., in the dissipative standard map [8], dissipative logistic map [9], semiclassical theory of quantum noise in open chaotic systems [10,11] and in the studies of decoherence in relation to chaos for analysis of quantum-classical correspondence [12,13].

In the semiclassical ($\hbar \to 0$) limit the dissipative quantum dynamics can be conveniently described by "WKB-like" ansatz (we refer to " WKB-like" since we are considering more that one dimension. Traditional WKB refer to one dimension only) [1,14] of Eq.(5) for Wigner function of the form

$$W(x,p,t) = Z(t)\exp(-\frac{s}{\hbar}) \quad . \qquad (6)$$

Here $Z(t)$ is a prefactor and $s(x,p,t)$ is a classical action which is a function of c-number variables x and p , satisfying the following Hamilton-Jacobi equation

$$\frac{\partial s}{\partial t}+p\frac{\partial s}{\partial x}-\frac{\partial s}{\partial x}\frac{\partial s}{\partial p}-\frac{\gamma}{2}\eta(x\frac{\partial s}{\partial x}+p\frac{\partial s}{\partial p})+\eta(\frac{\partial s}{\partial x})^2$$
$$+\eta\omega^2(\frac{\partial s}{\partial p})^2+\sum_{n\geq 1}\frac{x^{2n}(-1)^{3n+1}}{2^{2n}(2n)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}\frac{\partial s}{\partial p}=0 \quad . \qquad (7)$$

The derivation of Eq.(7) is based on the consideration of the "physical limit" of weak dissipation and slow fluctuations in the sense $\frac{D_1}{\hbar^2} \approx \frac{1}{\hbar}$ where $D_1 = \frac{D}{2\omega^2}$ (note that D_1 and $\hbar$ have same dimension). The above equation can be solved by integrating the Hamiltonian equations of motion,

$$\dot{x} = p - \frac{\gamma}{2}\eta x + 2\eta P$$
$$\dot{X} = P - \frac{\gamma}{2}\eta X$$
$$\dot{p} = V' + \frac{\gamma}{2}\eta p - 2\omega^2\eta X - \sum_{n\geq 1}\frac{(-1)^{3n+1}}{2^{2n}(2n)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}X^{2n}$$
$$\dot{P} = V''X + \frac{\gamma}{2}\eta P - \sum_{n\geq 1}\frac{(-1)^{3n+1}}{2^{2n}(2n+1)!}\frac{\partial^{2n+2}V}{\partial x^{2n+2}}X^{2n+1} \quad (8)$$

which are derived from the following Hamiltonian H_{eff}

$$H_{eff} = pP - V'X - \frac{\gamma}{2}\eta(xP+pX)+\eta(P^2+\omega^2X^2)$$
$$+\sum_{n\geq 1}\frac{(-1)^{3n+1}}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}X^{2n+1} \quad . \qquad (9)$$

Here we have put $\frac{\partial s}{\partial x} = P$ and $\frac{\partial s}{\partial p} = X$. The introduction of additional degree-of-freedom by incorporating the auxiliary momentum (P) the and co-ordinate (X) makes the system effectively a two-degree-of-freedom system. The origin of these two variables is the thermal fluctuations of the reservoir. It is easy to identify the environment-related terms containing η in Eqs.(7-9). The auxiliary Hamiltonian (9) is therefore not to be confused with the microscopic Hamiltonian (1) which describes a system in contact with a reservoir with infinite degrees of freedom. Although the phase space trajectories concern fluctuations of c-number variables in the formal sense, because of the equations of motion (8) described by a Hamiltonian (9), the motion is strictly deterministic. The experiments on the corresponding classical version of the problem by Luchinsky and McClintock [1] have demonstrated that a trajectory of fluctuation is indeed a part of physical reality. We emphasize, however, here a number of distinguishing features in this context. While the studies by Luchinsky and McClintock [1] and Graham and Tel [2] concern overdamped limit, we consider here a weakly dissipative system. Furthermore because of the quantum correction, the phase space trajectories of fluctuations are significantly modified by semiclassical features. The introduction of these *quantum features at a semiclassical level* through a c-number Hamiltonian description of a dissipative evolution in the physical limit of weak damping and slow diffusion due to thermal noise is the essential content of the present work. Since the Eq.(9) describes deterministic evolution under nonlinear potential, the pattern of trajectories of fluctuations may display chaotic behaviour. In what follows we investigate this dynamical aspect of the dissipative system.

The equations derived in the weak thermal noise limit for the weakly dissipative semiclassical systems are fairly general. For illustration we now consider a simple model system Hamiltonian H_0 (see Eq.2)

$$H_0 = \frac{p^2}{2} + ax^4 - bx^2 + gxcos\Omega_0 t \tag{10}$$

which describes a bistable potential driven by a time-periodic field. a and b are the constants of the potential $V(x)$. The fourth term in (10) includes the effect of coupling of the system as well as the strength of the field of frequency Ω_0. For the Hamiltonian (10) the Eqs.(8) read as

$$\begin{aligned}
\dot{x} &= p - \frac{\gamma}{2}\eta x + 2\eta P \\
\dot{X} &= P - \frac{\gamma}{2}\eta X \\
\dot{p} &= 4ax^3 - 2bx + g\cos\Omega t + \eta(\frac{\gamma}{2}p - 2\omega^2 X) - 3axX^2 \\
\dot{P} &= (12ax^2 - 2b)X + \frac{\gamma}{2}\eta P - aX^3 \qquad (11)
\end{aligned}$$

which are derivable from the auxiliary Hamiltonian

$$\begin{aligned}
H_{eff} = Pp - (4ax^3 - 2bx + g\cos\omega t)X - \frac{\gamma}{2}\eta(xP + pX) \\
+\eta(P^2 + \omega^2 X^2) + axX^3 \quad . \qquad (12)
\end{aligned}$$

The system Hamiltonian (10) has served as a standard paradigm for a number of theoretical and experimental investigations [1,10,15] over many years. For the present purpose we choose the following parameter values; $a = \frac{1}{4}$, $b = \frac{1}{2}$, $\Omega_0 = 6.07$ and $g = 10$. Since we are considering the physical limit of weak damping and small diffusion we take the initial conditions for the auxiliary variables X and P (which originate from the fluctuations due to environment) as $P = 0$ and $X \to 0$ (we have used $X = 1.5 \times 10^{-6}$). This ensures a vanishing Hamiltonian H_{eff} for the entire numerical investigation that follows below.

We first consider a specific trajectory with the initial condition $p = 0$ and $x = -2.512$ for small values of γ (typical 0.01). Under this condition($\eta = 1$) the system is vanishingly coupled to the surroundings and consequently the dynamical behaviour is effectively due to the weak dissipation only. We illustrate this situation in Fig. 1 in terms of a Poincare map for the phase space which exhibits strong global chaos. On the other hand when the parameter η is switched off ($\eta = 0$) the system displays typical weak chaos (Fig.2). The similar behaviour has been observed for other sets of initial conditions for x and p (We have not reproduced them here for the sake of brevity).

The effect of weak dissipation and slow diffusion due to thermal fluctuations from the surroundings can be seen in the case of other sets of initial conditions also. For the initial condition $p = 0$ and $x = -2.49$ one observes for $\gamma = 0.01$ dissipative strong chaos($\eta = 1$). This is illustrated in Fig.3. It is interesting to note that for $\eta = 0$ the same trajectory gets localized in the left well as a regular one as shown in Fig.4. The weakly chaotic and regular trajectories in Figs. 2 and 4, respectively, are purely semiclassical in nature (in the absence of any coupling to the surroundings). The strong global chaotic behaviour as shown in Figs. 1 and 3 has therefore its origin in the thermal fluctuations of the reservoir. In other words the chaotic behaviour or its enhancement is exclusively due to thermal fluctuations from the surroundings, which becomes appreciable even in the physical limit of weak damping and slow diffusion of the thermal noise within the semiclassical description. We have checked this assertion for some other values of the initial conditions for the system oscillator.

The reduction of the system-reservoir Hamiltonian description [H in Eq.1] for a dissipative quantum system to an auxiliary Hamiltonian description [H_{eff} in Eq.9] effectively reduces the infinite-degree-of-freedom system to a two-degree-of-freedom system where the auxiliary degree of freedom characterized by X and P owe their origin in the fluctuations of the reservoir. Since X and P appear as the multiplicative factors in the auxiliary Hamiltonian H_{eff}, the weak thermal noise limit makes H_{eff} a vanishing Hamiltonian. The observed semiclassical chaos may therefore be regarded as a dynamical manifestation of the interplay of nonlinearity and thermal fluctuations.

In this paper we have examined the weak thermal noise limit of a semiclassical dissipative nonlinear system. We have shown that the vanishing Hamiltonian method can be suitably extended to follow the phase space trajectories of fluctuations of c-number variables which exhibit dynamical chaos. In view of the accessibility of the model to analogue electronic circuits [1] we believe that the results discussed bear further experimental relevance in the semiclassical context.

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FIG. 1. Plot of x vs p on the Poincare surface of section ($X = 0$) for Eq.(11) with initial condition $x = -2.512$, $p = 0$, $X \to 0$, $P = 0$, $\gamma = 0.01$ $\eta = 1$. (Units are arbitrary).

FIG. 2. Same as in Fig.1 but for $\eta = 0$.

FIG. 3. Same as in Fig.1 but for $x = -2.49$.

FIG. 4. Same as in Fig.3 but for $\eta = 0$.

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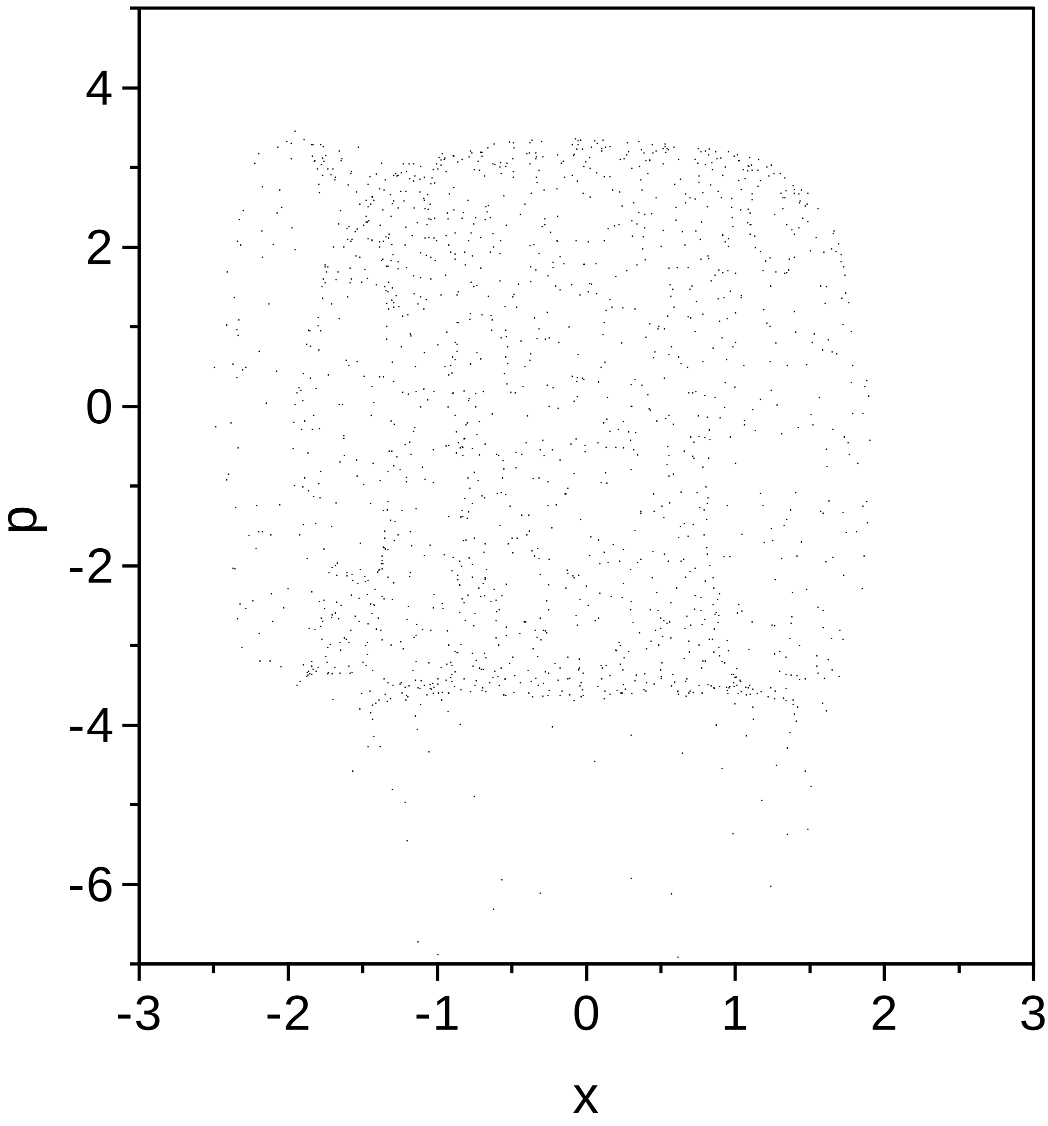

Fig. 1

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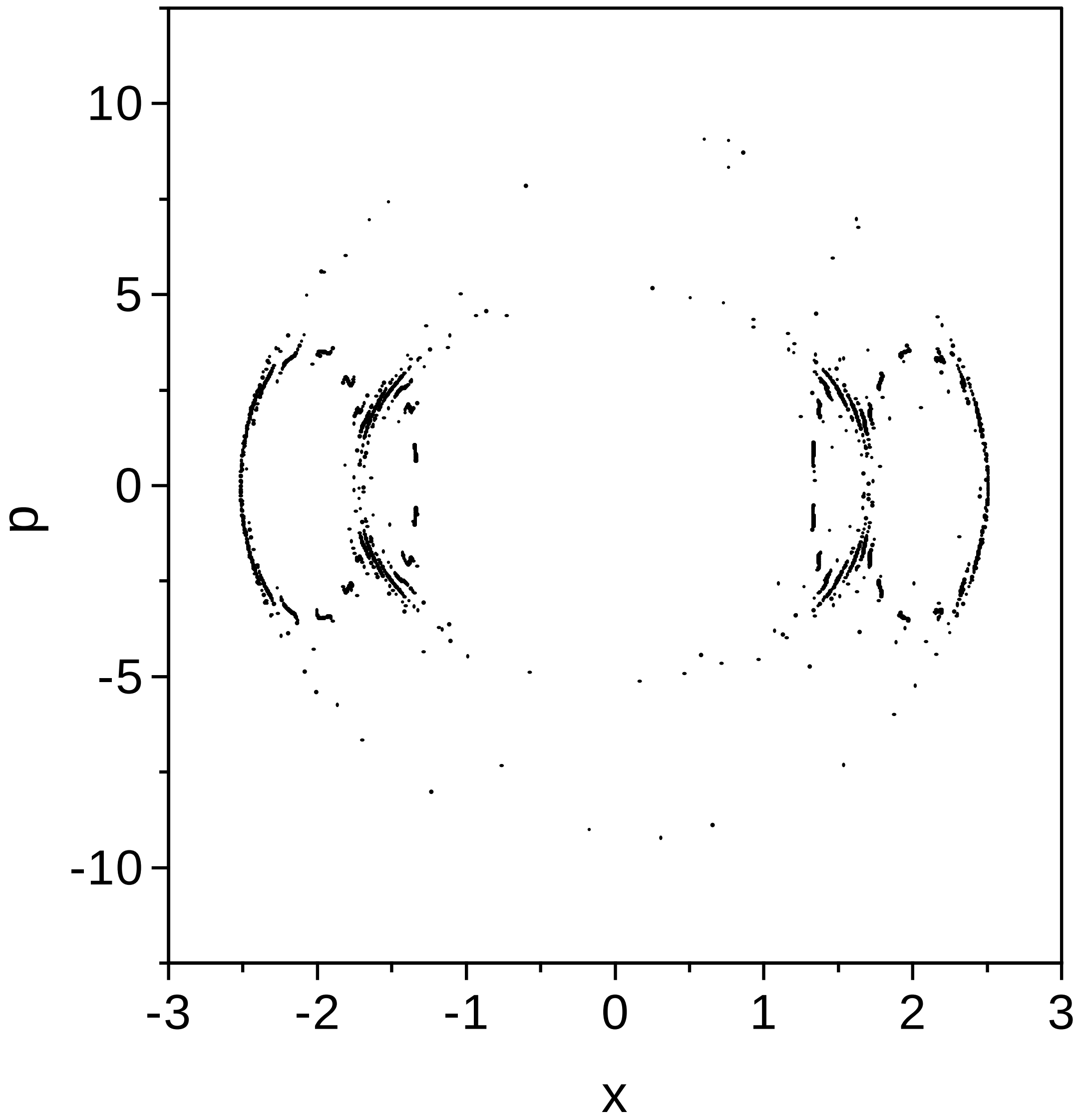

Fig. 2

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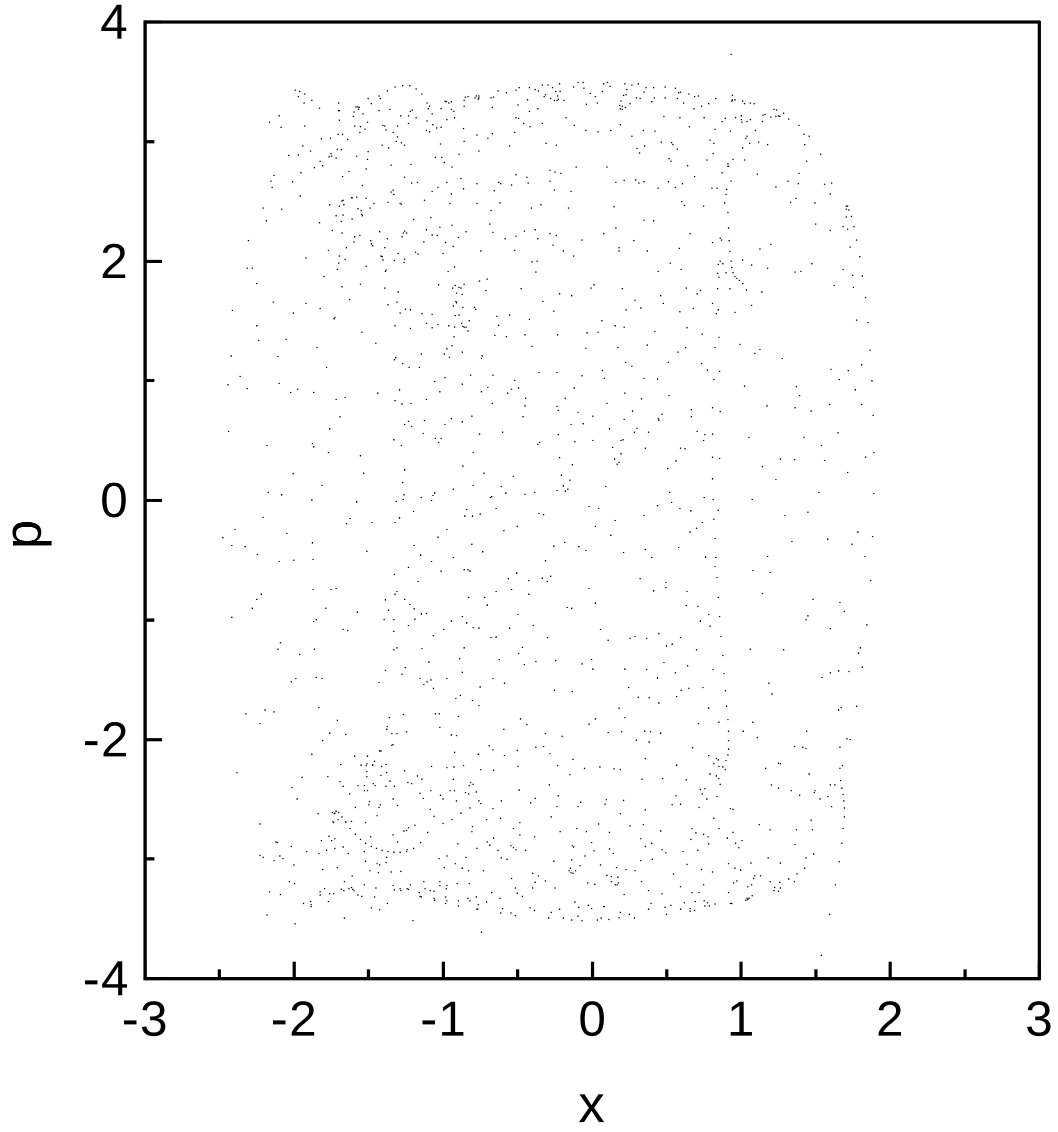

Fig. 3

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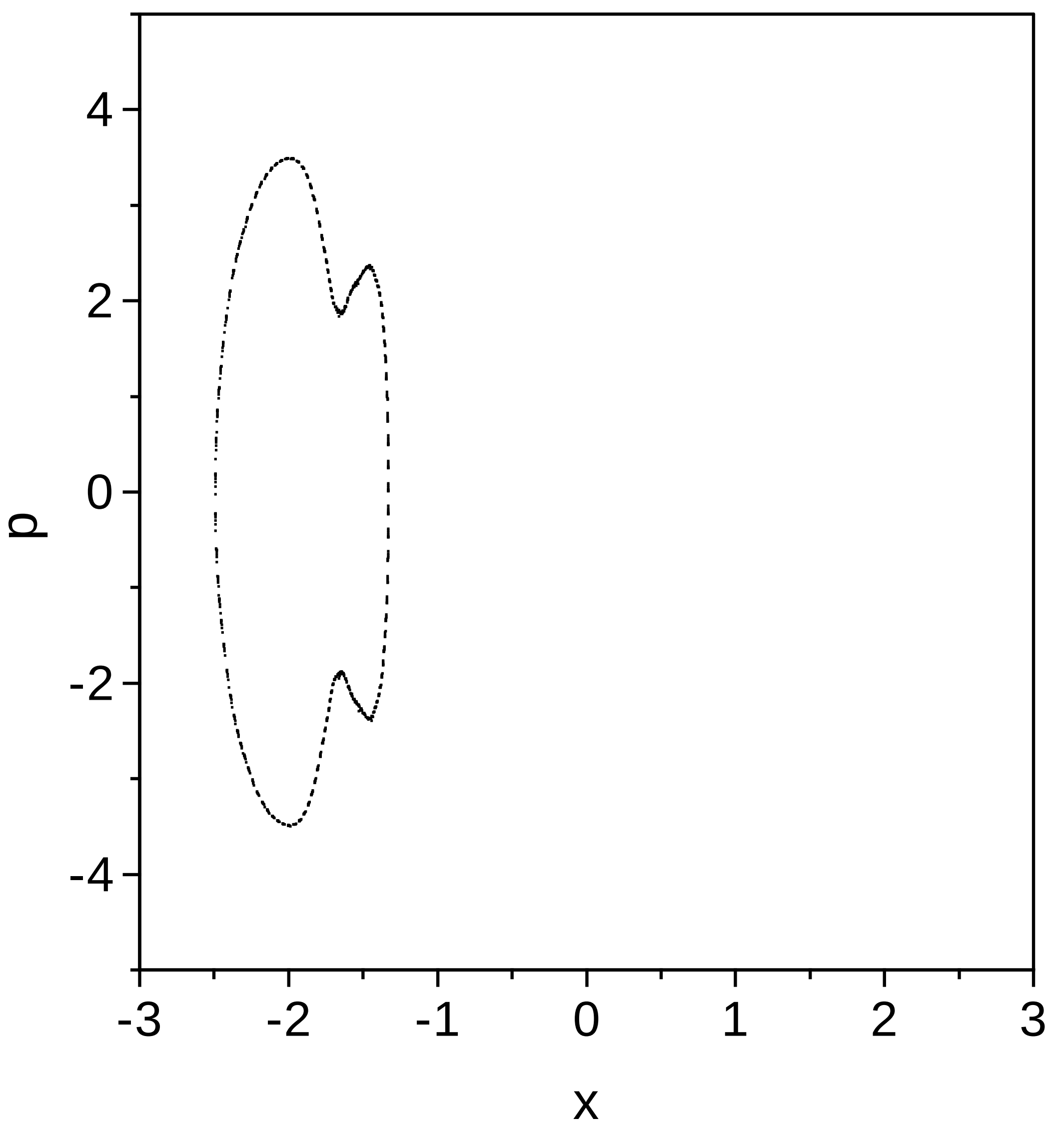

Fig. 4