A semiclassical theory of quantum noise in open chaotic systems

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Abstract

We consider the quantum evolution of classically chaotic systems in contact with surroundings. Based on $\hbar$-scaling of an equation for time evolution of the Wigner's quasi-probability distribution function in presence of dissipation and thermal diffusion we derive a semiclassical equation for quantum fluctuations. This identifies an early regime of evolution dominated by fluctuations in the curvature of the potential due to classical chaos and dissipation. A stochastic treatment of this classical fluctuations leads us to a Fokker-Planck equation which is reminiscent of Kramers' equation for thermally activated processes. This reveals an interplay of three aspects of evolution of quantum noise in weakly dissipative open systems; the reversible Liouville flow, the irreversible chaotic diffusion which is characteristic of the system itself, and irreversible dissipation induced by the external reservoir. It has been demonstrated that in the dissipation-free case a competition between Liouville flow in the contracting direction of phase space and chaotic diffusion sets a critical width in the Wigner function for quantum fluctuations. We also show how the initial quantum noise gets amplified by classical chaos and ultimately equilibrated under the influence of dissipation. We establish that there exists a critical limit to the expansion of phase space. The limit is determined by chaotic diffusion and dissipation. Making use of appropriate quantum-classical correspondence we verify the semiclassical analysis by the fully quantum simulation in a chaotic quartic oscillator.

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I. Introduction

Dissipation in quantum dynamical system has been one of the most intriguing issues in physics. Although much of our understanding of dissipative linear systems [1] forms a well-developed body of literature by now, the interplay of nonlinearity and dissipation [4-11] has drawn major attention in recent years, particularly in the problems relating to macroscopic quantum tunneling [2], multiphoton dissociation dynamics of molecules [3], quantum decoherence [4] etc. The class of nonlinear systems which are classically chaotic offers a good opportunity in this context, to understand the role of dissipation in quantum-classical correspondence [5]. The coupling of the system with its surroundings induces exchange of energy between them resulting in dissipation of energy of the system. This openness also imparts classicality in the quantum system to the extent that quantum localization in a classically chaotic system gets suppressed. The subject was analyzed early by Dittrich and Graham [5] on the basis of a quantized standard map. In an another issuse Cohen [5] considerd the problem of localization in the quantum kicked rotator model leading to nontrivial dynamical correlations where quantum chaos has an interesting bearing on the destruction of coherence. Quantum decoherence in the context of quantum-classical correspondence in several model systems, e. g. , circle and stadium billiards has been the subject of further investigation [4].

More recently a number of numerical experiments have demonstrated [6-8] that the initial growth of quantum variances of the canonical dynamical variables such as, position or momentum for a classically chaotic trajectory is exponential in nature. This has been identified as a typical signature of classical chaos on a generic quantum dynamical feature, or more precisely, a semiclassical manifestation of classical chaos. The manifestation of the classical chaotic spreading in the initial phase of a dissipative quantum dynamics has been treated recently in a general and elegant manner by Pattanayak and Brumer [11]. We have shown earlier that the fluctuation in the curvature of the potential [8-10] in the dynamical system, is amenable to a stochastic description in terms of the theory of multiplicative noise. The origin of classical instability and early divergence of quantum variances can be traced back to the correlation functions of fluctuations of the curvature of the classical potential. The present study focuses on two specific issues ; firstly we address the problem of evolution of quantum noise in an open system at the semiclassical level which identifies the interplay of three distinct aspects of evolution, e. g. , (a) deterministic Liouville flow (b) irreversible chaotic diffusion and (c) irreversible dissipation due to the external surroundings. Second, we explore the role of dissipation in the ultimate equilibration of the quantum noise in presence of classical chaos. Based

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on Wigner's quantum-classical correspondence we have derived the appropriate Fokker-Planck equation where the drift and diffusion terms have their origin in the dynamical properties of the fluctuations of the curvature of the classical potential and dissipation due to the coupling of the chaotic system to the surroundings. Our results show how the initial quantum noise gets amplified by chaotic diffusion and ultimately settles down to equilibrium with the passage of time under the influence of dissipation. In the dissipation-free case, an interplay of reversible Liouville flow and chaotic diffusion sets a limit on the width of Wigner function undergoing evolution. We also establish that there exists a critical limit to the expansion of phase space determined by dissipation and chaotic diffusion. A detailed analysis of classical and quantum mechanical calculations on a driven quartic double-well model has been carried out for numerical verification of our semiclassical analysis. We point out that the dissipative quantum dynamics of a similar system has recently been treated in a technically related approach by Dittrich, Oelschlaegel and Hänggi [12].

The organization of the paper is as follows ; In Sec. II we provide a background for quantum evolution of a classically chaotic system in presence of dissipation and thermal diffusion in the density matrix picture. A c-number formulation of the equation is described in Sec. III in terms of Wigner's quasi-classical probability function. $\hbar$-scaling of the equation of motion is then carried out to derive a semiclassical equation which identifies an early stage of evolution dominated by dissipation and curvature of the potential. Based on a cumulant expansion in $\alpha\tau_c$ where α is the strength and τ_c is the correlation time of fluctuations a Fokker-Planck equation is formulated. In Sec. IV we identify three distinct aspects of evolution in terms of a generic model driven quartic oscillator and analyze their interplay in two distinct situations. In Sec. V, the theoretical results have been compared with detailed fully quantum mechanical calculations. The paper is concluded in Sec. VI with a summary of the main results.

II.   Quantum dynamics in presence of dissipation and thermal diffusion

To study [1] the evolution of a quantum system in presence of weak dissipation and thermal diffusion from the reservoir modes we first consider the Hamiltonian of an N-degree-of-freedom system H_0.

$$H_0 = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(\{x_i\}) \ , \ \ i = 1 \cdots N \tag{1}$$

where $\{x_i, p_i\}$ represents the coordinates and momenta of the N-degree-of-freedom system.

The bare system is now coupled to an environment modeled by a reservoir of harmonic oscillator modes. The generation of quantum dynamics is given by the overall Hamiltonian operator for the system and the environment and the coupling

$$H = H_0 + \hbar \sum_i \omega_i b_i^\dagger b_i + \hbar \sum_i \left[K(\omega_i) b_i + K^*(\omega_i) b_i^\dagger \right] x \quad , \tag{2}$$

where x and p are position and momentum operator corresponding to a selected degree of freedom of the system; $b_i (b_i^\dagger)$ denotes the annihilation (creation) operator of the harmonic oscillator bath modes. The second and third terms correspond to reservoir modes and their linear coupling to the chaotic system. $K(\omega_i)$ is a c-function.

It is convenient to invoke the rotating wave approximation (RWA) so that one can use a symmetric coupling of the type $(b_i a^\dagger + b_i^\dagger a)$, where a and $a^\dagger$, are annihilation and creation operators corresponding to the system operator co-ordinate $x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger)$; ω refers to the frequency of the harmonic oscillator on the basis of which quantum calculations are performed as described in the latter part of the text.

Appropriate elimination of reservoir modes in the usual way, using Born and Markov approximations leads us to the following reduced density matrix equation for the evolution of the system [1]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] + \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + D(a^\dagger \rho a + a\rho a^\dagger - a^\dagger a\rho - \rho aa^\dagger) \tag{3}$$

Here the spectral density function of the reservoir is replaced by a continuous density $g(\omega)$ and we denote Boltzmann constant by k and $\gamma > 0$ is the limit of $2\pi|K(\omega)|^2 g(\omega)/\omega$ as $\omega \to 0_+$ and is assumed to be finite. γ is the relaxation or dissipation rate, $D(= \bar{n}\gamma)$ is the diffusion coefficient and $\bar{n}(= [exp\left(\frac{\hbar\omega}{kT}\right) - 1]^{-1})$ is the average thermal photon number of the reservoir. The terms analogous to Lamb and Stark shifts have been neglected.

The first term in Eq. (3) corresponds to the dynamical motion of the system that generates Liouville flow. The terms containing γ arise due to the interaction with the surroundings. The first term implies the loss of energy from the system to the reservoir, while the last term indicates the diffusion of fluctuations of the reservoir modes into the system of interest. The last term is responsible for quantum decoherence processes. In the limit $T \to 0$ the diffusion term in Eq. (3) vanishes, whereby the system decays primarily due to purely quantum noise.

We now make a few remarks on the approximations involved in Eq. (3) and its range of applicability in numerical simulation of full quantum dynamics as carried out in Sec. V.

(i) Since the system-reservoir dynamics as governed by the operator master Eq. (3) is

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based on Born-Markov approximation [the correlation time of the reservoir must be very short (Markov) for the interaction between the system and the reservoir to be sufficiently small (Born/weak coupling)], the underlying stochastic process due to the reservoir is *Markovian by construction*. We mention here that this has nothing to do with the stochasticity due to classical chaos which results in fluctuations of the curvature of the of the potential. This fluctuation has to be taken care of at a *non-Markovian* level of description because of its finite (but short) correlation time. While we note that there is a vast body of literature in condensed matter and chemical physics dealing with finite response time of the reservoir, which results in frequency dependence of friction coefficient γ, these and the related aspects of dissipative dynamics are outside the scope of Eq. (3). Our approach here is similar to that of Graham et. al. [5] in this regard.

(ii) Eq. (3) because of Born approximation is valid for weak damping case. It is necessary to take care of this limitation by choosing small values of γ while varying it in carrying out numerical simulation of the quantum master equation Eq. (3).

(iii) It must also be noted that γ and D terms in Eq. (3) are valid if the system operators pertain to a harmonic oscillator. When the system is nonlinear, as the present case, the usual practice is to add the additional contribution $-i[H_{non}, \rho]$ to the master equation [in the language of Fokker-Planck description this commutator, in general, contributes higher (third or more) order derivatives of the distribution] and to assume that the dissipative terms remain unaffected by the addition of commutator term, H_{non} being the nonlinear part of the Hamiltonian. The validity of this assumption was examined [3] earlier by Haake et. al. and also by us. It is now known that this assumption is quite satisfactory within the perview of weak damping and/or high temperature limit.

We note that Eq. (3) is a popular form of the operator master equation, as derived by Louisell [1], which is widely used in quantum optics. This equation had also been applied earlier by Graham et. al. [5] in the treatment of dissipative standard map and related problems of chaotic dynamics by others [13] . The correlation between different forms of operator master equations has been reviewed in Ref [2]. All of them, however, are not well-suited for numerical simulations. Eq. (3) suits this purpose well. We shall return to this issue in Sec. V to verify the theoretical propositions.

III. Semiclassical dynamics

A. $\hbar$-scaling and the semiclassical equation

Our next task is to go over from a full quantum operator problem to an equivalent 'classical' or more appropriately c-number problem described by the same Hamiltonian (2). Over the years the standard strategy of analysis of quantum-classical correspondence is the quasi-classical distribution function of Wigner[14], which is defined in phase space $\{x_i, p_i\}$ as follows;

$$
\begin{aligned}
W(\{x_i\}, \{p_i\}) &= \frac{1}{(\hbar\pi)^n} \int \cdots \int d\xi_1 \cdots d\xi_n \psi^\dagger(\{x_i + \xi_i\}) \\
&\times \psi(\{x_i - \xi_i\}) \exp\left[\frac{2i}{\hbar}\left(\sum_i p_i \xi_i\right)\right] \quad,
\end{aligned}
$$

where $\psi(x)$ refers the quantum wave function of the N-degree-of-freedom system.

The time evolution of Wigner function W of the dynamical system in presence of dissipation is now given by,

$$
\begin{aligned}
\frac{\partial W}{\partial t} &= \sum_{i=1}^{N}\left[-\frac{p_i}{2m_i}\frac{\partial W}{\partial x_i} + \left(\frac{\partial V}{\partial x_i}\right)\frac{\partial W}{\partial p_i}\right] \\
&+ \sum_{n_1 + n_3 + \cdots + n_N \ is \ odd \ and \ > 1} \left(\frac{\partial^{n_1 + \cdots n_N} V}{\partial x_1^{n_1} \cdots \partial x_N^{n_N}}\right)\frac{\left(\frac{\hbar}{2i}\right)^{n_1 + \cdots + n_N - 1}}{n_1! \cdots n_N!} \\
&\times \frac{\partial^{n_1 + \cdots + N_N}}{\partial p_1^{n_1} \cdots \partial p_N^{n_N}} W + 2\gamma\frac{\partial}{\partial p}pW + D\frac{\partial^2 W}{\partial p^2} \quad.
\end{aligned} \tag{4}
$$

The first term is the usual Poisson bracket which generates the Liouville flow. Both the Poisson bracket and the higher derivative terms result from an expansion of the Moyal bracket on the basis of an analytic $V(x)$. The last two terms are due to dissipation and diffusion induced by the external reservoir. It is important to note that the failure of correspondence between classical and quantum dynamics is predominantly due to higher derivative terms [14] which make their presence felt roughly beyond the Ehrenfest regime.

The above equation (4) is a full quantum mechanical equation as derived by Caldeira and Leggett [2]. The primary reasons for choosing Eq.(4) as our starting point for semiclassical analysis are: (i) in deriving Eq.(4) the rotating wave approximation (RWA) in the system-reservoir coupling has not been made. Had RWA been used Eq.(4) would have contained

additional contribution of terms such as $\frac{\partial x W}{\partial x}$ and $\frac{\partial^2 W}{\partial x^2}$. (ii) Eq.(4) is also free from Born approximation (or weak coupling approximation) ensuring that the theory is valid even in strong damping limit in contrast to Eq.(3) whose validity is restricted only to the weak damping regime. For a comparison over the entire range of dissipation one needs, however, other kinds of master equation which are free from weak coupling. Unfortunately, as we have already pointed out, most of them are not well-suited for numerical implementation. (iii) Eq.(4) reaches the correct classical limit when $\hbar \to 0$ and D reduces to thermal diffusion coefficient in the high temperature limit. Eq.(4) is thus likely to serve as a good description in our semiclassical analysis.

In the next two steps we invoke the symplectic structure of the Hamiltonian dynamics by defining

$$
z_i = \begin{cases} x_i & \text{for } i = 1 \cdots N \\ p_{i-N} & \text{for } i = N+1, \cdots 2N \ , \end{cases} \tag{5}
$$

and introduce the scaling of z_i in analogy to van Kampen's Ω - expansion as

$$
z_i = z_i(t) + \hbar^{1/2} \eta_i \tag{6}
$$

where

$$
\begin{aligned}
\eta_i &= \mu_i & \text{for } i = 1, \cdots N \\
&= \nu_{i-N} & \text{for } i = N+1 \cdots 2N.
\end{aligned}
$$

η-s refer to quantum noise variables in co-ordinate (μ_i) and momentum (ν_i). The equation of motion for quantum fluctuation distribution function $\phi\,(\eta, t)$ is given by (for details we refer to [8])

$$
\frac{\partial \phi}{\partial t} = [-\mathbf{F}(t) \cdot \nabla + 2N\gamma]\phi \ , \tag{7}
$$

where $\mathbf{F}(t) = \underline{J}(t)\eta - 2\gamma \underline{K}\eta$; ∇ refers to differentiation with respect to the components of η. $\underline{K}$ is a $2N \otimes 2N$ matrix defined as $k_{ij} = 0$ (for $i \neq j$), $k_{ii} = 0$ for $i = 1 \cdots N$ and $k_{ii} = 1$ for $i = N + 1 \cdots 2N$. $\underline{J}(\text{t})$ contains the second derivative of the potential as defined in the earlier paper [8].

We now make two important comments; (i) $\hbar$-scaling leads to a semiclassical description where the terms of higher powers of $\hbar$ have been left out and in the process we identify a

stage of early evolution only influenced by $\underline{\mathrm{J}}$ and γ (but not by thermal diffusion D). (ii) a key-point in determining the stability of motion rests on the jacobian matrix (or curvature of the potential) $\underline{\mathrm{J}}$ be it regular or chaotic. Since it depends explicitly on z_i (i. e. , x_i and p_i) it is a function of time and not a constant.

At this point we adapt the theory for the case when the trajectories in question are chaotic in nature. Thus, we consider a fully developed strong chaos such that the measure of regular region is sufficiently small so that $\mathbf{F}(t)$ which is governed by classical chaotic fluctuations in the curvature of the potential can be treated as a stochastic process.

Second, we *do not make any a priori assumption about the nature of the stochastic process* $\mathbf{F}(t)$. The special cases, such as, noise is Gaussian or Markovian or δ-correlated etc. have attracted so much attention in the literature that it is necessary to emphasize that these approximations have not been made. Eq.(7) may therefore be regarded as a stochastic differential equation with multiplicative noise.

B. A Fokker-Planck equation for probability distribution function of the quantum fluctuations

Our next task is to find out a differential equation whose average solution is given by $\langle \phi \rangle$ where the stochastic averaging has to be performed over the classical noise due to chaos. We denote, $\langle \phi \rangle = P(\eta, t)$ which defines a probability distribution of quantum fluctuation variables $\{\eta\}$ at time t. To this end we note that $\mathbf{F}(t) \cdot \nabla$ can be partitioned into two parts; a constant part $\mathbf{F}_0 \cdot \nabla$ and a fluctuating part $\mathbf{F}_1(t) \cdot \nabla$. Thus we write

$$\mathbf{F} \cdot \nabla = \mathbf{F}_0 \cdot \nabla + \mathbf{F}_1 \cdot \nabla \quad . \tag{8}$$

Making use of one of the main results for the theory of linear equation of the form (7) with multiplicative noise, we derive an equation for P as given by (for details, we refer to [15]);

$$
\begin{aligned}
\frac{\partial P}{\partial t} \;=\; & \left\{ -\mathbf{F}_0.\nabla + 2N\gamma - \langle \mathbf{F}_1.\nabla \rangle + \int_0^\infty d\tau \left| \frac{d\eta^{-\tau}}{d\eta} \right| \right. \\
& \left. \langle \langle \mathbf{F}_1(\eta, t).\nabla_\tau \mathbf{F}_1(\eta^{-\tau}, t-\tau) \rangle \rangle \cdot \nabla_{-\tau} \left| \frac{d\eta}{d\eta^{-\tau}} \right| \right\} P \quad ,
\end{aligned} \tag{9}
$$

where $\left| \frac{d\eta}{d\eta^{-\tau}} \right|$ is a jacobian of transformation as defined in Refs. [8, 15].

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Eq.(9) thus takes into account of two distinct stochastic processes. One is due to the external reservoir with infinite degrees of freedom which have been eliminated and the manybody effect is incorporated through diffusion coefficient D and dissipative term γ in Eq.(4). The other one concerns the classical fluctuations of the curvature of the potential $\underline{J}$ as embodied in $\mathbf{F}_1$ terms, which is due to classical chaos. While the former process is taken into consideration within *Markovian* description the second process is *non-Markovian* because of the finite correlation time τ_c of classical fluctuations. The construction of the associated Fokker-Planck equation is based on perturbative cumulant expansion in $\alpha\tau_c$. Following van Kampen [15] we have assumed that τ_c is short compared to the average time scale over which the probability distribution function $P(\eta, t)$ evolves in time. The convergence of the expansion in $\alpha\tau_c$ thus allows us to retain upto second order terms and as such one need not go over to higher order to describe the dynamics. If one takes care of τ^2 terms the theory can be appropriately extended [15]. We also point out that since we need not invoke any a priori approximation on the nature of noise (like Gaussian or δ correlated etc.) the values of correlation functions when calculated numerically are exact in this sense.

By $\hbar$-scaling one gets rid of thermal diffusion D in Eq.(9). Thus, the semiclassical description identifies an early stage of dynamical evolution dominated by dissipation of the system due to external surroundings and diffusion of fluctuations of the curvature of the potential due to classical chaos. The theory developed so far is valid for N-degree-of-freedom chaotic systems in presence of dissipation.

IV. An illustration

A. The Fokker-Planck equation

We now turn to a simple illustration of the general equation(9) ($N = 1$ case) in terms of a low dimensional dissipative chaotic system which allows us to solve the equation for probability distribution of quantum fluctuations analytically. The model is thus expected to capture some of the essential features of evolution of quantum fluctuations in presence of dissipation. We now consider the classical motion of a particle of mass m in a potential field $V(x)$ and driven by a classical field of frequency ω_0. The Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x) + gx\cos\omega_0 t, \tag{10}$$

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with $V(x) = ax^4 - bx^2$, where first and the second terms in Eq.(10) comprise the kinetic and potential energies of the particle, respectively. The third term is the driving term which includes the effect of coupling of the system with the field as well as the strength of the field.

The classical equation of motion of the particle in presence of damping (at a rate γ) are

$$\dot{x} = p \ ,$$
$$\dot{p} = -\gamma p - V'(x) - g \cos \omega_0 t \ . \tag{11}$$

Following the method as described in Refs. [8] and [15] the master equation (9) in the case of this model chaotic system can then be written down. This is

$$\frac{\partial P(\eta_1, \eta_2, t)}{\partial t} = \left[-\frac{\eta_2}{m} \frac{\partial}{\partial \eta_1} - \{(2b + c + c_2)\eta_1 - 2\eta_2\gamma\} \frac{\partial}{\partial \eta_2} \right.$$
$$\left. + 2\gamma + \left\{ c_2 \eta_1^2 \frac{\partial^2}{\partial \eta_2 \partial \eta_1} + \eta_1^2 c_1 \frac{\partial^2}{\partial \eta_2^2} - \frac{\eta_1 \eta_2}{m} c_2 \frac{\partial^2}{\partial \eta_2^2} \right\} \right] P(\eta_1, \eta_2, t) \ , \tag{12}$$

where

$$c = \langle \zeta(t) \rangle \ ,$$
$$c_1 = \int_0^\infty \langle\langle \zeta(t)\zeta(t - \tau) \rangle\rangle e^{-2\gamma\tau} d\tau \ ,$$
$$c_2 = \int_0^\infty \langle\langle \zeta(t)\zeta(t - \tau) \rangle\rangle e^{-2\gamma\tau} \tau d\tau \ . \tag{13}$$

where $\zeta(t) = 12ax^2$ represents the fluctuating part of the curvature of the potential $V(x)$. The above equation (12) is a Fokker-Planck equation for probability distribution of quantum fluctuations for the model chaotic dissipative system. It is evident that stochastic averaging over classical chaos leads us to the above equation and the correlation functions contained in c, c_1 and c_2. The correlation of fluctuations of curvature of the classical potential thus determines the drift and diffusion terms of the Fokker-Planck equation. The appearance of the variables η_1, η_2 in the diffusion terms precludes the possibility of an exact solution of this equation. One thus takes resort to weak noise approximation scheme (this is consistent with the assumption that fluctuations are not too large) under which the diffusion terms are assumed to be constant which are given by

$$A' = \eta_1^2(0)c_1 - \eta_1(0)\eta_2(0)c_2 \ ,$$
$$B = \eta_1^2(0)c_2 \ . \tag{14}$$

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Zeroes in η_1 and η_2 refer to their initial values corresponding to the initial preparation of the coherent wave packet.

We now use the abbreviation $2b + c + c_2 = \omega'^2$ and put $m = 1$ for the rest of the treatment. The Fokker-Planck equation can then be written in a more compact form as follows;

$$\begin{aligned}
\frac{\partial P(\eta_1, \eta_2, t)}{\partial t} &= \left[-\eta_2 \frac{\partial}{\partial \eta_1} - \omega'^2 \eta_1 \frac{\partial}{\partial \eta_2} + 2\gamma + 2\gamma \frac{\partial}{\partial \eta_2} \eta_2 \right. \\
&+ \left. A' \frac{\partial^2 \langle \phi \rangle}{\partial \eta_2^2} + B \frac{\partial^2 \langle \phi \rangle}{\partial \eta_1 \partial \eta_2} \right] P(\eta_1, \eta_2, t) \quad .
\end{aligned} \tag{15}$$

The above Fokker-Planck equation which governs the evolution of distribution of quantum fluctuations η_1 and η_2 corresponding to co-ordinate and momentum variables, respectively, in presence of dissipation has a formal similarity in structure to Kramers' equation [16] which describes the Brownian motion of a particle in phase space. While the stochasticity in Kramers' equation originates from the thermal fluctuations derived from the true statistical properties of the reservoir which is a many body system, the stochasticity in the present problem owes its origin to the dynamical properties of classical chaos in a low dimensional system.

Eq.(15) clearly illustrates three distinct aspects of evolution; (i) in the absence of γ and A' (and B) the evolution can be mapped into a purely deterministic and reversible Liouville flow under an inverted harmonic potential $-\frac{\omega'^2 \eta_1^2}{2}$. Note the ω' is essentially $2b$ (which is a parameter in the potential of the Hamiltonian (10)) appropriately modified by the average and correlation function of the fluctuations of the second derivative of the potential in (10). (ii) the reservoir-induced irreversible dissipation (γ) and (iii) the irreversible diffusion (A' and B) caused by the fluctuations of the jacobian or curvature of the potential (this diffusion is chaotic diffusion and is characteristic of the nonlinear system in question and not be confused with thermal diffusion D due to external reservoir). The structure of A' and B suggests that B in Eqs.(13) and (14) vanishes when $\tau_c \to 0$, and B can be identified as a non-Markovian small contribution [15], A' being the dominant Markovian part.

We shall now demonstrate in the following two sections (B) and (C) the interplay of three above-mentioned aspects of evolution in two distinct situations: (a) We first neglect the dissipative term by letting γ approach zero in Eq.(15) and consider a competitive effect between the reversible evolution and the chaotic diffusion. To this end we closely follow the analysis of Zurek and Paz [17] to establish that there exists a critical width of Wigner function for quantum noise determined by chaotic diffusion A' and ω'. (b) We then take full account of the dissipative terms by letting γ finite but small to show that how quantum noise gets amplified by chaotic diffusion in the early stage and then ultimately settles down to equilibrium under the

influence of dissipation. The equilibrium is characterized again by a critical width of Wigner function determined by chaotic diffusion and dissipation.

B. Reversible evolution of quantum fluctuations and chaotic diffusion

In order to study the interplay of reversible evolution of quantum fluctuations (due to Liouville flow expressed through 'reversible' operator L_{rev} as shown below) and chaotic diffusion (expressed in terms of irreversible contribution L_{irr}), we now rewrite the Fokker-Planck Eq.(15) (neglecting the γ-term) in the following form:

$$\frac{\partial P}{\partial t} = L_k P \ , \tag{16}$$

where

$$L_k = L_{\mathrm{rev}} + L_{\mathrm{irr}} \ , \tag{17}$$

$$L_{\mathrm{rev}}(\eta_1, \eta_2) = -\frac{\partial}{\partial \eta_1}\eta_2 + f'(\eta_1)\frac{\partial}{\partial \eta_2} \tag{18}$$

and

$$L_{\mathrm{irr}}(\eta_1, \eta_2) = A'\frac{\partial^2}{\partial \eta_2{}^2} + B\frac{\partial^2}{\partial \eta_1 \partial \eta_2} \tag{19}$$

Here

$$f(\eta_1) = -\frac{\omega'^2 \eta_1{}^2}{2} \tag{20}$$

is the inverted parabolic potential, since ω'^2 is expressed as

$$\omega'^2 = 2b + c + c_2 \ , \tag{21}$$

where $2b$ is associated with the saddle point of the quartic potential V in the Hamiltonian (10). ω' is the effective frequency afforded by the unstable inverted harmonic potential and is dressed by the average (c) and the correlation function (c_2) of fluctuations of the curvature of the potential V. It is important to realize that locally a chaotic flow pattern is similar to that around the saddle point from which both the stable and unstable manifolds emanate. The two second derivative terms in L_{irr} operator denote the chaotic diffusion of which A' term is the dominant Markovian contribution whereas B is a small non-Markovian addition. In the limit when the fluctuation is fast enough such that τ_c approaches zero, B vanishes [15].

To analysis the underlying reversible Liouville flow due to the effective unstable oscillator, it is now convenient to follow Zurek and Paz [17] and use contracting and expanding co-ordinates corresponding to stable and unstable directions of the flow, respectively, as follows:

$$
\begin{aligned}
r &= \eta_2 - \omega' \eta_1 \ , \\
s &= \eta_2 + \omega' \eta_1 \ .
\end{aligned}
$$
(22)

The flow generated by Eq.(16) (in absence of diffusion) causes exponential contraction in r and expansion in s. Expansion in s results in decrease in gradient in that direction. We neglect [17] the gradient along that direction to obtain the effective evolution of the Wigner function for quantum fluctuations, after sufficient number of stretching and folding,

$$
\frac{\partial P}{\partial t} = L_{\text{eff}}(r, s) P
$$
(23)

where

$$
L_{\text{eff}}(r, s) = \omega' \left(r \frac{\partial}{\partial r} - s \frac{\partial}{\partial s} + \frac{1}{2} \sigma_c^2 \frac{\partial^2}{\partial r^2} \right) \ .
$$
(24)

Here σ_c^2, the effective dispersion is given by

$$
\begin{aligned}
\sigma_c^2 &= 2 \left(\frac{A'}{\omega'} - B \right) \\
&\simeq \frac{2A'}{\omega'} \quad \text{(for B small)}
\end{aligned}
$$
(25)

Eq.(23) can be solved by considering the problem as eigenvalue equation [20] for the operator appearing on the right hand side of (23), i. e. ,

$$
L_{\text{eff}} P = \lambda P \ .
$$
(26)

L_{eff} can be cast into the form of an Hamiltonian for two interacting Boson operators [20]. The relevant eigenfunctions are $s^n F_m(\frac{r}{\sigma_c})$, where $F_m(x) = \exp(-\frac{x^2}{2}) H_{m-1} \left(\frac{x}{\sqrt{2}} \right)$ and $H_m(x)$ are Hermite polynomials of order m.

The corresponding eigenvalues are $-(n + m + 1)\omega'$; n and m being positive real numbers. (The constant term ω' in the eigenvalues has no dynamical significance).

The general solution of Eq.(23) is then given by

$$
P(r, s, t) = \sum_{\substack{n \geq 0 \\ m \geq 1}} C_{nm} s^n F_m(r) \exp -(n + m)\omega' t
$$
(27)

Taking into consideration [17] that P expands in s-direction through $s = s_0 \exp \omega' t$ and major contribution to the sum (27) can be attributed to $m = 1$ term we find that P approaches the Gaussian with a critical width σ_c as follows:

$$P \simeq \frac{1}{\sqrt{2\pi\sigma_c{}^2}} \exp(-\frac{r^2}{2\sigma_c{}^2}) \exp(-\omega' t) \int_{-\infty}^{+\infty} ds P(r_0, s_0, t = 0) \tag{28}$$

It is important to note that the existence of a critical width σ_c (expressed as $\frac{2A'}{\omega'}$) reveals an important interplay of evolution of quantum fluctuations through reversible dynamics (ω') and chaotic diffusion (A') which results due to correlation of fluctuations of the curvature of the potential. We emphasize once again that the chaotic diffusion coefficient A' (and B) is not to be confused with the usual thermal diffusion coefficient D that appeared in Eq.(4). While the analysis of Zurek and Paz [Z P] essentially considers a competition arising out of an interplay of reversible dynamics of the system and thermal diffusion due to surroundings, the present analysis in this section does not take into account of any external influence. This is an important point of departure from the analysis of Z P. Both quantum noise around the classical path following the reversible dynamics and chaotic diffusion due to the correlation of fluctuations of the curvature of the classical potential concern the system itself. It is apparent that the competition of evolution of quantum fluctuation (in the contracting direction wave packet gets squeezed) and chaotic diffusion (which causes the spread) as inherent in the expression for critical width provides a self-regulatory feature in the dynamics. This is characteristic of the chaotic system as demonstrated here and is not due to any external influence.

C. Evolution of quantum noise and its approach to equilibrium

We now switch on the dissipative term by letting $\gamma \to$ finite but small and show how quantum noise gets amplified by intrinsic classical stochasticity at early stage and then reaches a steady state under the influence of dissipation. We thus take into consideration of all the three aspects of evolution as identified earlier in the full Fokker-Planck equation (15).

We begin by applying the following transformations to fluctuation variables η_1, η_2 corresponding to position and momentum:

$$\begin{aligned} \bar{\eta}_1 &= \sqrt{\omega}\eta_1 \quad, \\ \bar{\eta}_2 &= \frac{1}{\sqrt{\omega}}\eta_2 \quad. \end{aligned} \tag{29}$$

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We then let

$$\begin{aligned}
\beta &= \bar{\eta}_1 + i\bar{\eta}_2 \quad, \\
\beta^* &= \bar{\eta}_1 - i\bar{\eta}_2 \quad.
\end{aligned} \tag{30}$$

Then the appropriate transformation of derivatives in Eq.(15) yields the following Fokker - Planck equation corresponding to Eq. (15).

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \left(-\frac{\beta\omega}{2i} + \frac{\beta^*\omega}{2i} - \frac{i\omega'^2\beta}{2\omega} - \frac{i\omega'^2\beta^*}{2\omega} + \gamma\beta - \gamma\beta^* \right)\frac{\partial p}{\partial \beta} \\
&+ \left(-\frac{\beta\omega}{2i} + \frac{\beta^*\omega}{2i} + \frac{i\omega'^2\beta}{2\omega} + \frac{i\omega'^2\beta^*}{2\omega} - \gamma\beta + \gamma\beta^* \right)\frac{\partial p}{\partial \beta^*} \\
&+ 2A\frac{\partial^2 p}{\partial\beta^*\partial\beta} + 2\gamma + (-A+iB)\frac{\partial^2 p}{\partial\beta^2} + (-A-iB)\frac{\partial^2 p}{\partial\beta^{*2}} \quad,
\end{aligned} \tag{31}$$

where for notational convenience we have used

$$\begin{aligned}
A &= \frac{A'}{\omega} \quad, \\
P(\bar{\eta}_1, \bar{\eta}_2, t) &= p(\beta, \beta^*, t) \quad.
\end{aligned} \tag{32}$$

We now search for the Green's function or conditional probability solution for the system at β and β^* at time t given that it had the values β' and β'^* at $t = 0$. The initial condition which is required to bring forth quantum-classical correspondence is represented by

$$p(\beta, \beta^*, t = 0) = \frac{\epsilon}{\pi}e^{-\epsilon(\beta-\beta^*)(\beta'-\beta^{*'})} \tag{33}$$

which corresponds to a coherent state. We then look for a solution of the equation (31) of the form

$$p(\beta, \beta^*, t)|\beta', \beta^{*'}, 0) = e^{G(t)}, \tag{34}$$

where

$$G(t) = -\frac{1}{\Gamma(t)}[\beta - \Omega(t)][\beta^* - \Omega(t)] + \ln\nu(t) \quad. \tag{35}$$

$G(t)$ is determined in terms of the time-varying parameters $\Gamma(t)$, $\Omega(t)$ and $\nu(t)$ which follow a set of ordinary differential equations given in Appendix A. The important relevant quantity required for the present analysis is $\Gamma(t)$ which is given by,

$$\Gamma(t) = \Gamma(0)e^{-2\gamma t} + \frac{A}{\gamma}(1 - e^{-2\gamma t}), \tag{36}$$

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The Green's function or the conditional probability solution (34) can then be employed to calculate the various theoretical quantities of which, the uncertainty in cordinate $\Delta\eta_1$ and that in momentum are obtained as follows,

$$\Delta\eta_1^2 = \langle\eta_1^2\rangle - \langle\eta_1\rangle^2 = \frac{1}{\omega}\left[\frac{\Gamma(t)}{2}\right] \quad , \tag{37}$$

$$\Delta\eta_2^2 = \langle\eta_2^2\rangle - \langle\eta_2\rangle^2 = \left[\frac{\Gamma(t)}{2}\right]\omega \quad . \tag{38}$$

We are led to two important results of this section.

(i) The uncertainty product $\Delta\eta_1\Delta\eta_2$ at any time is given by

$$\Delta\eta_1\Delta\eta_2 = \frac{1}{2}\Gamma(t) \tag{39}$$

where $\Gamma(t)$ is determined by Eq.(36) subject to initial conditions (A2). This implies that we choose $\epsilon = 1$ to satisfy the minimum uncertainty product condition for t=0, for the wave packet. Eq.(39) relates the evolution of quantum noise as a function of time in terms of $\Gamma(t)$ which by the virtue of Eq.(36) is determined by the initial condition $\Gamma(0)$ [Eq.(A2)] and the other two parameters A and γ. Note that A is the chaotic diffusion coefficient defined by Eqs.(14) and (32) [this is not to be confused with the thermal diffusion coefficient D in Eq. (3) which arises due to the interaction with the surroundings] and γ refers to the dissipation rate of the system in contact with the surroundings. Since A (and A') is related to the fluctuations of the curvature of the classical potential $\zeta(t)$ through c_2 and c_1 in Eq.(14), the origin of diffusion coefficient A is essentially the classical chaos. Eq.(39) thus illustrates how the initial quantum noise gets amplified by classical stochastisity.

(ii) The equilibrium condition is governed by the long time limit of the conditional probability function

$$P_{ss} = \lim_{t\to\infty} P(\bar\eta_1, \bar\eta_2, t|\bar\eta_1', \bar\eta_2', 0) \quad . \tag{40}$$

This is given by

$$P_{ss} = \bar\nu\exp\left[-\frac{(\bar\eta_1^2 + \bar\eta_2^2)}{A/\gamma}\right] \quad , \tag{41}$$

where

$$\bar\nu = \frac{\Gamma(0)\nu(0)\gamma}{A} \quad .$$

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Furthermore, the equilibrium condition implies that there exist a critical limit to the expansion of phase space. This is apparent from the uncertainty product relation (39) as follows;

$$\Delta\eta_1\Delta\eta_2|_{t\to\infty} = \frac{A}{\gamma} \quad . \tag{42}$$

The existence of critical limit which also appears as a width of equilibrium distribution (41) is a consequence of competition between chaotic diffusion which attempts to expand the wave packet and dissipation γ which has the opposite tendency and ultimately leads to a compromise steady state.

The relation (39) thus illustrates the effect of classical chaos on quantum fluctuations at the semiclassical level. In order to examine the initial divergence of quantum variances it is thus necessary to calculate the correlation functions in c_1 and c_2 numerically by solving the classical equations of motion (11) with specific initial conditions which admit chaos. In order to allow ourselves a fair comparison with fully quantum calculation and verify the theoretical propositions, we shall return to this issue in the next section.

V.   Numerical Calculations; Classical and Quantum

To analysis the growth of quantum fluctuations quantitatively [Eq. 39] we now consider the dissipative classical chaotic motion governed by Eq.(11). We choose the parameter values m=1, a=0.5, b=10. ω_0=6.07. The coupling constant cum field strength, g and the damping rate γ are the parameters which have been varied from set to set. These two quantities essentially determine the two competeting processes in the dynamics, e. g. , the fundamental strong coherent interaction (g) between the double-well oscillator and the external field, (strong coupling being responsible for classical chaos) and the irreversible weak decay of the oscillator (γ). Keeping in mind the approximations involved we thus consider a situation where $g >> \gamma$ (while scaled g varies over a range 12-18, the variation of scaled γ is of the order 10^{-4}-10^{-1}). In quantum mechanical terms the situation is somewhat reminiscent of a coherent regime in typical cavity quantum optical problem where the Rabi frequency far exceeds the damping rate (a typical Rabi frequency $\sim 10^{12}$/sec compared to damping rate $\sim 10^9$/sec). The parameter space in the dissipation-free version of the model is chosen from Lin and Ballentine [18]. We choose the initial condition x_0=-3.5 and p_0=0.0, which ensures strong global chaos [18]. Note that c_1 and c_2 as expressed in Eq.(13) are the integrals over the correlation function of $\zeta(t)$ [$\zeta(t)$ is the fluctuating part of the second derivative of the potential $V(x)$ and is given by $\zeta(t) = -12ax^2$].

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To calculate the correlation function $\langle\langle\zeta(t)\zeta(t-\tau)\rangle\rangle$ and the average $\langle\zeta(t)\rangle$ it is necessary to determine long time series in $\zeta(t)$ by numerically solving the classical equations of motion (11) for x and p. The next step is to carry out averaging over the time series. Since $\langle\zeta(t)\rangle$ and $\langle\langle\zeta(t)\zeta(t-\tau)\rangle\rangle$ are classical quantities to be calculated exactly and the underlying model is nonintegrable, one must have to take resort to numerical integration. Any approximation in this regard amounts to making specific assumption about the nature of the classical noise. We have already pointed out in Sec. III that a main virtue of the treatment of stochastic differential equations with multiplicative noise by van Kampen [15] is that one does not have to make any a priori assumption about the nature of the noise $\zeta(t)$. However it is essential that the time of decay of the correlation of classical quantities like $\langle\langle\zeta(t)\zeta(t-\tau)\rangle\rangle$ must not be too long (evidently the treatment of weak chaos which involves long correlation time is out of space of the present theory). The theoretical footing of our analysis thus remains intact in numerical calculation.

Once the average quantities like $\langle\zeta(t)\rangle$ and $\langle\langle\zeta(t)\zeta(t-\tau)\rangle\rangle$ are known, our next task is to calculate the integrals over time τ [in Eq.(13)] to determine c_1 and c_2. One of the primary considerations of the theory described in the earlier section is that one takes care of classical fluctuation of $\zeta(t)$ upto second order such that the correlation time τ_c is short but finite. On a coarse-grained time scale over which the distribution function $P(\eta, t)$ proceeds, the process is approximately Markovian. Numerical implementation of this near-Markovian character in our analysis rests on the fact that we consider the first fall of the correlation function to determine the cut off time for time integration in c_1 and c_2 . With these numerical values of c_1 and c_2 one obtains A and $\Gamma(t)$ of Eq.(39). In Figs 1 and 2 we plot (dotted curve) $ln\Delta\eta_1\Delta\eta_2$ as a function of time t for different values of g and γ.

For a full quantum-mechanical calculation to verify the basic theoretical propositions of semiclassical dynamics, we now return to Eq. (3). The eigenvectors $\{|n\rangle\}$ of a harmonic oscillator which satisfies $(\hat{p}^2/2m + (1/2)m\omega^2\hat{x}^2)|n\rangle = [(n + 1/2)\hbar\omega]|n\rangle$ are chosen as basis vectors to solve Eq. (3). The frequency ω is arbitrarily adjusted to economize the size of the basis set. For the present purpose we choose $\omega = 6.25$, $\hbar = 1$, and 120 basis vectors. In this representation the equation of motion for the reduced density [Eq. (3)] matrix elements is given by

$$
\begin{aligned}
\frac{d\rho_{nm}}{dt} &= -i/\hbar \left[\sum_k H_{nk}\rho_{km} - \sum_l \rho_{nl}H_{lm} \right] \\
&+ \frac{\gamma}{2}\left\{ \sqrt{(n+1)(m+1)}\rho_{n+1,m+1} - (n+m)\rho_{nm} \right\} \\
&+ D\left\{ \sqrt{(n+1)(m+1)}\rho_{n+1,m+1} - (n+m+1)\rho_{nm} \right\} \quad .
\end{aligned}
\tag{43}
$$

Here H_{mn} is as given in Ref[18]. H is given by Eq. 10.

The Eq.(43) describes the evolution of both population (diagonal elements) and coherence (off-diagonal elements). Without making any approximation regarding the separation of time scales for evolution of them, we have carried out numerical solution of 120x120 equations for density matrix elements of the reduced system as a typical initial value problem. We follow time scale of the period of driving force $T = 2\pi/\omega_0$, so that $t = \tau T$, where τ is a dimensionless quantity.

Quantum-classical correspondence is maintained through construction of minimum uncertainty wave packets $|\alpha_{x,p}\rangle$ of Gaussian form in position and momentum representations having position $\langle x \rangle$ and average momentum $\langle p \rangle$ such that

$$
\langle \alpha_{x,p}|n\rangle = \left[\exp(-0.5|\alpha|^2) \right] \frac{\alpha^n}{\sqrt{n!}} \quad ,
\tag{44}
$$

where,

$$
\alpha = \sqrt{m\omega/2}[\langle x \rangle + (i/m\omega)\langle p \rangle] \quad .
$$

The quantum evolution is followed by locating the average position and average momentum of the initial wave packet corresponding to the initial position and momentum of a classically chaotic trajectory. As a numerical check we have compared our results with those of Lin and Ballentine [18] in classical and quantum cases for D=0, γ=0. Another important check for the numerical calculation is to keep $Tr\rho = 1$ for the entire evolution.

Following Eq.(39) we plot the variation of $ln[\Delta\eta_1 \Delta\eta_2]$ ($\Delta\eta_1$ and $\Delta\eta_2$ are the quantum variances corresponding to position and momentum, respectively) as a function of time for several values of g but for a fixed γ (0.001) in Figs 1(a-c). It has already been pointed out that the major input for the theoretical quantity is the chaotic diffusion coefficient A which is further related to c_1 and c_2 , i.e., to classical correlation function of the curvature of the potential. The theoretical curves are denoted in Figs 1(a-c) by the dotted lines. It is evident that after a sharp initial growth the uncertainty product tends to settle down to some final value. The initial rate

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becomes large with the increase of g. The variation of rate constants with g is shown in table I. The theoretical analysis has been supplemented by the fully quantum mechanical calculations based on numerical integration of Eq.(43) by launching Gaussian wave packets centered around the classical position (x_0) and momentum (p_0) corresponding to the chaotic trajectory as described in the earlier part of this section. These numerical curves have been superimposed in Figs 1(a-c) for the corresponding values of g and γ (An initial flat plateau region has been cut off to make the rise more prominent [8]). It may be observed that the agreement between the theoretical and numerical curves is quite satisfactory so far as the initial growth part is concerned. The agreement is better for larger g-values. Table I also gives a relative comparison between the theoretical and numerical estimate of the rate constants.

The effect of relaxation rate γ is analyzed in Figs 2(a-c) where we have plotted $ln(\Delta\eta_1\Delta\eta_2)$ as a function time for a fixed g but for several values of γ. It is easy to see that the attainment of equilibrium significantly depends on γ. For low values of γ it takes larger time to attain the equilibrium state. To have a rough estimate of the time required to achieve equilibrium we have determined the time in which the $dln(\Delta\eta_1\Delta\eta_2)/dt$ reduces to $\approx 1\%$ of its initial value. These times have been tabulated in the table II for the theoretical and numerical curves in Figs 2(a-c). It is also apparent from the figures that with the increase of γ (within weak damping limit) the time for attainment of equilibrium gets shorter and one also finds a better agreement between the theoretical and numerical curves.

We thus observe an interesting interplay of chaotic diffusion and dissipation through A and γ. These are the two major factors that determine the initial rate of divergence (Eq.39) and approach to equilibrium (Eq.41) of quantum variances in a system which is classically chaotic. Since at very low values of γ, the approach to equilibrium gets slowed down significantly, one has to wait for a very long time for the steady state. Now if the time becomes too long then dispersion of the wave packet becomes too much so that the validity of a semiclassical analysis becomes questionable, where one takes into consideration of terms of lowest order in $\hbar$. At this point higher order derivative terms containing the potential in Eq.(4) become significant. Thus with the higher values of γ (within weak damping limit) and g one finds very good agreement between semiclassical theory and numerical analysis(e.g., Fig 2c).

Another pertinent point need be mentioned here. Apparently from the methods used one should expect that the agreement between quantum and semiclassical results improves as the damping is decreased. However in Fig.2 the opposite seems to be the case. To be more precise we first note that for smaller values of damping and short time both the results agree well. The discrepency arises, at smaller values damping and in the long time regime. It must

be understood that the full quantum simulation works in the entire time range of dynamics and is valid for weak dissipation. On the other hand the semiclassical analysis, although can be extended to large damping limit is not valid for a very long time regime since quantum effects begin to dominate. However, when damping is increased the time required to attain the equilibrium gets shorter and one finds a better agreement between semiclassical and quantum analysis (since one need not carry out the calculation of dynamics over a larger length of time when higher order quantum effects take over).

VI. Summary of the main results and conclusions

In this paper we have considered the quantum evolution of dissipative, chaotic systems whose classical limit is chaotic. Making use of appropriate $\hbar$-scaling (analogous to van Kampen's Ω-expansion) of the equation for Wigner's quasiprobability distribution function which takes into account of dissipation and thermal diffusion terms on the basis of a system - reservoir theory, we derive a semiclassical dynamical equation for probability distribution function for quantum fluctuations. The equation incorporates dissipation due to surroundings and fluctuations of the curvature of the classical potential (arising out of classical chaos) in addition to standard Liouville flow terms as the essential features of the semiclassical dynamics. Appropriate treatment of fluctuations of the curvature of the potential which is amenable to a theoretical analysis of multiplicative noise for short but finite correlation time leads us to a Fokker-Planck equation where the drift and diffusion terms have their origin in the intrinsic dynamical properties of the classical chaotic system as well as the dissipation of the system due to its contact with the surroundings. Formally, the equation is identical in structure with Kramers' equation [16] which describes a Brownian dynamics in phase space demonstrates the interplay of three distinct aspects of evolution (a) purely deterministic reversible Liouville flow (b) irreversible chaotic diffusion which is intrinsic to the system itself and (c) irreversible dissipation due to coupling of the system to external surroundings. We have corroborated our semiclassical analysis by numerical simulation of the full quantum operator master equation.

We now summarize the main conclusions of this study.

(i) $\hbar$-expansion and a stochastic treatment of the curvature of the classical potential in terms of a τ_c-expansion identify a specific stage of quantum evolution with three distinct aspects in an open system described by a Fokker-Planck equation. These are (a) reversible deterministic Liouville flow (b) irreversible chaotic diffusion which is intrinsic characteristic of the nonlinear system itself (c) irreversible dissipation of the system induced by external reservoir. This

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stage of evolution is less likely to be affected by thermal diffusion which is otherwise primarily responsible for decoherence processes.

(ii) Chaotic diffusion although an intrinsic property of the system imparts *a kind of irreversibility* in the dissipation-free evolution which has a truly deterministic origin.

(iii) In the dissipation-free case, an interplay of approximately reversible Liouville flow and chaotic diffusion sets a *critical limit* on the width of Wigner function for quantum noise undergoing time evolution.

(iv) Our results show how the initial quantum noise gets amplified by chaotic diffusion and then ultimately equilibrated with the passage of time under the influence of dissipation.

(v) We establish that there exists a *critical limit to the expansion of phase space*. This is determined by an interplay of chaotic diffusion and dissipation and has an important bearing on the evolution of entropy of an open system.

The present analysis is based on $\hbar$ -expansion and cumulant expansion in τ_c. Both being convergent perturbative schemes promise suitable extension to higher orders to reveal more details of subtlities of quantum evolution and long time memory effects due to weak classical chaos. Appropriate extension of the theory in this direction is worth-pursuing in future.

Appendix A
The solution for conditional probability(47)

We are to see that, by suitable choice of $\Omega(t)$, $\Omega^*(t)$, $\Gamma(t)$ and $\nu(t)$ Eq.(31) can be solved subject to the initial condition

$$p(\beta, \beta^*, 0)|\beta', \beta^{*'}, 0) = \frac{\epsilon}{\pi} e^{-\epsilon(\beta-\beta')(\beta^*-\beta^{*'})} . \qquad \text{A1}$$

Comparison of this with (34) with G(0) shows that

$$\Gamma(0) = \frac{1}{\epsilon} \ , \ \ \Omega(0) = \beta' \ , \ \ \Omega^*(0) = \beta^{*'} \ , \ \ \nu(0) = \frac{\epsilon}{\pi} \ . \qquad \text{A2}$$

If we put (34) in Eq.(31) and equate the coefficients of equal powers of β and β^* we obtain after some algebra the following set of equations

$$\frac{1}{\Gamma^2}\frac{d\Gamma}{dt} = -\frac{2\gamma}{\Gamma} + \frac{2A}{\Gamma^2} \ , \qquad \text{A3}$$

$$\frac{d\Omega}{dt} = \left[\frac{\omega}{2i} - \frac{i\omega'^2}{2\omega} - \gamma + \frac{2(A-iB)}{\Gamma}\right]\Omega^*$$

$$+ \left[\frac{\omega}{2i} + \frac{i\omega'^2}{2\omega} - \gamma\right]\Omega \ , \qquad \text{A4}$$

$$\frac{1}{\nu}\frac{d\nu}{dt} = 2\gamma - \frac{2A}{\Gamma} = -\frac{1}{\Gamma}\frac{d\Gamma}{dt} \ , \qquad \text{A5}$$

together with the conjugate of $d\Omega/dt$ equation. The relevant solutions of these which satisfy the initial conditions above and are necessary for the present analysis are seen to be

$$\Gamma(t) = \Gamma(0)e^{-2\gamma t} + \frac{A}{\gamma}(1 - e^{-2\gamma t}), \qquad \text{A6}$$

and

$$\nu(t) = \frac{\Gamma(0)\nu(0)}{\Gamma(0)e^{-2\gamma t} + \frac{A}{\gamma}(1 - e^{-2\gamma t})}. \qquad \text{A7}$$

The solution of $\Omega(t)$ can be similarly obtained after appropriate algebraic manipulations. Having known $\Gamma(t)$, $\nu(t)$, $\Omega(t)$ and $\Omega^*(t)$ one calculates the conditional probability function for quantum fluctuations given by Eq.(34).

We now calculate the quantum fluctuations of position and momentum variables. Since the conditional probability P is given by Eq.(34), this together with (35) and (32) may be employed to calculate first and second moments. Thus we express

$$\langle \bar{\eta}_1 \rangle = \frac{\int \int_{-\infty}^{\infty} P(\bar{\eta}_1, \bar{\eta}_2, t | \bar{\eta}_1', \bar{\eta}_2', 0) \bar{\eta}_1 d\bar{\eta}_1 d\bar{\eta}_2}{\int \int_{-\infty}^{\infty} P(\bar{\eta}_1, \bar{\eta}_2, t | \bar{\eta}_1', \bar{\eta}_2', 0) d\bar{\eta}_1 d\bar{\eta}_2} \qquad \text{A8}$$

in terms of conditional probability P using (30) and (34). Explicit calculation yields

$$\langle \bar{\eta}_1 \rangle = Re[\Omega(t)] \quad , \qquad \text{A9}$$

where $\Omega(t)$ is a solution of (A4). Similarly we obtain

$$\langle \bar{\eta}_1^2 \rangle = \frac{1}{2}\Gamma(t) + [Re\{\Omega(t)\}]^2 \quad . \qquad \text{A10}$$

The conjugate variable to $\bar{\eta}_1$ is $\bar{\eta}_2$ whose average is given by

$$\langle \bar{\eta}_2 \rangle = Im[\Omega(t)] \quad . \qquad \text{A11}$$

Similarly

$$\langle \bar{\eta}_2^2 \rangle = \frac{1}{2}\Gamma(t) + \{Im[\Omega(t)]\}^2 \quad . \qquad \text{A12}$$

The above expressions (A9-A12) for the averages can then be utilised to calculate the uncertainties in coordinate and momentum variables as given in Eq. (39).

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Table I. Comparison of the rate constant of initial divergence of uncertainty product calculated numerically (from fully quantum considerations, Eq. 43), $k_{\text{numerical}}$, with that calculated theoretically, $k_{\text{theoretical}}$, [from Eq. (39)].

g	$k_{\text{numerical}}$	$k_{\text{theoretical}}$
12	1.94	1.93
14	2.44	2.48
18	3.02	3.08

Table II. Comparison of the time required to reach equilibrium calculated numerically (from fully quantum considerations, Eq. 43), $t_{\text{numerical}}$, with that calculated theoretically, $t_{\text{theoretical}}$.

γ	$t_{\text{numerical}}$	$t_{\text{theoretical}}$
0.001	10.5	12.5
0.01	8	10
0.09	5.3	6.3

Figure Captions

1. Fig.1 Plot of log of uncertainty product with time (time scale corresponds to the time period of the driving force $\frac{2\pi}{\omega_0}$) for different values of g. The continuous line represents the numerical calculation (fully quantum). The dotted line refers to semiclassical calculation (Eq.(39)). (a) $g = 12.0$, (b) $g = 14.0$ and (c) $g = 18.0$. (Both units are arbitrary).
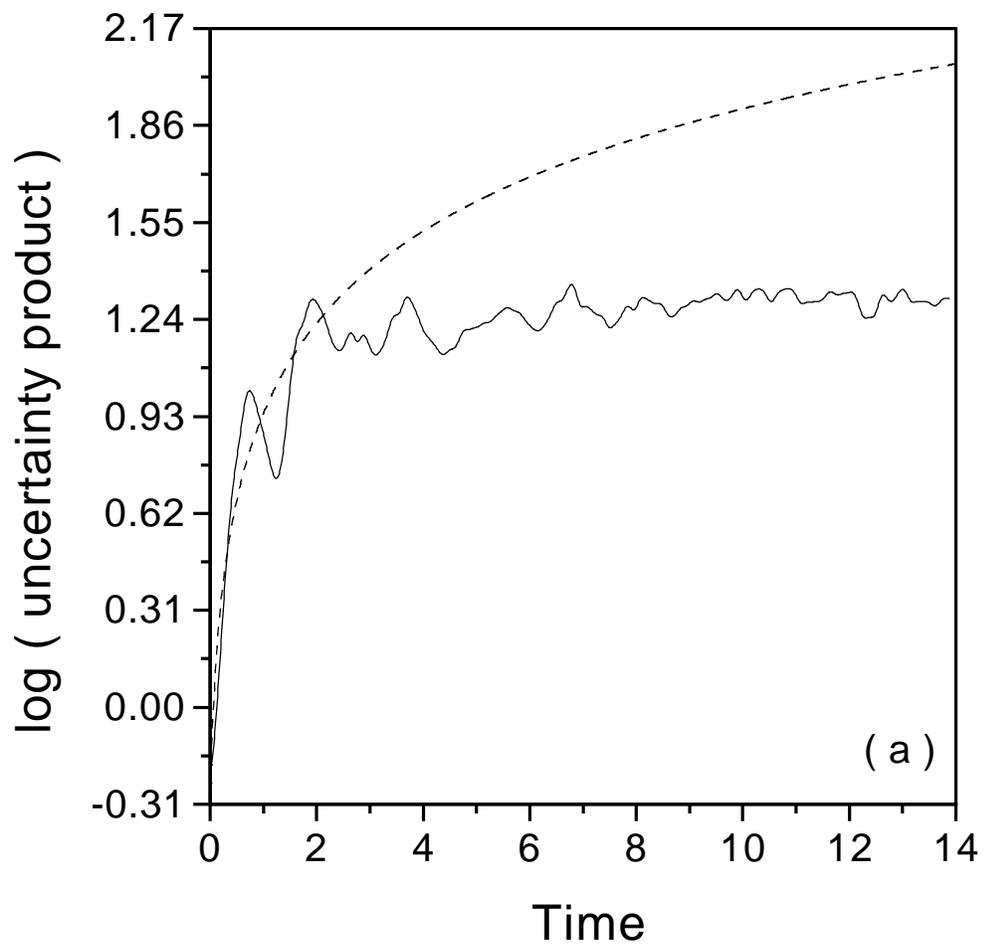
2. Fig.2 Plot of log of uncertainty product with time (time scale corresponds to the time period of the driving force $\frac{2\pi}{\omega_0}$) for different values of γ. The continuous line represents the numerical calculation (fully quantum). The dotted line refers to semiclassical calculation (Eq.(39)). (a) $\gamma = 0.0005$, (b) $\gamma = 0.005$ and (c) $\gamma = 0.045$. (Both units are arbitrary).
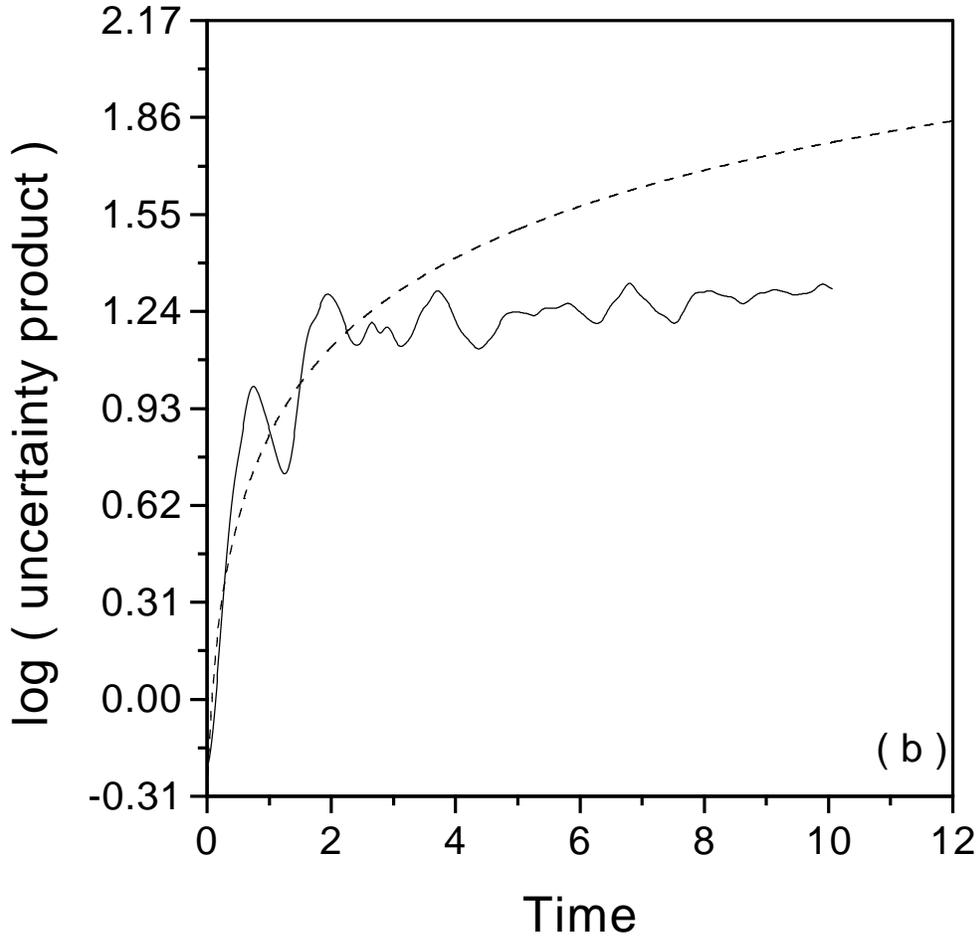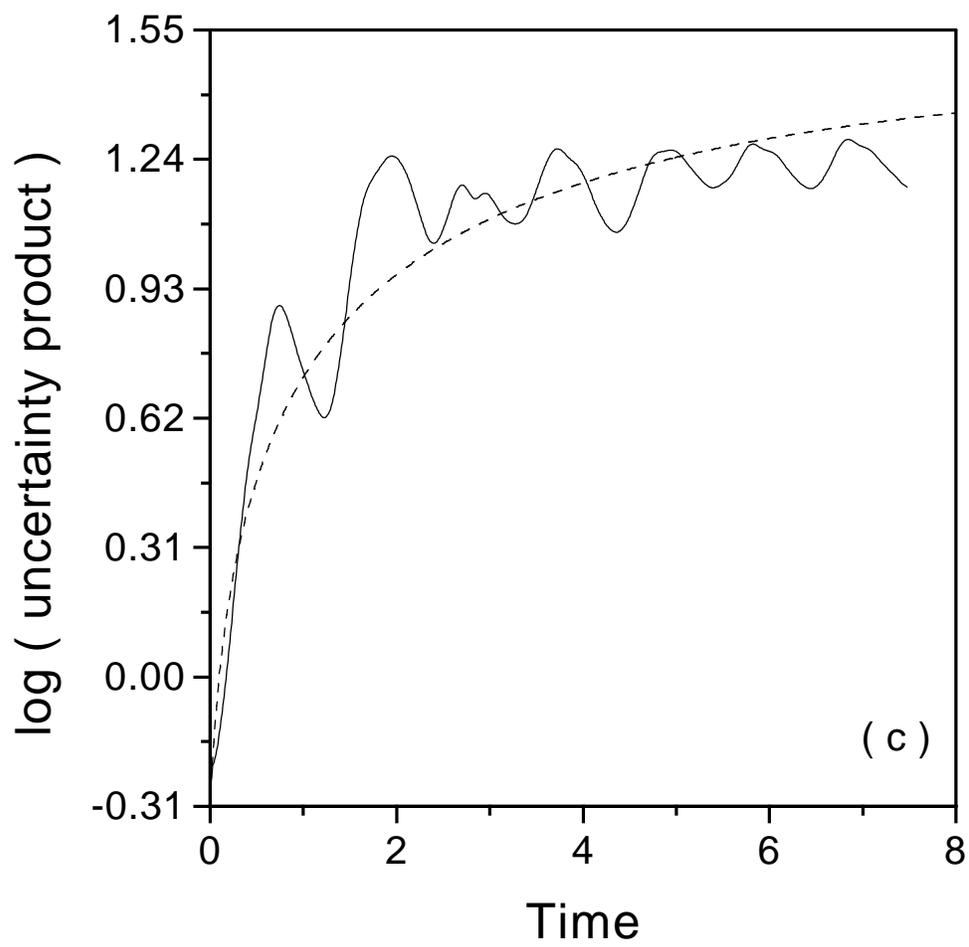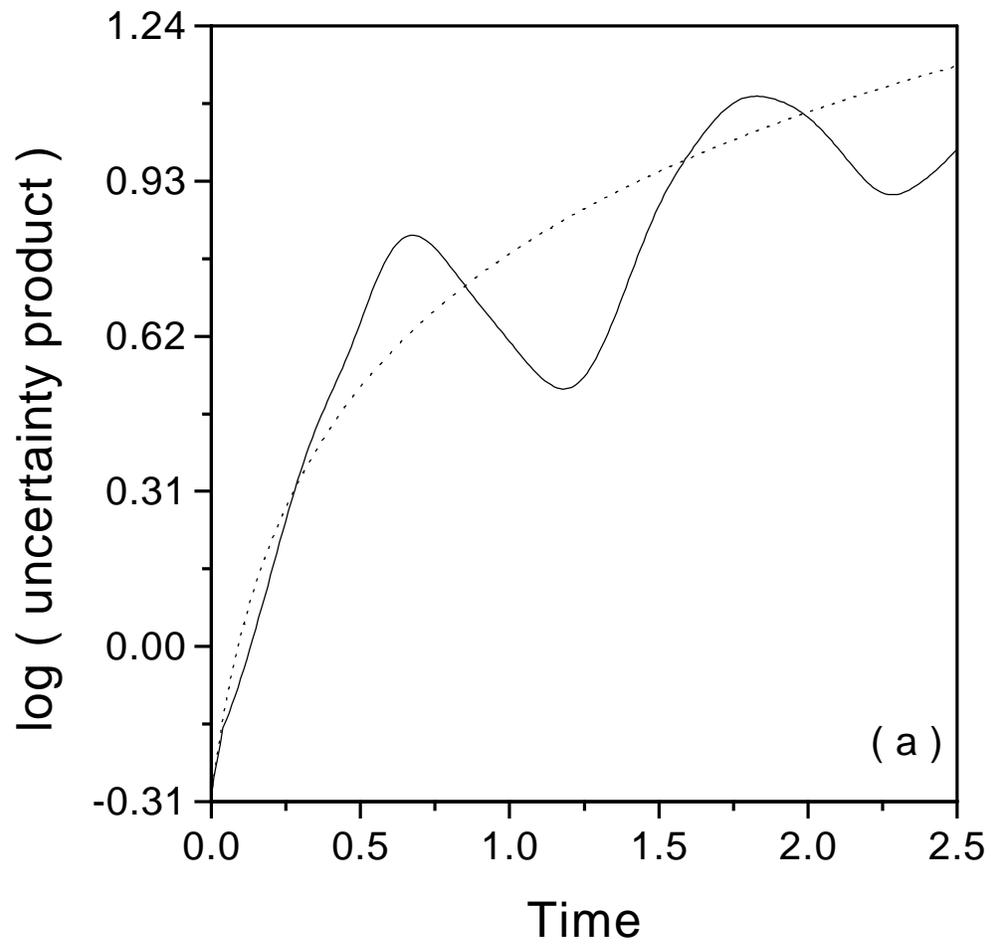
fig. (2)

fig. (2)

fig. (2)

fig. (1)

(b)

fig. (1)

fig. (1)