Gyroscopic Precession and Inertial Forces in Axially Symmetric Stationary Spacetimes

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Abstract

We study the phenomenon of gyroscopic precession and the analogues of inertial forces within the framework of general relativity. Covariant connections between the two are established for circular orbits in stationary spacetimes with axial symmetry. Specializing to static spacetimes, we prove that gyroscopic precession and centrifugal force both reverse at the photon orbits. Simultaneous non-reversal of these in the case of stationary spacetimes is discussed. Further insight is gained in the case of static spacetime by considering the phenomena in a spacetime conformal to the original one. Gravi-electric and gravi-magnetic fields are studied and their relation to inertial forces is established.

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1 Introduction

Recently, two general relativistic phenomena apparently related to each other in someway, have been investigated in considerable detail. These are gyroscopic precession and the general relativistic analogue of inertial forces. Iyer and Vishveshwara [1] have given a comprehensive treatment of gyroscopic precession in axially symmetric stationary spacetimes making use of the elegant Frenet-Serret(FS) formalism. This forms the basis for a covariant description of gyroscopic precession. At the same time, a general formalism defining inertial forces in general relativity has been presented by Abramowicz, Nurowski and Wex [2]. The motivation for this work stemmed from the earlier interest in centrifugal force and its reversal. Such reversal in the Schwarzschild spacetime at the circular photon orbit was first discussed by Abramowicz and Prasanna [3] and later in the case of the Ernst spacetime by Prasanna[4]. Abramowicz [5] showed that centrifugal force reversed at the photon orbit in all static spacetimes. He argued, on qualitative grounds, that gyroscopic precession should also reverse at the photon orbit. Taking the Ernst spacetime as a specific example of static spacetimes Nayak and Vishveshwara[6] have shown that, in fact, both centrifugal force and gyroscopic precession reverse at the photon orbits. A similar study by Nayak and Vishveshwara [7] in the Kerr-Newman spacetime indicates that the situation in the case of stationary spacetimes is much more complicated than in the case of static spacetimes. Neither centrifugal force nor gyroscopic precession reverses at the photon orbit.

The above studies raise some interesting questions. Is gyroscopic precession directly related to centrifugal force in all static spacetimes? If so, do they both necessarily reverse at the photon orbit? In the case of stationary spacetimes is it possible to make a covariant connection between the gyroscopic precession on the one hand and the inertial forces on the other, not necessarily just the centrifugal force? Does such a connecting formula reveal the individual non-reversal of gyroscopic precession and centrifugal force at the photon orbit? In this paper we consider these and related questions.

The contents of the present paper are organized in the following manner. In section 2 we sketch the Frenet-Serret description of gyroscopic precession. The formulae derived here form the basis for all subsequent considerations. They are further specialized to Killing trajectories in axisymmetric stationary spacetimes. Although most of the contents of this section may essentially be found in previous papers ([1], [6] and [7]), they are included here for the sake of completeness and ready readability of the present paper. Furthermore, we present here formulae expressed in terms of Killing vectors which have not appeared before. Section 3 comprises the formulation of general relativistic analogues of inertial forces and its specialization to Killing trajectories. In section 4 covariant connections are established between the precession and the forces in both static and stationary spacetimes. Section 5 examines the question of the reversal of gyroscopic precession in relation to inertial forces in static and stationary metrics in the light of these covariant connections. In section 6 and 7 we consider two related aspects. First, we treat gyroscopic precession as viewed in a spacetime conformal to the original static spacetime thereby factoring out the contribution due to the gravitational force. Secondly, we examine the idea of gravi-electric and gravi-magnetic fields in relation to the inertial forces. This is followed by concluding remarks of section 8.

2 Gyroscopic Precession

2.1 Frenet-Serret Description of Gyroscopic Precession

The Frenet-Serret (FS) formalism offers a covariant method of treating gyroscopic precession. It turns out to be quite a convenient and elegant description of the phenomenon when the worldlines along which the gyroscopes are transported follow spacetime symmetry directions or Killing vector fields. In fact, in most cases of interest orbits corresponding to such worldlines are considered for simplicity. In the FS formalism the worldlines are characterized in an invariant geometric manner by defining along the curve three parameters κ the curvature and the two torsions τ_1 and τ_2 and an orthonormal tetrad. As we shall see, the torsions τ_1 and τ_2 are directly related to gyroscopic precession. All the above quantities can be expressed in terms of the Killing vectors and their derivatives. These considerations apply to a single trajectory in any specific example. However, additional geometric insight may be gained by identifying the trajectory as a member of one or more congruences generated by combining different Killing vectors. For this purpose the FS formalism is generalized to what may be termed as quasi-Killing trajectories. For the sake of completeness we summarize below relevant formulae taken from reference [1].

Let us consider a spacetime that admits a timelike Killing vector ξ^a and a set of spacelike Killing vector $\eta_{(A)}$ (A=1,2,...m). Then a quasi-Killing vector may be defined as

$$\chi^a \equiv \xi^a + \omega_{(A)}\eta^a_{(A)}, \tag{1}$$

where (A) is summed over. The Lie derivative of the functions $\omega_{(A)}$ with respect to χ^a is assumed to vanish,

$$\mathcal{L}_\chi \omega_{(A)} = 0. \tag{2}$$

We adopt the convention that Latin indices $a, b, \ldots = 0-3$ and Greek indices $\alpha, \beta, \ldots = 1-3$ and the metric signature is $(+,-,-,-)$. Geometrized units with $c = G = 1$ are chosen. A congruence of quasi-Killing trajectories is generated by the integral curves of χ^a. As a special case we obtain a Killing congruence when $\omega_{(A)}$ are constants.

Assuming χ^a to be timelike, we may define the four velocity of a particle following χ^a by

$$e^a_{(0)} \equiv u^a \equiv e^\psi \chi^a, \tag{3}$$

so that

$$e^{-2\psi} = \chi^a\chi_a, \quad \psi_{,a}\chi^a = 0 \tag{4}$$

and

$$\dot{e}^a_{(0)} \equiv e^a_{(0);b} e^b_{(0)} = F^a{}_b e^b_{(0)}, \tag{5}$$

where

$$F_{ab} \equiv e^{\psi} \left(\xi_{a;b} + \omega_{(A)} \eta_{(A)a;b} \right). \tag{6}$$

The derivative of $\omega_{(A)}$ drops out of the equation. The Killing equation and the equation $\xi_{a;b;c} \equiv R_{abcd}\xi^d$ satisfied by any Killing vector lead to

$$F_{ab} = -F_{ba} \; and \; \dot{F}_{ab} = 0. \tag{7}$$

Now, the FS equations in general are given by

$$\begin{aligned}
\dot{e}^a_{(0)} &= \kappa e^a_{(1)}, \\
\dot{e}^a_{(1)} &= \kappa e^a_{(0)} + \tau_1 e^a_{(2)}, \\
\dot{e}^a_{(2)} &= -\tau_1 e^a_{(1)} + \tau_2 e^a_{(3)}, \\
\dot{e}^a_{(3)} &= -\tau_2 e^a_{(2)},
\end{aligned} \tag{8}$$

As mentioned earlier κ,τ_1 and τ_2 are respectively the curvature, and the first and second torsions while $e^a_{(i)}$ form an orthonormal tetrad. The six quantities describe the worldline completely. In the case of the quasi-Killing trajectories one can show that κ, τ_1 and τ_2 are constants and that each of $e^a_{(i)}$ satisfies a Lorentz like equation:

$$\dot{\kappa} = \dot{\tau}_1 = \dot{\tau}_2 = 0, \tag{9}$$

$$\dot{e}^a_{(i)} = F^a{}_b e^b_{(i)}. \tag{10}$$

Further, κ, τ_1, τ_2 and $e^a_{(\alpha)}$ can be expressed in terms of $e^a_{(0)}$ and $F^n_{ab} \equiv F_a{}^{a_1} F_{a_1}{}^{a_2} \cdots F_{a_{n-1}b}$.

$$\begin{aligned}
\kappa^2 &= F^2_{ab} e^a_{(0)} e^b_{(0)} \\
\tau_1^2 &= \kappa^2 - \frac{F^4_{ab} e^a_{(0)} e^b_{(0)}}{\kappa^2} \\
\tau_2^2 &= \frac{F^6_{ab} e^a_{(0)} e^b_{(0)}}{\kappa^2 \tau_1^2} - \frac{\left(\kappa^2 - \tau_1^2\right)^2}{\tau_1^2}
\end{aligned} \tag{11}$$

$$\begin{aligned}
e^a_{(1)} &= \frac{1}{\kappa} F^a{}_b e^b_{(0)} \\
e^a_{(2)} &= \frac{1}{\kappa \tau_1} \left[F^{2a}{}_b - \kappa^2 \delta^a_b \right] e^b_{(0)} \\
e^a_{(3)} &= \frac{1}{\kappa \tau_1 \tau_2} \left[F^{3a}{}_b + \left(\tau_1^2 - \kappa^2 \right) F^a{}_b \right] e^b_{(0)}
\end{aligned} \tag{12}$$

The above equations were first derived by Honig, Schücking and Vishveshwara [8] to describe charged particle motion in a homogeneous electromagnetic field. Interestingly, they are identical to those that arise in the case of quasi-Killing trajectories.

Next let us consider an inertial frame of tetrad $\left(e^a_{(0)}, f^a_{(\alpha)}\right)$ which undergoes Fermi-Walker (FW) transport along the worldline. The triad $f_{(\alpha)}$ may be physically realized by a set of three mutually orthogonal gyroscopes. Then, the angular velocity of the FS triad $e^a_{(\alpha)}$ with respect to the FW triad $f^a_{(\alpha)}$ is given by [1]

$$\omega^a_{FS} = \tau_2 e^a_{(1)} + \tau_1 e^a_{(3)}. \tag{13}$$

Or the gyroscopes precess with respect to the FS frame at a rate given by $\Omega_{(g)} = -\omega_{FS}$. In case of the Killing congruence ω_{FS} is identical to the vorticity of the congruence.

Other important relations which relate the gyroscopic precession and the acceleration and its derivatives can also be derived. The transport law for vector $e^a_{(1)}$ $\left(\kappa e^a_{(1)} = a^a\right)$ can be decomposed as

$$\frac{de^a_{(1)}}{d\tau} = \left[\left(a^a u^b - a^b u^a\right) + \varepsilon^{cdab} u_c \omega_d\right] e_{(1)b} \tag{14}$$

Here $(a^a u^b - a^b u^a)$ gives the Fermi-Walker part. The second term gives the spatial rotation with respect to the Fermi-Walker frame precessing with angular velocity ω^a which is orthogonal to u^a,

$$u^a \omega_a = 0 \tag{15}$$

Since $\kappa e^a_{(1)} = a^a$ and $\dot{\kappa} = 0$, We can show

$$\dot{a}^a = \kappa^2 u^a + \varepsilon^{cdab} u_c \omega_d a_b \tag{16}$$

From this after some simplification we get

$$\omega_p = \frac{1}{\kappa^2}\left[\varepsilon_{pqra}\dot{a}^a u^r a^q - a_p(\omega_q a^q)\right] \tag{17}$$

2.2 Axially Symmetric Stationary Spacetimes

An axially symmetric stationary metric admits a timelike Killing vector ξ^a and a spacelike Killing vector η^a with closed circular orbits around the axis of symmetry. Assuming orthogonal transitivity, in coordinates $(x^0 \equiv t, \quad x^3 \equiv \phi)$ adapted to ξ^a and η^a respectively the metric takes on its canonical form

$$ds^2 \;=\; g_{00}dt^2 \;+\; 2g_{03}dtd\phi \;+\; g_{33}d\phi^2 \;+\; g_{11}dr^2 \;+\; g_{22}d\theta^2 \tag{18}$$

with g_{ab} functions of $x^1 \equiv r$ and $x^2 \equiv \theta$ only. The quasi-Killing vector field

$$\chi^a = \xi^a + \omega\eta^a \tag{19}$$

generates closed circular orbits around the symmetry axis with constant angular speed ω along each orbit. The FS parameters and the tetrad can be determined either by the direct substitution of χ^a or by transforming to a rotating coordinate frame as discussed in [1]. They can be written in terms of the Killing vectors and their derivatives as follows.

$$\kappa^2 = -g^{ab} a_a a_b \tag{20}$$

$$\tau_1^2 = \left[g^{ab} a_a d_b\right]^2 \tag{21}$$

$$\tau_2^2 = \left[\frac{\varepsilon^{abcd}}{\sqrt{-g}} n_a \tau_b a_c d_d\right]^2 \tag{22}$$

$$\begin{aligned}
e^a_{(0)} &= \frac{1}{\sqrt{\mathcal{A}}}(1, 0, 0, \omega) \\
e^a_{(1)} &= -\frac{1}{\kappa}(0, g^{11} a_1, g^{22} a_2, 0) \\
e^a_{(2)} &= \frac{1}{\sqrt{\mathcal{A}}\sqrt{-\Delta_3}}(\mathcal{B}, 0, 0, -\mathcal{C}) \\
e^a_{(3)} &= \frac{\sqrt{g^{11} g^{22}}}{\kappa}(0, -a_2, a_1, 0)
\end{aligned} \tag{23}$$

In the above,

$$\begin{aligned}
d_a &= \left(\frac{\mathcal{B}}{2\sqrt{-\Delta_3}\kappa}\right)\left[\frac{\mathcal{B}_a}{\mathcal{B}} - \frac{\mathcal{A}_a}{\mathcal{A}}\right] = \left(\frac{\mathcal{B}}{\sqrt{-\Delta_3}\kappa}\right)[b_a - a_a] \\
a_a &= \frac{\mathcal{A}_a}{2\mathcal{A}} \\
b_a &= \frac{\mathcal{B}_a}{2\mathcal{B}} \\
\mathcal{A} &= (\xi^a \xi_a) + 2\omega\,(\eta^a \xi_a) + \omega^2\,(\eta^a \eta_a) \\
\mathcal{B} &= (\eta^a \xi_a) + \omega\,(\eta^a \eta_a) \\
\mathcal{C} &= (\xi^a \xi_a) + \omega\,(\eta^a \xi_a) \\
\mathcal{A}_a &= (\xi^b \xi_b)_{,a} + 2\omega(\eta^b \xi_b)_{,a} + \omega^2(\eta^b \eta_b)_{,a}; \quad a = 1, 2. \\
\mathcal{B}_b &= (\eta^a \xi_a)_{,b} + \omega(\eta^a \eta_a)_{,b}; \quad b = 1, 2. \\
\Delta_3 &= (\xi^a \xi_a)(\eta^b \eta_b) - (\eta^a \xi_a)^2
\end{aligned} \tag{24}$$

where n^a is the unit vector along $\zeta_a = \xi_a - \frac{(\xi^b \eta_b)}{(\eta^c \eta_c)} \eta_a$ and τ^i is the unit vector along the rotational Killing vector η^a. We may note that all the above equations can be specialized to a static spacetime by setting $\xi^a \eta_a = 0$ and $\zeta^a \equiv \xi^a$.

3 Inertial Forces

3.1 General Formalism

As has been mentioned earlier, in a recent paper Abramowicz *et. al.* [2] have formulated the general relativistic analogues of inertial forces in an arbitrary spacetime. The particle four velocity u^a is decomposed as

$$u^a = \gamma \left(n^a + v\tau^a\right) \tag{25}$$

In the above, n^a is a globally hypersurface orthogonal timelike unit vector, τ^a is the unit vector orthogonal to it along which the spatial three velocity v of the particle is aligned and γ is the normalization factor that makes $u^a u_a = 1$.

Then the forces acting on the particle are written down as

$$\begin{array}{rrcl} Gravitational \quad force \; G_k & = & \phi_{,k} \\ Centrifugal \quad force \; Z_k & = & -\left(\gamma v\right)^2 \tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k \\ Euler \quad force \; E_k & = & -\dot{V}\tilde{\tau}_k \\ Coriolis - Lense - Thirring \quad force \; C_k & = & \gamma^2 v X_k \end{array} \tag{26}$$

where

$$\begin{array}{rcl} \dot{V} & = & (ve^{\phi}\gamma)_{,i}u^i \\ X_k & = & n^i \left(\tau_{k;i} - \tau_{i;k}\right) \\ \phi_{,k} & = & -n^i n_{k,i} \end{array} \tag{27}$$

Here $\tilde{\tau}^i$ is the unit vector along τ^i in the conformal space orthogonal to n^i with the metric

$$\tilde{h}_{ik} = e^{-2\phi}\left(g_{ik} - n_i n_k\right) \tag{28}$$

One can show that the covariant derivatives in the two spaces are related by

$$\tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k = \tau^i \nabla_i \tau_k - \tau^i \tau_k \nabla_i \phi - \nabla_k \phi. \tag{29}$$

We shall now apply this formalism to axially symmetric stationary space-times.

3.2 Inertial Forces in Axially Symmetric Stationary Spacetimes

As has been shown by Greene, Schücking and Vishveshwara [9], axially symmetric stationary spacetimes with orthogonal transitivity admit a globally hypersurface orthogonal timelike vector field

$$\zeta^a = \xi^a + \omega_0 \eta^a, \tag{30}$$

where the fundamental angular speed of the irrotational congruence is

$$\omega_0 \;=\; -(\xi^a \eta_a)/(\eta^b \eta_b) \tag{31}$$

The unit vector along ζ^a is identified with n^a. Further, if u^a follows a quasi-Killing circular trajectory, then τ^i is along the rotational Killing vector η^a. In this case it is easy to show that $\dot{V} = 0$ and hence the Euler force does not exist.

More specifically,

$$u^a \;=\; e^\psi(\xi^a + \omega\eta^a) \;=\; e^\psi \chi^a \;=\; \gamma(n^a + v\tau^a). \tag{32}$$

Then we have

$$\begin{aligned} n^a &= e^{-\phi}\zeta^a \\ \tau^a &= e^{-\alpha}\eta^a \\ \gamma &= e^{\psi+\phi} \\ v &= e^{-\phi+\alpha}\,(\omega - \omega_0) \end{aligned} \tag{33}$$

where

$$\phi \;=\; \frac{1}{2}\ln(\zeta^a\zeta_a), \quad \alpha \;=\; \frac{1}{2}\ln(-\eta^a\eta_a), \quad \psi \;=\; \frac{1}{2}\ln(\chi^a\chi_a) \tag{34}$$

From the above relations, we can write down the inertial forces from their definitions as follows.
Gravitational force

$$G_k \;=\; \phi_{,k} \tag{35}$$

Centrifugal force

$$Z_k = \frac{1}{2}e^{2(\psi+\phi)}\tilde{\omega}^2 \left(\frac{\eta^a\eta_a}{\zeta^b\zeta_b}\right)_{,k} \tag{36}$$

Coriolis-Lense-Thirring force

$$C_k = e^{2(\psi+\alpha)}\tilde{\omega}\left(\frac{\xi^a\eta_a}{\eta^b\eta_b}\right)_{,k} \tag{37}$$

where $\tilde{\omega} = (\omega - \omega_0)$. For a particle following a quasi-Killing trajectory, inertial forces are proportional to gradiants of functions.

3.3 Specialization to Static Spacetimes:

In a static spacetime the global timelike Killing vector ξ^a itself is hypersurface orthogonal. The unit vector n^a is now aligned along ξ^a,

$$n^a \;=\; e^{-\phi}\xi^a. \tag{38}$$

Then we have the inertial forces as follows:
Gravitational force

$$G_k \ = \ \phi_{,k} \tag{39}$$

where $\phi \ = \ \frac{1}{2} ln(\xi^a \xi_a)$
Centrifugal force

$$Z_k \ = \ -\frac{\omega^2}{2} e^{2(\psi+\alpha)} \ \left[\ln\left(\frac{\eta^i \eta_i}{\xi^j \xi_j}\right)\right]_{,k} \tag{40}$$

Coriolis-Lense-Thirring force is identically zero,

$$C_k = 0 \tag{41}$$

4 Covariant Connections

In the preceding section we have derived expressions for τ_1 and τ_2 which give gyroscopic precession rate in terms of the Killing vectors. Similarly, inertial forces in an arbitrary axisymmetric stationary spacetime have also been written down in terms of the Killing vectors. All these quantities have been defined in a completely covariant manner. We shall now proceed to establish covariant connections between gyroscopic precession, *i.e.* the FS torsions τ_1 and τ_2, on the one hand and the inertial forces on the other. First, we shall consider the simpler case of static spacetimes.

4.1 Static Spacetimes

We have derived in equation (21) and (22), the FS torsions τ_1 and τ_2 for a stationary spacetime. As has been mentioned earlier, for a static spacetime $\xi^a \eta_a \ = \ 0$ and $\zeta^a \ = \ \xi^a$ in the above equations as well as in the expressions for inertial forces. With this specialization, centrifugal force can be written from equation (40) as

$$Z_b \ = \ e^{-(\phi-\alpha)} \omega \kappa d_b \tag{42}$$

Substituting equation (42) in equations (21) and (22) we arrive at the relations

$$\tau_1^2 \ = \ \frac{\beta^2}{\omega^2} \left[a^b Z_b\right]^2 \tag{43}$$

and

$$\tau_2^2 \ = \ \frac{\beta^2}{\omega^2} \left[\frac{\varepsilon^{abcd}}{\sqrt{-g}} n_a \tau_b a_c Z_d\right]^2 \tag{44}$$

where

$$\beta \;=\; \frac{e^{(\phi-\alpha)}}{\kappa} \tag{45}$$

The equations above relate gyroscopic precession directly to the centrifugal force. The two torsions τ_1 and τ_2, equivalent to the two components of precession, are respectively proportional to the scalar and cross products of acceleration and the centrifugal force. We shall discuss the consequences of these relations later on.

4.2 Stationary Spacetimes

From equation (24) we have

$$\begin{aligned}\mathcal{A} \;&=\; (\xi^a\xi_a) \;+\; 2\omega\;(\eta^a\xi_a) \;+\; \omega^2\;(\eta^a\eta_a)\\ \mathcal{B} \;&=\; (\eta^a\xi_a) \;+\; \omega\;(\eta^a\eta_a)\end{aligned}$$

We decompose the angular speed ω with reference to the fundamental angular speed of the irrotational congruence $\omega_0 = -\frac{(\xi^a\eta_a)}{(\eta^a\eta_a)}$,

$$\omega = \tilde{\omega} + \omega_0. \tag{46}$$

Then we have

$$\begin{aligned}\mathcal{A} \;&=\; \zeta^a\zeta_a \;+\; \tilde{\omega}^2\;\eta^a\eta_a\\ \mathcal{B} \;&=\; \tilde{\omega}\;\eta^a\eta_a\end{aligned} \tag{47}$$

Similarly, we get

$$\begin{aligned}\mathcal{A}_a \;&=\; (\zeta^b\zeta_b)_{,a} \;+\; 2\tilde{\omega}\;\mathcal{C}_a \;+\; \tilde{\omega}^2\;(\eta^b\eta_b)_{,a}\\ \mathcal{B}_a \;&=\; \mathcal{C}_a \;+\; \tilde{\omega}\;(\eta^b\eta_b)_{,a}\end{aligned} \tag{48}$$

where

$$\mathcal{C}_a \;\equiv\; (\xi^b\eta_b)_{,a} \;+\; \omega_0\;(\eta^b\eta_b)_{,a} \tag{49}$$

or equivalatly

$$\mathcal{C}_a \;=\; -(\xi^b\eta_b)\omega_{0,a} \tag{50}$$

From equation (24),(47) and (48) we can show

$$d_a \;=\; -e^{2\psi}\frac{e^{-(\phi+\alpha)}\tilde{\omega}}{2\kappa}\Big\{(\zeta^p\zeta_p)\mathcal{C}_a \;+\; \tilde{\omega}\,[(\zeta^p\zeta_p)(\eta^q\eta_q)_{,a} \;-\; (\eta^p\eta_p)(\zeta^q\zeta_q)_{,a}] \;-\; \tilde{\omega}^2\;(\eta^p\eta_p)\mathcal{C}_a\Big\} \tag{51}$$

Further, it is is easy to see that $\mathcal{C}_a$ is directly proportional to the Coriolis force,

$$\mathcal{C}_a \;= -\; e^{-2\psi}\;\tilde{\omega}^{-1}C_a \tag{52}$$

where C_a is the Coriolis-Lense-Thirring force. Then equation (51) takes on the form

$$d_a \; = \; \frac{e^{(\phi-\alpha)}}{\tilde{\omega}\kappa}\left\{Z_a \; - \; \frac{1}{2}\left[1+\tilde{\omega}^2 e^{2(\alpha-\phi)}\right] C_a\right\} \tag{53}$$

Where Z_a is the centrifugal force.

Substituting this in equation(21) for τ_1^2 we get the relation,

$$\tau_1^2 \; = \; \frac{\beta^2}{\tilde{\omega}^2}\left[g^{ab} a_a \left(Z_b \; + \; \beta_1 C_a\right)\right]^2 \tag{54}$$

where

$$\begin{aligned} \beta \; &= \; \frac{e^{(\phi-\alpha)}}{\kappa} \\ \beta_1 \; &= \; -\frac{1}{2}\left[1+\tilde{\omega}^2 e^{2(\alpha-\phi)}\right] \end{aligned} \tag{55}$$

Again, from equation (22) and (51), we obtain the expression

$$\tau_2^2 \; = \; \frac{\beta^2}{\tilde{\omega}^2}\left[\frac{\varepsilon^{abcd}}{\sqrt{-g}} n_a \tau_b a_c \left(Z_d \; + \; \beta_1 C_d\right)\right]^2 \tag{56}$$

These relations are more complicated than those we have derived in the static case. Nevertheless, they closely resemble the latter with the centrifugal force replaced by the combination of centrifugal and Coriolis forces $(Z_a + \beta_1 C_a)$. The static case formulae are obtained from those of stationary case by setting the Coriolis force to zero.

A formula for gyroscopic precession in the case of circular orbits in axially symmetric stationary spacetimes was derived by Abramowicz, Nurowski and Wex[11] within a different framework. We note that gyroscopics precession does not involve the gravitational force. In case of geodetic orbits, total force is zero but not the centrifugal and Coriolis force individually. Therefore gyroscopic precession is also nonzero even for geodetic orbits.

We may also note that the definition of inertial force is not unique. For instance, De Felice[12], Semerak[13] Barrabes, Boisseau and Israel [14] have suggested approaches different from the one employed in the present paper. The gyroscopics precession was solved in an alternative formalism by Semerak[15, 16] (see also De Felice[17]). A recent paper by Bini, Carini and Jantzen[18] has established the geometrical basis and interrelation of various modes of force splitting while they apply their general results to circular orbits in axially symmetric stationary spactimes[19].

5 Reversal of Gyroscopic Precession and Inertial Forces:

The condition for the reversal of gyroscopic precession is given by

$$\omega^a_{FS} \; = \; \tau_1 e^a_{(3)} \; + \; \tau_2 e^a_{(1)} \; = \; 0 \tag{57}$$

Since $e^a_{(1)}$ and $e^a_{(3)}$ are linearly independent vector fields at each point, this condition is the same as requiring

$$\tau_1 \ = \ \tau_2 \ = \ 0 \tag{58}$$

By considering the actual structure of τ_2, it is easy to show that τ_2 becomes zero on a plane about which the metric components are reflection invariant. The equatorial plane in the black hole spacetime is an example of this.

We shall now examine the vanishing of the FS torsions expressed as above and in relation to the inertial forces.

5.1 Static Spacetimes

In what follows, we shall prove a theorem that relates the simultaneous reversal of centrifugal force and gyroscopic precession to the existence of a null circular orbit. We start from the condition for gyroscopic precession reversal, *i.e.* $\tau_1 = \tau_2 = 0$, and show that at the point where this occurs a null circular geodesic must exist.

Setting $\tau_2 = 0$ in equation (22) and noting that the only nonvanishing components of n_a and τ_a are respectively n_0 and τ_3 we arrive at the condition

$$\mathcal{A}_1\mathcal{B}_2 \ = \ \mathcal{A}_2\mathcal{B}_1 \tag{59}$$

Further setting $\tau_1 = 0$ in equation (21), we obtain

$$g^{11}\left(\frac{\mathcal{A}_1\mathcal{B}_1}{\mathcal{B}} - \frac{\mathcal{A}_1^2}{\mathcal{A}}\right) \ + \ g^{22}\left(\frac{\mathcal{A}_2\mathcal{B}_2}{\mathcal{B}} - \frac{\mathcal{A}_2^2}{\mathcal{A}}\right) \ = 0. \tag{60}$$

We shall now assume that the gyroscope is transported along a circular orbit which is not a geodesic, *i.e.* $\kappa \neq 0$. This we do in anticipation of the result that a null geodesic, not a timelike one, exists with its spatial trajectory identical to that of this timelike orbit. Now $\kappa \neq 0$ implies $\mathcal{A}_1 \neq 0$ and $\mathcal{A}_2 \neq 0$ from equation (24). Then from equation (59) and (60) we arrive at

$$\mathcal{A}\mathcal{B}_1 - \mathcal{B}\mathcal{A}_1 \ = \ 0 \tag{61}$$

and

$$\mathcal{A}\mathcal{B}_2 - \mathcal{B}\mathcal{A}_2 \ = \ 0 \tag{62}$$

Combining the above two equations,

$$\mathcal{A}\mathcal{B}_a - \mathcal{B}\mathcal{A}_a \ = \ 0$$

Then, equation (24) reduces this condition to

$$(\xi^b\xi_b)(\eta^c\eta_c)_{,a} \ - \ (\xi^b\xi_b)_{,a}(\eta^c\eta_c) \ = \ 0 \tag{63}$$

With the help of this equation we can show that, if a circular geodesic exists where precession reverses, then it has to be null as follows.

The condition for circular geodesic is

$$(\xi^b \xi_b)_{,a} + \omega^2 (\eta^b \eta_b)_{,a} \ = \ 0 \tag{64}$$

This can be proved from the geodesic equation, assuming that the four velocity u^a is proportional to $\xi^a + \omega\eta^a$. Using condition (63), this reduces to

$$\frac{(\xi^b \xi_b)_{,a}}{(\xi^c \xi_c)} \left[(\xi^d \xi_d) \ + \ \omega^2 (\eta^d \eta_d) \right] = 0 \tag{65}$$

Since $\frac{(\xi^b \xi_b)_{,a}}{2(\xi^c \xi_c)}$ is the gravitational force, which is assumed to be nonzero, this is equivalent to

$$(\xi^d \xi_d) \ + \ \omega^2 (\eta^d \eta_d) \ = \ 0 \tag{66}$$

This means that the geodesic, if one exists, is null. Now we shall show that in fact a geodesic must exist at the point of precession reversal.

If a geodesic does not exist at the point of reversal, then

$$(\xi^b \xi_b)_{,a} \ + \ \omega^2 (\eta^b \eta_b)_{,a} \ \neq \ 0 \tag{67}$$

for all values of ω. However, equation (63) may be recast as

$$(\xi^b \xi_b)_{,a} \ - \left(\frac{\xi^c \xi_c}{\eta^c \eta_c} \right) (\eta^b \eta_b)_{,a} \ = \ 0. \tag{68}$$

This shows that the geodesic condition is satisfied for $\omega^2 \ = \ -\left(\frac{\xi^c \xi_c}{\eta^c \eta_c} \right)$. Therefore there does exist a geodesic and we have already shown that it has to be null. We shall now prove the converse, *i.e.* if a circular null geodesic exists then τ_1 and τ_2 are zero at the null geodesic.

The condition for a circular null geodesic is given by equation (63). Dividing this equation by $(\xi^a \xi_a)(\eta^b \eta_b)$, we see that it reduces to $\left[\ln \left(\frac{\eta^b \eta_b}{\xi^c \xi_c} \right) \right]_{,k}$ which is proportional to Z_a from equation (40) and is equal to zero. Further, from the dependence of τ_1 and τ_2 on Z_a from equations (43) and (44) we see that $\tau_1 \ = \ \tau_2 \ = \ 0$. We may note the fact that both gyroscopic precession and centrifugal force reverse simultaneously as is evident from equations (43) and (44). We have therefore proved the following theorem.

Theorem: In the case of circular orbits in static spacetimes reversal of gyroscopic precession and centrifugal force takes place at some point, if and only if a null geodesic exists at that point.

5.2 Stationary Spacetimes

In section 4 we have derived expressions for τ_1 and τ_2, that embody gyroscopic precession, in terms of inertial forces, namely the centrifugal force Z_a and Coriolis-Lense-Thirring force C_a. These are complicated expressions and ω does not stand out as an overall multiplicative coefficient. Consequently, reversal of gyroscopic precession is not related directly to that of these forces individually. As has been discussed in reference [7], these reversals occur at different places and also not at the null geodesic. Nevertheless, one can see from equation (54) and (56) that gyroscopic precession reverses at a point where the combination of centrifugal and Coriolis forces given by $(Z_a + \beta_1 C_a)$ goes to zero.

We shall derive the angular velocity of a timelike orbit whose three dimensional trajectory coincides with a null geodesic in terms of inertial forces. Although there are no reversals at the null geodesic, this should give an idea of how these forces are structured along the null trajectory.

Conditions for the existence of circular null geodesic are

$$\mathcal{A} \;\equiv\; (\xi^a\xi_a) \;+\; 2\bar{\omega}\;(\eta^a\xi_a) \;+\; \bar{\omega}^2\;(\eta^a\eta_a) \;=\; 0 \tag{69}$$

and

$$\mathcal{A}_a \;\equiv\; (\xi^b\xi_b)_{,a} + 2\bar{\omega}(\eta^b\xi_b)_{,a} + \bar{\omega}^2(\eta^b\eta_b)_{,a} \;=\; 0 \tag{70}$$

The expression for $\mathcal{A}$ can also be written as

$$\mathcal{A} \;=\; \zeta^a\zeta_a \;+\; \tilde{\bar{\omega}}^2\eta^a\eta_a, \tag{71}$$

where

$$\bar{\omega} \;=\; \tilde{\bar{\omega}} - \frac{\xi^b\eta_b}{\eta^c\eta_c} \tag{72}$$

Then $\mathcal{A} \;=\; 0$ implies

$$\tilde{\bar{\omega}} \;=\; \pm\sqrt{-\frac{\zeta^a\zeta_a}{\eta^a\eta_a}} \tag{73}$$

Further, from equation (70)

$$\mathcal{A}_a \;=\; \frac{1}{\eta^p\eta_p}\left\{(\eta^q\eta_q)(\zeta^b\zeta_b)_{,a} \;-\; (\zeta^b\zeta_b)(\eta^q\eta_q)_{,a}\right\} \;\pm\; \sqrt{-\frac{\zeta^b\zeta_b}{\eta^q\eta_q}}\mathcal{C}_a \;=\; 0 \tag{74}$$

This has to be zero for a null geodesic. For a timelike curve with the same spatial orbit, but having angular velocity $\tilde{\omega}$ with respect to n^a,we have from equation (36) and (52)

$$Z_k \;=\; \frac{e^{2\psi}}{2}\;(\omega-\omega_0)^2\;\frac{1}{\zeta^a\zeta_a}\;\Big[\;(\zeta^b\zeta_b)\;(\eta^c\eta_c)_{,k} \;-\; (\eta^b\eta_b)\;(\zeta^c\zeta_c)_{,k}\Big] \tag{75}$$

$$\mathcal{C}_a = -e^{-2\psi}\tilde{\omega}^{-1}C_a \tag{76}$$

Substituting in equation (74), we get

$$\frac{1}{\eta^p\eta_p}\left\{2e^{-2\psi}(\zeta^p\zeta_p)\tilde{\omega}^{-2}Z_a\right\} \mp \left\{2\sqrt{-\frac{\zeta^b\zeta_b}{\eta^q\eta_q}}e^{-2\psi}\tilde{\omega}^{-1}C_a\right\} = 0 \tag{77}$$

This reducess to the equation

$$\sqrt{-\frac{\zeta^p\zeta_p}{\eta^q\eta_q}}Z_a \mp \tilde{\omega}C_a = 0 \tag{78}$$

which gives $\tilde{\omega}$ in terms of centrifugal and Coriolis forces.

6 Gyroscopic Precession and Inertial Forces in Conformal Static Spacetimes

Some further insight into gyroscopic precession and inertial forces may be gained by considering them in a space conformal to the original one as given in Abramowicz, Carter and Lasota [10]. In the case of the static metric we carry out the conformal transformation

$$\hat{g}_{ab} = e^{-2\phi}g_{ab} \tag{79}$$

If we choose

$$e^{-2\phi} = g^{00} = \frac{1}{g_{00}} = \frac{1}{\hat{\phi}} \tag{80}$$

then, $\hat{g}_{00} = \hat{g}^{00} = 1$. Spatial part of metric $\hat{g}_{ab}$ corresponds to optical geometry defined in reference [10] for identifying inertial forces in such geometry. Purely in the conformal space, with out referring to original g_{ab}, we have

$$\hat{u}^a\hat{\nabla}_a\hat{u}^b = 0 \tag{81}$$

for a stationary observer with four velocity $\hat{u}^a = (1,0,0,0)$, where $\hat{\nabla}_a$ is covariant derivative with respect to the conformal metric $\hat{g_{ab}}$. The two four velocities u^a and $\hat{u}^a$ are related by $u^a = e^{-\phi}\hat{u}^a$. Equation (81) indicates that because of dilation, $\hat{u}^a$ follows a geodesic trajectory in the conformal metric. This is equivalent to the statement that the only force acting on a particle at rest in the original space is gravitational force which is not felt in the conformal space. Since gravitational force is independent of velocity, no particle will experience it in the conformal space. In other words, gravitational force is effectively removed to some extent by dilation given in equation (80). Consequently, if a particle is moving in a circular trajectory, then the only force acting on it is the centrifugal force.

If ξ^a is a Killing vector in the original space then ξ^a is also a Killing vector in the conformal space if

$$\mathcal{L}_\xi \ \hat{\phi} = 0 \tag{82}$$

This is trivially true in coordinates adapted to Killing vector ξ^a. Then the Killing vectors in the original spacetime are also Killing vectors in the conformal spacetime. Therefore, $\hat{\xi^a} = (1,0,0,0)$ is the timelike Killing vector and $\hat{\eta^a} = (0,0,0,1)$ is the spacelike Killing vector which generates circular orbits in the conformal spacetime. The quasi-Killing trajectories

$$\hat{\chi}^a \ = \ \hat{\xi^a} \ + \ \omega \ \hat{\eta^a} \tag{83}$$

generate circular orbits and the only force acting on these particles are centrifugal force. It is easy to prove that the expression for centrifugal force is now

$$\hat{Z}_a \ = \ \hat{u}^b \hat{\nabla}_b \hat{u}_a \tag{84}$$

where

$$\hat{u}_a \ = \ e^{\hat{\psi}} \hat{\chi}_a \ \ and \ \ e^{-2\hat{\psi}} \ = \ \hat{\chi}_a \hat{\chi}^a \tag{85}$$

6.1 Gyroscopic Precession in The Conformal Space

The gyroscopic precession in the conformal spacetime can be computed exactly as before. The Frenet-Serret parameters for circular quasi-Killing trajectories can be written as

$$\hat{\kappa}^2 \ = \ -\frac{1}{4}\left(\frac{\hat{g}^{11}\hat{\mathcal{A}}_1^2 + \hat{g}^{22}\hat{\mathcal{A}}_2^2}{\hat{\mathcal{A}}^2}\right) \tag{86}$$

$$\begin{aligned}\hat{\tau}_1^2 \ &= \ \left(\frac{\hat{\mathcal{B}}^2}{4\hat{\Delta}_3\left(\hat{g}^{11}\hat{\mathcal{A}}_1^2 + \hat{g}^{22}\hat{\mathcal{A}}_2^2\right)}\right) \cdot \\ &\cdot \left(\frac{\hat{g}^{11}\hat{\mathcal{A}}_1\hat{\mathcal{B}}_1 + \hat{g}^{22}\hat{\mathcal{A}}_2\hat{\mathcal{B}}_2}{\hat{\mathcal{B}}} - \frac{\hat{g}^{11}\hat{\mathcal{A}}_1^2 + \hat{g^{22}}\hat{\mathcal{A}}_2^2}{\hat{\mathcal{A}}}\right)^2\end{aligned} \tag{87}$$

$$\hat{\tau}_2^2 \ = \ \frac{\hat{g}^{11}\hat{g}^{22}\left(\hat{\mathcal{A}}_1\hat{\mathcal{B}}_2 - \hat{\mathcal{A}}_2\hat{\mathcal{B}}_1\right)^2}{4\hat{\Delta}_3\left(\hat{g}^{11}\hat{\mathcal{A}}_1^2 + \hat{g}^{22}\hat{\mathcal{A}}_2^2\right)} \tag{88}$$

where

$$\hat{\mathcal{A}} \ = \ \hat{\xi}^a\hat{\xi}_a \ + \ \omega^2\hat{\eta}^a\hat{\eta}_a \ = \ \frac{\mathcal{A}}{\hat{\phi}}$$

$$\hat{\mathcal{B}} \ = \ \omega \ \hat{\eta}^a\hat{\eta}_a \ = \ \frac{\mathcal{B}}{\hat{\phi}} \tag{89}$$

$$\hat{\Delta}_3 \ = \ (\hat{\xi}^a\hat{\xi}_a)(\hat{\eta}^b\hat{\eta}_b) \tag{90}$$

and

$$\begin{aligned}
\hat{\mathcal{A}}_a &= \omega^2(\hat{\eta}^b\hat{\eta}_b)_{,a} \quad ; a = 1,2 \\
\hat{\mathcal{B}}_a &= \omega(\hat{\eta}^b\hat{\eta}_b)_{,a} \quad ; a = 1,2
\end{aligned} \tag{91}$$

One can then show that

$$\begin{aligned}
\hat{\mathcal{A}}_a &= \frac{\hat{\phi}\mathcal{A}_a - \mathcal{A}\hat{\phi}_{,a}}{\hat{\phi}^2} \\
\hat{\mathcal{B}}_a &= \frac{\hat{\phi}\mathcal{B}_a - \mathcal{B}\hat{\phi}_{,a}}{\hat{\phi}^2} = \frac{\hat{\mathcal{A}}_a}{\omega}
\end{aligned} \tag{92}$$

With the help of the above equations, $\hat{\kappa}^2$ can be related to κ^2. After some simplification we have,

$$\hat{\kappa}^2 = \hat{\phi}\kappa^2 - \frac{1}{4\hat{\phi}}(g^{ab}\ \hat{\phi}_{,a}\ \hat{\phi}_{,b}) + \frac{1}{2\mathcal{A}}(g^{ab}\ \mathcal{A}_a\ \hat{\phi}_{,b}) \tag{93}$$

From the definition of $\hat{\kappa}$ and the expression for the centrifugal force as in (84), it is clear that the two are one and the same. This is because the contribution from the gravitational force has been removed and the acceleration that appears is due to the centrifugal force alone. We can relate $\hat{\tau}_1^2$ to $\hat{\kappa}^2$ by using the expression for $\hat{\tau}_1^2$ to obtain

$$\hat{\tau_1}^2 = -\frac{\hat{\kappa}^2}{\hat{\Delta}_3\omega^2} \tag{94}$$

The above equation is similar to equation(43) which relates τ_1 to the centrifugal force. It can also be shown that

$$\hat{\tau}_2^2 = 0 \tag{95}$$

everywhere in the conformal spacetime.

From equations (94) and (95) it is clear that gyroscopic precession also reverses when $\hat{\kappa} = 0$ and that in turn corresponds to the centrifugal force reversal. Also $\hat{\kappa} = 0$ corresponds to the geodesic condition in the conformal space, which represents the null geodesics in the original space as given in reference [10].

To sum up, we have factored out the contribution due to the gravitational force by conformal transformation and have shown in a simple manner the simultaneous reversal of both the gyroscopic precession and the centrifugal force at the photon orbit.

7 Gravi-electric and Gravi-magnetic fields

Gravielectric and gravi-magnetic fields are closely related to the idea of inertial forces. These fields with respect to observers following the integral curves of n^a can be defined as follows.

Gravi-electric field:

$$E^a = F^{ab}n_b \tag{96}$$

Gravi-magnetic field:

$$H^a = \tilde{F}^{ab}n_b \tag{97}$$

where $\tilde{F}^{ab}$ is the dual of F^{ab},

$$\tilde{F}^{ab} = \frac{1}{2}(\sqrt{-g})^{-1}\varepsilon^{abcd}F_{cd} \tag{98}$$

In the above, as before, $F^{ab} = e^{\psi}(\xi_{a;b} + \omega\eta_{a;b})$. The equation of motion is

$$\dot{u}^a = F^{ab}u_b \tag{99}$$

Projecting onto the space orthogonal to n^a with $h_{ab} = g_{ab} - n_a n_b$ and decomposing u_a as given in (25), we get

$$\dot{u}_{\perp a} = \gamma\left[F_{ac}n^c + v(F_{ac}\tau^c - n_a F_{bc}n^b\tau^c)\right] \tag{100}$$

where γ is the normalization factor. This equation can be written in the form

$$\dot{u}_{\perp a} = \gamma\left[F_{ac}n^c + v\sqrt{-g}\varepsilon_{abcd}n^b\tau^c H^d\right], \tag{101}$$

or

$$\dot{u}_{\perp a} = \gamma\left[E + v \times H\right] \tag{102}$$

We can therefore define
Gravi-electric force:

$$f_{GEa} = \gamma F_{ac}n^c \tag{103}$$

Gravi-magnetic force:

$$f_{GHa} = \gamma v\sqrt{-g}\varepsilon_{abcd}n^b\tau^c H^d = \gamma v(F_{bc}\tau^c - n_a F_{bc}n^b\tau^c) \tag{104}$$

7.1 Relations among Gravi-electric , Gravi-magnetic and Inertial forces

7.1.1 Static Case

We have defined the gravi-electric field E_a by

$$\gamma E_a = \gamma F_{ac}n^c$$

If we substitute for $F_{ab} = e^{\psi}(\xi_{a;b} + \omega\eta_{a;b})$, we get

$$f_{GEa} = \gamma E_a = \gamma F_{ac}n^c = -e^{2(\psi+\phi)}G_a \tag{105}$$

So,

$$E_a = -e^{(\psi+\phi)} G_a \tag{106}$$

Here G_a is the gravitational force. Similarly we have for the gravi-magnetic field

$$f_{GHa} = \gamma v (F_{ac}\tau^c - n_a n^b F_{bc}\tau^c)$$

The second term in this equation is identically zero because the Killing vector field ξ^a and η^a commute and we get

$$\begin{aligned} f_{GHa} &= \gamma v \sqrt{-g}\varepsilon_{abcd} n^b \tau^c H^d \\ &= \gamma v F_{ac}\tau^c \\ &= \left[e^{2(\psi+\alpha)}\omega^2 G_a - Z_a\right] \end{aligned} \tag{107}$$

The above relation clearly shows the connection between the gravi-magnetic force on the one hand and the gravitational and centrifugal forces on the other.

7.1.2 Stationary Case

In the stationary case, n^a is given by equation (34). As before we decompose $\omega = \tilde{\omega} + \omega_0$, where ω_0 is given by (31). Then a straightforward computation gives the expression for the gravi-electric field.

$$E_a = -e^{(\psi+\phi)} G_a \ + \ e^{-(\psi+\phi)} C_a \tag{108}$$

and the gravi-electric force,

$$f_{GEa} = \gamma E_a = -e^{2(\psi+\phi)} G_a \ + \ C_a \tag{109}$$

This shows the relation of gravi-electric field or force to both gravitational and centrifugal forces. In the stationary case also we have

$$n_a n^b F_{bc}\tau^c \ \equiv \ 0 \tag{110}$$

Then it follows

$$\begin{aligned} f_{GHa} &\equiv \gamma v \sqrt{-g}\varepsilon_{abcd} n^d \tau^c H^d \\ &= \gamma v F_{ac}\tau^c \\ &= \left[\frac{C_a}{2} \ + \ e^{2(\psi+\alpha)}\tilde{\omega}^2 G_a \ - \ Z_a\right] \end{aligned} \tag{111}$$

Hence gravi-magnetic force is related to all the three inertial forces – gravitational, centrifugal and Coriolis.

7.2 Gravi-electric and Gravi-magnetic fields with respect to comoving frame

In the previous section we have defined gravi-electric and gravimagnetic fields with respect to the irrotational congruence. Similarly these fields can be defined with respect to the four velocity u^a of the particle as follows.
Gravi-electric field:

$$\tilde{E}^a = F^{ab} u_b \tag{112}$$

Gravi-magnetic field:

$$\tilde{H}^a = \tilde{F}^{ab} u_b \tag{113}$$

Where $\tilde{F}^{ab}$ is dual to F^{ab} as before. The equation of motion takes the form

$$a^a = \tilde{E}^a \tag{114}$$

Precession frequency can be written simply as

$$\omega^a = \tilde{H}^a \tag{115}$$

Following Honig, Schücking and Vishveshwara [8], Frenet-Serret parameters κ, τ_1 and τ_2 can be expressed in terms of gravi-electric and gravi-magnetic fields.

$$\kappa = |\tilde{E}| \tag{116}$$

where

$$|\tilde{E}| = \sqrt{-\tilde{E}^a \tilde{E}_a} \tag{117}$$

$$\tau_1 = \frac{|\tilde{P}|}{|\tilde{E}|} \tag{118}$$

where

$$\tilde{P}^a = \varepsilon^{abcd} \tilde{E}_b \tilde{H}_a u_d = \tilde{E} \times \tilde{H} \tag{119}$$

$$|\tilde{P}| = \sqrt{-\tilde{P}^a \tilde{P}_a} \tag{120}$$

and

$$\tau_2 = -\frac{\tilde{H}^a \tilde{E}_a}{|\tilde{E}|} \tag{121}$$

Frenet-Serret tetrad components can also be expressed in terms of $\tilde{E}^a$, $\tilde{H}^a$ and $\tilde{P}^a$,

$$\begin{aligned} e^a_{(1)} &= \frac{\tilde{E}^a}{|\tilde{E}|} \\ e^a_{(2)} &= \frac{\tilde{P}^a}{|\tilde{P}|} \\ e^a_{(3)} &= \frac{\varepsilon^{abcd} \tilde{E}_b \tilde{P}_c u_d}{\tilde{P}^r \tilde{E}_r} \end{aligned} \tag{122}$$

In reference [8], these expressions had been derived for charged particle motion in a constant electromagnetic field. We have now demonstrated the exact analogues in the case of gravi-electric and gravi-magnetic fields. The one-to-one correspondence is indeed remarkable.

Several authors have discussed gravi-electromagnetism in earlier papers with application to gyroscopic precession. We may cite as examples the papers by Embacher[20], Thorne and Price[21], Jantzen, Carini and Bini[22], and Ciufolini and Wheeler[23].

8 Conclusion

The main purpose of the present paper was to establish covariant connection between gyroscopic precession on the one hand and the analogues of inertial forces on the other. This has been accomplished in the case of axially symmetric stationary spacetimes for circular orbits. In the special case of static spacetimes gyroscopic precession can be directly related to the centrifugal force. From this we have been able to prove that both precession and centrifugal force reverse at a photon orbit, provided the latter exists. In the case of stationary spacetimes, the corresponding relations are more complicated. The place of centrifugal force is now taken by a combination of centrifugal and Coriolis-Lense-Thirring forces. As a result, gyroscopic precession and centrifugal force do not reverse in general at the photon orbit. We have also studied some of the above aspects in the spacetime conformal to the original static spacetime. In this approach part of the gravitational effect is factored out thereby achieving certain degree of simplicity and transparency in displaying interrelations and the reversal phenomenon. Closely related to these considerations is the idea of gravi-electric and gravi-magnetic fields. We have covariantly defined these with respect to the globally hypersurface orthogonal vector field that constitutes the general relativistic equivalent of Newtonian rest frame. In this instance, these fields can be related to the inertial forces. When these fields are formulated with respect to the orbit under consideration, they lead to a striking similarity to the corresponding physical quantities that arise for a charge moving in an actual, constant electromagnetic field. We have thus established connections and correspondences among several interesting general relativistic phenomena.

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