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ON THE GLOBAL DENSITY WAVES IN SELF-GRAVITATING FLAT DISKS

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ABSTRACT

The problem of global stability and structure of a class of disk models, both cold and "warm," against axisymmetric and nonaxisymmetric perturbations has been reconsidered in the form of an eigenvalue problem involving a triply infinite matrix. Some large wavelength (or open), nonaxisymmetric modes are found to stabilize in centrally concentrated disks. Unstable "leading" modes of cold disks become, gradually, "trailing" as the thermal energy of the disk is increased. The growth rates of short wavelength modes are lowered significantly by pressure effects in warm disks.

Subject headings: galaxies: internal motions — galaxies: structure — hydrodynamics

I. INTRODUCTION

The problem of global or large-scale stability and spiral structure of self-gravitating flat disks has been reconsidered here. The spiral structure of galaxies was first interpreted as global modes of oscillations at least as far back as Hunter (1963) who analytically studied the stability of a class of uniformly rotating disk models. Later Hunter (1965) considered axisymmetric modes of various cold disks by an eigenvalue problem involving infinite matrices. More recently, Bardeen (1975), Iye (1978), Takahara (1978), and Aoki, Noguchi, and Iye (1979) have studied the stability of "hot" disks against axisymmetric as well as nonaxisymmetric perturbations essentially along Hunter's formulation. All such studies involve extensive use of numerical methods since analytic solutions are, in general, not possible. Hence, in order to derive general characteristics of the global modes, it is essential to examine the problem for various disk models using different approaches.

We consider axisymmetric and nonaxisymmetric global modes, using the formulation developed by Yabushita (1969*a*), in cold as well as hot disks. Employing appropriate cylindrical functions, Yabushita (1969*a*) studied the stability of Saturnian rings and later considered the problem of a cold galactic disk model (Yabushita 1969*b*). The surface density and corresponding gravitational potential are obtained without any singularities throughout the disk using this approach. Both Yabushita (1969*b*) and Hunter (1969*a*) found that cold disks are unstable against short wavelength nonaxisymmetric perturbations and the corresponding patterns are "leading" in nature, as demonstrated, analytically, by Hunter (1969*a*), provided the angular velocity of the disk decreases away from the center. However, pressure effects are expected to change the results significantly, especially for short wavelength perturbations.

The main results, obtained here for the perturbations in cold disks, are in good agreement with earlier works. In particular, the behavior of nonaxisymmetric, unstable modes in centrally concentrated cold disks is examined, and it is found that, by increasing the central surface density concentration in the disk, large-scale (or open) nonaxisymmetric modes are suppressed in a manner similar to the axisymmetric modes (Hunter 1965). On the other hand, the growth rates of short wavelength perturbations are enhanced.

As expected, it is found that the short wavelength perturbations are significantly stabilized, while comparatively large wavelength modes remain rather unaffected in moderately warm disks. However, all the unstable modes are suppressed in sufficiently hot disk models under consideration. Most importantly, it is found that the leading patterns associated with the unstable surface density perturbations in cold disks become gradually *trailing* in warm disks. Also, much less thermal energy is required to quench all unstable modes, compared to the results obtained by other such global studies.

Basic hydrodynamic equations are described in § II. Equilibria of some finite and infinite radius disk models are considered in § III. The stability of these disks is discussed in § IV, and, finally, the results are discussed in § V.

II. BASIC EQUATIONS

Let us consider a flat, self-gravitating disk rotating about an axis perpendicular to its plane and passing through the center. The hydrodynamic equations describing the dynamics of the disk, in cylindrical coordinates, are

$$\frac{\partial \sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\sigma u)+\frac{1}{r}\frac{\partial}{\partial \theta}(\sigma v)=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}=-\frac{1}{\sigma}\frac{\partial p}{\partial r}+\frac{\partial \Psi}{\partial r}, \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}=-\frac{1}{r\sigma}\frac{\partial p}{\partial \theta}+\frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \tag{3}$$

where σ, u, v, p, and Ψ, representing surface density, radial and azimuthal velocities, pressure and gravitational potential, respectively, are all functions of position (r,θ) on the disk and time, t. The surface density and potential are related through Poisson's equation, viz.,

$$\nabla^2\Psi=-4\pi G\delta(z)\sigma. \tag{4}$$

The equations (1)–(4), along with a suitable equation of state for closure, describe the dynamics of the system completely. Here we assume a polytropic equation of state, given by

$$p=c\sigma^{\gamma}. \tag{5}$$

Thus, the acoustic velocity, a, is

$$a=\left(\frac{dp}{d\sigma}\right)^{1/2}=\left(\gamma c\sigma^{\gamma-1}\right)^{1/2}, \tag{6}$$

where c is a constant and γ is the polytropic index of the disk.

One can expand the surface density and the gravitational potential in the form

$$\sigma(r,\theta,t)=\sum_{m=0}^{\infty}\sigma_m(r)\exp\left[i(\omega t+m\theta)\right], \tag{7}$$

$$\Psi(r,\theta,z,t)=\sum_{m=0}^{\infty}\psi_m(r,z)\exp\left[i(\omega t+m\theta)\right], \tag{8}$$

where

$$\psi_m(r,z)=2\pi G\int_0^{\infty}A_m(\varpi)J_m(\varpi r)\exp\left(-\varpi|Z|\right)d\varpi, \tag{9}$$

and $A_m(\varpi)$ is the Hankel transform of $\sigma_m(r)$, such that

$$A_m(\varpi)=\int_0^{\infty}\sigma_m(r)J_m(\varpi r)\,r\,dr \tag{10}$$

(see Clutton-Brock 1972).

Now we expand the radial part of surface density, σ, in terms of Bessel functions as

$$\sigma_m(r)=\sum_{j=0}^{\infty}a_j^{(m)}J_m\left(\lambda_j^{(m)}r\right), \qquad r\leqslant R;$$

$$=0, \qquad r>R. \tag{11}$$

Following Yabushita (1969*a*), the potential in the plane of the disk, corresponding to the density distribution (11), can be obtained in the form

$$\psi_m(r)=2\pi G\sum_{j=0}^{\infty}a_j^{(m)}\left[\frac{J_m\left(\lambda_j^{(m)}r\right)}{\lambda_j^{(m)}}+\frac{4}{\pi^2Y_m\left(\lambda_j^{(m)}R\right)}\int_0^{\infty}\frac{I_m(k'r)K_m(k'R)\,dk'}{\lambda_j^{(m)2}+k'^2}\right], \qquad r\leqslant R;$$

$$=2\pi G\sum_{j=0}^{\infty}a_j^{(m)}\left[\frac{4}{\pi^2Y_m\left(\lambda_j^{(m)}R\right)}\int_0^{\infty}\frac{I_m(k'R)K_m(k'r)\,dk'}{\lambda_j^{(m)2}+k'^2}\right]; \qquad r>R. \tag{12}$$

In equations (11)–(12), R represents the radius of the disk and $\lambda_j^{(m)}$ the roots of the transcendental equation

$$J_m(\lambda R) = 0. \tag{13}$$

Also, $J_m(x), Y_m(x)$ are Bessel functions, and $I_m(x), K_m(x)$ are modified Bessel functions of first and second kind, respectively. The density distribution and potential expressed in the form discussed above are free from any singularities throughout the disk. In general, any arbitrary disk model can be constructed using equations (11)–(12).

III. EQUILIBRIUM OF THE DISK

Having defined the basic equations governing the system, we now proceed to construct the time-independent, axisymmetric state of the disk. The azimuthal velocity, $V_0(r)$, of the disk is obtained from the radial component of the momentum conservation equation, in the absence of any radial flow [i.e., $U_0(r) = 0$], as

$$-\frac{V_0^2}{r} = -\frac{1}{\sigma_0(r)}\frac{dP_0(r)}{dr} + \frac{d\Psi_0(r)}{dr}. \tag{14}$$

For realistic models, the following conditions must hold throughout the disk:

$$\Omega^2(r) \geqslant 0, \qquad K^2(r) \geqslant 0, \quad \text{and} \quad \Omega'(r) \leqslant 0. \tag{15}$$

Here $\Omega(r)$ and $K(r)$ are the angular velocity and epicyclic frequency of the disk, respectively. One can introduce various nondimensionalized quantities, following Hunter (1965), as

$$\sigma(r) = \left(\frac{M}{2\pi R^2}\right)\hat{\sigma}(\hat{r}), \qquad p(r) = \left(\frac{GM^2}{2R^3}\right)\hat{p}(\hat{r}), \qquad \psi(r) = \left(\frac{\pi GM}{R}\right)\hat{\psi}(\hat{r}),$$

$$V(r) = \left(\frac{\pi GM}{R}\right)^{1/2}\hat{V}(\hat{r}), \quad \text{and} \quad t = \left(\frac{\pi GM}{R^3}\right)^{-1/2}\tau, \tag{16}$$

where $\hat{r} = r/R$. The nondimensional energies of the disk are

$$\hat{E}_\psi = -\frac{R}{\pi GM^2}\int_0^R 2\pi r\,\sigma(r)\Psi(r)\,dr,$$

$$\hat{E}_R = \frac{R}{\pi GM^2}\int_0^R 2\pi r\,\sigma(r)V^2(r)\,dr,$$

and

$$\hat{E}_T = \frac{R}{\pi GM^2}\int_0^R 2\pi r\,2P(r)\,dr,$$

where $\hat{E}_\psi$, $\hat{E}_R$, and $\hat{E}_T$ are the potential, rotational, and thermal energies, respectively. The "hotness" of the disk is defined by

$$\beta = \frac{E_T}{|E|} = \frac{2E_T}{|E_\psi|}, \tag{17}$$

E being total energy.

In what follows we consider some density distributions generally studied in the literature and discuss their potential and velocity profiles using equations (9), (10), and (14).

a) *Finite Disk Models*

i) *Bessel Disks*

Consider an axisymmetric density distribution of the form

$$\hat{\sigma}_0(\hat{r}) = \sum_{j=0}^{\infty} a_j J_0\left(\hat{\lambda}_j^{(0)}\hat{r}\right), \qquad \hat{r} \leqslant 1;$$
$$= 0, \qquad \hat{r} > 1. \tag{18}$$

Correspondingly, the nondimensionalized gravitational potential is obtained from equation (12) (putting $m = 0$):

$$\hat{\Psi}_0(\hat{r}) = \frac{1}{\pi}\sum_{j=0}^{\infty} a_j\left[\frac{J_0\left(\hat{\lambda}_j\hat{r}\right)}{\hat{\lambda}_j} + \frac{4}{\pi^2 Y_0\left(\hat{\lambda}_j\right)}\int_0^{\infty}\frac{I_0(\hat{k}'\hat{r})K_0(\hat{k}')\,d\hat{k}'}{\hat{k}'^2 + \hat{\lambda}_j^2}\right]. \tag{19}$$

The rotational profile is obtained from equation (14). Although the contribution to the potential, in equation (19), from the infinite integrals is generally small, it is important near the edge of the disk since the first term vanishes at $\hat{r} = 1$. Yabushita (1969*b*) has considered the stability of a cold disk of this class.

ii) *Hunter's Disks*

The class of disks, given by

$$\hat{\sigma}_0(\hat{r}) = (2N+1)(1-\hat{r}^2)^{(2N-1)/2}, \qquad \hat{r} \leqslant 1,$$
$$= 0, \qquad \hat{r} > 1, \tag{20}$$

covers a wide range of disk models, including the $N = 1$ disk with solid body rotation [i.e., constant $\hat{\Omega}(\hat{r})$] and a disk with uniform azimuthal velocity, $\hat{V}_0$, as $N \to \infty$. The gravitational potential for these disks is given by

$$\hat{\Psi}_0(\hat{r}) = \frac{(2N+1)}{2\sqrt{\pi}}\,\frac{\Gamma[(2N+1)/2]}{N!}\,F\left(\frac{1}{2}, -N;\, 1, \hat{r}^2\right), \tag{21}$$

where $F(\alpha', \beta', \gamma', \hat{r}^2)$ represents hypergeometric function, which reduces into a polynomial with $N+1$ terms since β' is a negative integer. An increase in N leads to a larger central density concentration. The conditions (15) restrict the allowed values of parameter β ($\equiv E_T/|E_\psi|$) in a small range in centrally more concentrated disks. Takahara (1978) has considered nonpolytropic pressure distributions to avoid this problem. In order that $\hat{K}(\hat{r})$ is not singular at $\hat{r} = 1$ and $\hat{\Omega}(\hat{r})$ decreases uniformly when pressure coefficient c increases, polytropic indices for disks with $N \geqslant 2$ should be such that

$$\gamma \geqslant (2N+3)/(2N-1)$$

(See Iye 1978).

b) *Infinitely Extended Disks*

The luminosity curves of the disk components of spiral galaxies exhibit, usually, an exponential nature (Freeman 1975) which suggests, correspondingly, a density distribution of the form

$$\hat{\sigma}_0(\hat{r}) = \mu \exp(-\alpha\hat{r}), \tag{22}$$

where α, μ are constants. For these disks, using equations (9) and (10), one has

$$\hat{\Psi}_0(\hat{r}) = \frac{\hat{r}\mu}{2\pi}\left[I_0\left(\frac{\alpha\hat{r}}{2}\right)K_1\left(\frac{\alpha\hat{r}}{2}\right) - I_1\left(\frac{\alpha\hat{r}}{2}\right)K_0\left(\frac{\alpha\hat{r}}{2}\right)\right], \tag{23}$$

and

$$\hat{\Omega}_0^2(\hat{r}) = \frac{\alpha\mu}{2\pi}\left[K_0\left(\frac{\alpha\hat{r}}{2}\right)I_0\left(\frac{\alpha\hat{r}}{2}\right) - K_1\left(\frac{\alpha\hat{r}}{2}\right)I_1\left(\frac{\alpha\hat{r}}{2}\right)\right]. \tag{24}$$

Similarly, for a Gaussian distribution

$$\hat{\sigma}_0(\hat{r}) = \mu \exp(-\alpha^2\hat{r}^2), \tag{25}$$

potential and angular velocity can be obtained as

$$\hat{\Psi}_0(\hat{r}) = \frac{\mu}{2\sqrt{\pi}\,\alpha} \exp\left(-\frac{\alpha^2\hat{r}^2}{2}\right) I_0\left(\frac{\alpha^2\hat{r}^2}{2}\right), \tag{26}$$

and

$$\hat{\Omega}_0^2(\hat{r}) = \frac{\mu\alpha}{2\sqrt{\pi}} \exp\left(-\frac{\alpha^2\hat{r}^2}{2}\right)\left[I_0\left(\frac{\alpha^2\hat{r}^2}{2}\right) - I_1\left(\frac{\alpha^2\hat{r}^2}{2}\right)\right]. \tag{27}$$

Here $\hat{\Omega}_0(\hat{r})$ represents the angular velocity in cold (pressureless) disks in all the above cases. However, the contributions from the "pressure" term can be obtained using the equation of state (eq. [5]).

We consider here the stability of the class of finite disks represented by equations (18) and (20). However, infinite models, given by equations (22) and (25), can also be studied by the approach discussed in § IV.

IV. THE NORMAL MODE ANALYSIS

We wish to study the stability of axisymmetric, equilibrium disks against infinitesimally small perturbations of the form

$$\tilde{A}(\hat{r}, \theta, \tau) = A(\hat{r}) \exp[i(\omega\tau + m\psi)], \tag{28}$$

where ω is the frequency and m is the azimuthal wavenumber of the perturbations. In a linear analysis, the modes of oscillation (m, ω) are uncoupled and can be studied separately.

In most asymptotic studies for tightly wrapped perturbations, based on WKBJ type of approximation, one can solve the problem analytically. Lin and Shu (1964) thus obtained a "local" dispersion relation which allows the radial wavenumber to take on a continuum of values. However, boundary conditions are incorporated subsequently by the method of matched asymptotic expansions (see Mark 1977), leading to phase integral constraints which provide asymptotic approximations of the discrete global modes. Pannatoni and Lau (1979) extended this approach further by a numerical scheme to evaluate exact discrete global modes by solving an integro-differential equation along with certain radiation conditions (see Lau, Lin, and Mark 1976).

Alternatively, the allowed modes of oscillations in the plane of the disk can also be obtained by an indirect method, first suggested by Hunter (1965), according to which the radial parts $A(\hat{r})$ of the perturbed quantities are expressed in terms of certain infinite expansions and the problem is essentially posed in a matrix eigenvalue eigenfunction form. We here expand functions $A(\hat{r})$ in equation (28) as

$$\sigma(\hat{r}) = \sum_{k=0}^{\infty} C_k J_m(\hat{\lambda}_k^{(m)}\hat{r}), \qquad \psi(\hat{r}) = \sum_{k=0}^{\infty} C_k \psi_m(\hat{r}), \qquad u(\hat{r}) = \frac{i}{2}\sum_{k=0}^{\infty}\left[A_k J_{m+1}(\hat{\lambda}_k^{(m)}\hat{r}) + B_k J_{m-1}(\hat{\lambda}_k^{(m)}\hat{r})\right],$$

and

$$v(\hat{r}) = \frac{1}{2}\sum_{k=0}^{\infty}\left[A_k J_{m+1}(\hat{\lambda}_k^{(m)}\hat{r}) - B_k J_{m-1}(\hat{\lambda}_k^{(m)}\hat{r})\right] \tag{29}$$

(see Yabushita 1969*a*). Here A_k, B_k, and C_k are certain expansion coefficients to be determined later. The quantities ψ_m are as defined in equation (12). We shall drop caps from λ, k', and r in what follows.

Now we linearize the set of hydrodynamic equations (1)–(3) to obtain the equations describing the perturbed quantities of the form (28). Using the radial functions expressed by equation (29), one obtains an infinite set of equations:

$$\sum_{k=0}^{\infty}\left\{-A_k J_{m+1}(\lambda_k r)\left[\omega+\left(m+\frac{3}{2}\right)\frac{V_0}{r}+\frac{V_0'}{2}\right]+B_k J_{m-1}(\lambda_k r)\left[\frac{V_0}{2r}-\frac{V_0'}{2}\right]\right\}$$
$$=-\sum_{k=0}^{\infty}C_k\left\{\frac{\gamma P_0}{\sigma_0^2}\left[-\lambda_k J_{m+1}(\lambda_k r)+(\gamma-2)\frac{\sigma_0'}{\sigma_0}J_m(\lambda_k r)\right]+\frac{1}{\pi}\left[J_{m+1}(\lambda_k r)-I_{k,m+1}^{(\infty)}\right]\right\}, \tag{30}$$

$$\sum_{k=0}^{\infty}\left\{A_k J_{m+1}(\lambda_k r)\left[\frac{V_0'}{2}-\frac{V_0}{2r}\right]+B_k J_{m-1}(\lambda_k r)\left[\frac{V_0'}{2}-\left(m-\frac{3}{2}\right)\frac{V_0}{r}-\omega\right]\right\}$$
$$=\sum_{k=0}^{\infty}C_k\left\{\frac{\gamma P_0}{\sigma_0^2}\left[\lambda_k J_{m-1}(\lambda_k r)+(\gamma-2)\frac{\sigma_0'}{\sigma_0}J_m(\lambda_k r)\right]+\frac{1}{\pi}\left[J_{m-1}(\lambda_k r)-I_{k,m-1}^{(\infty)}\right]\right\}, \tag{31}$$

$$\sum_{k=0}^{\infty}\left\{A_k\frac{\sigma_0}{2}\left[\lambda_k J_m(\lambda_k r)+\frac{\sigma_0'}{\sigma_0}J_{m+1}(\lambda_k r)\right]+B_k\frac{\sigma_0}{2}\left[\frac{\sigma_0'}{\sigma_0}J_{m-1}(\lambda_k r)-\lambda_k J_m(\lambda_k r)\right]+C_k J_m(\lambda_k r)\left[\omega+m\frac{V_0}{r}\right]\right\}=0, \tag{32}$$

where we have defined the modifications arising from the pressure distribution (5) in the disk and where the superscripts m of the roots have been dropped. The infinite integrals $I_{k,m+1}^{(\infty)}, I_{k,m-1}^{(\infty)}$ are defined as

$$I_{k,m+1}^{(\infty)}=(2/\pi)\lambda_k J_{m+1}(\lambda_k)\int_0^{\infty}\left[k'I_{m+1}(k'r)K_m(k')\,dk'/(\lambda_k^2+k'^2)\right],$$

$$I_{k,m-1}^{(\infty)}=(2/\pi)\lambda_k J_{m-1}(\lambda_k)\int_0^{\infty}\left[k'I_{m-1}(k'r)K_m(k')\,dk'/(\lambda_k^2+k'^2)\right].$$

Now, multiplying equations (30)–(32) by $rJ_{m+1}(\lambda_k r)$, $rJ_{m-1}(\lambda_k r)$ and $rJ_m(\lambda_k r)$, respectively, and integrating over the disk, one gets

$$\sum_{k=0}^{\infty}\left(A_k P_{kj}+B_k Q_{kj}+C_k R_{kj}\right)=\omega A_j, \tag{33}$$

$$\sum_{k=0}^{\infty}\left(A_k S_{kj}+B_k T_{kj}+C_k U_{kj}\right)=\omega B_j, \tag{34}$$

$$\sum_{k=0}^{\infty}\left(A_k V_{kj}+B_k W_{kj}+C_k X_{kj}\right)=\omega C_j, \tag{35}$$

where the elements P_{kj}, Q_{kj}, etc., are defined in the Appendix. The set of algebraic equations thus obtained can, conveniently, be written in a matrix form

$$\begin{bmatrix} P & Q & R\\ S & T & U\\ V & W & X\end{bmatrix}\begin{bmatrix}A\\ B\\ C\end{bmatrix}=\omega\begin{bmatrix}A\\ B\\ C\end{bmatrix}, \tag{36}$$

or

$$\mathbf{MZ}=\omega\mathbf{Z},$$

which defines an eigenvalue problem with ω as eigenfrequencies and $\mathbf{Z}$ the corresponding eigenfunctions. The triply infinite dimensional matrix **M** is, in general, real and nonsymmetric with infinite submatrices **P**,**Q**,...,etc. It is not possible to evaluate the eigenvalues and eigenfunctions without truncating the matrix **M**. Since the nonaxisymmetric, truncated matrix **M** in equation (36) cannot in general be diagonalized, an exact evaluation of the eigenmodes is not possible. However, a satisfactory convergence of some (if not all) of the modes can be achieved by using an adequately large matrix. Since, in the linear analysis, each mode satisfies the governing equations independently of others, convergence of an eigenmode would ensure its validity, even if other modes remain doubtful.

If the expansions (29) retain n terms, the resulting eigenmodes possess, at most, $n-1$ nodes in the disk, and shorter wavelength modes (with a larger number of nodes) would not appear. As one increases n, previously determined modes are refined, while new modes with shorter wavelengths appear. However, our interest is, mainly, in rather open modes, since the thin disk approximation is violated for extremely short wavelength perturbations. In fact, for short modes, the asymptotic theory itself would provide reasonably accurate results.

We have examined the convergence of various eigenmodes admitted by the disk models under study here, and it is found that comparatively large wavelength modes converge adequately with the expansions in equation (29) retaining only nine terms. However, the growth rates of highly unstable, short wavelength modes in cold disks appear to be rather overestimated. The convergence of the open modes is good in hot disks also.

a) *Cold Disks* ($\beta = 0$)

It has been shown, analytically, that short wavelength, nonaxisymmetric unstable modes in cold disks exhibit leading patterns provided $\Omega'(r) < 0$ (Hunter 1969*a*). We find here that relatively less unstable, large wavelength (open) modes too are leading.

Figure 1 shows unstable modes for various values of m in the $N = 2$ disk. Real (or purely oscillatory) and damped modes are not shown in the figure. Growth rates of corresponding modes for different values of m are comparable; hence, the bisymmetric modes are not preferred over others. Although the growth rates of various modes for a fixed m vary considerably, pattern velocities, Ω_p ($\equiv -\omega_r/m$) are more or less the same, especially for smaller values of m (as in Hunter 1969*a*).

The patterns of oscillations associated with the perturbed surface density for $m = 2$ and 3 are shown in the frames of Figures 2 and 3, respectively, in the $N = 2$ disk. The complex frequencies for these unstable modes are given in the brackets. Trailing patterns would be associated with the damped modes, while the neutral modes allow sectorial patterns (Lynden-Bell and Ostriker 1967). Similar diagrams have been obtained for disks with larger N values and also for Bessel disks described in § III, and, in all cases, unstable modes are associated with smooth leading patterns. However, Iye (1978) and Aoki, Noguchi, and Iye (1979) find only highly irregular and ill-converged, tightly wrapped leading patterns in extremely cold disks.

We consider various disk models to examine nonaxisymmetric modes in centrally concentrated disks. It is found that some axisymmetric modes are stabilized in such disks (Hunter 1965). A similar result is found here for the nonaxisymmetric modes, as illustrated by Figure 4 in the case of $m = 2$ modes in various disks. As the central concentration (i.e., N) is enhanced, some modes with rather large wavelength (and smaller growth rates) become less unstable, and, eventually, some are even stabilized. On the other hand, short wavelength modes grow much faster in centrally condensed disks.

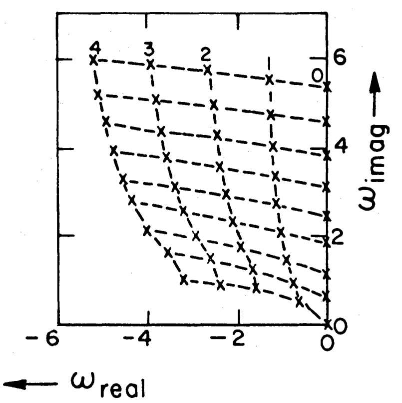

FIG. 1.—Eigenvalues of the $N = 2$ (cold) disk for $0 \leqslant m \leqslant 4$. Only growing modes are shown.

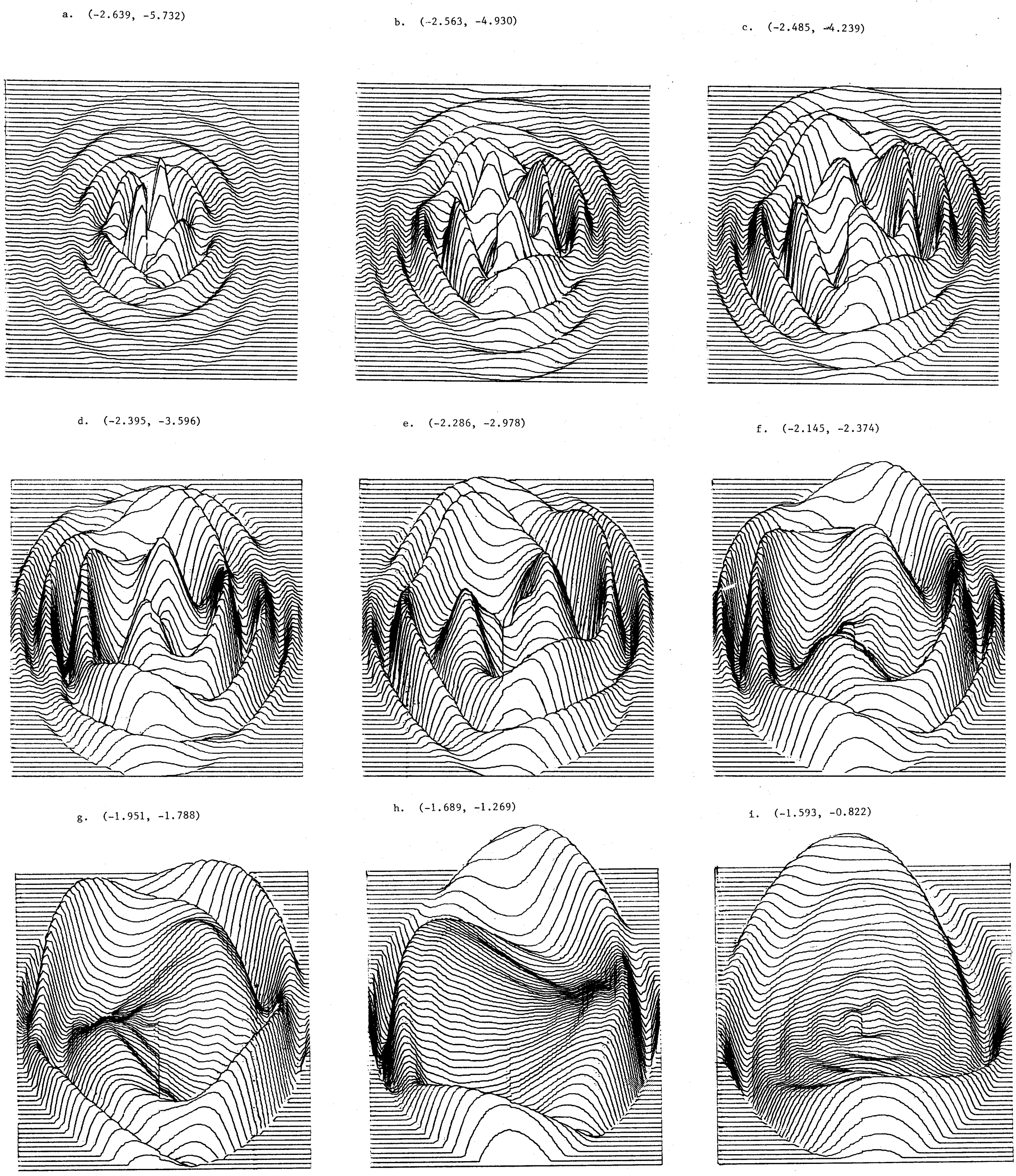

FIG. 2.—Eigenpatterns of perturbed surface density $\tilde{\sigma}(\hat{r}, \theta, \tau = 0)$ associated with some $m = 2$ (bisymmetric) modes of the $N = 2$ (cold) disk. Numbers in brackets are the real and imaginary parts of eigenvalues.

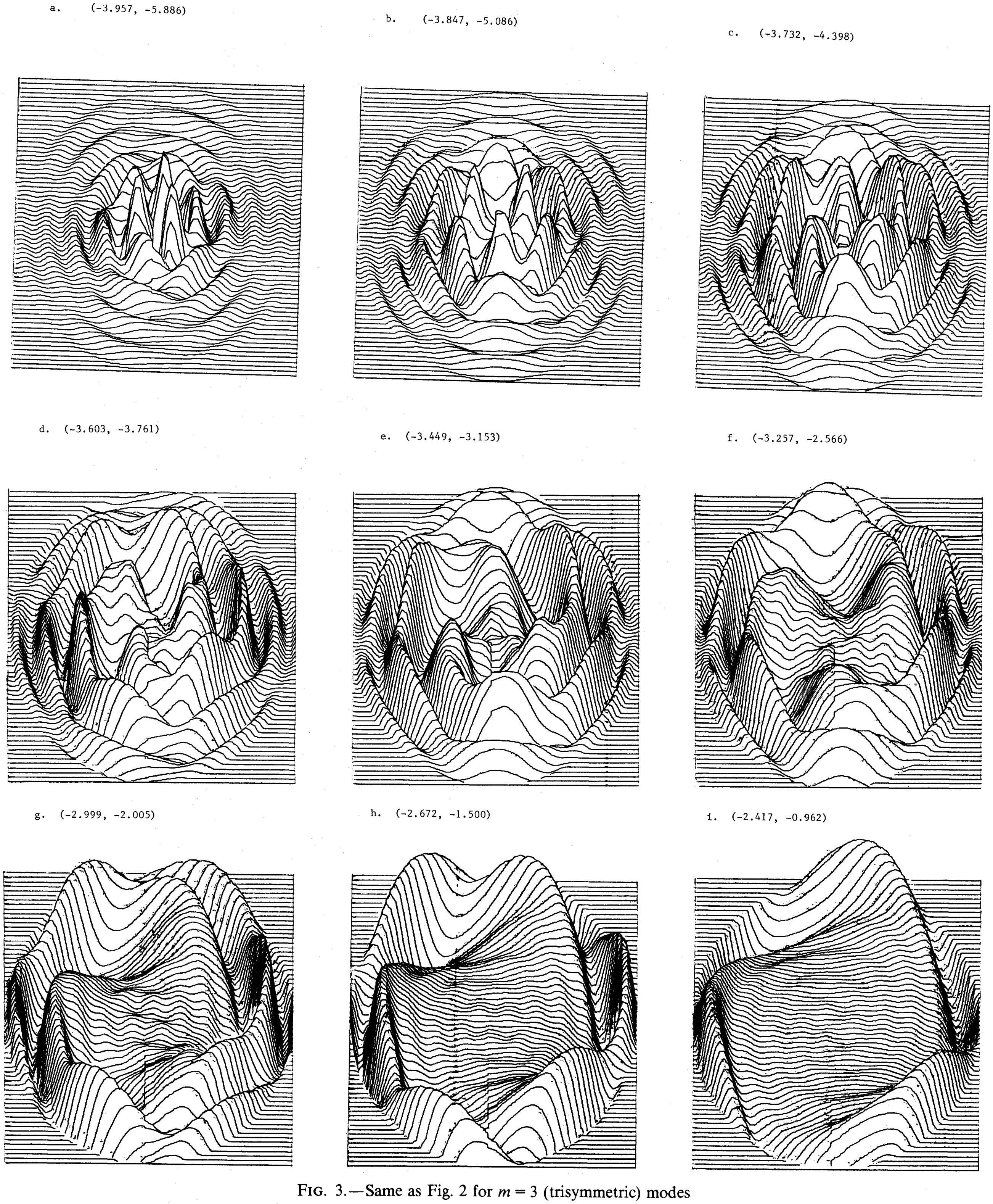

FIG. 3.—Same as Fig. 2 for $m = 3$ (trisymmetric) modes

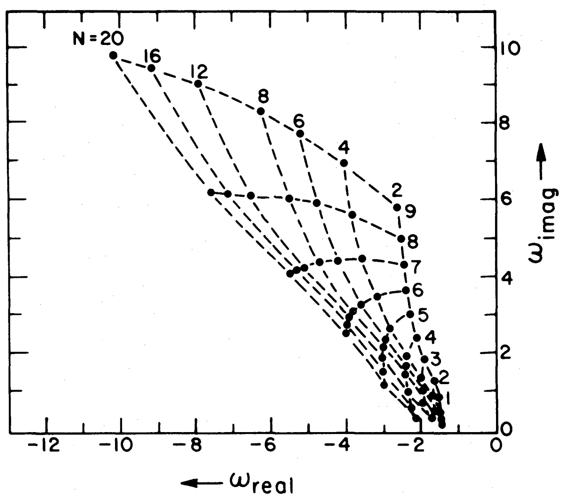

FIG. 4.—Eigenvalues of growing ($m = 2$) modes in various pressureless disks

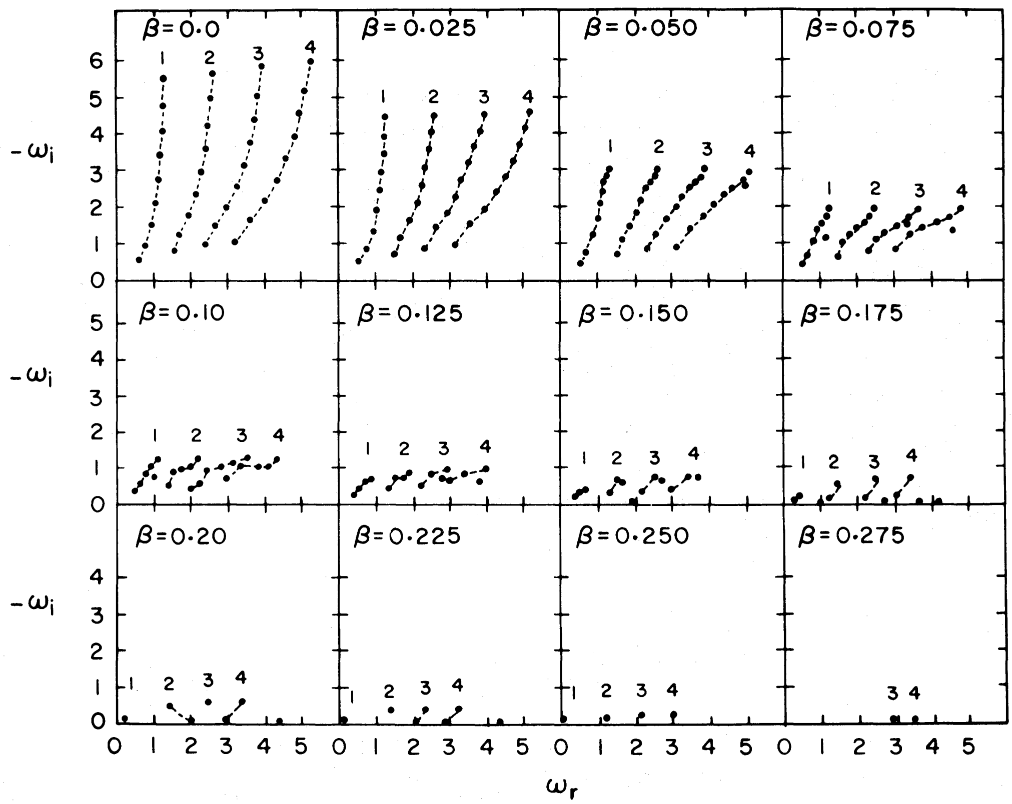

FIG. 5.—Eigenvalues of growing modes for $1 \leqslant m \leqslant 4$ in $N = 2$ (warm) disks

b) *Warm Disks* ($\beta \neq 0$)

According to the "local" density wave theory, short wavelength perturbations are suppressed by pressure, and no unstable modes exist if the thermal velocity is large enough that $Q = 1$ is satisfied everywhere in the disk (Toomre 1964). At the solar neighborhood, $Q \approx 1.5$ (Toomre 1974) in the Milky Way. According to the local theory, the region of any conceivable waves extends all the way from the inner Lindblad resonance to the outer; however, with $Q \approx 1.5$,

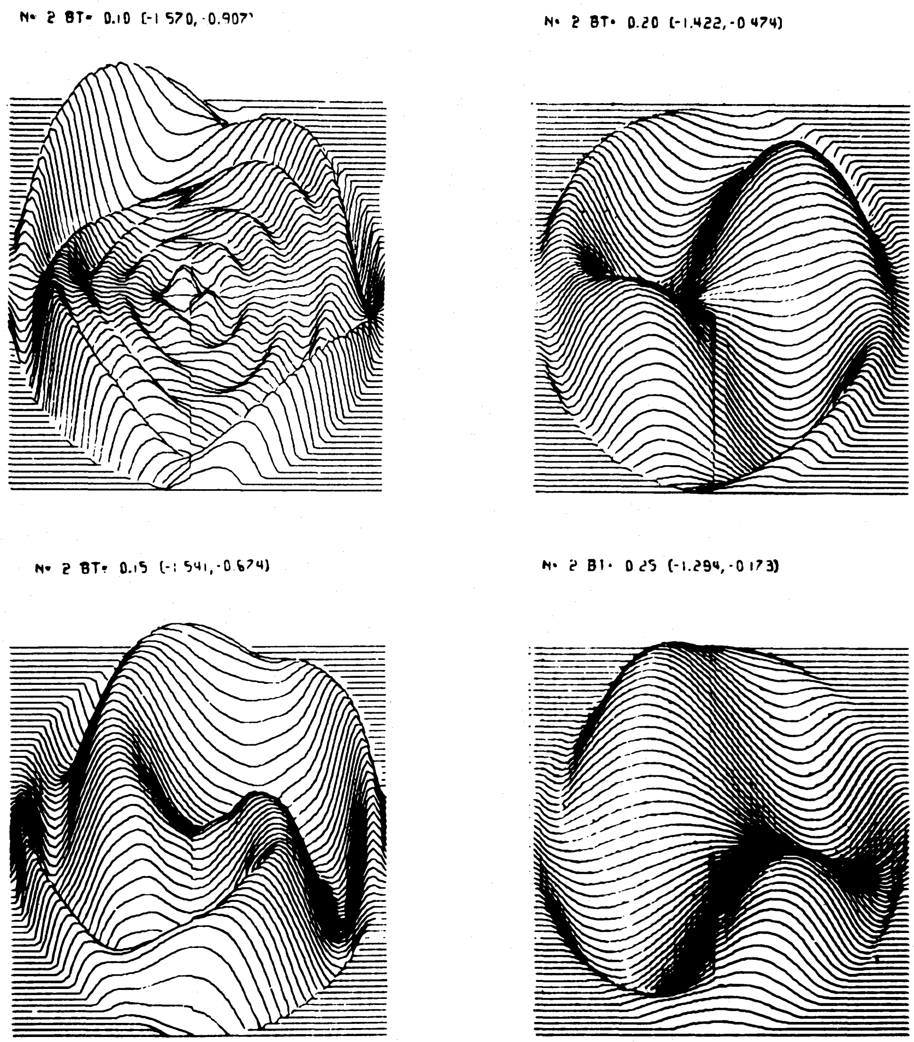

FIG. 6.—Eigenpatterns of perturbed surface density, $\tilde{\sigma}(\hat{r}, \theta, \tau = 0)$, for one of the slowest growing modes of the $N = 2$ (cold) disk as β is increased.

the Lin-Shu dispersion relation no longer admits waves within a fairly extensive annulus, and the tightly wound spirals become cramped up.

On the other hand, computer experiments reveal that even if $Q \gtrsim 1$, the system remains unstable against global bar modes (Hohl 1970; Miller, Prendergast, and Quirk 1970), and large velocity dispersions comparable to, or greater than, the rotational velocity are required to stabilize the disk completely (Ostriker and Peebles 1973). The global mode studies also require comparatively large thermal velocities for a complete stability.

Here various frames of Figure 5 show the stabilizing trend of various nonaxisymmetric modes in $N = 2$ warm disks, β measuring the "hotness" as defined in equation (17). We have considered $\gamma = 7/3$ for $N = 2$ according to § III*a*.

All the unstable modes lie on a smooth branch in the complex ω-plane in moderately warm disks. A moderate increase in β reduces the growth rates of more unstable, short wavelength modes, while relatively slowly growing, large wavelength modes are rather unaffected. At $\beta = 0.275$, most of the unstable modes are already suppressed completely. However, there remain several globally unstable, nonaxisymmetric modes even when the local stability criterion, $Q = 1.0$, holds in major portions of the disk (corresponding to $\beta \approx 0.135$ in the $N = 2$ disk here). All axisymmetric as well as nonaxisymmetric modes are stabilized when $\beta \approx 0.35$, which corresponds to $Q \approx 1.5$. It is found here, as in Smith (1979), that nonaxisymmetric modes with larger values of m are more difficult to suppress by pressure. The bar modes in computer simulations require β as large as 0.72 for a complete stability (Ostriker and Peebles 1973), which is not entirely surprizing because in such studies the nonlinearities of the system are retained.

This study reveals a connection between leading and trailing natures of the global modes. It is found that various unstable leading modes of cold disks gradually become trailing in warm disks. In each case, the smooth leading patterns break into trailing segments in the central and leading segments in the outer regions of the disk as β is increased, and, ultimately, smooth trailing patterns occupy the entire disk. The spiral segments of the trailing patterns shorten and open up in more "hot" disks, and, finally, they become almost "sectorial" or spokelike when the modes are nearly stabilized. It is important to note that relatively open (or large wavelength) unstable modes are rather difficult to suppress by pressure.

Figure 6 illustrates the patterns of oscillations for the perturbed surface density associated with the slowest growing mode of $m = 2$ perturbation in the $N = 2$ disk as β is increased. (This mode at $\beta = 0$ is shown in frame h of Fig. 2.) At $\beta = 0.1$, the pattern is rather ill defined and irregular. However, at $\beta = 0.15$, it has already turned into smooth, trailing mode; the spiral segments are shortened as β increases further.

V. DISCUSSION

Global spiral patterns appear as self-excited, natural modes of oscillations in the plane of flat, differentially rotating disks. The main results for these modes are, essentially, similar to those obtained by earlier authors; however, there are some differences.

Iye (1978) and Aoki, Noguchi, and Iye (1979) found rather ill-defined and irregular leading patterns in extremely cold disks. However, both open and tightly wrapped, smooth leading modes with varying growth rates appear in cold disks studied here. With the inclusion of thermal pressure, the results change significantly, and short wavelength modes are completely stabilized. Also, the leading mode becomes, gradually, trailing in warm disks and almost sectorial when the modes are nearly stabilized.

Nonaxisymmetric modes in centrally concentrated, cold disks have been found to stabilize, as is the case with axisymmetric modes (Hunter 1965).

Pannatoni and Lau (1979) have obtained primarily trailing patterns by a different approach. They impose certain radiation conditions to account for an outward angular momentum transfer in gaseous disks; consequently, they consider perturbations asymptotic to short trailing waves in the far field. However, unstable trailing modes appear in warm disks considered here even when no angular transport is explicit in the analysis. Thus, it is not clear that the transport of angular momentum, occasioned by dissipative processes, alone would decide the nature of the spiral patterns—trailing versus leading.

The nonlinear evolution of the global modes, which may result in the global transport of mass, angular momentum, and energy, would be very interesting. At present such studies are restricted only to tightly wound spiral waves (Kaplan, Khodataev, and Tsytovitch 1977; Norman 1978).

Hunter (1969b) has shown that the frequency spectra of nonaxisymmetric modes of oscillations in the plane of cold, differentially rotating flat disks can have discrete and continuous parts. Since the present study concerns an eigenvalue problem involving a truncated matrix, only a finite number of real and complex conjugate pairs of discrete eigenfrequencies appear, even though the untruncated, triply infinite matrix may allow a continuum in certain intervals. Some conclusions may, however, be drawn by the study of eigenmodes of truncated problems, as follows.

As the size of the matrix is increased, the number of modes, both complex as well as real, increases. A poor convergence of strongly unstable (and damped) short wavelength modes coupled with an increase in the number of such modes may suggest a continuum in short wavelength eigenmodes. A similar continuum may be expected for real (i.e., stable) modes also. However, rather large-scale complex modes are found to converge rapidly to well-defined values; hence, they are valid discrete modes.

The spiral structure of disk galaxies appears to be very sensitive to various morphological properties, such as the gaseous and stellar contents, the disk-to-bulge ratio, etc. In a separate study, we have investigated the stability of a gaseous disk under the influence of a fixed central "bulge" component (Ambastha and Varma 1982). A transition from tight spirals to rather open patterns is obtained as the bulge content diminishes. The "explosive" instabilities (i.e., $\omega_i \gtrsim \omega_r$), especially in cold disks of present study, do not occur in disks with massive bulge components.

APPENDIX

The matrix elements, constituting the submatrices of $\mathbf{M}$ in equation (36), are defined here as follows:

$$P_{kj} = \Big\{ -\frac{1}{2} V_0(1) J_{m+1}(\lambda_j) J_{m+1}(\lambda_k) - 2(m+1)\int_0^1 V_0(r) J_{m+1}(\lambda_j r) J_{m+1}(\lambda_k r)\, dr + \frac{1}{2}\int_0^1 V_0(r)\big[\lambda_j r J_m(\lambda_j r) J_{m+1}(\lambda_k r) + \lambda_k r J_{m+1}(\lambda_j r) J_m(\lambda_k r)\big]\, dr \Big\} \Big/ M_j; \quad \text{(A1)}$$

$$Q_{kj}=\left\{-\frac{1}{2}V_0(1)J_{m+1}(\lambda_j)J_{m-1}(\lambda_k)+\frac{1}{2}\int_0^1 V_0(r)\left[\lambda_j rJ_m(\lambda_j r)J_{m-1}(\lambda_k r)-\lambda_k rJ_m(\lambda_k r)J_{m+1}(\lambda_j r)\right]dr\right\}\Big/M_j; \tag{A2}$$

$$R_{kj}=\left\{-\gamma c\int_0^1\left[\lambda_j r\sigma_0^{\gamma-2}J_m(\lambda_j r)J_m(\lambda_k r)\right]dr\right.$$
$$\left.+\frac{1}{\pi}\left[\frac{1}{2}J_{m+1}(\lambda_k)J_{m+1}(\lambda_k)\delta_{jk}-\frac{2}{\pi}\lambda_k J_{m+1}(\lambda_k)J_{m+1}(\lambda_j)\int_0^\infty\frac{k'^2I_m(k')K_m(k')\,dk'}{(\lambda_k^2+k'^2)(\lambda_j^2+k'^2)}\right]\right\}\Big/M_j; \tag{A3}$$

$$S_{kj}=\left\{\frac{1}{2}V_0(1)J_{m-1}(\lambda_j)J_{m+1}(\lambda_k)+\frac{1}{2}\int_0^1 V_0(r)\left[\lambda_j rJ_m(\lambda_j r)J_{m+1}(\lambda_k r)-\lambda_k rJ_{m-1}(\lambda_k r)J_m(\lambda_j r)\right]dr\right\}\Big/M_j'; \tag{A4}$$

$$T_{kj}=\left\{\frac{1}{2}V_0(1)J_{m-1}(\lambda_j)J_{m-1}(\lambda_k)+\frac{1}{2}\int_0^1 V_0(r)\left[\lambda_j rJ_m(\lambda_j r)J_{m-1}(\lambda_k r)+\lambda_k rJ_m(\lambda_k r)J_{m-1}(\lambda_j r)\right]dr\right.$$
$$\left.-2(m-1)\int_0^1 V_0(r)J_{m-1}(\lambda_j r)J_{m-1}(\lambda_k r)\,dr\right\}\Big/M_j'; \tag{A5}$$

$$U_{kj}=\left\{\gamma c\int_0^1\lambda_j r\sigma_0^{\gamma-2}J_m(\lambda_j r)J_m(\lambda_k r)\right.$$
$$\left.-\frac{1}{\pi}\left[\frac{1}{2}J_{m-1}(\lambda_k)J_{m-1}(\lambda_k)\delta_{jk}-\frac{2}{\pi}\lambda_k J_{m-1}(\lambda_k)J_{m-1}(\lambda_j)\int_0^\infty\frac{k'^2I_m(k')K_m(k')}{(\lambda_k^2+k'^2)(\lambda_j^2+k'^2)}dk'\right]\right\}\Big/M_j'; \tag{A6}$$

$$V_{kj}=\left[-\frac{1}{2}\lambda_j\int_0^1 r\sigma_0(r)J_{m+1}(\lambda_j r)J_{m+1}(\lambda_k r)\,dr\right]\Big/M_j''; \tag{A7}$$

$$W_{kj}=\left[\frac{1}{2}\lambda_j\int_0^1 r\sigma_0(r)J_{m-1}(\lambda_j r)J_{m-1}(\lambda_k r)\,dr\right]\Big/M_j''; \tag{A8}$$

and

$$X_{kj}=\left[-m\int_0^1 V_0(r)J_m(\lambda_j r)J_m(\lambda_k r)\,dr\right]\Big/M_j''; \tag{A9}$$

where

$$\delta_{kj}=\begin{cases}0, & k\neq j\\ 1, & k=j\end{cases};\qquad M_j=\tfrac{1}{2}\left[J_{m+1}(\lambda_j)\right]^2;\qquad M_j'=\tfrac{1}{2}\left[J_{m-1}(\lambda_j)\right]^2;\qquad M_j''=-\tfrac{1}{2}J_{m-1}(\lambda_j)J_{m+1}(\lambda_k), \tag{A10}$$

and $J_m(x)$, $I_m(x)$, $K_m(x)$ are Bessel functions of first kind and modified Bessel functions of first and second kind, respectively.

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