WAHL'S CONJECTURE HOLDS IN ODD CHARACTERISTICS FOR SYMPLECTIC AND ORTHOGONAL GRASSMANNIANS

V. LAKSHMIBAI, K. N. RAGHAVAN, AND P. SANKARAN

Abstract. It is shown that the proof by Mehta and Parameswaran of Wahl's conjecture for Grassmannians in positive odd characteristics also works for symplectic and orthogonal Grassmannians.

Let X be a non-singular projective variety over an algebraically closed field $\boldsymbol{k}$. For ample line bundles L and M over X, consider the natural restriction map (called the *Gaussian*)

$$H^0(X \times X, \mathcal{I}_\Delta \otimes p_1^* L \otimes p_2^* M) \to H^0(X, \Omega^1_{X/\boldsymbol{k}} \otimes L \otimes M) \tag{1}$$

where $\mathcal{I}_\Delta$ denotes the ideal sheaf of the diagonal Δ in $X \times X$, p_1 and p_2 the first and second projections of $X \times X$ on X, and $\Omega^1_{X/\boldsymbol{k}} = \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ the sheaf of differential 1-forms of X over $\boldsymbol{k}$. Wahl conjectured in [11] that this map is surjective when X is a homogeneous space for the action of a semisimple group G, that is, when $X = G/P$ for G a semisimple and simply connected linear algebraic group over $\boldsymbol{k}$ and P a parabolic subgroup of G. The original conjecture was perhaps meant only over the field of complex numbers, and in fact it has been proved in that case by Kumar [3] using representation theoretic techniques, but following [4, 7] we use the term 'Wahl's conjecture' to refer to the surjectivity of the Gaussian without any restriction on the characteristic. The truth of the conjecture in infinitely many positive characteristics would imply its truth in characteristic zero, for the Gaussian is defined over the integers.

Assume from now on that the base field $\boldsymbol{k}$ has positive characteristic. Lakshmibai, Mehta, and Parameswaran [4] show that, in odd characteristic, Wahl's conjecture holds if there is a Frobenius splitting of $X \times X$ that compatibly splits the diagonal Δ and has maximal multiplicity along Δ (the definitions are recalled below). Moreover they conjecture that such a Frobenius splitting exists (in any characteristic). Mehta and Parameswaran [7] prove that this latter conjecture holds for Grassmannians. In the present paper it is shown that their proof also works for symplectic and orthogonal Grassmannians—see Theorem 8 and the conclusion in §6 below.

This paper is organized as follows: notation is fixed in §1, some basic definitions and results about Frobenius splittings are recalled in §2, the results of [4] about splittings for blow-ups are recalled in §3, some results about canonical splittings are

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recalled in §4, the main result (Theorem 8) is proved in §5, and Wahl's conjecture for ordinary, symplectic, and orthogonal Grassmannians is deduced from the main theorem in §6 following the argument in [7].

1. Notation

The following notation remains fixed throughout:

- $\boldsymbol{k}$ an algebraically closed field of positive characteristic p;
- $V^{[m]}$ where V is a $\boldsymbol{k}$-vector space and m an integer denotes the $\boldsymbol{k}$-vector space obtained by pulling back V via the automorphism $\lambda \mapsto \lambda^{p^m}$ of $\boldsymbol{k}$.
- G a semisimple, simply connected group linear algebraic over $\boldsymbol{k}$;
- B a Borel subgroup of G;
- T a maximal torus of G such that $T \subseteq B$;
- P a *standard* parabolic subgroup of G (standard means $B \subseteq P$);
- the roots are taken with respect to T;
- the positive roots are taken to be the characters for the adjoint action of T on the Lie algebra of the unipotent part U of B;
- ρ denotes half the sum of all positive roots (equivalently, the sum of all fundamental weights);
- ℓ is the rank of G; $\varpi_1, \ldots, \varpi_\ell$ are the fundamental weights with respect to T and B; these are assumed to be ordered as in Bourbaki [1] for G simple.
- w_0 denotes the longest element of the Weyl group;
- ι denotes the Weyl involution $\lambda \mapsto -w_0\lambda$ on characters of T;
- for a character λ of T,
  - λ denotes also its extension to B via the isomorphism $B/U \cong T$ (induced by the inclusion of T in B);
  - $\boldsymbol{k}_\lambda$ denotes the one dimensional B-module defined by the character λ;
  - $\mathcal{L}(\lambda)$ denotes the line bundle $G \times_B k_{-\lambda}$ on G/B;
  - $H^0(G/B, \mathcal{L}(\lambda))$ denotes the G-module of global sections of $\mathcal{L}(\lambda)$.

Observe that $H^0(G/B, \mathcal{L}(\lambda))$ can be identified as a G-module with the space of regular functions f on G that transform thus:

$$f(gb) = \lambda(b)f(g) \quad \forall\, g \in G \quad \forall\, b \in B \tag{2}$$

the action of G on functions being given by $(gf)(x) := f(g^{-1}x)$; the highest and lowest weights of $H^0(G/B, \mathcal{L}(\lambda))$ are respectively $-w_0\lambda$ and $-\lambda$. Observe also that the anti-canonical bundle K^{-1} of G/B is $\mathcal{L}(2\rho)$, for $G \times^B \mathfrak{g}/\mathfrak{b} \to G/B$ is the tangent bundle of G/B, where $\mathfrak{g}$ and $\mathfrak{b}$ denote the Lie algebras of G and B respectively.

2. Frobenius Splittings

Let X be a scheme over $\boldsymbol{k}$, separated and of finite type. Denote by F the *absolute Frobenius* map on X: this is the identity map on the underlying topological space X and is the p-th power map on the structure sheaf $\mathcal{O}_X$. We say that X is *Frobenius split* if the p-th power map $F^\# : \mathcal{O}_X \to F_*\mathcal{O}_X$ splits as a map of $\mathcal{O}_X$-modules (see [8, §1, Definition 2], [2, Definition 1.1.3]). A splitting $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ *compatibly splits* a closed subscheme Y of X if $\sigma(F_*\mathcal{I}_Y) \subseteq \mathcal{I}_Y$ where $\mathcal{I}_Y$ is the ideal sheaf of Y (see [8, §1, Definition 3], [2, Definition 1.1.3]).

Now suppose that X is a non-singular projective variety and denote by K its canonical bundle. Using Serre duality and the observation that $F^*\mathcal{L} \cong \mathcal{L}^p$ for an invertible sheaf $\mathcal{L}$ on X (applied to K), we get a canonical isomorphism of $H^0(X, \mathcal{H}\mathrm{om}(F_*\mathcal{O}_X, \mathcal{O}_X)) = \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ with $H^0(X, K^{1-p})^{[1]}$ (see [8, Page 32] or [2, Lemma 1.2.6 and §1.3]; without the superscript '[1]', the isomorphism would only be $\boldsymbol{k}$-semilinear). To say that σ *splits* X for $\sigma \in H^0(X, K^{1-p}) \cong \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)^{[-1]}$ means that the underlying $\mathcal{O}_X$-module homomorphism $F_*\mathcal{O}_X \to \mathcal{O}_X$ is a splitting of $F^{\#}$. Set

$$\boxed{\mathrm{End}_F(X) := \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)^{[-1]}}$$

3. Splittings and Blow-ups

Let Z be a non-singular projective variety and σ a section of K^{1-p} (where K is the canonical bundle) that splits Z. Let Y be a closed non-singular subvariety of Z of codimension c. Let $\mathrm{ord}_Y\sigma$ denote the order of vanishing of σ along Y. Let $\tilde{Z}$ denote the blow up of Z along Y and E the exceptional divisor (the fiber over Y) in $\tilde{Z}$.

A splitting $\tilde{\tau}$ of $\tilde{Z}$ induces a splitting τ on Z, by virtue of $\pi_*\mathcal{O}_{\tilde{Z}} \leftarrow \mathcal{O}_Z$ being an isomorphism where $\pi : \tilde{Z} \to Z$ is the natural map (see the result to this effect quoted in §4 below). We say that σ *lifts* to a splitting of $\tilde{Z}$ if it is induced thus from a splitting $\tilde{\sigma}$ of $\tilde{Z}$. The lift of σ to $\tilde{Z}$ is unique if it exists, since $\tilde{Z} \to Z$ is birational and two global sections of the locally free sheaf $\mathcal{H}\mathrm{om}_{\mathcal{O}_{\tilde{Z}}}(F_*\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}})$ that agree on an open set must be equal.

Proposition 1. *With notation as above, we have*

(1) $\mathrm{ord}_Y\sigma \leq c(p-1)$.
(2) *If* $\mathrm{ord}_Y\sigma = c(p-1)$ *then* Y *is compatibly split.*
(3) $\mathrm{ord}_Y\sigma \geq (c-1)(p-1)$ *if and only if* σ *lifts to a splitting* $\tilde{\sigma}$ *of* $\tilde{Z}$*; moreover,* $\mathrm{ord}_Y\sigma = c(p-1)$ *if and only if the splitting* $\tilde{\sigma}$ *is compatible with* E.

Proof: These statements appear as exercises in [2, 1.3.E.12]. In any case, (1) and (2) are elementary to see from the local description as in [8, Proposition 5] of the functorial isomorphism between $\mathrm{End}_F(X)$ and $H^0(X, K^{1-p})$. Statement (3) is Proposition 2.1 of [4]. □

The proposition above justifies the definition below:

Definition 2. ([4, Remark 2.3]) We say that Y is *compatibly split by σ with maximal multiplicity* if $\mathrm{ord}_Y\sigma = c(p-1)$.

Now let $Z = G/P \times G/P$, and Y the diagonal copy of G/P in Z. We have:

Theorem 3. ([4, pages 106–7]) *Assume that the characteristic p is odd. If E is compatibly split in $\tilde{Z}$, or, equivalently, if there is a splitting of Z compatibly splitting Y with maximal multiplicity, then the Gaussion map (1) is surjective for $X = G/P$.*

Conjecture 4. ([4, page 106]) For any G/P, there exists a splitting of Z that compatibly splits the diagonal copy of G/P with maximal multiplicity.

4. Canonical splitting

For a B-scheme X there is the notion of a *B-canonical element* in $\mathrm{End}_F(G/P)$ ([10, Definition 4.3.5], [2, Definitions 4.1.1, 4.1.4]). We can take the following characterization to be the definition:

Proposition 5. ([2, Lemma 4.1.6]) *For a B-scheme X, an element ϕ belonging to $End_F(X)$ is B-canonical if and only if there exists a $\boldsymbol{k}$-linear B-module map*

$$\Theta_\phi : \mathrm{St} \otimes \boldsymbol{k}_{(p-1)\rho} \to End_F(X) \quad \text{with } \Theta_\phi(f_- \otimes f_+) = \phi$$

where $0 \neq f_+ \in \boldsymbol{k}_{(p-1)\rho}$ and f_- is a non-zero lowest weight vector of the Steinberg module $\mathrm{St} := H^0(G/B, \mathcal{L}((p-1)\rho)$.

Theorem 6. ([5]; [2, Theorem 4.1.15]) *There is a unique (up to non-zero scalar multiples) non-zero B-canonical element in $End_F(G/P)$. Moreover, this element is a splitting of G/P and compatibly splits all Schubert and opposite Schubert subvarieties.*

Since, on the one hand, splittings of X are mapped to splittings of Y under f_* for any morphism $f : X \to Y$ of schemes such that $f_*\mathcal{O}_X \leftarrow \mathcal{O}_Y$ is an isomorphism ([8, Proposition 4], [2, Proposition 1.1.8]), and, on the other hand, as is readily seen (see also [2, Exercise 4.1.E.3]), B-canonical elements of $\mathrm{End}_F(Y)$ are mapped to B-canonical elements of $\mathrm{End}_F(X)$ by f_* for such f that are B-morphisms of B-schemes, it follows that the B-canonical splitting of G/B is mapped to the B-canonical splitting of G/P under the natural map $G/B \to G/P$.

Theorem 7. ([2, Theorem 4.1.17, Remark 4.1.18], [6]) *For a B-scheme X there is a natural injective association $\sigma \mapsto \tilde{\sigma}$ of B-canonical elements in $End_F X$ to B-canonical elements of $End_F(G \times^B X)$. Splittings are mapped to splittings under this association. Moreover, for a B-canonical splitting σ of X, the splitting $\tilde{\sigma}$ is the unique B-canonical splitting of $G \times^B X$ that compatibly splits $X \cong e \times X \subseteq G \times^B X$ (the fiber over the identity coset of G/B) and restricts on $e \times X$ to σ.*

Consider the isomorphism $G \times^B G/B \cong G/B \times G/B$ defined by the association $(g_1, g_2B) \mapsto (g_1B, g_1g_2B)$. It follows from the above theorems that there exists a unique B-canonical splitting of $G/B \times G/B$ that compatibly splits $e \times G/B$ and restricts to the canonical splitting of $G/B \cong e \times G/B$—here $G/B \times G/B$ is a B-variety by the diagonal action. This we call *the canonical splitting* of $G/B \times G/B$. The splitting of $G/P \times G/P$ obtained as the push forward of this under the natural map $G/B \times G/B \to G/P \times G/P$ is called *the canonical splitting* of $G/P \times G/P$.

5. The theorem

Theorem 8. *The B-canonical splitting of G/B has maximal multiplicity along P/B in the following cases*[1]

(1) *$G = SL_n$ and P is any maximal parabolic (the set of such G/P are precisely all Grassmannians).* (This is already in [7], but we prove it again below.)
(2) *$G = Sp_{2n}$ and $P = P_n$ (the set of such G/P are precisely all symplectic Grassmannians).*
(3) *the characteristic is ≥ 3, $G = SO_{2n}$ and $P = P_n$ (the set of such G/P are precisely all orthogonal Grassmannians).*[2]

[1] Our convention that G be simply connected is violated in case (3). The violation is however not serious and we trust the reader can make the appropriate adjustments.

[2] As is well known $SO_{2n+1}/P_n \cong SO_{2n+2}/P_{n+1}$; but, as one can verify for oneself following the lines of the proof below, the B-canonical splitting of SO_{2n+1}/B does not have maximal multiplicity along P_n/B.

PROOF: Global sections of K^{-1} can be identified with regular functions on G that transform thus:

$$f(gb) = \rho(b)^2 f(g) \quad \forall\, g \in G, \quad \forall\, b \in B \tag{3}$$

We will explicitly write, as such a function, the section of K^{-1} whose $(p-1)^{\mathrm{st}}$ power gives the B-canonical splitting of G/B.

5.1. **The case $G = SL_n$ and P any maximal parabolic.** Although this case is dealt with already by Mehta and Parameswaran [7], rewriting their proof as below is helpful. We take the Borel subgroup B to be the subgroup consisting of upper triangular matrices, and the maximal torus T to be the subgroup consisting of diagonal matrices. We denote by ε_k the character of T that maps elements of T to their (k,k)-entries. The simple roots in order are $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$, …, $\varepsilon_{n-1} - \varepsilon_n$, and the corresponding fundamental weights are

$$\varpi_1 = \varepsilon_1, \quad \varpi_2 = \varepsilon_1 + \varepsilon_2, \quad \ldots, \quad \varpi_{n-1} = \varepsilon_1 + \cdots + \varepsilon_{n-1}.$$

For $1 \leq k \leq n-1$ and $1 \leq a_1 < \ldots < a_k \leq n$, let $\det(a_1, \ldots, a_k \,|\, 1, \ldots, k)$ denote the function on the space of $n \times n$ matrices (and so also on the group SL_n by restriction) obtained by taking the determinant of the submatrix formed by the first k columns and the rows numbered a_1, …, a_k.

Proposition 9. (1) $\det(a_1, \ldots, a_k \,|\, 1, \ldots, k)$ *is a global section of the line bundle* $\mathcal{L}(\varpi_k)$*; it is a weight vector of weight* $-(\varepsilon_{a_1} + \cdots + \varepsilon_{a_k})$.

(2) $\det(1, \ldots, k \,|\, 1, \ldots, k)$ *is a weight vector of weight* $-(\varepsilon_1 + \cdots + \varepsilon_k)$*; the line through it is* B^- *stable;*

(3) $\det(n, \ldots, n-k+1 \,|\, 1, \ldots, k)$ *is a weight vector of weight* $-(\varepsilon_n + \cdots + \varepsilon_{n-k+1})$*; the line through it is* B*-stable.*

The proof of the proposition consists of elementary verifications.

We continue with the proof of the theorem. It follows from (1) of the proposition that

$$\begin{aligned} \mathfrak{p} &:= \det(1\,|\,1)\ \det(1,2\,|\,1,2)\ \cdots\ \det(1,\ldots,n-1\,|\,1,\ldots,n-1) \\ \mathfrak{q} &:= \det(n\,|\,1)\ \det(n,n-1\,|\,1,2)\ \cdots\ \det(n,n-1,\ldots,2\,|\,1,\ldots,n-1) \end{aligned}$$

are both sections of $\mathcal{L}(\rho)$. We claim that the $(p-1)^{\mathrm{st}}$ power of $\mathfrak{pq}$ is the section of K^{1-p} that gives the B-canonical splitting of G/B. By Theorem 6, it is enough to check that this section is a B-canonical element, and for this we make use of the characterization in Proposition 5.

Apply the proposition with $X = SL_n/B$. As observed in §2 above, $\mathrm{End}_F(X)$ is naturally isomorphic to $H^0(X, K^{1-p})$ (when X is a non-singular projective variety). By Proposition 9 (2), we can take f_- to be $\mathfrak{p}^{p-1}$ (observe that $\sum_{k=1}^{n-1} -(\varepsilon_1 + \cdots + \varepsilon_k) = \sum_{k=1}^{n-1} -\varpi_k = -\rho$, and that the Steinberg module has lowest weight $-(p-1)\rho$); by Proposition 9 (3), we can take f_+ to be $\mathfrak{q}^{p-1}$ (observe that $\sum_{k=1}^{n-1} -(\varepsilon_n + \cdots + \varepsilon_{n-k+1}) = \sum_{k=1}^{n-1} \iota(\varepsilon_1 + \cdots + \varepsilon_k)$ where ι is the Weyl involution (whose action in the case of SL_n on characters of T is given by $\iota(\varepsilon_k) = -\varepsilon_{n-k+1}$), and so $\mathfrak{q}$ has weight $\sum_{k=1}^{n-1} \iota(\varpi_k) = \iota(\rho) = \rho$). Observe also that when global sections of line bundles on G/B are identified as functions on the group G that transform according to (2), the natural map $H^0(G/B, \mathcal{L}_1) \otimes H^0(G/B, \mathcal{L}_2) \to H^0(G/B, \mathcal{L}_1 \otimes \mathcal{L}_2)$ is just the ordinary multiplication of functions. This completes the proof that $(\mathfrak{pq})^{p-1}$ is B-canonical.

Now fix a standard maximal parabolic P of SL_n. Given the claim of the previous paragraph, it is enough, in order to prove the theorem, to show that the order of vanishing of the section $\mathfrak{pq}$ along P/B equals the codimension of P/B in SL_n/B (which is the same as the dimension of SL_n/P). Since the identity coset eB (here e denotes the identity element of the group SL_n) belongs to P/B and $\mathfrak{p}$ does not vanish at eB, we have only to be concerned with $\mathfrak{q}$. By Proposition 9 (3), $\mathfrak{q}$ is a weight vector of weight ρ for the action of B, that is, $\mathfrak{q}(bg) = \rho(b) \cdot \mathfrak{q}(g)$ for $b \in B$ and $g \in SL_n$. Thus the order of vanishing of $\mathfrak{q}$ along P/B is the same as that of $\mathfrak{q}$ at the T-fixed point $w_0^P B \in P/B$ (where w_0^P denotes the longest element in the Weyl group of P).

Let now P be the maximal parabolic subgroup corresponding to ϖ_r. Then $w_0^P = (r, \ldots, 1, n, \ldots, n-r+1)$. The affine patch of SL_n/B centered around $w_0^P B$ given by $w_0^P B^- B$ consists of matrices of the following explicit form:

$$\begin{array}{r} \\ 1 \\ \\ r \\ r+1 \\ \\ \\ n \end{array}
\begin{array}{c} \begin{array}{cc} 1 \qquad\quad r & r+1 \qquad\qquad n \end{array} \\
\left[\begin{array}{c|c}
\begin{matrix} 0 & & 1 \\ & \iddots & \\ 1 & & \star \end{matrix} & 0 \\ \hline
\star & \begin{matrix} 0 & & 1 \\ & \iddots & \\ 1 & & \star \end{matrix}
\end{array}\right] \end{array}$$

It is now elementary to check that $\det(n, \ldots, n-k+1 \,|\, 1, \ldots, k)$ vanishes at w_0^P to order

$$\mathrm{ord}_{P/B}\left(\det(n, \ldots, n-k+1 \,|\, 1, \ldots, k)\right) = \begin{cases} k & \text{if } k \leq r \text{ and } k+r \leq n \\ r & \text{if } k \geq r \text{ and } k+r \leq n \\ n-k & \text{if } k \geq r \text{ and } k+r \geq n \\ n-r & \text{if } k \leq r \text{ and } k+r \geq n \end{cases}$$

An elementary calculation using this shows $\mathrm{ord}_{P/B}(\mathfrak{q}) = r(n-r) = \dim SL_n/P$. This finishes the proof of the theorem in case (1).

5.2. **The case of $G = Sp_{2m}$ and $P = P_n$.** Let V be a $\boldsymbol{k}$-vector space with a non-degenerate skew-symmetric form $\langle\ ,\ \rangle$. The dimension of V is then even, say $2n$. For $1 \leq k \leq 2n$, let $k^\star := 2n+1-k$. Fix a basis $e_1, \ldots, e_{2n}$ of V such that

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } j = i^\star \text{ and } i < j \\ -1 & \text{if } j = i^\star \text{ and } i > j \\ 0 & \text{if } j \neq i^\star \end{cases}$$

and think of elements of Sp_{2n} as $2n \times 2n$ matrices with respect to this basis (that preserve the form $\langle\ ,\ \rangle$). The advantage of such a choice of basis is this: matrices in Sp_{2n} that are diagonal form a maximal torus (in Sp_{2n}) and matrices that are upper triangular form a Borel subgroup. We take T and B to be these. We continue to denote by ε_k the restriction to T of the character ε_k of the diagonal torus of SL_{2n}. An easy verification shows that $\varepsilon_k = -\varepsilon_{k^\star}$. The simple roots in order are $\varepsilon_1 - \varepsilon_2$,

$\varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n$, and $\varepsilon_n - \varepsilon_{n+1} = 2\varepsilon_n$, and the corresponding fundamental weights are

$$\varpi_1 = \varepsilon_1, \quad \varpi_2 = \varepsilon_1 + \varepsilon_2, \quad \ldots, \quad \varpi_n = \varepsilon_1 + \cdots + \varepsilon_n.$$

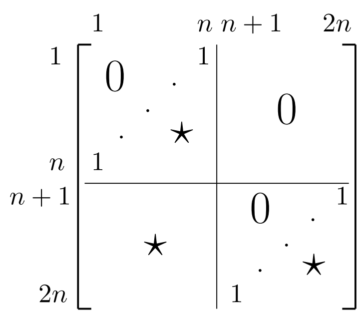
Consider the following functions on Sp_{2n} (the symbols on the right hand side denote functions on SL_{2n} in the notation defined in §5.1 above, and now they also denote the restriction to Sp_{2n} of those functions):

$$\mathfrak{p} \;:=\; \det(1\,|\,1)\,\det(1,2\,|\,1,2)\cdots\det(1,2,\ldots,n\,|\,1,2,\ldots,n)$$
$$\mathfrak{q} \;:=\; \det(2n\,|\,1)\,\det(2n,2n-1\,|\,1,2)\cdots\det(2n,2n-1,\ldots,n\,|\,1,2,\ldots,n)$$

Just as in the case of SL_n, an easy verification using Proposition 9 shows that $\mathfrak{p}$, $\mathfrak{q}$ are sections of $\mathcal{L}(\rho)$ and that the $(p-1)^{\mathrm{st}}$ power of $\mathfrak{pq}$ is the section of K^{1-p} that gives the B-canonical splitting of Sp_{2n}/B.

Let $P = P_n$ be the maximal parabolic subgroup corresponding to the fundamental weight $\varpi_n = \varepsilon_1 + \cdots + \varepsilon_n$. We calculate the order of vanishing of $\mathfrak{pq}$ along P/B and show that it equals the codimension of P/B in Sp_{2n}/B (which equals $\sum_{k=1}^n k = \binom{n+1}{2}$). This calculation too runs parallel to the case of SL_n. Just as in that case, we reduce to considering the order of vanishing of $\mathfrak{q}$ at the point $w_0^P B$ of P/B.

The affine patch around $w_0^P B$ of Sp_{2n}/B given by $w_0^P B^- B$ consists of matrices having the following explicit form:



The inderminates in positions $\star$ are not algebraically independent.

We claim that the order of vanishing of the section $\det(2n,\ldots,2n-k+1\,|\,1,\ldots,k)$ (for $1 \le k \le n$) at $w_0^P B$ is k. The order of vanishing being k for SL_{2n}, it follows immediately that now it is no less than k. To see that is is no more than k, we specialize: set all the variables $\star$ equal to 0 except those on the anti-diagonal in the $n \times n$ matrix in the bottom left corner and take those on the anti-diagonal to be algebraically independent variables (this is a valid specialization in the sense that the resulting matrices are inside Sp_{2n}). The restriction of $\det(2n,\ldots,2n-k+1\,|\,1,\ldots,k)$ to this closed set (which is an affine n-space) is given by the product of k indeterminates and so has order of vanishing exactly k at the origin. This finishes the proof of the theorem in case (2).

5.3. **The case $G = SO_{2n}$ and $P = P_n$.** This proof runs mostly parallel to that for Sp_{2n}/P_n, but there are two notable differences—firstly, the number of factors in the definitions below of the functions $\mathfrak{p}$ and $\mathfrak{q}$ does not equal the number of fundamental weights; secondly, the specialization when n is odd does not work in the same fashion as when it is even.

Let V be a $\boldsymbol{k}$-vector space of even dimension, say $2n$, with a non-degenerate symmetric form $(\ ,\ )$. For $1 \leq k \leq 2n$, let $k^\star := 2n+1-k$. Fix a basis $e_1, \ldots, e_{2n}$ of V such that

$$(e_i, e_j) = \begin{cases} 1 & \text{if } j = i^\star \\ 0 & \text{if } j \neq i^\star \end{cases}$$

and think of elements of SO_{2n} as $2n \times 2n$ matrices with respect to this basis (that preserve the form $(\ ,\ )$ and have determinant 1). The advantage of such a choice of basis is this: matrices in SO_{2n} that are diagonal form a maximal torus (in SO_{2n}) and matrices that are upper triangular form a Borel subgroup. We take T and B to be these. We continue to denote by ε_k the restriction to T of the character ε_k of the diagonal torus of SL_{2n}. An easy verification shows that $\varepsilon_k = -\varepsilon_{k^\star}$. The simple roots in order are $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$, $\ldots$, $\varepsilon_{n-1} - \varepsilon_n$, and $\varepsilon_{n-1} - \varepsilon_{n+1} = \varepsilon_{n-1} + \varepsilon_n$, and the corresponding fundamental weights are

$$\varpi_1 = \varepsilon_1, \quad \varpi_2 = \varepsilon_1 + \varepsilon_2, \quad \ldots, \quad \varpi_{n-2} = \varepsilon_1 + \cdots + \varepsilon_{n-2},$$

$$\varpi_{n-1} = \frac{1}{2}\left(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n\right), \quad \varpi_n = \frac{1}{2}\left(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n\right).$$

Consider the following functions on SO_{2n} (the symbols on the right hand side denote functions on SL_{2n} in the notation defined in §5.1, and now they represent the restriction to SO_{2n} of those functions):

$$\begin{aligned} \mathfrak{p} &:= \det(1\,|\,1)\,\det(1,2\,|\,1,2)\cdots\det(1,2,\ldots,n-1\,|\,1,2,\ldots,n-1) \\ \mathfrak{q} &:= \det(2n\,|\,1)\,\det(2n,2n-1\,|\,1,2)\cdots\det(2n,2n-1,\ldots,n+1\,|\,1,2,\ldots,n-1) \end{aligned}$$

Note that the weight $\varepsilon_1 + \cdots + \varepsilon_{n-1}$ of the $(n-1)^{\text{st}}$ factor of $\mathfrak{q}$ for the action of T is precisely the sum of ϖ_{n-1} and ϖ_n (the weight of $\mathfrak{p}$ is the negative of that of $\mathfrak{q}$). Using this and Proposition 9, it follows, just as in the case of SL_n, that $\mathfrak{p}$, $\mathfrak{q}$ are sections of $\mathcal{L}(\rho)$ and that the $(p-1)^{\text{st}}$ power of $\mathfrak{pq}$ is the section of K^{1-p} that gives the B-canonical splitting of SO_{2n}/B.

Let $P = P_n$ be the maximal parabolic subgroup corresponding to the fundamental weight $\varpi_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$. We calculate the order of vanishing of $\mathfrak{pq}$ along P/B and show that it equals the codimension of P/B in SO_{2n}/B (which is $\sum_{k=1}^{n-1} k = \binom{n}{2}$). This calculation too runs parallel to the case of SL_n. Just as in that case, we reduce to considering the order of vanishing of $\mathfrak{q}$ at the point $w_0^P B$ of P/B.

The affine patch around $w_0^P B$ of SO_{2n}/B given by $w_0^P B^- B$ consists of matrices having the following explicit form:

$$\begin{array}{r|ccc|ccc|}
 & 1 & & n & n+1 & & 2n \\
\hline
1 & 0 & \cdot & 1 & & & \\
 & & \cdot & & & 0 & \\
 & \cdot & \star & & & & \\
n & 1 & & & & & \\
\hline
n+1 & & & & 0 & \cdot & 1 \\
 & & \star & & & \cdot & \\
 & & & & \cdot & \star & \\
2n & & & & 1 & & \\
\hline
\end{array}$$

The inderminates in positions $\star$ are not algebraically independent.

Since $w_0^P B$ belongs to SO_{2n}/B, it follows that the ideal defining the affine patch of SO_{2n}/B as a closed subset of SL_{2n}/B is contained in the maximal ideal generated by the indeterminates $\star$. We can therefore conclude that the order of vanishing of $\det(2n,\ldots,2n-k+1\,|\,1,\ldots,k)$ at $w_0^P B$ is no less than k, so that the order of vanishing of $\mathfrak{q}$ at $w_0^P B$ is no less than $\sum_{k=1}^{n-1} k = (n-1)n/2 = \dim SO_{2n}/P$. To see that the order of vanishing equals this lower bound, we specialize. We first do this when n is even. Set all the variables $\star$ equal to 0 except those on the anti-diagonal in the $n\times n$ matrix in the bottom left corner; set the variables $\star$ that are on the anti-diagonal on rows $3n/2$ through $2n$ to be algebraically independent variables, say $X_{3n/2}, \ldots, X_{2n}$; and set the variables $\star$ that are on the anti-diagonal on rows $n+1$ through $3n/2-1$ to be $-X_{2n}, \ldots, -X_{3n/2}$ (this is a valid specialization in the sense that the resulting matrices are inside SO_{2n}). The restriction of $\det(2n,\ldots,2n-k+1\,|\,1,\ldots,k)$ to this closed set (which is an affine $n/2$-space) is given by the product of k (possibly repeated and with signs) indeterminates and so has order of vanishing exactly k at the origin. This finishes the proof of the theorem in case (3) when n is even.

Now suppose that n is odd. Specialize as follows: set equal to 0 all variables $\star$ not in the $n\times n$ matrix in the lower left hand corner; take the $n\times n$ matrix in the lower left hand corner to be a generic skew-symmetric matrix. This is a valid specialization. The resulting space is an affine $(n-1)n/2$-space. Our goal is to show that the restriction of $\det(2n,\ldots,2n-k+1\,|\,1,\ldots,k)$ to this affine space does not lie in the $(k+1)^{\text{st}}$ power of the maximal ideal corresponding to the origin, and this will be reached once we prove the following claim: let V be an n-dimensional vector space with a skew-symmetric form $\langle\ ,\ \rangle$ of rank $n-1$ (such a form exists); for k an integer, $1\leq k\leq n-1$, there exist vectors $e_1,\ldots,e_k$ and $e_{n-k+1},\ldots,e_n$ such that the matrix $(\langle e_i, e_j\rangle)$, $n-k+1\leq j\leq n$ and $1\leq i\leq k$, is invertible. (To see why it suffices to prove the claim, we think of the skew-symmetric $n\times n$ matrix in the bottom left hand corner as defining the form with respect to some basis.)

To prove the claim, let W be a k-dimensional subspace of V that does not meet the radical[3] (which is 1 dimensional by our hypothesis). Then $W^\perp$ has dimension $n-k$. Let W' to be a k-dimensional subspace that meets $W^\perp$ trivially and intersects W in only a subspace of dimension $2k-n$. Choose $e_{n-k+1},\ldots,e_k$ to be a basis of $W\cap W'$, and extend it to a basis $e_1,\ldots,e_k$ for W and to a basis $e_{n-k+1},\ldots,e_n$ of W'. □

6. Conclusion: Wahl's conjecture holds for the G/P of Theorem 8

We assume in this section that the characteristic p is odd. In order to prove Wahl's conjecture for G/P, it is enough, by Theorem 3, that there exist a splitting of $G/P\times G/P$ which compatibly splits the diagonal copy of G/P with maximal multiplicity. We now argue that such a splitting exists for the G/P as in Theorem 8. In fact, we show that the canonical splitting of $G/P\times G/P$ defined in §4 above has the desired property. Our argument follows that in [7].

First recall the following explicit description of the canonical splitting of $G/B\times G/B$. Let D be the union of the Schubert divisors in G/B and $\tilde{D}$ the union of the opposite Schubert divisors. Let p_1 and p_2 denote respectively the first and second projections of $G/B\times G/B$ onto G/B. Then, as shown in [9], σ^{p-1} is a

[3]Recall that the radical consists of the elements w of W such that $\langle w, x\rangle = 0\ \forall\ x\in W$.

splitting of $G/B \times G/B$ (that compatibly splits the diagonal copy of G/B), where σ denotes the section of the the canonical bundle of $G/B \times G/B$ given by the divisor $p_1^*D + G \times^B D + p_2^*\tilde{D}$. The presence of p_1^*D means that this splitting splits compatibly $X \times G/B$ for X any Schubert variety in G/B; in particular, $e \times G/B$ is compatibly split. Thus, in order show that σ^{p-1} is the canonical splitting, it suffices, by Theorem 7, to show that σ^{p-1} is B-canonical.

Before turning to the proof of this, let us finish the proof of Wahl's conjecture. It has been proved in §5 that the order of vanishing of D along P/B equals the codimension of P/B (in G/B). The occurrence of $G \times^B D$ in the divisor defining σ and the smoothness of the natural map $G/B \times G/B \to G/P \times G/P$ along $G \times^B P/B$ together imply (by [7, Lemma 3.1] or [2, 1.3.E.13]) that the image of $G \times^B P/B$ is compatibly split with maximal multiplicity (under the induced splitting of $G/P \times G/P$). But the image of $G \times^B P/B$ is the diagonal. This completes the proof of Wahl's conjecture.

To prove that σ^{p-1} is B-canonical, we use the characterization in Proposition 5. As observed in §2, we have

$$\mathrm{End}_F(G/B \times G/B) \;\cong\; H^0(G/B \times G/B, \mathcal{L}(2(p-1)\rho, 2(p-1)\rho))$$

The right hand side above is naturally isomorphic to

$$H^0(G_1/B_1, \mathcal{L}(2(p-1)\rho_1)) \;\otimes\; H^0(G_2/B_2, \mathcal{L}(2(p-1)\rho_2))$$

where the subscripts 1 and 2 are used to denote respectively objects associated to the copies $G \times e$ and $e \times G$ of G in $G \times G$. And there are natural maps

$$\mathrm{St}_1 \otimes \mathrm{St}_1 \;\to\; H^0(G_1/B_1, \mathcal{L}(2(p-1)\rho_1) \qquad \mathrm{St}_2 \otimes \mathrm{St}_2 \;\to\; H^0(G_2/B_2, \mathcal{L}(2(p-1)\rho_2)$$

So we have a natural map

$$\mathrm{St}_1 \otimes \mathrm{St}_1 \otimes \mathrm{St}_2 \otimes \mathrm{St}_2 \;\longrightarrow\; \mathrm{End}_F(G/B \times G/B)$$

As G-modules, we have $\mathrm{St}_1 \cong \mathrm{St} \cong \mathrm{St}_2$. And, since St is irreducible and self-dual, there exists a unique G-invariant element—call it $\mathfrak{r}$—in $\mathrm{St}_1 \otimes \mathrm{St}_2$. On the other hand, as seen in §5, the sections $\mathfrak{p}^{p-1}$ and $\mathfrak{q}^{p-1}$ (with $\mathfrak{p}$ and $\mathfrak{q}$ being defined as there) can respectively be taken to be a (non-zero) lowest weight vector in St_2 and a (non-zero) highest weight vector in St_1.

Putting all the above together, we get a $\boldsymbol{k}$-linear B-module map

$$\boldsymbol{k}_{(p-1)\rho} \otimes \mathrm{St} \longrightarrow \mathrm{St}_1 \otimes \mathrm{St}_1 \otimes \mathrm{St}_2 \otimes \mathrm{St}_2$$

sending $f_+ \otimes f_-$ to $\mathfrak{q}^{p-1} \otimes \mathfrak{r} \otimes \mathfrak{p}^{p-1}$. But the splitting section σ^{p-1} is just the natural image in $\mathrm{End}_F(G/B \times G/B)$ of $\mathfrak{q}^{p-1} \otimes \mathfrak{r} \otimes \mathfrak{p}^{p-1}$: p_1^*D corresponds to $\mathfrak{q}^{p-1}$, $\mathfrak{r}$ to $G \times^B D$, and $p_2^*\tilde{D}$ to $\mathfrak{p}^{p-1}$. This finishes the proof that σ^{p-1} is B-canonical.

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