LIMITS OVER CATEGORIES OF EXTENSIONS

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Abstract. We consider limits over categories of extensions and show how certain well-known functors on the category of groups turn out as such limits. We also discuss higher (or derived) limits over categories of extensions.

1. Introduction

Let k be a commutative ring with identity, and let $\mathcal{C}$ be one of the following categories: the category Gr of groups, the category Ab of abelian groups, the category Ass_k of associative algebras over k. Given an object $G \in \mathcal{C}$, let $\mathsf{Ext}_{\mathcal{C}}(G)$ be the category whose objects are the extensions $H \rightarrowtail F \twoheadrightarrow G$ in $\mathcal{C}$ with G as the cokernel, and the morphisms are the commutative diagrams of short exact sequences of the form

$$\begin{array}{ccccc} H_1 & \hookrightarrow & F_1 & \twoheadrightarrow & G \\ \downarrow & & \downarrow & & \| \\ H_2 & \hookrightarrow & F_2 & \twoheadrightarrow & G. \end{array}$$

It is clearly natural to consider also the full subcategory $\mathsf{Fext}_{\mathcal{C}}(G)$ of $\mathsf{Ext}_{\mathcal{C}}(G)$ which consists of the short exact sequences $H \rightarrowtail F \twoheadrightarrow G$ where F is a free object in $\mathcal{C}$. The category $\mathsf{Ext}_{\mathsf{Gr}}(G)$ has been studied extensively from the point of view of the theory of cohomology of groups (see, for example, [6], [7]). The general question which we wish to address here can be formulated as follows: how can one study the properties of objects $G \in \mathsf{Ob}(\mathcal{C})$ from the properties of the category $\mathsf{Fext}_{\mathcal{C}}(G)$?

Let C be a small category and $\mathfrak{F} : \mathsf{C} \longrightarrow \mathcal{C}$ a covariant functor. The inverse limit $\varprojlim \mathfrak{F}$ of $\mathfrak{F}$, by definition, consists of those families $(x_c)_{c\in\mathsf{C}}$ in the direct product $\prod_{c\in\mathsf{C}} \mathfrak{F}(c)$ which are compatible in the following sense: For any two objects $c, c' \in \mathsf{C}$ and any morphism $a \in \mathrm{Hom}_{\mathsf{C}}(c, c')$, we have $\mathfrak{F}(a)(x_c) = x_{c'} \in \mathfrak{F}(c')$. Let $\mathfrak{F}$, $\mathfrak{G}$ be two functors from C to the category $\mathcal{C}$. Then a natural transformation $\eta : \mathfrak{F} \longrightarrow \mathfrak{G}$ induces a homomorphism

$$\varprojlim \eta : \varprojlim \mathfrak{F} \longrightarrow \varprojlim \mathfrak{G},$$

by mapping any element $(x_c)_{c\in\mathsf{C}} \in \varprojlim \mathfrak{F}$ onto $(\eta_c(x_c))_{c\in\mathsf{C}} \in \varprojlim \mathfrak{G}$. In this way, $\varprojlim$ itself becomes a functor from the functor category $\mathcal{C}^{\mathsf{C}}$ to the category $\mathcal{C}$.

Our aim in this paper is to consider the categories $\mathsf{Ext}_{\mathcal{C}}(G)$, $\mathsf{Fext}_{\mathcal{C}}(G)$ and limits $\varprojlim \mathfrak{F}$ for functors $\mathfrak{F}_G : \mathsf{Ext}_{\mathcal{C}}(G) \to \mathsf{Gr}$, $\mathfrak{F}_G : \mathsf{Fext}_{\mathcal{C}}(G) \to \mathsf{Gr}$. Suppose these functors are natural in the following sense:

Given a morphism $\alpha : G_1 \to G_2$ in $\mathcal{C}$, every commutative diagram of the form

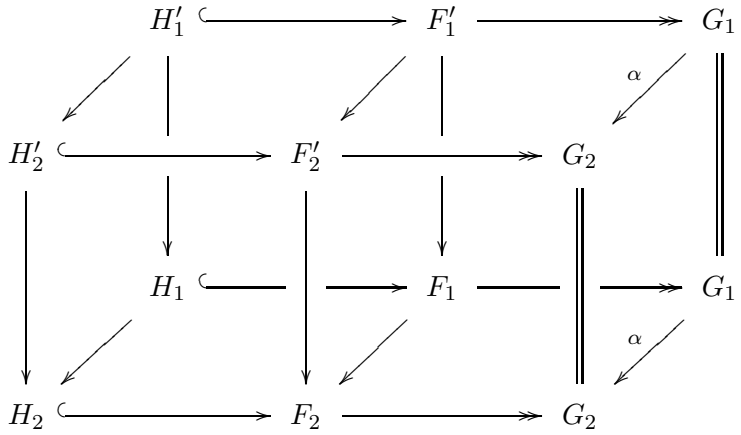
$$\begin{array}{ccccc} H_1 & \hookrightarrow & F_1 & \twoheadrightarrow & G_1 \\ \downarrow & & \downarrow & & \downarrow \alpha \\ H_2 & \hookrightarrow & F_2 & \twoheadrightarrow & G_2 \end{array}$$

induces a homomorphism of groups

$$\mathfrak{F}_{G_1}\{H_1 \rightarrowtail F_1 \twoheadrightarrow G_1\} \to \mathfrak{F}_{G_2}\{H_2 \rightarrowtail F_2 \twoheadrightarrow G_2\},$$

compatible with morphisms in $\mathsf{Ext}_{\mathcal{C}}(G_1)$ and $\mathsf{Ext}_{\mathcal{C}}(G_2)$, i.e., every commutative diagram of the form

(1)



induces the following commutative diagram of groups

(2)
$$\begin{array}{ccc} \mathfrak{F}_{G_1}\{H_1' \rightarrowtail F_1' \twoheadrightarrow G_1\} & \longrightarrow & \mathfrak{F}_{G_1}\{H_1 \rightarrowtail F_1 \twoheadrightarrow G_1\} \\ \downarrow & & \downarrow \\ \mathfrak{F}_{G_2}\{H_2' \rightarrowtail F_2' \twoheadrightarrow G_2\} & \longrightarrow & \mathfrak{F}_{G_2}\{H_2 \rightarrowtail F_2 \twoheadrightarrow G_2\}. \end{array}$$

In that case the limit of the functors over categories of extensions defines a functor $\mathcal{C} \to \mathsf{Gr}$ by setting $G \mapsto \varprojlim \mathfrak{F}_G$.

To clarify our point of view further, let us recall some known examples which have motivated our present investigation. Let $\mathcal{C} = \mathsf{Gr}$, G a group, $\mathbb{Z}[G]$ its integral group ring and M a $\mathbb{Z}[G]$-module. For $n \geq 1$, define the functor

(3)
$$\mathfrak{F}_n : \mathsf{Fext}_{\mathcal{C}}(G) \to \mathsf{Ab}$$

by setting

$$\mathfrak{F}_n : \{R \rightarrowtail F \twoheadrightarrow G\} \mapsto (R_{ab})^{\otimes n} \otimes_{\mathbb{Z}[G]} M,$$

where R_{ab} denotes the abelianization of R, the action of G on $R_{ab}^{\otimes n}$ is diagonal and is defined via conjugation in F. It is shown by I. Emmanouil and R. Mikhailov in [4] that $\varprojlim \mathfrak{F}_n$ is

isomorphic to the homology group $H_{2n}(G,\, M)$:

$$\varprojlim \mathfrak{F}_n \simeq H_{2n}(G,\, M). \tag{4}$$

For any group G and field k of characteristic 0 viewed as a trivial G-module, the homology groups $H_n(G,\, k)$, $n \geq 2$, appear as inverse limits for suitable natural functors defined on the category $\mathsf{Ext}_{\mathsf{Gr}}(\mathsf{G})$ (see [5] for details).

Consider the category $\mathcal{C} = \mathsf{Ass}_{\mathbb{Q}}$ of associative algebras over $\mathbb{Q}$, the field of rationals. For integers $n \geq 1$, and an associative algebra A over $\mathbb{Q}$, consider the functors

$$\mathfrak{F}_n : \mathsf{Ext}_{\mathcal{C}}(A) \to \mathbb{Q}\text{-modules}$$

given by setting

$$\mathfrak{F}_n : \{I \rightarrowtail R \twoheadrightarrow A\} \mapsto R/(I^{n+1} + [R,\, R]), \tag{5}$$

where $[R,\, R]$ is the $\mathbb{Q}$-submodule of R, generated by the elements $rs - sr$ with $r,\, s \in R$. It has been shown by D. Quillen in [11] that the inverse limit $\varprojlim \mathfrak{F}_n$ is isomorphic to the even cyclic homology group $HC_{2n}(A,\, \mathbb{Q})$:

$$HC_{2n}(A,\, \mathbb{Q}) \simeq \varprojlim \mathfrak{F}_n. \tag{6}$$

Furthermore, the Connes suspension map $S : HC_{2n}(A,\, \mathbb{Q}) \to HC_{2n-2}(A,\, \mathbb{Q})$ can be obtained as follows:

$$\begin{array}{ccc} HC_{2n}(A,\, \mathbb{Q}) & = & \varprojlim \mathfrak{F}_n \\ \downarrow S & & \downarrow \\ HC_{2n-2}(A,\, \mathbb{Q}) & = & \varprojlim \mathfrak{F}_{n-1} \end{array}$$

where the right hand side map is induced by the natural projection

$$R/(I^{n+1} + [R,\, R]) \to R/(I^n + [R,\, R]).$$

The motivation for our investigation should now be clear from the above examples: one takes quite simple functors, like (3) and (5), on the appropriate categories of extensions and asks for the corresponding inverse limits. It is then also natural to consider the derived functors

$$\varprojlim{}^{i} : \mathsf{Ab}^{\mathsf{Fext}(G)} \to \mathsf{Ab}$$

of the limit functor. Every functor $\mathfrak{F} : \mathsf{Fext}_{\mathcal{C}}(G) \to \mathsf{Ab}$ which is natural in the above-mentioned sense (see diagrams (1) and (2)) determines a series of functors $\mathcal{C} \to \mathsf{Ab}$ by setting $G \mapsto \varprojlim{}^{i}\mathfrak{F} \in \mathsf{Ab},\ i \geq 0$.

We now briefly describe the contents of the present paper. We begin by recalling in Section 2 two properties of the limits given in [4] and [5]. The first (Lemma 2.1) provides a set of vanishing conditions for $\varprojlim \mathfrak{F}$ while, the second states that $\varprojlim \mathfrak{F}$ embeds in $\mathfrak{F}(c_0)$ for every quasi-initial object c_0. In Section 3 (Theorems 3.1 and 3.4) we show how the derived functors in the sense of A. Dold and D. Puppe [3] of certain standard non-additive functors on Ab, like

tensor power, symmetric power, exterior power, are realized as limits of suitable functors over extension categories. In Section 4 we discuss higher limits and prove (Theorem 4.4) that for the functor

$$\mathfrak{F} : \mathsf{Fext}_{\mathsf{Gr}}(G) \to \mathsf{Ab}, \quad (R \rightarrowtail F \twoheadrightarrow G) \mapsto R/[F, R],$$

$\varprojlim{}^1\mathfrak{F}$ is non-trivial if G is not a perfect group. We conclude with some remarks and possibilities for further work in Section 5.

2. Properties of limits recalled

Recall that the coproduct of two objects a and b in a category C is an object $a \star b$ which is endowed with two morphisms $\iota_a : a \longrightarrow a \star b$ and $\iota_b : b \longrightarrow a \star b$ having the following universal property:

> For any object c of C and any pair of morphisms $f : a \longrightarrow c$ and $g : b \longrightarrow c$, there is a unique morphism $h : a \star b \longrightarrow c$, such that $h \circ \iota_a = f$ and $h \circ \iota_b = g$. The morphism h is usually denoted by (f, g).

We recall from [4] the following Lemma which provides certain conditions which imply the triviality of the inverse limit.

Lemma 2.1. [4] *Let C be a small category and $\mathfrak{F} : \mathsf{C} \longrightarrow \mathsf{Ab}$ a functor to the category of abelian groups, and suppose that the following conditions are satisfied:*

(i) Any two objects a, b of C have a coproduct $(a \star b, \iota_a, \iota_b)$ as above.

(ii) For any two objects a, b of C the morphisms $\iota_a : a \longrightarrow a \star b$ and $\iota_b : b \longrightarrow a \star b$ induce a monomorphism

$$(\mathfrak{F}(\iota_a), \mathfrak{F}(\iota_b)) : \mathfrak{F}(a) \oplus \mathfrak{F}(b) \longrightarrow \mathfrak{F}(a \star b)$$

of abelian groups.

Then, the inverse limit $\varprojlim \mathfrak{F}$ is the zero group.

Indeed, let $(x_c)_{c \in Ob(\mathsf{C})} \in \varprojlim \mathfrak{F}$ be a compatible family and fix an object a of C. We consider the coproduct $a \star a$ of two copies of a and the morphisms $\iota_1 : a \longrightarrow a \star a$ and $\iota_2 : a \longrightarrow a \star a$. Then, we have

$$\mathfrak{F}(\iota_1)(x_a) = x_{a \star a} = \mathfrak{F}(\iota_2)(x_a)$$

and hence the element $(x_a, -x_a)$ is contained in the kernel of the additive map

$$(\mathfrak{F}(\iota_1), \mathfrak{F}(\iota_2)) : \mathfrak{F}(a) \oplus \mathfrak{F}(a) \longrightarrow \mathfrak{F}(a \star a).$$

In view of our assumption, this latter map is injective and hence $x_a = 0$. Since this is the case for any object a of C, we conclude that the compatible family $(x_c)_{c \in Ob(\mathsf{C})}$ vanishes, as asserted.

We next mention another property of the limit functor. Recall that an object c_0 of a category C is called *quasi-initial* if the set $\mathrm{Hom}_{\mathsf{C}}(c_0, c)$ is non-empty for every object c of C.

Lemma 2.2. [5] *Let* C *be a category with a quasi-initial object* c_0*. Then, for any functor* $\mathfrak{F} : \mathsf{C} \longrightarrow \mathsf{Ab}$*, the natural map*

$$\varprojlim \mathfrak{F} \to \mathfrak{F}(c_0)$$

is injective, whereas its image consists of those elements $x \in \mathfrak{F}(c_0)$ *which equalize any pair of maps* $\mathfrak{F}(f_i) : \mathfrak{F}(c_0) \to \mathfrak{F}(c),\ i = 1, 2$*, where* $f_1, f_2 \in Hom_{\mathsf{C}}(c_0, c)$ *(i.e.,* $\mathfrak{F}(f_1)(x) = \mathfrak{F}(f_2)(x)$*).*

Observe that given the category $\mathcal{C}$ and an object $G \in \mathsf{Ob}(\mathcal{C})$, the category $\mathsf{Fext}_{\mathcal{C}}(G)$ consists of quasi-initial objects.

3. Derived functors of certain functors on Ab

In this section we study the limits $\varprojlim \mathfrak{F}$ for certain functors $\mathfrak{F} \in \mathsf{Ab}^{\mathsf{Fext}_{\mathsf{Ab}}(A)}$ for abelian groups A.

To fix notation, let $\otimes^n : \mathsf{Ab} \to \mathsf{Ab}$, $n \geq 1$, be the *n-th tensor power functor* $A \mapsto A^{\otimes n} := \underbrace{A \otimes \cdots \otimes A}_{n \text{ terms}}$. The symmetric group Σ_n of degree n acts naturally on $A^{\otimes n}$:

$$\sigma(x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},\ x_i \in A,\ \sigma \in \Sigma_n. \tag{7}$$

We thus have the *n-th symmetric power functor* $SP^n : \mathsf{Ab} \to \mathsf{Ab}$ with $SP^n(A) = A^{\otimes n}$ modulo the subgroup generated by the elements $\sigma(x_1 \otimes \cdots \otimes x_n) - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},\ x_i \in A,\ \sigma \in \Sigma_n$. The *n-th exterior power functor* $\Lambda^n : \mathsf{Ab} \to \mathsf{Ab}$ is defined by $A \mapsto A^{\otimes n}$ modulo the subgroup generated by the elements $x_1 \otimes \ldots \otimes x_n$ with $x_1, \ldots, x_n \in A$ and $x_i = x_{i+1}$ for some i. The *n-th divided power functor* $\Gamma_n : \mathsf{Ab} \to \mathsf{Ab}$ is defined, for $A \in \mathsf{Ab}$, to the n-th homogeneous component of the graded group $\Gamma_*(A)$ generated by symbols $\gamma_i(x)$ of degree $i \geq 0$ satisfying the following relations for all $x, y \in A$:

$$\begin{aligned}
&(i)\ \gamma_0(x) = 1\\
&(ii)\ \gamma_1(x) = x\\
&(iii)\ \gamma_s(x)\gamma_t(x) = \binom{s+t}{s}\gamma_{s+t}(x)\\
&(iv)\ \gamma_n(x+y) = \sum_{s+t=n} \gamma_s(x)\gamma_t(y),\ n \geq 1\\
&(v)\ \gamma_n(-x) = (-1)^n\gamma_n(x),\ n \geq 1.
\end{aligned}$$

In particular, the canonical map $A \to \Gamma_1(A)$ is an isomorphism. It is known that, for a free abelian group A, there is a natural isomorphism

$$\Gamma_n(A) \simeq (A^{\otimes n})^{\Sigma_n},\ n \geq 1$$

where the action of the symmetric group Σ_n on $A^{\otimes n}$ is defined as in (7).

Let $n \geq 0$ be an integer, and T an endofunctor on the category Ab of abelian groups. The doubly indexed family $L_iT(-, n)$ of derived functors, in the sense of Dold-Puppe [3], of T are

defined by

$$L_iT(A,\, n) = \pi_i T N^{-1} P_*[n],\ i \geq 0,\ A \in \mathsf{Ab},$$

where $P_*[n] \to A$ is a projective resolution of A of level n, and N^{-1} is the Dold-Kan transform, which is the the inverse of the Moore normalization functor

$$N : \mathcal{S}(\mathsf{Ab}) \to \mathcal{C}h(\mathsf{Ab})$$

from the category of simplicial abelian groups to the category of chain complexes (see, for example, [10], pp. 306, 326; or [12], Section 8.4). For any functor T, we set

$$L_iT(A) := L_iT(A,\, 0),\ i \geq 0.$$

For abelian groups $B_1, \ldots, B_n$, let the group $\mathrm{Tor}_i(B_1, \ldots, B_n)$ denote the i-th homology group of the complex $P_1 \otimes \cdots \otimes P_n$, where P_j is a $\mathbb{Z}$-flat resolution of B_j for $j = 1, \ldots, n$. We clearly have

$$\mathrm{Tor}_0(B_1, \ldots, B_n) = B_0 \otimes \cdots \otimes B_n,\ \mathrm{Tor}_i(B_1, \ldots, B_n) = 0,\ i \geq n.$$

It turns out from the Eilenberg-Zilber theorem that the derived functors of the n-th tensor power can be described as

$$L_i \otimes^n (A) = \mathrm{Tor}_i(\underbrace{A, \cdots, A}_{n \text{ terms}}),\ 0 \leq i \leq n-1.$$

We will use the following notation:

$$\mathrm{Tor}^{[n]}(A) := \mathrm{Tor}_{n-1}(\underbrace{A, \cdots, A}_{n \text{ terms}}),\ n \geq 2.$$

Theorem 3.1. *For $n \geq 2$, there is an isomorphism of abelian groups*

$$\mathrm{Tor}^{[n]}(A) \simeq \varprojlim (F^{\otimes n}/H^{\otimes n}),\ A \in \mathsf{Ab},$$

where the limit is taken over the category $\mathsf{Fext}_{\mathsf{Ab}}(A)$ of free extensions $H \rightarrowtail F \twoheadrightarrow A$ in the category Ab.

To proceed with the proof, we first recall the following result which is well-known.

Lemma 3.2. *Let $A = F_1/H_1,\ B = F_2/H_2$, where F_1, F_2 are free abelian groups. Then there is an isomorphism of abelian groups*

$$\mathrm{Tor}(A,\, B) = \frac{(H_1 \otimes F_2) \cap (F_1 \otimes H_2)}{H_1 \otimes H_2},$$

where the intersection is taken in $F_1 \otimes F_2$.

Indeed, the above result follows directly from the exact sequence of abelian groups

$$0 \to \mathrm{Tor}(A,\, B) \to H_1 \otimes B \to F_1 \otimes B \to A \otimes B \to 0$$

and isomorphisms $H_1 \otimes B \simeq (H_1 \otimes F_2)/(H_1 \otimes H_2),\ F_1 \otimes B \simeq (F_1 \otimes F_2)/(F_1 \otimes H_2)$.

Lemma 3.3. *Let $A = F/H$, where F is a free abelian group. Then, for every $n \geq 2$, there is an isomorphism of abelian groups*

$$\mathrm{Tor}^{[n]}(A) \simeq \frac{\bigcap_{i=1}^{n}(H^{\otimes i-1} \otimes F \otimes H^{\otimes n-i})}{H^{\otimes n}}$$

where the intersection is taken in $F^{\otimes n}$.

Proof. Observe that

$$\mathrm{Tor}^{[n]}(A) \simeq \mathrm{Tor}(\mathrm{Tor}^{[n-1]}(A),\, A),\ n \geq 3. \tag{8}$$

To see (8), one can apply the Künneth formula to the tensor product of the chain complexes $P \otimes \cdots \otimes P$ ($n-1$ times) and P, where P is a projective resolution of A. The Lemma follows by inductive argument and Lemma 3.2. □

Proof of Theorem 3.1. Given $A = F/H$, where F is a free abelian group, consider the following exact sequence of abelian groups:

$$0 \to \mathrm{Tor}^{[n]}(A) \to F^{\otimes n}/H^{\otimes n} \to F^{\otimes n}/\bigcap_{i=1}^{n}(H^{\otimes i-1} \otimes F \otimes H^{\otimes n-i}) \to 0. \tag{9}$$

The sequence (9) is natural in the following sense: any morphism in $\mathsf{Fext}_{\mathsf{Ab}}(A)$, say $f : (F_1, \pi_1) \to (F_2, \pi_2)$ (with $H_1 = \ker(\pi_1)$, $H_2 = \ker(\pi_2)$) implies the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Tor}^{[n]}(A) & \longrightarrow & F_1^{\otimes n}/H_1^{\otimes n} & \longrightarrow & F_1^{\otimes n}/\bigcap_{i=1}^{n}(H_1^{\otimes i-1} \otimes F_1 \otimes H_1^{\otimes n-i}) \\
 & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Tor}^{[n]}(A) & \longrightarrow & F_2^{\otimes n}/H_2^{\otimes n} & \longrightarrow & F_2^{\otimes n}/\bigcap_{i=1}^{n}(H_2^{\otimes i-1} \otimes F_2 \otimes H_2^{\otimes n-i})
\end{array}$$

Since the inverse limit functor is left exact in $\mathsf{Ab}^{\mathsf{Fext}_{\mathsf{Ab}}(A)}$, we obtain a natural monomorphism

$$\mathrm{Tor}^{[n]}(A) \hookrightarrow \varprojlim F^{\otimes n}/H^{\otimes n},\ n \geq 2. \tag{10}$$

Given a free presentation $A = F/H$ in Ab, consider the two morphisms in $\mathsf{Fext}_{\mathsf{Ab}}(A)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f_1,\, f_2} & & \| & & \\
0 & \longrightarrow & F \oplus H & \longrightarrow & F \oplus F & \longrightarrow & A & \longrightarrow & 0
\end{array} \tag{11}$$

given by setting:

$$f_1 : g \mapsto (0,\, g),\ g \in F,$$
$$f_2 : g \mapsto (g,\, g),\ g \in F.$$

Let $\alpha \in F^{\otimes n}/H^{\otimes n}$ be an element which belongs to the equalizer of the maps

$$f_1^*,\ f_2^* : F^{\otimes n}/H^{\otimes n} \to \frac{(F\oplus F)^{\otimes n}}{(F\oplus H)^{\otimes n}}$$

induced by $f_1,\ f_2$ respectively. Express α as a coset

$$\alpha = (\sum_i g_1^i \otimes \cdots \otimes g_n^i) + H^{\otimes n},\ g_j^i \in F.$$

Identifying $\frac{(F\oplus F)^{\otimes n}}{(F\oplus H)^{\otimes n}}$ with $\bigoplus_{(i_1,\ldots,i_n)\in\{0,1\}^n} \frac{F^{\otimes n}}{C_{i_1}\otimes\cdots\otimes C_{i_n}}$ where $C_0 = F$ and $C_1 = H$, we can describe $f_i^*(\alpha),\ i = 1, 2$, as

$$f_1^*(\alpha) = (0, \ldots, 0, \sum_i g_1^i \otimes \cdots \otimes g_n^i)$$

$$f_2^*(\alpha) = (\sum_i g_1^i \otimes \cdots \otimes g_n^i, \ldots, \sum_i g_1^i \otimes \cdots \otimes g_n^i)$$

Since α lies in the equalizer of f_1^* and f_2^*, we conclude that

$$\sum_i g_1^i \otimes \cdots \otimes g_n^i \in \bigcap_{i=1}^{n} (H^{\otimes i-1} \otimes F \otimes H^{\otimes n-i}).$$

The category $\mathsf{Fext}_{\mathsf{Ab}}(A)$ clearly consists of quasi-initial objects; hence Lemma 2.2 implies that the natural map (10) is an isomorphism and the theorem is proved. □

Theorem 3.4. *For every abelian group A and integer $n \geq 2$, there are natural isomorphisms*

$$L_{n-1}SP^n(A) \simeq \varprojlim \Lambda^n(F)/\Lambda^n(H) \tag{12}$$

$$L_{n-1}\Lambda^n(A) \simeq \varprojlim \Gamma_n(F)/\Gamma_n(H) \tag{13}$$

where the limits are taken over the category $\mathsf{Fext}_{\mathsf{Ab}}(A)$ of free extensions $H \rightarrowtail F \twoheadrightarrow A$.

Proof. Given a free extension

$$0 \to H \xrightarrow{f} F \to A \to 0$$

the Koszul complexes

$$0 \to \Lambda^n(H) \xrightarrow{\kappa_n} \Lambda^{n-1}(H)\otimes F \xrightarrow{\kappa_{n-1}} \cdots \xrightarrow{\kappa_2} H \otimes SP^{n-1}(F) \xrightarrow{\kappa_1} SP^n(F) \tag{14}$$

and

$$0 \to \Gamma_n(H) \xrightarrow{\kappa^n} \Gamma_{n-1}(H)\otimes F \xrightarrow{\kappa^{n-1}} \cdots \xrightarrow{\kappa^2} H \otimes \Lambda^{n-1}(F) \xrightarrow{\kappa^1} \Lambda^n(F) \tag{15}$$

represent models of the objects $LSP^n(A)$ and $L\Lambda^n(A)$ in the derived category (see [9], Proposition 2.4 and Remark 2.7). In these complexes, the maps

$$\kappa_{k+1} : \Lambda^{k+1}(H)\otimes SP^{n-k-1}(F) \to \Lambda^k(H)\otimes SP^{n-k}(F),\ k = 0, \ldots, n-1$$
$$\kappa^{k+1} : \Gamma_{k+1}(H)\otimes \Lambda^{n-k-1}(F) \to \Gamma_k(H)\otimes \Lambda^{n-k}(F),\ k = 0, \ldots, n-1$$

are defined by setting:

$$\kappa_{k+1} : p_1 \wedge \cdots \wedge p_{k+1} \otimes q_{k+2} \ldots q_n \mapsto$$
$$\sum_{i=1}^{k+1} (-1)^{k+1-i} p_1 \wedge \cdots \wedge \hat{p}_i \wedge \cdots \wedge p_{k+1} \otimes f(p_i)\, q_{k+2} \ldots q_n$$
$$p_1, \ldots, p_{k+1} \in H,\ q_{k+2}, \ldots, q_n \in F.$$

and

$$\kappa^{k+1} : \gamma_{r_1}(p_1) \ldots \gamma_{r_k}(p_k) \otimes q_1 \wedge \ldots \wedge q_{n-k-1} \mapsto$$
$$\sum_{j=1}^{k} \gamma_{r_1}(p_1) \ldots \gamma_{r_{j-1}}(p_j) \ldots \gamma_{r_k}(p_k) \otimes f(p_j) \wedge q_1 \wedge \ldots \wedge q_{n-k-1},\ p_1, \ldots, p_k \in H,\ q_1, \ldots, q_{n-k-1} \in F$$

In particular, the homology groups of complexes (14) and (15) are isomorphic to the derived functors $L_i SP^n(A)$ and $L_i \Lambda^n(A)$ respectively. If $f : H \to F$ is the identity map and $A = 0$, the complexes (14) and (15) are acyclic complexes. The commutative diagram

$$\begin{array}{ccc} H & \xhookrightarrow{f} & F \\ \downarrow{\scriptstyle f} & & \| \\ F & = & F \end{array}$$

implies the following diagrams with exact columns:

(16)
$$\begin{array}{ccccccccc}
\Lambda^n(H) & \xhookrightarrow{\kappa_n} & \Lambda^{n-1}(H) \otimes F & \xrightarrow{\kappa_{n-1}} & \cdots & \xrightarrow{\kappa_2} & H \otimes SP^{n-1}(F) & \xrightarrow{\kappa_1} & SP^n(F) \\
\downarrow & & \downarrow & & & & \downarrow & & \| \\
\Lambda^n(F) & \hookrightarrow & \Lambda^{n-1}(F) \otimes F & \longrightarrow & \cdots & \longrightarrow & F \otimes SP^{n-1}(F) & \longrightarrow & SP^n(F) \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\frac{\Lambda^n(F)}{\Lambda^n(H)} & \longrightarrow & \frac{\Lambda^{n-1}(F)}{\Lambda^{n-1}(H)} \otimes F & \longrightarrow & \cdots & \longrightarrow & A \otimes SP^{n-1}(F), & &
\end{array}$$

(17)
$$\begin{array}{ccccccccc}
\Gamma_n(H) & \xhookrightarrow{\kappa^n} & \Gamma_{n-1}(H) \otimes F & \xrightarrow{\kappa^{n-1}} & \cdots & \xrightarrow{\kappa^2} & H \otimes \Lambda^{n-1}(F) & \xrightarrow{\kappa^1} & \Lambda^n(F) \\
\downarrow & & \downarrow & & & & \downarrow & & \| \\
\Gamma_n(F) & \hookrightarrow & \Gamma_{n-1}(F) \otimes F & \longrightarrow & \cdots & \longrightarrow & F \otimes \Lambda^{n-1}(F) & \longrightarrow & \Lambda^n(F) \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\frac{\Gamma_n(F)}{\Gamma_n(H)} & \longrightarrow & \frac{\Gamma_{n-1}(F)}{\Gamma_{n-1}(H)} \otimes F & \longrightarrow & \cdots & \longrightarrow & A \otimes \Lambda^{n-1}(F). & &
\end{array}$$

Since the middle horizontal sequences in the diagrams (16) and (17) are exact, we obtain the following exact sequences:

$$0 \to L_{n-1}SP^n(A) \to \frac{\Lambda^n(F)}{\Lambda^n(H)} \to \frac{\Lambda^{n-1}(F)}{\Lambda^{n-1}(H)} \otimes F \tag{18}$$

$$0 \to L_{n-1}\Lambda^n(A) \to \frac{\Gamma_n(F)}{\Gamma_n(H)} \to \frac{\Gamma_{n-1}(F)}{\Gamma_{n-1}(H)} \otimes F \tag{19}$$

Since the inverse limit functor is left exact, we obtain the following natural sequences:

$$0 \to L_{n-1}SP^n(A) \to \varprojlim \frac{\Lambda^n(F)}{\Lambda^n(H)} \to \varprojlim \frac{\Lambda^{n-1}(F)}{\Lambda^{n-1}(H)} \otimes F \tag{20}$$

$$0 \to L_{n-1}\Lambda^n(A) \to \varprojlim \frac{\Gamma_n(F)}{\Gamma_n(H)} \to \varprojlim \frac{\Gamma_{n-1}(F)}{\Gamma_{n-1}(H)} \otimes F \tag{21}$$

where the limits are taken, as usual, over the category of extensions $H \rightarrowtail F \twoheadrightarrow A$. We claim that for $n \geq 2$, and every $k \geq 1$,

$$\varprojlim \left(\frac{\Lambda^{n-1}(F)}{\Lambda^{n-1}(H)} \otimes F^{\otimes k} \right) = \varprojlim \left(\frac{\Gamma_{n-1}(F)}{\Gamma_{n-1}(H)} \otimes F^{\otimes k} \right) = 0. \tag{22}$$

For $n = 2$, the functor

$$\{H \rightarrowtail F \twoheadrightarrow A\} \mapsto A \otimes F^{\otimes k}$$

satisfies the conditions of Lemma 2.1 and the assertion follows. Now assume that (22) is proved for a fixed $n \geq 1$. Consider the tensor products of sequences (18) (for $n+1$) and (19) with $F^{\otimes k}$:

$$0 \to L_nSP^{n+1}(A) \otimes F^{\otimes k} \to \frac{\Lambda^{n+1}(F)}{\Lambda^{n+1}(H)} \otimes F^{\otimes k} \to \frac{\Lambda^n(F)}{\Lambda^n(H)} \otimes F^{\otimes k+1}$$

$$0 \to L_n\Lambda^{n+1}(A) \otimes F^{\otimes k} \to \frac{\Gamma_{n+1}(F)}{\Gamma_{n+1}(H)} \otimes F^{\otimes k} \to \frac{\Gamma_n(F)}{\Gamma_n(H)} \otimes F^{\otimes k+1}$$

By induction,

$$\varprojlim \left(\frac{\Lambda^n(F)}{\Lambda^n(H)} \otimes F^{\otimes k+1} \right) = \varprojlim \left(\frac{\Gamma_n(F)}{\Gamma_n(H)} \otimes F^{\otimes k+1} \right) = 0$$

and the functors

$$\{H \rightarrowtail F \twoheadrightarrow A\} \mapsto L_nSP^{n+1}(A) \otimes F^{\otimes k}$$

$$\{H \rightarrowtail F \twoheadrightarrow A\} \mapsto L_n\Lambda^{n+1}(A) \otimes F^{\otimes k}$$

satisfy the conditions of Lemma 2.1. Hence (22) follows. The statement of the theorem now follows from sequences (20) and (21). □

4. Higher limits

Let C be a small category. The first derived functor

$$\varprojlim{}^1 : \mathsf{Gr}^{\mathsf{C}} \to \text{pointed sets}$$

can be defined via cosimplicial replacement in the category Gr^{C} described in [2]. Given $\mathfrak{F} \in \mathsf{Gr}^{\mathsf{C}}$, define a cosimplicial replacement $\prod^*\mathfrak{F}$, a cosimplicial group, with

$$\prod{}^n\mathfrak{F} = \prod_{u \in I_n} \mathfrak{F}(i_0),\ u = \{i_0 \overset{\alpha_1}{\leftarrow} \cdots \overset{\alpha_n}{\leftarrow} i_n\}$$

and coface and codegeneracy maps induced by

$$\begin{aligned} d^0 &: \mathfrak{F}(i_1) \overset{\mathfrak{F}(\alpha_1)}{\to} \mathfrak{F}(i_0), \\ d^j &: \mathfrak{F}(i_0) \overset{id}{\to} \mathfrak{F}(i_0),\ 0 < j \leq n, \\ s^j &: \mathfrak{F}(i_0) \overset{id}{\to} \mathfrak{F}(i_0),\ 0 \leq j \leq n. \end{aligned}$$

One can check that there is a natural isomorphism (see [2])

$$\varprojlim \mathfrak{F} = \pi^0 \prod{}^*\mathfrak{F}.$$

The derived functor of the inverse limit can be defined as

$$\varprojlim{}^1\mathfrak{F} = \pi^1 \prod{}^*\mathfrak{F} \in \text{pointing sets}.$$

We then have the following:

Proposition 4.1. *Let* 1 *be an identity functor in* Gr^{C} *and let*

$$1 \to \mathfrak{F}_1 \to \mathfrak{F}_2 \to \mathfrak{F}_3 \to 1$$

be a short exact sequence in Gr^{C}*. There is a natural long exact sequence of groups and pointed spaces:*

$$1 \to \varprojlim \mathfrak{F}_1 \to \varprojlim \mathfrak{F}_2 \to \varprojlim \mathfrak{F}_3 \to \varprojlim{}^1\mathfrak{F}_1 \to \varprojlim{}^1\mathfrak{F}_2 \to \varprojlim{}^1\mathfrak{F}_3. \tag{23}$$

In the case of the category Ab^{C}, the functor $\varprojlim{}^1$ has values in Ab and the sequence (23) is a long exact sequence of abelian groups. In this case there is a cochain complex of abelian groups defined by

$$\prod{}^\bullet\mathfrak{F} : \prod{}^0\mathfrak{F} \overset{\delta^0}{\to} \prod{}^1\mathfrak{F} \overset{\delta^1}{\to} \dots$$

with

$$\delta^n(a^n)\{i_0 \overset{\alpha_1}{\leftarrow} \cdots \overset{\alpha_{n+1}}{\leftarrow} i_{n+1}\} =$$

$$\mathfrak{F}(i_0 \overset{\alpha_1}{\leftarrow} i_1)a^n\{i_1 \overset{\alpha_1}{\leftarrow} \cdots \overset{\alpha_{n+1}}{\leftarrow} i_{n+1}\} + \sum_{j=1}^{n+1}(-1)^j a^n\{i_0 \overset{\alpha_1}{\leftarrow} \cdots \leftarrow \hat{i}_j \leftarrow \cdots \overset{\alpha_{n+1}}{\leftarrow} i_{n+1}\},$$

$$a^n \in \prod{}^n\mathfrak{F},$$

such that the derived functors of the inverse limit are the cohomology groups:

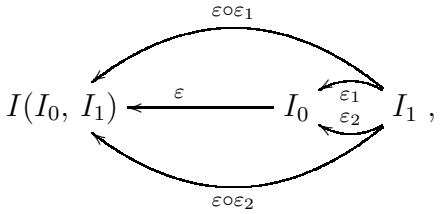
$$\varprojlim{}^n\mathfrak{F} = H^n(\prod{}^\bullet\mathfrak{F}),\ n \geq 0$$

(see [8], Theorem 4.1). Clearly,

$$\varprojlim\mathfrak{F} = \varprojlim{}^0\mathfrak{F} = \ker(\delta^0).$$

The question of vanishing of higher limits of functors defined on small categories in general reduces to the computation of local cohomology of nerves of these categories. We give a simple condition for the vanishing of $\varprojlim{}^1$.

Proposition 4.2. *Let* C *be a category with a quasi-initial object and* $\mathfrak{F} : \mathsf{C} \longrightarrow \mathsf{Ab}$ *a functor. Suppose that every pair of morphisms in* C *has a coequalizer, i.e., for every pair of morphisms* $\varepsilon_1, \varepsilon_2 : I_1 \to I_0,\ I_1, I_0 \in \mathsf{Ob}(\mathsf{C})$ *there is a morphism* $\varepsilon : I_0 \to I(I_0, I_1)$ *in* C *such that the following diagram is commutative*



i.e., $\varepsilon \circ \varepsilon_1 = \varepsilon \circ \varepsilon_2$ *and the induced map* $\mathfrak{F}(\varepsilon) : \mathfrak{F}(I_0) \to \mathfrak{F}(I(I_0, I_1))$ *is injective. Then* $\varprojlim{}^1\mathfrak{F} = 0$.

Proof. Let $a^1 \in \prod^1\mathfrak{F} = \prod^1_{i_0 \leftarrow i_1}\mathfrak{F}$ be a 1-cocycle, i.e.,

$$\delta^1 a^1(i_0 \leftarrow i_1 \leftarrow i_2) = \mathfrak{F}(i_0 \leftarrow i_1)a^1(i_1 \leftarrow i_2) + a^1(i_0 \leftarrow i_1) - a^1(i_0 \leftarrow i_2) = 0 \tag{24}$$

for every diagram $i_0 \leftarrow i_1 \leftarrow i_2$. Given two morphisms $\varepsilon_1, \varepsilon_2 : i_1 \to i_0$, consider a morphism $\varepsilon : i_0 \to I(i_0, i_1)$, such that $\mathfrak{F}(\varepsilon) : \mathfrak{F}(i_0) \to \mathfrak{F}(I(i_0, i_1))$ is a monomorphism of abelian groups and $\varepsilon \circ \varepsilon_1 = \varepsilon \circ \varepsilon_2$. The cocycle condition (24) implies that

$$\mathfrak{F}(\varepsilon)a^1(i_0 \overset{\varepsilon_1}{\leftarrow} i_1) = a^1(I(i_0, i_1) \overset{\varepsilon\circ\varepsilon_1}{\leftarrow} i_1) - a^1(I(i_0, i_1) \overset{\varepsilon}{\leftarrow} i_0)$$
$$\mathfrak{F}(\varepsilon)a^1(i_0 \overset{\varepsilon_2}{\leftarrow} i_1) = a^1(I(i_0, i_1) \overset{\varepsilon\circ\varepsilon_2}{\leftarrow} i_1) - a^1(I(i_0, i_1) \overset{\varepsilon}{\leftarrow} i_0)$$

and, therefore,

$$\mathfrak{F}(\varepsilon)a^1(i_0 \overset{\varepsilon_1}{\leftarrow} i_1) = \mathfrak{F}(\varepsilon)a^1(i_0 \overset{\varepsilon_2}{\leftarrow} i_1)$$

in $\mathfrak{F}(I(i_0, i_1))$. Since $\mathfrak{F}(\varepsilon)$ is a monomorphism, we conclude that

$$a^1(i_0 \overset{\varepsilon_1}{\leftarrow} i_1) = a^1(i_0 \overset{\varepsilon_2}{\leftarrow} i_1) \tag{25}$$

in $\mathfrak{F}(i_0)$. Now we can take a quasi-initial object $i \in \mathsf{Ob}(\mathsf{C})$ and define an element $a^0 \in \prod^0\mathfrak{F}$ by setting

$$a^0(i_0) = a^1(i_0 \leftarrow i)$$

for arbitrary map $i_0 \leftarrow i$ (such a map exists, since i is a quasi-initial object). The equality (25) implies that this is a well-defined element. By definition, we have

$$-a^1(i_0 \leftarrow i_1) = \mathfrak{F}(i_0 \leftarrow i_1)a^0(i_1) - a^0(i_0) = \delta^0 a^0(i_0 \leftarrow i_1),$$

and the proof is complete. □

At the moment we are not able to compute higher limits over categories of free extensions. We present here an approach towards this problem and illustrate it with an application. In particular, we show that higher limits 'cover' certain homology functors.

Given a category $\mathcal{C}$, object $G \in \mathsf{Ob}(\mathcal{C})$, and the category of free extensions $\mathsf{Fext}_{\mathcal{C}}(G)$, suppose we have two pairs of functors

$$\mathcal{H}_1,\, \mathcal{H}_2 : \mathcal{C} \to \mathsf{Ab}$$
$$\mathcal{F}_1,\, \mathcal{F}_2 : \mathsf{Fext}(G) \to \mathsf{Ab}$$

such that, for every $\alpha \in \mathsf{Fext}_{\mathcal{C}}(G)$, there is a natural 4-term exact sequence

$$0 \to \mathcal{H}_2(G) \to \mathcal{F}_1(\alpha) \to \mathcal{F}_2(\alpha) \to \mathcal{H}_1(G) \to 0 \tag{26}$$

which is natural in the sense that every morphism $\beta \to \alpha$ in $\mathsf{Fext}_{\mathcal{C}}(G)$ induces the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{H}_2(G) & \hookrightarrow & \mathcal{F}_1(\beta) & \longrightarrow & \mathcal{F}_2(\beta) & \twoheadrightarrow & \mathcal{H}_1(G) \\
\| & & \downarrow & & \downarrow & & \| \\
\mathcal{H}_2(G) & \hookrightarrow & \mathcal{F}_1(\alpha) & \longrightarrow & \mathcal{F}_2(\alpha) & \twoheadrightarrow & \mathcal{H}_1(G)
\end{array} .$$

Suppose further that

$$\varprojlim \mathcal{F}_2 = 0. \tag{27}$$

The condition (27) implies the following exact sequences of abelian groups:

(28)
$$\begin{array}{ccccccc}
 & & \varprojlim{}^1 \mathcal{F}_1 & & & & \\
 & & \downarrow & & & & \\
\mathcal{H}_1(G) & \hookrightarrow & \varprojlim{}^1_{\alpha \in \mathsf{Fext}_{\mathcal{C}}(G)} C(\alpha) & \longrightarrow & \varprojlim{}^1 \mathcal{F}_2 & \longrightarrow & \varprojlim{}^1 \mathcal{H}_1(G) \\
 & & \downarrow & & & & \\
 & & \varprojlim{}^2 \mathcal{H}_2(G) & & & &
\end{array}$$

where $C(\alpha) = \operatorname{coker}\{\mathcal{H}_2(G) \to \mathcal{F}_1(\alpha)\} = \ker\{\mathcal{F}_2(\alpha) \to \mathcal{H}_1(G)\},\ \alpha \in \mathsf{Fext}_{\mathcal{C}}(G)$.

For every $\alpha \in \mathsf{Fext}_{\mathcal{C}}(G)$, fix sections $s_\alpha : \mathcal{H}_1(G) \to \mathcal{F}_2(\alpha)$ and $t_\alpha : C(\alpha) \to \mathcal{F}_1(\alpha)$. To describe the map f, let $a \in \mathcal{H}_1(G)$ and let $\gamma \to \beta \to \alpha$ be a diagram in $\mathsf{Fext}_{\mathcal{C}}(G)$. Consider

the following diagram

$$\begin{array}{ccccccc}
\mathcal{H}_2(G) & \hookrightarrow & \mathcal{F}_1(\gamma) & \longrightarrow & \mathcal{F}_2(\gamma) & \twoheadrightarrow & \mathcal{H}_1(G) \\
\| & & \downarrow{\scriptstyle \mathcal{F}_1(\gamma\to\beta)} & & \downarrow{\scriptstyle \mathcal{F}_2(\gamma\to\beta)} & & \| \\
\mathcal{H}_2(G) & \hookrightarrow & \mathcal{F}_1(\beta) & \longrightarrow & \mathcal{F}_2(\beta) & \twoheadrightarrow & \mathcal{H}_1(G) \\
\| & & \downarrow{\scriptstyle \mathcal{F}_1(\beta\to\alpha)} & & \downarrow{\scriptstyle \mathcal{F}_2(\beta\to\alpha)} & & \| \\
\mathcal{H}_2(G) & \hookrightarrow & \mathcal{F}_1(\alpha) & \longrightarrow & \mathcal{F}_2(\alpha) & \twoheadrightarrow & \mathcal{H}_1(G)
\end{array}$$

Define

$$a^2(\gamma\to\beta\to\alpha) := \mathcal{F}_1(\beta\to\alpha)t_\beta(\mathcal{F}_2(\gamma\to\beta)s_\gamma(a) - s_\beta(a)) - \\ t_\gamma\mathcal{F}_2(\beta\to\alpha)(\mathcal{F}_2(\gamma\to\beta)s_\gamma(a) - s_\beta(a))$$

The 2-cocycle condition can be checked directly; moreover, note that the element $\xi \in \varprojlim{}^2\mathcal{H}_2(G)$ defined by the cocycle $a^2(\gamma\to\beta\to\alpha)$ does not depend on the choice of sections s_α, t_α. The map $f : \mathcal{H}_1(G) \to \varprojlim{}^2\mathcal{H}_2(G)$ is thus the one given by $a \mapsto \xi$.

We are interested in finding conditions which imply the triviality of the map f.

Proposition 4.3. *Suppose we have functors*

$$\mathcal{F}_3,\ \mathcal{F}_4,\ \mathcal{F} :\ \mathsf{Fext}_{\mathcal{C}}(G) \to \mathsf{Gr}$$

such that the following conditions are satisfied:
(1) *There is a natural diagram*

$$\begin{array}{ccccccc}
\mathcal{H}_2(G) & \hookrightarrow & \mathcal{F}_1(\alpha) & \longrightarrow & \mathcal{F}_3(\alpha) & \twoheadrightarrow & \mathcal{F}_4(\alpha) \\
\| & & \| & & \downarrow & & \downarrow \\
\mathcal{H}_2(G) & \hookrightarrow & \mathcal{F}_1(\alpha) & \longrightarrow & \mathcal{F}_2(\alpha) & \twoheadrightarrow & \mathcal{H}_1(G)
\end{array} \qquad (29)$$

(2) *The natural map*

$$\varprojlim \mathcal{F}_4 \to \mathcal{H}_1(G)$$

is an epimorphism.
(3) *For every $\alpha \in \mathsf{Fext}_{\mathcal{C}}(G)$, there is a natural monomorphism*

$$\mathcal{F}_1(\alpha) \to \mathcal{F}(\alpha)$$

and natural short exact sequences

$$1 \to \mathcal{F}_1(\alpha) \to \mathcal{F}(\alpha) \to \mathcal{F}_4(\alpha) \to 1$$
$$1 \to \mathcal{H}_2(G) \to \mathcal{F}(\alpha) \to \mathcal{F}_3(\alpha) \to 1.$$

Then the natural map

$$\varprojlim \mathcal{F}_4 \to \varprojlim {}^2\mathcal{H}_2(G)$$

is the trivial map and therefore the map

$$f : \mathcal{H}_1(G) \to \varprojlim {}^2\mathcal{H}_2(G)$$

is the zero map.

Proof. The proof follows from the functoriality of the considered constructions and the following natural commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{H}_2(G) & = & \mathcal{H}_2(G) & \xrightarrow{0} & \mathcal{F}_4(\alpha) & = & \mathcal{F}_4(\alpha) \\
\| & & \downarrow & & \downarrow & & \| \\
\mathcal{H}_2(G) & \to & \mathcal{F}(\alpha) & \to & \mathcal{F}_3(\alpha) \oplus \mathcal{F}_4(\alpha) & \xrightarrow{(0,id)} & \mathcal{F}_4(\alpha) \\
\| & & \uparrow & & \uparrow & & \| \\
\mathcal{H}_2(G) & \hookrightarrow & \mathcal{F}_1(\alpha) & \to & \mathcal{F}_3(\alpha) & \twoheadrightarrow & \mathcal{F}_4(\alpha)
\end{array} .$$

□

We next give examples of functors satisfying (26).

Examples.

1. Let G be a group, $n \geq 1$, $\{R \rightarrowtail F \twoheadrightarrow G\} \in \mathsf{Fext}_{\mathsf{Gr}}(G)$, then there is a natural exact sequence of abelian groups (see [4]):

$$0 \to H_{2n}(G) \to H_0(F,\, R_{ab}^{\otimes n}) \to H_1(F,\, R_{ab}^{\otimes n-1}) \to H_{2n-1}(G) \to 0. \tag{30}$$

2. Let A be an associative algebra over $\mathbb{Q}$, $n \geq 1$, $\{I \rightarrowtail R \twoheadrightarrow A\} \in \mathsf{Fext}_{\mathsf{Ass}_{\mathbb{Q}}}(A)$. There is a natural exact sequence (see [11]):

$$0 \to HC_{2n}(A) \to HH_0(R/I^{n+1}) \to H_1(R,\, R/I^n) \to HC_{2n-1}(A) \to 0.$$

Lemma 2.1 implies that, for the example 1 above, the condition (27) is satisfied for the functor

$$\{R \rightarrowtail F \twoheadrightarrow G\} \mapsto H_1(F,\, R_{ab}^{\otimes n-1})$$

for $n \geq 1$ (see [4] for details), hence there is an isomorphism $H_{2n}(G) \simeq \varprojlim H_0(F,\, R_{ab}^{\otimes n})$. For the simplest case, namely for $n = 1$, functors from the exact sequence (30), the diagram (29)

can be chosen to be the one given below (with the obvious maps):

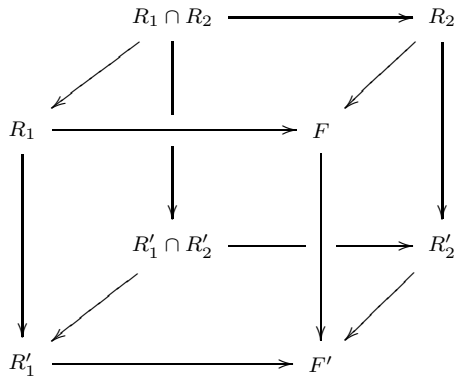
$$\begin{array}{ccccccc} H_2(G) & \hookrightarrow & R/[R,\,F] & \longrightarrow & F/(R\cap[F,\,F]) & \twoheadrightarrow & G \\ \| & & \| & & \downarrow & & \downarrow \\ H_2(G) & \hookrightarrow & R/[F,\,R] & \longrightarrow & F/[F,\,F] & \twoheadrightarrow & H_1(G) \end{array}$$

Proposition 4.3 then implies that the natural map $H_1(G) \to \varprojlim{}^2 H_2(G)$ is the zero map. Consequently, the diagram (28) implies that $H_1(G)$ is contained in a group, which is an epimorphic image of $\varprojlim{}^1(R/[F,\,R])$. We have thus proved the following:

Theorem 4.4. *If G is not a perfect group, then $\varprojlim{}^1(R/[F,\,R])$ is non-trivial.*

5. Concluding remarks and questions

Observe that given an object $G \in \mathsf{Ob}(\mathcal{C})$, one can consider the category $\mathsf{Fext}_2(G)$ of double (resp. triple etc) presentations of G. For simplicity, let us assume that we work in the category of groups. The objects of $\mathsf{Fext}_2(G)$ are triples $(F,\,R_1,\,R_2)$, where F is a group, $R_1,\,R_2$ normal subgroups in F, such that $F/R_1R_2 = G$. The morphisms in $\mathsf{Fext}_2(G)$ are the diagrams of the form



which induce the identity isomorphism $F/R_1R_2 \to F'/R_1'R_2'$. It would be of interest to examine limits of functors over the category $\mathsf{Fext}_2(G)$. For example, note that, given a group G, there is a natural homomorphism

$$H_3(G) \to \varprojlim_{(F,\,R_1,\,R_2)\in\mathsf{Fext}_2(G)} \frac{R_1\cap R_2}{[R_1,\,R_2][F,\,R_1\cap R_2]};$$

for the construction of this map see the homology exact sequence in [1].

In the same way, one can make variations of Quillen's description (6) of cyclic homology. Given an associative algebra A over $\mathbb{Q}$, consider the category $\mathsf{Fext}_2(A)$ whose objects are the triples $(R,\,I_1,\,I_2)$, where R is a free algebra $I_1,\,I_2$ are ideals in R and $R/(I_1+I_2) = A$. The

description (6) implies, for example, that for $n \geq 2,\ n > k \geq 1$, there is a natural morphism

$$\lim_{\longleftarrow\ (R, I_1, I_2) \in \mathsf{Fext}_2(A)} \frac{R}{I_1^{n+1-k} I_2^k + I_1^k I_2^{n+1-k} + [R, R]} \to HC_{2n}(A)$$

and its investigation may be of interest for cyclic homolgy.

Finally, one now knows how to define even-dimensional homology of groups, Lie algebras, cyclic homology of associative algebras as limits of certain functors over the categories of extensions. What can one say about higher limits of the functors yielding this relationship?

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