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Exact Relativistic Model for a Superdense Star

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Abstract. Assuming that the physical 3-space t = const in a superdense star is spheroidal, a static spherically symmetric model based on an exact solution of Einstein's equations is given which will permit densities of the order of 2×10^{14} gm cm^{-3}, radii of the order of a few kilometers and masses up to about four times the solar mass.

Key words: neutron stars—general relativity

1. Introduction

General relativity establishes a correspondence between matter-energy and the geometry of the physical 3-space, the presence of the former inducing curvatures in the latter. The space-times associated with Schwarzschild's interior metric representing the gravitational field within a sphere containing homogeneous fluid at rest, or with Einstein's metric representing a static model of the universe, or de Sitter's and Robertson-Walker metrics representing models of the expanding universe, contain enough matter-energy to curve up the physical 3-space t= const into 3-spheres. The radius R of the 3-sphere is related to the density of matter in each of the above cases.

This suggests that it would be worthwhile investigating the gravitational situations described by space-times which give physical spaces t=const as simple closed 3-spaces. We have explored here the gravitational significance of space-times which have hypersurfaces t=const as 3-spheroids. It is shown that these space-times can be used to develop static models describing the gravitational field in the interior of superdense condensations of matter like white dwarfs and neutron stars. The astrophysical significance of this result is apparent in the light of the fact that only a few out of a large number of static solutions of Einstein's field equations for spherical matter-distributions entertain these possibilities.

In the next section, the metric on the 3-spheroid is deduced from the metric of the four-dimensional flat space in which it is immersed. It is observed that spheroidal 3-space exhibits central symmetry. The spherically symmetric form of the space-time metric for which the hypersurface t = const will be the given spheroidal 3-space, can be easily written down. The geometry of the space-time is governed by two parameters denoted by R and K.

Following Tolman's (1939) approach, Einstein's field equations for spherical distributions of matter in the form of a perfect fluid at rest are written out explicitly. The relationship between the matter-density and the geometry of the associated physical 3-space governed by the parameters R and K is critically examined in the next section. If λ denotes the ratio of the matter-density at the boundary of the star to the density at its centre, then it is possible to estimate the size of the configuration for different values of λ and K, if the order of magnitude of the density on the boundary is known. The gravitational field in the exterior region is uniquely described by Schwarzschild's exterior solution. The continuity of metric potentials across the boundary enables one to estimate the total mass inside the configuration.

An exact solution of the field equations is derived in Section 4 corresponding to the particular value $K = -2$ and this solution is used to develop the model for a superdense star. Assuming the matter-density on the boundary surface of the configuration to be $\rho_a \simeq 2 \times 10^{14}$ gm cm^{-3} *i.e.* of the order of the average matter density in a neutron star (Rees, Ruffini and Wheeler 1975), the size and the total mass of the configuration is computed by the scheme outlined in the previous section.

For different values of density-variation parameter λ, the respective relevant quantities are displayed in a tabular form in Section 6. These estimates clearly indicate that the solution gives physically viable static models for superdense stars. If we apply these results to describe a neutron star, we find that a maximum mass 3·575 $M_\odot$ is obtained with a radius of 18 ·371 km when λ attains the minimum permissible value 0·4598. This maximum permissible mass is higher than the maximum upper limit 3.2 $M_\odot$ obtained by Rhoades and Ruffini (1974) without any recourse to nuclear theory. The consequences of relaxing the requirement $\rho - 3p/c^2 \geqslant 0$ to $\rho - p/c^2 \geqslant 0$ are also examined and it is found that the mass limit can be raised even further. This is an interesting feature of this model because raising the mass limit of a neutron star is of significance in deciding whether the unseen component of a binary star system is a black hole or a neutron star.

2. The field equations

Consider a 4-dimensional Euclidean flat space with the metric

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2.$$

A 3-spheroid immersed in this 4-dimensional flat space will have the 'Cartesian' equation

$$(w^2/b^2) + (x^2 + y^2 + z^2)/R^2 = 1.$$

The sections w=const of the 3-spheroid are concentric spheres while sections x=const y=const or z=const represent systems of confocal ellipsoids. The parametrization

$x=R \sin\lambda \sin\alpha \cos\beta,$

$y=R \sin\lambda \sin\alpha \sin\beta,$

$z=R \sin\lambda \cos\alpha,$

$w=b \cos\lambda\ ,$

of the 3-spheroid leads to the metric on the 3-space as

$$d\sigma^2=(R^2 \cos^2\lambda + b^2 \sin^2\lambda)\, d\lambda^2 + R^2 \sin^2\lambda\, (d\alpha^2 + \sin^2\alpha d\beta^2). \tag{1}$$

On introducing the space variable r through

$r = R \sin\lambda,$

metric (1) transforms to

$$d\sigma^2 = [1 - K(r^2/R^2)]\,[1 - (r^2/R^2)]^{-1}\, dr^2 + r^2 (d\alpha^2 + \sin^2\alpha d\beta^2) \tag{2}$$

where

$$K = 1 - (b^2/R^2). \tag{3}$$

For $K < 1$, the metric (2) is regular and positive definite at all points $r^2 < R^2$. In the case K=1, the spheroidal 3-space degenerates into flat 3-space and in the case K=0 (*i.e.* b=R) the spheroidal 3-space becomes spherical.

We shall study the space-time with the metric

$$\begin{aligned} ds^2 &= e_\nu\, dt^2 - d\sigma^2 \\ &= e^\nu\, dt^2 - [1 - K(r^2/R^2)]\,[1 - (r^2/R^2)]^{-1}\, dr^2 - r^2 (d\alpha^2 + \sin^2\alpha d\beta^2) \end{aligned} \tag{4}$$

where

$$\nu = \nu(r);\ x^1 = r,\ x^2 = \alpha,\ x^3 = \beta,\ x^4 = t.$$

It is clear that when K=0, the physical 3-space t = const becomes spherical and so

(i) when $e^\nu = \{A + B\,[1 - (r^2/R^2\,)]^{1/2}\}^2$, K= 0, metric (4) gives the metric of Schwarzschild's interior solution;
(ii) when $\nu = 0$, $K = 0$, metric (4) gives the metric of Einstein's Universe and
(iii) when $e^\nu = 1 - (r^2/R^2)$, $K = 0$, metric (4) gives the metric of de Sitter Universe.

Our aim is to consider the perfect fluid distribution represented by metric (4) when $K < 1$, $K \neq 0$ *i.e.* when in metric (4) the physical 3-space t=const is spheroidal and not spherical. For a perfect fluid the energy-momentum tensor is given by

$$T_{ik} = \left(\rho + \frac{p}{c^2}\right) u_i\, u_k - \frac{p}{c^2} g_{ik}$$

where ρ and p denote the density and pressure respectively of the fluid and u^i stands for the four-velocity, $u^i u_i = 1$. For an equilibrium situation with the metric (4) we can take

$$u^i = (0, 0, 0, e^{-\nu/2}).$$

The field equations of Einstein,

$$R_{ik} - \frac{1}{2} g_{ik}\, R = -\frac{8\,\pi G}{c^2}\, T_{ik}$$

will now give, for the space-time metric (4)

$$(8\pi G/c^2)\,\rho = \frac{3\,(1-K)}{R^2}\, \frac{\left(1 - K\dfrac{r^2}{3R^2}\right)}{\left(1 - K\dfrac{r^2}{R^2}\right)^2}, \tag{5}$$

$$(8\pi G/c^2)\,\frac{p}{c^2} = \frac{\left(1 - \dfrac{r^2}{R^2}\right)}{\left(1 - K\dfrac{r^2}{R^2}\right)} \left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} \tag{6}$$

together with the consistency condition ($T^1_1 = T^2_2$) which reads as

$$\left(1 - \frac{r^2}{R^2}\right)\left(1 - K\frac{r^2}{R^2}\right)\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{\nu'}{r}\right) - \frac{2\,(1-K)\,r}{R^2}\left(\frac{1}{2}\nu' + \frac{1}{r}\right) + \frac{2\,(1-K)}{R^2}\left(1 - K\frac{r^2}{R^2}\right) = 0. \tag{7}$$

Here and in what follows an overhead prime implies differentiation with respect to r($\nu' \equiv d\nu/dr$ *etc*).

Equation (5) shows that the density of the fluid is determined by the curvature of the physical 3-space. Thus, in our approach we have replaced the usual equation of state of matter by the geometrical requirement that the physical 3-space be spheroidal which predetermines the law of variation of density with r. The field equation (6) gives the variation of pressure with r when ν is chosen to satisfy Equation (7). In the next section we shall see how the law of variation of the density given by Equation (5) enables us to estimate the mass and the radius of the configuration.

3. Law of variation of density

Expression (5) for the matter density implies that at $r = 0$,

$$(8\pi G/c^2)\,\rho_0 = 3\,(1 - K)/R^2. \tag{8}$$

As $K < 1$, the central density ρ_0 is positive. From Equation (5) we see that if $0 < K < 1$, ρ remains positive in the spherical region $r^2 < 3R^2/K$ which imposes a restriction on the size of the configuration. However, if $K < 0$, there is no such restriction and ρ remains positive throughout the configuration. It can be verified that ρ' is negative so that as r increases, the density decreases from the maximum value ρ_0 at the centre. On the boundary $r = a$, it attains the value

$$(8\pi G/c^2)\,\rho_a = 3\,(1 - K)\left(1 - K\frac{a^2}{3R^2}\right)\left(1 - K\frac{a^2}{R^2}\right)^{-2} R^{-2}. \tag{9}$$

We find that the ratio ρ_α to ρ_0 is given by

$$\lambda = \rho_a/\rho_0 = \left(1 - K\frac{a^2}{3R^2}\right)\bigg/\left(1 - K\frac{a^2}{R^2}\right)^2 < 1. \tag{10}$$

The field in the exterior region $r \geqslant a$ is described by the Schwarzschild's exterior metric

$$ds^2 = (1 - 2m/r)\,dt^2 - (1 - 2m/r)^{-1}\,dr^2 - r^2\,(d\alpha^2 + \sin^2 \alpha d\beta^2). \tag{11}$$

Hence the metric (4) should be continuous with the metric (11) as we cross the boundary $r = a$. This is achieved by stipulating the continuity of the metric coefficients g_{11} and g_{44} as also the continuity of the fluid pressure p across the boundary. The continuity of g_{11} at $r = a$ implies that

$$m/a = [(1 - K)\,a^2/2R^2]/(1 - Ka^2/R^2). \tag{12}$$

Equation (10) determines a^2/R^2 in terms of K and λ as

$$a^2/R^2 = [6\lambda - 1 - \sqrt{(1 + 24\lambda)}]/6\mathrm{K}\lambda \tag{13}$$

and subsequently Equation (12) determines m/a in terms of K and λ.

Given ρ_α, λ and K, the parameter R can be calculated from Equation (8). The radius a and the total mass m of the configuration can subsequently be calculated from Equations (13) and (12). It is thus seen that the. knowledge of the curvature of the physical 3-space is sufficient to form estimates about the size and the mass of the star-model. However it is necessary to solve the field Equations (6) and (7) in order to see that the physical conditions like $p \geqslant 0$, $\rho - 3p/c^2 \geqslant 0$ are satisfied throughout the configuration. We now go to study these conditions after solving the field equations in the next section.

4. Solutions of field equations

To solve the linear differential Equation (7) for ν, we make the following changes in the variables ν and r.

$$\psi^2 = e^{\nu},$$

$$u^2 = K(1 - r^2/R^2)/(K-1) \quad \text{if} \qquad K < 0$$

or

$$u^2 = K(1 - r^2/R^2)/(1-K) \quad \text{if} \quad 0 < K < 1.$$

Then Equation (7) reduces to the convenient form

$$(1 - u^2)\frac{d^2\psi}{du^2} + u\frac{d\psi}{du} + (1-K)\psi = 0.$$

The points $u = \pm 1$ are regular singular points and all other points are regular points for this linear equation. If we seek a series solution of this equation in the form $\psi = \Sigma A_k u^k$, one gets the following recurrence relation for the coefficients A_k.

$$(n+1)(n+2)A_{n+2} = [n^2 - 2n + K - 1] A_n.$$

If the parameter K has value such that the equation

$$n^2 - 2n + K - 1 = 0$$

admits integral values of n as solutions, either of the two sets $(A_0, A_2, A_4, \ldots)$, $(A_1, A_3, A_5 \cdots)$ contains finite number of elements and the corresponding terms in the solution-series constitute a finite polynomial. It can be verified that if K is to be in the range $K < 1$, the simplest value of K is -2 which corresponds to $n = 3$. With $K = -2$ one obtains the following solution

$$\psi = A_0\left(1 - \frac{3}{2}u^2 + \frac{3}{8}u^4 + \frac{1}{16}u^6 + \ldots\right) + A_1 u\left(1 - \frac{2}{3}u^2\right)$$

It will be observed that the infinite series with A_0 as the coefficient is the expansion of $(1 - u^2)^{3/2}$. Hence we find the closed-form solution of Equation (7) with $K = -2$ as

$$\exp(\nu/2) = \psi = Az\left(1 - \frac{4}{9}z^2\right) + B\left(1 - \frac{2}{3}z^2\right)^{3/2}$$

where $z^2 = 1 - r^2/R^2$, and A and B are undetermined constants of integration.

Closed-form solutions of Equation (7) have also been obtained for $K = -7$, $K = -14$ *etc.* but are not reported here.

5. Particular solution $K = -2$

We shall now write down the solution of the field equations corresponding to $K = -2$ in explicit form and verify that the fluid content satisfies all the relevant physical restrictions. The metric describing this solution explicitly is

$$ds^2 = \left[B\left(1-\frac{2}{3}z^2\right)^{3/2} + Az\left(1-\frac{4}{9}z^2\right)\right]^2 dt^2 - \frac{3-2z^2}{z^2}dr^2 - r^2(d\alpha^2 + \sin^2\alpha d\beta^2) \tag{14}$$

where

$$z^2 = 1 - r^2/R^2.$$

The matter density and the fluid pressure are found to be

$$(8\pi G/c^2)\rho = \frac{3}{R^2}\frac{(5-2z^2)}{(3-2z^2)^2}, \tag{15}$$

$$(8\pi G/c^2)\frac{p}{c^2} = \frac{3}{R^2}\frac{B(2z^2-1)(1-\frac{2}{3}z^2)^{1/2} - \frac{1}{3}Az(5-4z^2)}{(3-2z^2)[B(1-\frac{2}{3}z^2)^{3/2} + Az(1-\frac{4}{9}z^2)]}. \tag{16}$$

The density ρ and the pressures p at the centre ($r = 0$) attain the values

$$(8\pi G/c^2)\rho_0 = \frac{9}{R^2}, \tag{17}$$

$$(8\pi G/c^2)\frac{p_0}{c^2} = \frac{9}{R^2}\frac{B\sqrt{3}-A}{B\sqrt{3}+A}. \tag{18}$$

If the fluid sphere is of radius a, for $r \geqslant a$ the field is described by Schwarzschild's exterior solution (11). Making the metric (14) continuous with metric (11) at $r=a$, one gets

$$(1-a^2/R^2)(1+2a^2/R^2)^{-1} = 1 - 2m/a, \tag{19}$$

$$B\sqrt{3}(1+2a^2/R^2)^{3/2} + A(1-a^2/R^2)^{1/2}(5+4a^2/R^2) = 9(1-2m/a)^{1/2}. \tag{20}$$

A further boundary condition is that the fluid pressure must vanish at $r = a$. This leads to

$$B\sqrt{3}(1-2a^2/R^2)(1+2a^2/R^2)^{1/2} - A(1-a^2/R^2)(1+4a^2/R^2) = 0. \tag{21}$$

Equations (19), (20) and (21) give A, B and m as

$$A = \frac{3}{2}(1-2a^2/R^2)(1-a^2/R^2)^{-1/2}(1-2m/a)^{1/2}, \tag{22}$$

$$B = \frac{\sqrt{3}}{2}(1 + 4a^2/R^2)(1 + 2a^2/R^2)^{-1/2}(1 - 2m/a)^{1/2}, \tag{23}$$

$$m = \frac{3}{2}(a^2/R^2)(1 + 2a^2/R^2)^{-1}\, a. \tag{24}$$

It is clear that the constants B and m are always positive. However if $a^2/R^2 > \frac{1}{2}$, A will be negative. But it can be verified that $A < 0$ would imply $\rho_0 < p_0/c^2$, an unphysical situation. So A is also positive and we have the restriction $a^2/R^2 < \frac{1}{2}$, as a restriction on the boundary radius a.

In order to ensure that at the centre $\rho_0 \geqslant 3p_0/c^2$, it is easy to see that we should further restrict the boundary radius a by

$$a^2/R^2 \leqslant 0{\cdot}31$$

or

$$a/R \leqslant 0{\cdot}5567. \tag{25}$$

As a matter of fact, with the restriction (25) on the boundary, we ensure that at the centre

$$\rho_0 > 0, \quad p_0 > 0, \quad \rho_0 - 3p_0/c^2 \geqslant 0.$$

In order to ensure that ρ and p are well behaved throughout the configuration we impose the restriction $dp/d\rho < c^2$. (This will also ensure that the velocity of sound in the fluid is smaller than c). The calculations leading to this restriction are rather stiff but straight-forward and we have found that for $A/B \geqslant 0{\cdot}26$ the condition $dp/dp < c^2$ will be satisfied for all r such that $0 < r \leqslant a$.

A physically viable model can be constructed to satisfy all the restrictions and we give in the concluding section a model for a neutron star based on this particular solution.

6. Discussion: superdense stars

When thermonuclear sources of energy in the interior of a star are exhausted, the star undergoes gravitational contraction. As it contracts, its mass density goes on increasing and it ultimately ends up as a white dwarf or a neutron star or a black hole. It is this last stage of stellar evolution which leads to the formation of superdense condensations of matter. The model that we have presented here can describe the hydrostatic equilibrium conditions in such a superdense star with densities in the range of $10^{14} - 10^{16}$ gm cm $^{-3}$.

Using the scheme outlined in Section 3, we take the matter density on the boundary $r=a$ of the star as $\rho_a = 2\times10^{14}$ gm cm^{-3}. Again, we choose different values for the ratio $\lambda = \rho_a/\rho_0$ and for each chosen value of λ and the assumed value, ρ_a we calculate ρ_0. Equation (8) is then used to calculate R^2. Equation (13) then gives us an

Table 1. Masses and equilibrium radii of superdense-star models corresponding to $K = -2$ and $\rho_a = 2 \times 10^{14}$ gm cm^{-3}

No.	λ	a/R	m/a	R km	a km	m km	$m/M_\odot$	A	B	A/B
1	0·9	0·181	0·046	46·63	8·44	0·39	0·2644	1·3581	0·9043	1·5018
2	0·8	0·269	0·095	43·96	11·82	1·12	0·7593	1·1987	0·8780	1·3653
3	0·7	0·348	0·146	41·12	14·32	2·10	1·4237	1·0202	0·9705	1·0512
4	0·6	0·429	0·202	38·07	16·33	3·29	2·2305	0·8101	0·9924	0·8163
5	0·5	0·517	0·261	34·75	17·98	4·70	3·1864	0·5639	1·0001	0·5638
6	0·4598	0·5567	0·287	33·33	18·37	5·273	3·575	0·4481	0·9946	0·4506
7	0·4	0·622	0·327	31·08	19·33	6·32	4·2847	0·2526	0·975	0·2616
8	0·3384	0·70	0·3712	28·58	20·00	7·426	5·005	0·0213	0·9253	0·023
9	0·3	0·757	0·401	26·92	20·38	8·16	5·5322			
10	0·2	0·959	0·486	21·98	21·08	10·25	6·9492			
11	0·1	1·367	0·592	15·54	21·25	12·58	8·53			

Note:
The mass 'm' recorded in the table is measured in km. The corresponding value in gm is $M = mc^2/G$. $1\ M_\odot = 1{\cdot}475$ km.

estimate of a, the radius of the star and finally Equation (12) gives the mass of the star. The value of m as given by Equation (12) will be in km. The mass M of the star in gm is obtained by $MG/c^2 = m$ or $M = mc^2/G$. It is easier to express the mass of the star as a multiple of one solar mass $M_{\odot}$. The results of the calculations for various values of λ are given in Table 1.

It will be seen from Table 1 that a/R is a decreasing function of λ. Now the physical requirement that ρ, p and $\rho - 3p/c^2$ be all $\geqslant 0$ restricts a/R to the condition (25) *viz.* $a/R \leqslant 0{\cdot}5567$.
Therefore the corresponding restriction on λ is $\lambda \geqslant 0{\cdot}4598$. Thus the first six values of λ in the table give a series of physically viable star-models. The equilibrium radius of each of these models is of the order of the radius of a neutron star. The maximum mass for the configuration is $3{\cdot}573\ M_{\odot}$ and is obtained at the radius of 18·37 km. Both m and a are decreasing functions of λ.

If however we relax the physical requirement to $\rho > 0$, $p \geqslant 0$, $\rho - p/c^2 \geqslant 0$, the restriction on a/R is $a^2/R^2 < \frac{1}{2}$ or $a/R \leqslant 0{\cdot}7$ and we can go as far as the first eight values of λ in the table. The models will admit higher values of masses and radii, the maximum mass being $5{\cdot}037\ M_{\odot}$ occurring for a radius of 20 km.

Thus the space-time with metric (14) with space-sections t=const as spheroids (of eccentricity $\sqrt{2/3}$) gives us a series of equilibrium configurations each having surface density, mass and radius of the same order as in a neutron star. Without recourse to any equation of state, but using purely geometrical properties, we have been able to derive these models. One good feature of these models is that they permit higher values of maximum mass of a neutron star than the values permitted by a similar non-nuclear analysis by Rhoades and Ruffini (1974).

Though numerical calculations have been carried out for the exact solution corresponding to $K = -2$, the method is quite general and can be used for a whole series of models with $K < 1$.

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