Non-relativistic Fermions, Coadjoint Orbits of W_∞ and String Field Theory at $c = 1$

Avinash Dhar[*], Gautam Mandal[†] and Spenta R. Wadia[‡]

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

ABSTRACT

We apply the method of coadjoint orbits of W_∞-algebra to the problem of non-relativistic fermions in one dimension. This leads to a geometric formulation of the quantum theory in terms of the quantum phase space distribution of the fermi fluid. The action has an infinite series expansion in the string coupling, which to leading order reduces to the previously discussed geometric action for the classical fermi fluid based on the group w_∞ of area-preserving diffeomorphisms. We briefly discuss the strong coupling limit of the string theory which, unlike the weak coupling regime, does not seem to admit of a two dimensional space-time picture. Our methods are equally applicable to interacting fermions in one dimension.

[*] adhar@tifrvax.bitnet.
[†] mandal@tifrvax.bitnet.
[‡] wadia@tifrvax.bitnet

1. **Introduction:**

Non-relativistic fermions in one dimension have recently been investigated in connection with models of two-dimensional string theory. The connection proceeds by realizing that two-dimensional string theory (in flat spacetime and linear dilaton background) is perturbatively equivalent to two-dimensional "Liouville gravity" coupled to one-dimensional matter [1-5]. The lattice formulation of the latter is described by a hermitian matrix model in one dimension [6], which in turn is exactly mapped onto a theory of nonrelativistic fermions in one dimension [7]. There are various reasons why it is of interest to write down an exact bosonization of this model. It would provide us with an exact field theory action of two-dimensional string theory, with manifest invariance principles. It would make possible a description of stringy non-perturbative behaviour. It would also be of interest from the viewpoint of condensed matter physics, where this problem was first posed and approximately solved by Tomonaga [8]. An exactly solvable version of Tomonaga's model was formulated by Luttinger [9] where fermions obey the relativistic dispersion relation $E(k) = \pm|k|$. This version of the model has been the subject of much study and elaboration [10,11]. There are also connections with quantum Hall effect in two dimensions. In a completely different area of activity connected with the study of the large N limit of matrix models, this problem was studied by the collective field formulation [12], which was also adopted for the study of two-dimensional string field theory [13]. A perturbative (low energy) expansion and a treatment of the turning point problem was given in [14]. In a series of papers we have studied the nonrelativistic fermion problem from the viewpoint of W_∞ symmetry and its classical limit w_∞ (= the area-preserving diffeomorphisms in two dimensions) [15-18]. See also [19]. In [18] we discussed the classical limit as an incompressible fermi fluid in two dimensions. Using the method of the co-adjoint orbits of w_∞ we presented a geometrical action and a string picture in terms of the classical phase space of the fermi fluid. We have also made precise statements about the limitations of the collective field method. In this paper we extend the results of [18] and give an exact discussion of the bosonization, using the method

of coadjoint orbits of the group W_∞. The nonrelativistic fermions give rise to specific coadjoint orbits of W_∞. These coadjoint orbits are specified by a quadratic constraint and by the number of particles or equivalently by the fermi level. There is a close analogy with the problem of an $SU(2)$ spin in a magnetic field, where the coadjoint orbits of $SU(2)$ are specified by the values of the total angular momentum. In case the spin is formed out of a two-state fermi system, the coadjoint orbit corresponds to spin half if the number of fermions is one (half filled) and to spin zero if the number of fermions is zero (unfilled) or two (completely filled). In exact analogy with the spin problem we present an action functional on those coadjoint orbits of W_∞ (specified by an appropriate set of constraints) which correspond to non-relativistic fermions in one dimension. This action is manifestly invariant under the W_∞ transformations that are a symmetry of the original fermionic action. We emphasize that the symmetry group is W_∞ and not w_∞. The latter is obtained only in the limit $\hbar$ (string coupling) $\to 0$. There is a way of writing the action in terms of a scalar field in $2 + 1$ dimensions. This field can be interpreted as the "phase space" distribution function of the original one-dimensional fermi fluid. A novel feature of the action is that it can be formally expanded in an infinite series in $\hbar$ or the string coupling. The leading term reduces to the geometric action presented in [18], which is based on w_∞ symmetry. In the strong coupling limit, $\hbar \to \infty$, however, this picture clearly breaks down. Indeed, it seems that an interpretation in terms of a two-dimensional target space theory does not exist. This seems to suggest that the standard reasoning that the dynamical metric on the world-sheet is equivalent to one conformal (Liouville) mode which in turn gives rise to one additional target space dimension does not work in the strong coupling limit.

The bosonization technique we have developed here is also applicable to the case of interacting fermions in one dimension.

The plan of the paper is as follows. In the next section we review some aspects of the formulation of fermion field theory and W_∞ algebra as developed in [15-18]. This will also serve to set up our notation. In Sec. 3 we discuss in detail the analogy

of the present problem with that of a spin in a magnetic field. Indeed, the problems are identical, except that the "rotation" group in the present case is W_∞. We show that the bilocal operator, which is the analogue of the spin operator in the present case, satisfies a constraint that determines the representation to which the W_∞ spin belongs, analogous to the constraint $\vec{S}^2 = \text{constant}$ for the rotation group which determines the spin content. We write down the "classical" bosonized action in Sec. 4, in exact analogy with that for a spin in a magnetic field. The group for which the action is written down is W_∞, which is a one-parameter deformation of w_∞, the group of area-preserving diffeomorphisms in two dimensions. The parameter is $\hbar$ and in the present case is identified with the string coupling. The "classical" action may, therefore, be thought of as an infinite series in string coupling. In Sec. 5 we discuss solutions to the classical equation of motion which satisfy the constraints on the bilocal operator. The constraints can be solved only perturbatively in $\hbar$, the string coupling constant. We show that at the lowest order in $\hbar$ the solutions are characteristic functions, as one might expect for a classical fermi fluid. In Sec. 6 we discuss how in the $\hbar \to 0$ limit the results of [18] are reproduced. In Sec. 7 we indicate how <u>interacting</u> fermions in one dimension can be treated by our bosonization technique. Finally, in Sec. 8 we end with some concluding remarks.

2. Fermion Field Theory and W-infinity algebra:

In the gauge theory formulation of [15-16], the action for the fermion field theory which is equivalent to the $c = 1$ matrix model, is

$$S[\Psi, \Psi^\dagger, \bar{A}] = \int dt \, \langle\Psi(t)|(i\hbar\partial_t + \bar{A}(t))|\Psi(t)\rangle \tag{1}$$

where $\bar{A}(t)$ is some given background field. The fermion field $|\Psi(t)\rangle$ is a ket vector in the single-particle Hilbert space with components $\langle x|\Psi(t)\rangle \equiv \psi(x,t)$ in the coordinate basis. In the same basis, the matrix elements of $\bar{A}(t)$ will be denoted

4

by $\langle x|\bar{A}(t)|y\rangle \equiv \bar{A}(x, y, t)$. For the $c = 1$ matrix model,

$$\bar{A}(x, y, t) = \frac{1}{2}(\hbar^2\partial_x^2 - V(x))\delta(x - y), \quad V(x) = -x^2 + \frac{g_3}{\sqrt{N}}x^3 + \cdots \quad (2)$$

In writing (1)-(2) we have chosen the zeroes of the energy and x-axis appropriately such that the (quadratic) maximum of the potential occurs at $x = 0$ and $V_{\text{max}} = V(0) = 0$. We have also introduced appropriate rescalings suitable for the double scaling limit. The parameter N that appears in (2) is the total number of fermions,

$$N = \langle \Psi(t)|\Psi(t)\rangle = \int dx\,\psi^\dagger(x, t)\psi(x, t) \quad (3)$$

which is taken to infinity in the double scaling limit. The other parameter that appears in (1) and (2), i.e. $\hbar$, is the string coupling constant (see $e.g$ [20]). The quantum theory is defined by the functional integral

$$Z = \int \mathcal{D}\Psi, \mathcal{D}\Psi^\dagger \exp\frac{i}{\hbar}S(\Psi, \Psi^\dagger, \bar{A}) \quad (4)$$

The action (1) has the background gauge invariance

$$\begin{aligned} |\Psi(t)\rangle &\rightarrow V(t)|\Psi(t)\rangle \\ \bar{A}(t) &\rightarrow V(t)\bar{A}(t)V^\dagger(t) + i\hbar V(t)\partial_t V^\dagger(t) \end{aligned} \quad (5)$$

where $V(t)$ is a unitary operator in the single-particle Hilbert space. For a given fixed $\bar{A}(t)$, the residual gauge symmetry is determined by

$$i\hbar\partial_t V(t) + [\bar{A}(t), V(t)] = 0 \quad (6)$$

with the solution

$$V(t) = \mathcal{U}(t)V_0\mathcal{U}^\dagger(t), \quad \mathcal{U}(t) = \mathcal{P}\exp[\frac{i}{\hbar}\int^t d\tau\,\bar{A}(\tau)]. \quad (7)$$

Thus the residual symmetry, for any given $\bar{A}(t)$, is parametrized by an arbitrary constant unitary operator V_0. The set of all the V_0's forms the group W_∞.

The W_∞ algebra is the algebra of differential operators in the single-particle Hilbert space [21,22]. A convenient way to describe it is by introducing the generating function,

$$\hat{g}(\alpha, \beta) \equiv \exp i(\alpha\hat{x} - \beta\hat{p}), \quad [\hat{x}, \hat{p}] = i\hbar \tag{8}$$

The product law

$$\hat{g}(\alpha, \beta)\hat{g}(\alpha', \beta') = \exp[\frac{i\hbar}{2}(\alpha\beta' - \alpha'\beta)]\hat{g}(\alpha + \alpha', \beta + \beta'). \tag{9}$$

is a well-known consequence of the Heisenberg algebra. The W_∞ algebra is a straightforward consequence of (9):

$$[\hat{g}(\alpha, \beta), \hat{g}(\alpha', \beta')] = 2i \sin[\frac{\hbar}{2}(\alpha\beta' - \alpha'\beta)]\hat{g}(\alpha + \alpha', \beta + \beta'). \tag{10}$$

The $\hat{g}(\alpha, \beta)$ form an "orthogonal" basis for the W_∞ algebra. That is,

$$\text{tr}[\hat{g}(\alpha, \beta)\hat{g}(\alpha', \beta')] = \frac{2\pi}{\hbar}\delta(\alpha + \alpha')\delta(\beta + \beta') \tag{11}$$

This can be easily proved, for example by evaluating the trace in the coordinate basis and by using the fact that the matrix elements of $\hat{g}(\alpha, \beta)$ are

$$\langle x|\hat{g}(\alpha, \beta)|y\rangle = \delta(x - y + \hbar\beta) \exp(i\alpha\frac{x + y}{2}) \tag{12}$$

The notation 'tr' in (11) stands for integration over x, y etc.

A general element Θ of W_∞ algebra may, therefore, be written as

$$\Theta = \int d\alpha \, d\beta \, \theta(\alpha, \beta)\hat{g}(\alpha, \beta) \tag{13}$$

Since $\hat{g}(\alpha, \beta)$ satisfies the hermiticity condition $\hat{g}(\alpha, \beta) = \hat{g}(-\alpha, -\beta)$, we see from (13) that for hermitian Θ we must have $\theta^*(\alpha, \beta) = \theta(-\alpha, -\beta)$. Because of this

6

hermiticity condition $\theta(\alpha, \beta)$ can be expressed in terms of a real function $u(p, q)$:

$$\theta(\alpha, \beta) = \int \frac{dp}{2\pi} \frac{dq}{2\pi} u(p, q) \exp i(p\beta - q\alpha) \tag{14}$$

Equations (13) and (14) define the Weyl correspondence between functions in phase space $(u(p, q))$ and operators (Θ). As we shall see later, the functions $u(p, q)$ will later turn out to be closely related to the phase space density of the fermion theory.

The unitary operators V_0 appearing in (7) may now be constructed by exponentiating the general element of the W_∞ algebra in (13). To end this section we note that the algebra in (10) reduces to the algebra of area-preserving diffeomorphisms in two dimensions in the limit $\hbar \to 0$. The W_∞ group that we are dealing with therefore is a quantum deformation of the group of area-preserving diffeomorphisms in two dimensions, the parameter of deformation being $\hbar$ or the string coupling.

3. The Bilocal Operator, the Constraint and Analogy with Spin in a Magnetic Field:

The analogy between the present problem and that of a spin in a magnetic field has already been pointed out by us in [16] and [18]. In this section we will elaborate on that analogy further and show that, in fact, the two problems are closely related. The "rotation" group in this case is W_∞.

The appropriate "spin" variable in the present context is [15-17] the fermion bilocal operator $\Phi(t)$ defined as follows

$$\Phi(t) \equiv |\Psi(t)\rangle\langle\Psi(t)| \tag{15}$$

In the coordinate basis, the xy-component is given by

$$\Phi(x, y, t) \equiv \langle x|\Phi(t)|y\rangle = \Psi(x, t)\Psi^\dagger(y, t) \tag{16}$$

Under W_∞ "rotations" of the fermion field, the bilocal operator, which is gauge-

covariant by construction, transforms by the adjoint action of the group:

$$|\Psi(t)\rangle \rightarrow V|\Psi(t)\rangle \quad \Rightarrow \Phi(t) \rightarrow V\Phi(t)V^\dagger \tag{17}$$

We may expand $\Phi(t)$ in the basis $\hat{g}(\alpha, \beta)$ provided by the Heisenberg-Weyl group. We have,

$$\Phi(t) = \frac{\hbar}{2\pi} \int d\alpha \, d\beta \, W(\alpha, \beta, t)\hat{g}(\alpha, \beta) \tag{18}$$

where the fermion bilocal operator

$$W(\alpha, \beta, t) \equiv \int dx \, \psi(x + \frac{1}{2}\hbar\beta, t)\psi^\dagger(x - \frac{1}{2}\hbar\beta, t) \exp(i\alpha x) \tag{19}$$

provides a field theoretic representation of W_∞ algebra:

$$[W(\alpha, \beta, t), W(\alpha', \beta', t)] = 2i \sin[\frac{\hbar}{2}(\alpha\beta' - \alpha'\beta)]W(\alpha + \alpha', \beta + \beta', t) \tag{20}$$

Finally, using the equation of motion for the fermion ket $|\Psi(t)\rangle$, which can be obtained by varying action (1), one can easily obtain the equation of motion for $\Phi(t)$:

$$i\hbar\partial_t\Phi(t) + [\bar{A}(t), \Phi(t)] = 0. \tag{21}$$

Equations (18), (20) and (21) are exactly like the corresponding equations for a spin in a magnetic field. Let $S^i(t)$ be the spin variable, T^i the generators of $SU(2)$ (in the appropriate representation). Then, the operator $S(t) = \sum_i S^i(t)T^i$ is like $\Phi(t)$, $S^i(t)$ being like $W(\alpha, \beta, t)$ and T^i like $\hat{g}(\alpha, \beta)$. The algebra of $W(\alpha, \beta, t)$'s is like the spin algebra $[S^i(t), S^j(t)] = i\epsilon^{ijk}S^k(t)$. The equation of motion $\partial_t S^i(t) = (1/i)[\vec{B}.\vec{S}, S^i] = \epsilon^{ijk}B^j S^k(t)$ can be rewritten in terms of $S(t)$ and $B \equiv \sum_i B^i T^i$ and reads $i\partial_t S(t) = [B, S(t)]$, which is like (21) with B playing the role of $-\bar{A}$. The analogy between the two cases is therefore complete. In the case of the $SU(2)$ spin the problem is completely specified by further specifying the representation to

8

which the spin belongs. This may be done, for example, by specifying the value of $\sum_i [S^i(t)]^2$. This is equivalent to giving a quadratic equation for the matrix $S(t)$, as may be easily verified. Another way of specifying the representation to which the spin belongs is by giving an explicit representation for $S^i(t)$ in terms of more elementary objects. For example, the spin $1/2$ (and spin 0) representation can be constructed in terms of a spin-$1/2$ fermi system. Let us study this representation in more detail since this is what happens for the W_∞ spin that is of interest to us in this work.

Let us assume that the spin variable $S^i(t)$ has a more microscopic representation in terms of spin $1/2$ fermions $\psi_a(t)$ $(a = 1, 2)$:

$$S^i(t) = \psi^\dagger(t) \frac{\sigma^i}{2} \psi(t) \tag{22}$$

where σ^i are the Pauli matrices satisfying

$$\begin{aligned}
[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] &= i\epsilon^{ijk} \frac{\sigma^k}{2} \\
\{\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\} &= \frac{1}{2}\delta^{ij}
\end{aligned} \tag{23}$$

Using equal-time fermion anticommunication relations it is easy to verify that (22) satisfies $[S^i(t), S^j(t)] = i\epsilon^{ijk} S^k(t)$. Further, it can be easily verified, using the identity

$$\sum_i (\frac{\sigma^i}{2})_{ab} (\frac{\sigma^i}{2})_{a'b'} = \frac{1}{2}(\delta_{ab'}\delta_{a'b} - \frac{1}{2}\delta_{ab}\delta_{a'b'}) \tag{24}$$

that the $S^i(t)$ are characterized by the relation

$$\sum_i [S^i(t)]^2 = \frac{3}{4} n_f (2 - n_f) \tag{25}$$

where $n_f = \sum_a \psi_a^\dagger(t)\psi_a(t)$ is a Casimir operator (since it commutes with all $S^i(t)$). It simply measures the total number of filled levels in any state, which is a fixed

9

number for all the states of the system and equals the total number of fermions. So, in this simple case of a two-level system we are led to the constraint (25). For half-filling, $n_f = 1$, we find $[S^i(t)]^2 = 3/4$, which is the correct value of the Casimir for spin $1/2$. This accords with the fact that in this case of a two-level system, half-filling corresponds to a two-state system—one, the fermi vacuum in which the lower of the two states is occupied, and the other one is the excited state in which the fermion in the vacuum is excited to the higher level. For no filling (or equivalently complete filling) there is only the fermi vacuum and no excited states. Therefore, $\sum_i [S^i(t)]^2 = 0$ is appropriate for this case. So, we see that information regarding which spin representation the system belongs to is contained in the constraint (25) (which follows from the representation (22)) and depends only on the filling of the fermi sea. An identical situation arises in our present case of interest of W_∞ spin. Before we discuss that, note that in the $SU(2)$ case the constraint (25) is equivalent to a quadratic equation for $S(t)$. In fact, one can show that $4S(t)^2 + S(t) = \sum_i [S^i(t)]^2$. We have mentioned this because there are an infinite number of Casimirs for W_∞. Since the above type of quadratic equation contains information about all of them, it is easier to deduce this type of relation in this case.

The above line of argument can be applied identically to the present case of W_∞ spin. The "spin variable" $\Phi(t)$ has a microscopic representation in terms of fermions (15). Thus,

$$
\begin{aligned}
\langle x|[\Phi(t)]^2|y\rangle &= \int dz \langle x|\Phi(t)|z\rangle\langle z|\Phi(t)|y\rangle \\
&= \int dz\, \psi(x,t)\psi^\dagger(z,t)\psi(z,t)\psi^\dagger(y,t) \\
&= \langle x|\Phi(t)|y\rangle(1+N),
\end{aligned}
$$

i.e.

$$(\Phi(t))^2 = (1+N)\Phi(t) \tag{26}$$

where now $N = \int dx\, \psi^\dagger(x,t)\psi(x,t)$ and is again a Casimir since it commutes with all $W(\alpha,\beta,t)$. It is the total number of filled levels in any state, that is, the total

10

number of fermions. The constraint (26), together with the constraint on total number of fermions, fixes the representation to which the W_∞-spin $\Phi(t)$ belongs.

4. The Action

The most elegant way of arriving at an action for this problem is to follow Kirillov's method of coadjoint orbits [23]. We will briefly outline the procedure first for the case of spin in a magnetic field. (For details see [18]). The configuration space here is the space of classical spins which we describe by three-dimensional vectors of a given length (the length ultimately gets related to the Casimir of the $SU(2)$-representation). This space is naturally embedded in $\mathbf{R}^3$; we consider the latter to be the dual space, Γ, to the Lie algebra $su(2)$ under the following scalar product. Let $\{x^i\}$ label the points of $\mathbf{R}^3$ and let $\sum_{i=1}^3 a^i T^i$ denote the elements of $su(2)$ Lie algebra where T^i are generators of $su(2)$. We define a natural scalar product between the two: $\sum_i x^i a^i$. Equivalently, in the matrix notation $X \equiv \sum_i x^i T^i$, $A \equiv \sum_i a^i T^i$, we may write the scalar product as $\mathrm{tr}(XA)$.

The above scalar product has a natural interpretation in terms of expectation value of the spin operator in a coherent state. Consider a coherent state of $SU(2)$, $|\vec{x}\rangle$, belonging to the spin-s representation, which satisfies the well-known property

$$\langle \vec{x}|S^i|\vec{x}\rangle = -sx^i. \tag{27}$$

Here S^i are the components of the quantum spin operator in the spin-s representation. The above scalar product can then be interpreted as expectation value of the operator $\sum_i a^i S^i$ in the coherent state $|\vec{x}\rangle$.

Using the above scalar product one can define the coadjoint action of $SU(2)$ on Γ. This simply rotates the vector $\{x^i\}$. Hence, the coadjoint orbits of $SU(2)$ in $\mathbf{R}^3$ are spheres of different radii. In terms of the matrix X this means $X^2 = $ constant. We may now write down the action by Kirillov's construction:

$$S[X] = i \int ds\, dt\, \mathrm{tr}(X[f_s, f_t]) + \int dt\, \mathrm{tr}(XB) \tag{28}$$

where f_t and f_s are two tangent vectors on some given coadjoint orbit at the point

11

$X(t, s)$ and may be computed from

$$\partial_t X = [f_t, X], \ \partial_s X = [f_s, X].\qquad(29)$$

The f_t and f_s are easily obtained from (29) using the constraint $X^2 = $ constant. If we rescale X for convenience to cast the constraint in the form $X^2 = \mathbf{1}$, then we find that

$$f_t = \frac{1}{4}[\partial_t X, X], \ f_s = \frac{1}{4}[\partial_s X, X]\qquad(30)$$

so that the action can be rewritten as

$$S[X] = \frac{i}{4}\int ds\, dt\, \mathrm{tr}(X[\partial_t X, \partial_s X]) + \int dt\, \mathrm{tr}(XB), \quad X^2 = \mathbf{1}\qquad(31)$$

Quantization is done by the path integral

$$Z \sim \int \mathcal{D}X(t) \prod_t \delta[X(t)^2 - 1] \exp[i\lambda S[X]]\qquad(32)$$

where λ is a constant. It is well-known that for the path-integral to be well-defined λ must be quantized. Different values of λ correspond to different spin representations. The theory then knows about the underlying fermionic structure by the specific choice of λ corresponding to the spin-1/2 representation. Note that in the limit $\lambda \to \infty$, the semiclassical method is exact.

The above procedure can be followed step-by-step for the present case of W_∞ spin. The natural dual space, Γ, is the W_∞-algebra itself, which is the set of single-particle operators, or equivalently, is the space of (generalized) functions on the phase space related in a one-to-one fashion to the operators by the Weyl correspondence (13) and (14). Let us denote the points in Γ by ϕ. Let us explicitly

write it out in terms of the Heisenberg-Weyl basis

$$\phi = \int d\alpha \, d\beta \, \tilde{u}(\alpha, \beta) \, \hat{g}(\alpha, \beta) \tag{33}$$

where

$$\tilde{u}(\alpha, \beta) = \int \frac{dp}{2\pi} \frac{dq}{2\pi} e^{i(p\beta - q\alpha)} u(p, q) \tag{34}$$

The analogy with the $SU(2)$ spin case is that $\hat{g}(\alpha, \beta)$ are like the generators T^i, and $\tilde{u}(\alpha, \beta)$ (or $u(p, q)$) are like the components x^i of the point $\vec{x}$ in $\mathbf{R}^3$, and ϕ is like the matrix X. Moreover, ϕ can be interpreted in terms of the expectation value of the bilocal operator Φ in a coherent state of the W_∞ algebra, just like the interpretation of x^i as the expectation value of the spin operator in an $SU(2)$-coherent state.

Just like in the $SU(2)$ case, there is a natural scalar product between the points ϕ and elements Θ ((13) and (14)) of W_∞ Lie algebra. This scalar product is

$$\langle \phi | \Theta \rangle = \mathrm{tr}(\phi\Theta) \tag{35}$$

Under this scalar product, the coadjoint action on ϕ is defined in the standard way. That is, corresponding to the infinitesimal transformation $\delta_\epsilon \Theta = \frac{i}{\hbar}[\epsilon, \Theta]$, ϕ transforms as $\delta_\epsilon \phi = -\frac{i}{\hbar}[\epsilon, \phi]$. The compatibiliy of this coadjoint action with the scalar product is obvious.

In terms of the phase space function $u(p, q)$ introduced in (33) and (34), the coadjoint action is easily deduced using (10) and is given by the Moyal bracket [24],

$$\delta_\epsilon u(p, q) = \{\epsilon, u\}_{MB}(p, q) \tag{36}$$

which is defined by

$$\{A, B\}_{MB}(p, q) = \frac{2}{\hbar} \sin\frac{\hbar}{2}(\partial_q\partial_{p'} - \partial_p\partial_{q'})[A(p, q)B(p', q')]_{p'=p, q'=q} \tag{37}$$

In the $\hbar \to 0$ limit it reduces to the Poisson bracket.

The specific coadjoint orbits of interest to us will be picked out by imposing the constraints

$$\phi^2 = \phi, \quad \operatorname{tr} \phi = N \tag{38}$$

in the dual space Γ. These constraints reflect an underlying fermionic structure and can be understood as follows. The "configuration" ϕ is related to the fermion bilocal operator Φ as

$$\phi = \langle \{\phi\} | \mathbf{1} - \Phi | \{\phi\} \rangle \tag{39}$$

where $|\{\phi\}\rangle$ is a coherent state of W_∞, analogous to the state $|\vec{x}\rangle$ in the $SU(2)$ case. The reason for the appearance of $\mathbf{1} - \Phi$ instead of just Φ can be traced to the definition (16), according to which it is the trace of $\mathbf{1} - \Phi$ which equals the number of fermions. This is also the origin of the constraint $\operatorname{tr} \phi = N$ in (38). The origin of the other constraint, $\phi^2 = \phi$, can be traced to the operator constraint (26), as can be seen by analyzing in detail its expectation value in any coherent state. There is, however, a more direct way to see that ϕ must satisfy the quadratic constraint. Let us evaluate the expectation value (39) in the fermi ground state (which is a coherent state in a trivial sense). The corresponding configuration $\phi = \phi_0$ is given by

$$\phi_0 = \sum_{i \leq N} |i\rangle\langle i| \tag{40}$$

where $|i\rangle, i = 1, 2, \cdots, \infty$ denote the energy eigenstates of the single-particle hilbert space. Clearly ϕ_0 satisfies the constraints (38). Moreover, it is clear from (39) that different configurations ϕ are related to each other by similarity transformations (i.e. by W_∞-coadjoint transformations). Therefore, once we have shown that one point of the orbit satisfies (38), we have proved it for the entire orbit. From the form (40) the fermionic character of our coadjoint orbit is clear.

Now that we know the dual space and characterization of the coadjoint orbits

14

of interest, we can apply Kirillov's method to construct the boson action

$$S[\phi, \bar{A}] = \frac{i}{\hbar} \int ds\, dt\, \mathrm{tr}(\phi[f_t, f_s]) - \int dt\, \mathrm{tr}(\phi\bar{A}) \tag{41}$$

where f_t and f_s are the hamiltonians on the coadjoint orbit that lead to the motions

$$i\hbar \partial_t \phi = [f_t, \phi], \quad i\hbar \partial_s \phi = [f_s, \phi]. \tag{42}$$

The crucial point is that using the constraint $\phi^2 = \phi$ and equations (42) we can easily prove that

$$S[\phi, \bar{A}] = i\hbar \int ds\, dt\, \mathrm{tr}(\phi[\partial_t\phi, \partial_s\phi]) - \int dt\, \mathrm{tr}(\phi\bar{A}) \tag{43}$$

This action can be written more explicitly in terms of the phase space function $u(p, q, t, s)$ defined as in (33) and (34):

$$S[u, \bar{A}] = \int ds\, dt \int \frac{dpdq}{2\pi\hbar} u(p, q, t, s)[\hbar^2 \{\partial_s u(p, q, t, s), \partial_t u(p, q, t, s)\}_{MB}]$$
$$+ \int dt \int \frac{dpdq}{2\pi\hbar} h(p, q) u(p, q, t) \tag{44}$$

where $h(p, q)$ is the classical hamiltonian, $h(p, q) = \frac{1}{2}(p^2 + V(q))$, obtained from the background gauge field (2), using the Weyl correspondence. In terms of the u-variable the constraints read

$$\int \frac{dpdq}{2\pi\hbar} u(p, q) = N \tag{45}$$

$$\cos\frac{\hbar}{2}(\partial_q\partial_{p'} - \partial_{q'}\partial_p)[u(p, q)u(p', q')]_{p'=p, q'=q} = u(p, q) \tag{46}$$

At this point we wish to mention that in writing down the action (41) and in the subsequent manipulations with it, we have made use of the trace identity

15

$\text{tr}(\phi_1\phi_2) = \text{tr}(\phi_2\phi_1)$. Since ϕ's are infinite dimensional matrices, this identity is not satisfied unless we put some restrictions on them. The simplest statement of the restriction is in terms of the corresponding phase space functions in terms of which the trace identity is equivalent to the condition $\int \frac{dpdq}{2\pi\hbar}\{u_1(p,q), u_2(p,q)\}_{MB} = 0$. Since the integrand can be written as a total derivative involving at least one derivative on the u-function, we can satisfy the above condition by requiring $u(p,q) \to$ constant as $p,q \to \infty$.

5. Classical equation of motion and its solutions:

The most general variation of ϕ, consistent with the constraints $\phi^2 = \phi$ and $\text{tr}\,\phi = N$, is

$$\phi \to V\phi V^\dagger, \qquad VV^\dagger = \mathbf{1}. \tag{47}$$

That is, the independent variables are the W_∞ "angles". To obtain the classical equation of motion from the action (44), therefore, we make the above variation (47) in ϕ, with $V = 1 + i\Theta$, Θ infinitesimal. The change in the action is

$$\delta S[\phi, \bar{A}] = -\hbar \int ds\, dt\, [\partial_s\{\text{tr}(\Theta\partial_t\phi)\} - \partial_t\{\text{tr}(\Theta\partial_s\phi)\}] + i \int dt\, \text{tr}(\Theta[\bar{A}, \phi]) \tag{48}$$

We shall take time t to be non-compact. Then the (s,t) space is a half plane, with $-\infty \le s \le 0$, $-\infty \le t \le +\infty$ and the boundary conditions $\phi(t, s = -\infty) = \mathbf{1}$ and $\phi(t, s = 0) = \phi(t)$. Also, assuming that $\phi(t) \to \mathbf{1}$ as $t \to \pm\infty$, only the s-boundary term contributes in (48) and we get

$$\delta S = -\int dt\, \text{tr}[\Theta(\hbar\partial_t\phi - i[\bar{A}, \phi])]. \tag{49}$$

This gives the equation of motion

$$i\hbar\partial_t\phi + [\bar{A}, \phi] = 0, \quad \phi^2 = \phi,\ \text{tr}\phi = N \tag{50}$$

Classically, therefore, the W_∞ spin system under consideration is completely defined by (50). We will now solve this equation and show that the constraints $\phi^2 = \phi$, $\text{tr}\,\phi = N$ keep track of the underlying fermionic structure.

16

Expanding $\bar{A}$ and ϕ in the Heisenberg-Weyl basis, we may write

$$\bar{A} = \int d\alpha \, d\beta \, [\frac{1}{2}(\partial_\beta^2 - \partial_\alpha^2 + i\frac{g_3}{\sqrt{N}}\partial_\alpha^3 + \cdots)\delta(\alpha)\delta(\beta)]\hat{g}(\alpha, \beta) \tag{51}$$

and (33) and (34) for ϕ. The equation of motion for $u(p, q, t)$ now becomes

$$\partial_t u = \{h, u\}_{MB} \tag{52}$$

where

$$h(p, q) = \frac{1}{2}(p^2 - q^2 + \frac{g_3}{\sqrt{N}}q^3 + \cdots). \tag{53}$$

Time-independent case:

In this case the equation of motion is solved by any u which depends on p, q only through the function $h(p, q)$. That is to say, in the phase space u takes the same value on curves of constant classical energy. Out of all such u's, the classical problem is solved only by those that satisfy the quadratic constraint (46). This constraint cannot be solved exactly, except in the limit $\hbar \to 0$. Denoting u by $u^{(0)}$ in this limit, (46) leads to

$$(u^{(0)}(p, q))^2 = u^{(0)}(p, q) \tag{54}$$

Thus $u^{(0)}$ takes the same value (1 or 0) on curves of constant energy in phase space. For example, one may choose

$$u^{(0)}(p, q) = \theta(\epsilon_F - h(p, q)). \tag{55}$$

ϵ_F is a parameter of this classical solution. Finally $u^{(0)}(p, q)$ must satisfy the fermion number constraint (45); for a $u^{(0)}$ of the above form this fixes ϵ_F in terms of $\hbar$ and N. For the hamiltonian (53) we find $-\epsilon_F \sim 1/(\hbar N)$ which is consistent with the fact that in the double scaling limit we treat $-\epsilon_F N \equiv \mu$ as the inverse string coupling. It is clear that (55) is just the classical phase space density of fermions in the fermi vacuum. We have thus once again arrived at the underlying fermionic picture.

<u>Time-dependent case</u>:

As in the time-independent case we are able to solve the equations only in the $\hbar \to 0$ limit. Let us denote the solution of the constraint in this limit by the characterisitic function $\chi_{R(t)}(p, q)$, which satisfies (54) and defines a region $R(t)$ of phase space. There is a time-dependence in $R(t)$ because in the present case the region changes with time. The region $R(0)$ can in principle be quite complicated involving several fluid blobs or droplets of the fermi fluid. Since we are working in the $\hbar \to 0$ limit, the equation of motion satisfied by $u(p, q, t)$ reduces to the classical one: $\partial_t u = \{h(p, q), u(p, q)\}_{PB}$. It can be easily shown that $u = \chi_{R(t)}$ satisfies the equation of motion if the region $R(t)$ is given by

$$\chi_{R(t)}(p, q) = \chi_{R(0)}(\bar{p}(t), \bar{q}(t)) \tag{56}$$

where $(\bar{p}(t), \bar{q}(t))$ denote the classical trajectory evolving according to the hamiltonian $-h(p, q)$ with the initial conditions $\bar{p}(t = 0) = p$, $\bar{q}(t = 0) = q$. In other words, the region $R(t)$ is obtained by evolving each point in the region $R(0)$ for time t under the classical hamiltonian h. For the hamiltonian (53) we can write the classical trajectories explicitly if we ignore the $O(1/\sqrt{N})$ terms. This leads to

$$\chi_{R(t)}(p, q) = \chi_{R(0)}(p \cosh t - q \sinh t, -p \sinh t + q \cosh t). \tag{57}$$

6. Correspondence with Geometric Action for Fluid Profiles

In this section we would like to show how the geometric action for fluid profiles [18] may be obtained from the exact classical action in the limit $\hbar \to 0$. As we mentioned in the last section the characteristic function of a region R in phase space satisfies the constraint (46) in the limit $\hbar \to 0$. This reflects the fact that as $\hbar \to 0$ the phase space density $u(p, q)$ corresponds to that of an incompressible fermi fluid whose density is 1 in some region R and 0 outside. When we consider a two-parameter deformation (in (t, s)) of the phase space density $u(p, q, t, s)$, classically

it corresponds to a two-parameter deformation $R(t, s)$ of the fluid region R. In the following we shall therefore put

$$u(p, q, t, s) = \chi_{R(t,s)}(p, q) + \hbar \text{ corrections.} \tag{58}$$

The correspondence with the fluid-profile action [18] is most directly made by rewriting the action (41) and the equations (42) in terms of the phase space variables. Let us denote the first term of (41) by S_0. In terms of the phase space variables it reads

$$S_0 = \int ds\, dt \int \frac{dp\, dq}{2\pi\hbar} u(p, q, t, s)\{f_t, f_s\}_{MB} \tag{59}$$

The hamiltonians f_t and f_s are defined by (42). In terms of phase space variables equations (42) read

$$\partial_t u = \{f_t, u\}_{MB}, \quad \partial_s u = \{f_s, u\}_{MB} \tag{60}$$

In the limit $\hbar \to 0$, the Moyal bracket goes over to the Poisson bracket. Using this fact and equation (58) we get

$$S_0 = \int ds\, dt \int \frac{dp\, dq}{2\pi\hbar}[\chi_{R(t,s)}\{f_t, f_s\}_{PB} + o(\hbar^2)] \tag{61}$$

where

$$\partial_t \chi_R = \{f_t, \chi_R\}_{PB}, \quad \partial_s \chi_R = \{f_s, \chi_R\}_{PB} \tag{62}$$

It is simple to see that (61) is the same as the action S_0 written in equation (62) of [18]. To facilitate the comparison, let us recall that equation (62) of [18] is

$$S_0 = \int dt\, ds \, \langle \chi_{R(t,s)}(p, q) \,|\, [\partial_t U U^{-1}, \partial_s U U^{-1}]\rangle \tag{63}$$

which is equivalent to

$$S_0 = \int dt\, ds \int \frac{dp\, dq}{2\pi\hbar} \chi_{R(t,s)}(p, q)\{f_t, f_s\}_{PB} \tag{64}$$

where we have written out the definition of the scalar product used in the last paper, and used the fact that $\partial_a U U^{-1}, a = s, t$ are Lie algebra elements corresponding to

the functions f_s, f_t satisfying the property (62) (we have explained in [18] how the commutatator in the w_∞ Lie algebra is equivalent to Poisson bracket of functions in phase space).

The second term in (41) in the limit $\hbar \to 0$ becomes the classical energy contained in the fluid region $R(t,s)$, which is the same as equation (66) of [18]. Therefore we see that the action written in the present paper agrees with the one in [18] in the limit $\hbar \to 0$. A different approach to the classical limit is discussed in [25]. A different approach to coadjoint orbits of w_∞, the group of area-preserving diffeomorphisms, is discussed in [26].

7. Interacting Fermions

In this section we indicate how the bosonization technique described so far can be applied to a wide class of interacting fermi systems.

Let us try to use the bilocal operator ϕ or equivalently the phase space density $u(p,q)$ again as the basic dynamical variable. The first point to realize is that the constraints (38)(equivalently (45) and (46)) have been derived above using purely kinematic reasoning without considering the equation of motion of the fermi field. This is obvious for (45), which simply states that the total number of fermions is N, a condition that is satisfied by any closed system of fermions, interacting or otherwise. The second constraint, (46), originates from the operator constraint (26), *viz.*, $\Phi^2 = (N+1)\Phi$. The only ingredient that went into the derivation of this constraint is the anticommutation relation of the fermi field which again does not depend on the dynamics of the fermi system. Besides the constraints, the first term (the symplectic form) in the action (44) is also purely kinematic, and it does not depend on the choice of the many-body hamiltonian. With these remarks, it is now easy to see that the coadjoint orbits of W_∞ that we have constructed are suitable for representing interacting fermions also, provided the interaction can be expressed in terms of ϕ or equivalently $u(p,q)$.

20

Let us now give some examples. The most general interaction involving quadrat-∎
ics of ϕ (or u) is

$$S_{\text{int}} = \int dt \int dx\, dy\, dz\, dw\, \mathcal{A}_{xyzw}\phi(x,y)\phi(z,w) \tag{65}$$

The $SU(2)$-spin analog of such a term would be $\sum_{ij} S_i S_j \mathcal{B}_{ij}$ which can be thought of as coupling to some "generalized magnetic fields" which have tensorial trans-formation properties under $SU(2)$ rotations (instead of vector transformations which are true of usual magnetic fields). The above interaction term would be the bosonized form of the following fermion interaction:

$$S_{\text{int}} = \int dt \int dx\, dy\, dz\, dw\, \mathcal{A}_{xyzw}\psi(x,t)\psi^\dagger(y,t)\psi(z,t)\psi^\dagger(w,t) \tag{66}$$

Clearly, the standard four-fermi interaction $\int dt\, dx\, [\psi^\dagger(x)\psi(x)]^2$ is included in this list.

The generalization to cubic and higher interaction terms in ϕ is obvious; they just involve introduction of higher external tensor fields of W_∞ in the sense ex-plained above. It would be extremely interesting to understand emergence of new collective excitations like plasmons arising out of interacting bose theories such as the ones mentioned above.

8. Concluding Remarks

In this paper we have presented solution of the bosonization problem of non-relativistic fermions in 1-dimension. We believe that this formulation will give us a handle on some important issues of two-dimensional string theory. For instance, using our action (37) and the constraints (45) and (46), we can look for stringy non-perturbative effects $\sim \exp(-1/\hbar)$ that have been discussed by Shenker [27]. We should note in this context that both our classical action and the constraint (46) contain explicit factors of $\hbar$. The second point is that we have seen that a $(1+1)$ dimensional target space picture emerges from the $c = 1$ matrix model

perturbatively in $\hbar$, the string coupling constant. Since our formulation is valid for all values of $\hbar$ it is clearly important to ask what happens to this picture for large $\hbar$, i.e. in the strong coupling limit. We have not been able to solve the constraint (46) in this limit, but it seems to us that the above picture of a $(1+1)$-dimensional target space theory cannot be valid in this limit. The approximation to a single fluid blob, valid for small $\hbar$, must necessarily break down as the string coupling constant increases, which is also accompanied with the loss of incompressiblity of the fermi fluid on account of large quantum corrections to step-function-like densities. One presumably then has numerous fluid blobs all over the phase space indicating that in the limit $\hbar \to \infty$, one may have to deal with the full $(2+1)$-dimensional theory described by the action (44). Such a scenario implies that the standard reasoning from the viewpoint of continuum quantum gravity that the dynamical metric is equivalent to one conformal mode (Liouville) which in turn is equivalent to <u>one</u> additional target space dimension, breaks down in the strong coupling limit. It is clearly very important to make this discussion more concrete.

Another interesting aspect of the strong coupling limit is the following. From the viewpoint of fermions moving in the inverted harmonic oscillator, $\hbar \to \infty$ limit is equivalent to $\mu \equiv -N\epsilon_F \to 0^+$ (we are measuring ϵ_F with respect to the top of the potential and using the convention that μ is positive for energies below the top). In this limit the fermi level moves to the top of the potential. Since the potential barrier is negligible here, the fluid will freely move between the two "classical worlds" described by the inverted harmonic oscillator potential. In order to gain some insight into the description of the $\mu \to 0^+$ limit, it may be useful to consider a generalized model in which we consider the entire range $\mu \in (-\infty, +\infty)$. For $\mu \to 0^-$, this model does not correspond to a string theory, in the sense that the perturbation expansion of the matrix model fails to exist. However, negative μ's make perfect sense as a theory of fermions. The generalized model has another classical limit as $\mu \to -\infty$ in addition to the weak coupling string theory ($\mu \to \infty$). It would be interesting to see if the two different signs of μ correspond to two different phases of the fermi system. Such phase transitions

are known to occur in $2+1$ dimensional fermi systems in the disussion of quantum Hall effect*. We have also seen that our bosonization techniques can be applied to interacting fermion systems in one dimension. It would be worthwhile if some of the techniques introduced in this paper can deepen our understanding of $(1+1)$- and $(2+1)$- dimensional condensed matter systems.

REFERENCES

1. S. Das, S. Naik and S.R. Wadia, Mod. Phys. Lett. A4 (1989) 1033.

2. A. Dhar, T. Jayaraman, K.S. Narain and S.R. Wadia, Mod. Phys. Lett. A5 (1990) 863.

3. S. Das, A. Dhar and S.R. Wadia, Mod. Phys. Lett. A5 (1990) 799.

4. J. Polchinski, Nucl. Phys. B234 (1989) 123.

5. T. Banks and J. Lykken, Nucl. Phys. B331 (1990) 173.

6. V.A. Kazakov and A.A. Migdal, Nucl. Phys. B320 (1989) 654.

7. E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, Comm. Math. Phys. 59 (1978) 35.

8. S. Tomonaga, Prog. Theor. Phys. 5 (1950) 544.

9. J.M. Luttinger, J. Math Phys. 4 (1963) 1154.

$\star$ We thank F.D.M. Haldane for pointing this out to us.

10. Lieb and Mattis, J. Math. Phys. 6 (1965) 304.

11. F.D.M. Haldane, J. Phys. C14 (1981) 2585.

12. A. Jevicki and B. Sakita, Nucl. Phys. B165 (1980) 511.

13. S.R. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639.

14. A.M. Sengupta and S.R. Wadia, Int. J. Mod. Phys. A6 (1991) 1961; G. Mandal, A.M. Sengupta and S.R. Wadia, Mod. Phys. Lett. A6 (1991) 1465.

15. S. Das, A. Dhar, G. Mandal and S.R. Wadia, ETH, IAS and Tata preprint, ETH-TH-91/30, IASSNS-HEP-91/52 and TIFR-TH-91/44 (Sept. 1991), to appear in Int. J. Mod. Phys. A (1992).

16. S. Das, A. Dhar, G. Mandal and S.R. Wadia, Mod. Phys. Lett. A7 (1992) 71.

17. S. Das, A. Dhar, G. Mandal and S.R. Wadia, Mod. Phys. Lett. A7 (1992) 937.

18. A. Dhar, G. Mandal and S.R. Wadia, IAS and Tata preprint, IASSNS-HEP-91/89 and TIFR-TH-91/61, to appear in Int. J. Mod. Phys. A (1992).

19. U.H. Danielsson, *A Study of Two Dimensional String Theory*, Ph. D. Dissertation, Princeton University, 1992.

20. J. Polchinski, Nucl. Phys. B346 (1990) 253.

21. C.N.Pope, L.J.Romans and X.Shen, "A brief History of W_∞," in *Strings 90*, ed. R.Arnowitt et al (World Scientific, 1991), and references therein.

22. I. Bakas and E.B. Kiritsis, Int. J. Mod. Phys. A6 (1991) 2871.

23. See for example, A.A. Kirillov, *Elements of the Theory of Representations* (1976); A. Alekseev and S. Shatasvili, Nucl. Phys. B 323 (1989) 719; A. Alekseev, L. Faddeev and S. Shatasvili, J. Geom. Phys. 1 (1989) 3; B. Rai and V.G.J. Rogers, Nucl. Phys. B341 (1990) 119.

24. J.E. Moyal, Proc. Cambridge Phil. Soc. 45 (1949) 99.

25. S. Iso, D. Karabali and B. Sakita, City College preprint, 1992.

26. J. Avan and A. Jevicki, Mod. Phys. Lett. A7 (1992) 357.

27. S. Shenker, in Proceedings of the Cargese Workshop on Random Surfaces, Quantum Gravity and Strings, 1990, edited by O. Alvarez, E. Marinari and P. Windey (Plenum Press).