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A Time-dependent Classical Solution of $c = 1$ String Field Theory and Non-perturbative Effects

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ABSTRACT

We describe a real-time classical solution of $c = 1$ string field theory written in terms of the phase space density, $u(p, q, t)$, of the equivalent fermion theory. The solution corresponds to tunnelling of a single fermion above the filled fermi sea and leads to amplitudes that go as $\exp(-C/g_{\rm str})$. We discuss how one can use this technique to describe non-perturbative effects in the Marinari-Parisi model. We also discuss implications of this type of solution for the two-dimensional black hole.

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1. **Introduction:**

If we call $g_{\rm str}$ the coupling constant of string theory then we would expect that, at weak coupling, non-perturbative effects would go as $\exp(-C/{g_{\rm str}}^2)$. However, Shenker [1] has made the remarkable observation that in closed string theories there can be non-perturbative effects which go as $\exp(-C/g_{\rm str})$ and that this is a generic feature of string theory. Such a non-perturbative behaviour can be argued for on the basis of large orders of string perturbation theory and can be seen in the solutions of string theory with $c < 1$. Their existence in the $c = 1$ model (two-dimensional string theory) [2] and in the Marinari-Parisi model [3] is also argued for in terms of 'eigenvalue tunnelling'. For a detailed discussion of these we refer the reader to [1].

In this paper we discuss this phenomenon in the non-perturbative formulation of string field theory at $c = 1$ deveolped in [4]. This formulation is based on the mapping of the $c = 1$ matrix model onto a theory of free non-relativistic fermions moving in one dimension in a potential [5, 2]. The central object in the formulation developed in [4] is the fermion density operator, $u(p, q, t) = \int dx \psi^\dagger(q - x/2)\psi(q + x/2)\exp(-ipx)$, whose expectation value in any state is the fermion distribution function in phase space in that state. The string field theory action of [4] is written in terms of $u(p, q, t)$ and has a nontrivial dependence on $g_{\rm str} \sim \hbar$. Formally, the action can be written as an infinite series in $g_{\rm str}$. We present a real-time classical solution of this theory which can be seen to correspond to quantum tunnelling of a single fermion above the filled fermi sea. This classical solution has a nontrivial dependence on $g_{\rm str}$ and leads to amplitudes that go as $\exp(-C/g_{\rm str})$ rather than $\exp(-C/{g_{\rm str}}^2)$. A comparison with collective field theory shows that our classical solution is not a classical solution of the collective field theory even in the limit $g_{\rm str} \to 0$. We explain the discrepancy. Our classical solution can be generalized to the Marinari-Parisi model and has a possible application to supersymmetry breaking in that model. Also, in the context of identification [6, 7, 8] with black hole physics in two dimensions [9, 10] we find that the "hyperbolic" transform [6] of our solution corresponds to a rather interesting time-dependent tachyon solution in the black hole background.

The plan of the paper is as follows. In Sec. 2 we briefly review our formulation of the $c = 1$ string field theory and set up the notation. In Sec. 3 we discuss an exact solution of the classical equations of motion of the $c = 1$ string field theory which describes a single fermion tunnelling. In Sec. 4 we combine this solution with the phase space density corresponding to the filled fermi sea of $N - 1$ fermions to get a time-dependent solution of the full theory and show that it leads to amplitudes $\propto \exp(-C/g_{\rm str})$. In Sec. 5, we discuss how this technique can be applied to find non-perturbative effects in the Marinari-Parisi Model. In Sec. 6, we make a comparsion with collective field theory. In Sec. 7, we discuss our solution in the black hole context.

2. $c = 1$ String Field Theory:

We briefly review the non-perturbative formulation of the $c = 1$ string field theory [4]. As is well-known, this theory is exactly described by non-relativistic fermions moving in a background hamiltonian [5, 2]. The double scaled field theory corresponds to the hamiltonian $h(p, q) = \frac{1}{2}(p^2 - q^2)$. Since the fermion number is held fixed, the basic excitations are described by the bilocal operator $\phi(x, y, t) = \psi(x, t)\psi^\dagger(y, t)$ or equivalently its transform

$$u(p, q, t) = \int_{-\infty}^{+\infty} dx \; \psi^\dagger \left(q - \hbar x/2, t\right) e^{-ipx} \psi \left(q + \hbar x/2, t\right) \tag{1}$$

Here and in the following we have used the notation $\hbar$ for $g_{\rm str}$ as in [4]. The expectation value of this operator in a state is the phase space fermion distribution function in that state. Eqn. (1) also has the important property that given a "classical function" $f(p, q, t)$ in the phase space, we have an operator in the fermion field theory

$$\mathcal{O}_f = \int \frac{dpdq}{(2\pi)^2} f(p,q,t) u(p,q,t) = \frac{1}{2\pi}\int dx\ \psi^\dagger(x,t)\hat{f}(\hat{x},\hat{p},t)\psi(x,t) \tag{2}$$

where $\hat{f}(\hat{x},\hat{p},t)$ is the Weyl-ordered operator corresponding to the classical function $f(p,q,t)$. For example, vector fields corresponding to the functions $f_{\alpha\beta}(p,q) = e^{i(p\beta - q\alpha)}$ satisfy the classical algebra ω_∞ of area-preserving diffeomorphisms. The corresponding quantum operators in the fermion field theory

$$\tilde{u}(\alpha,\beta,t) = \int \frac{dp\ dq}{(2\pi)^2} e^{i(p\beta - q\alpha)} u(p,q,t) \tag{3}$$

satisfy the W_∞ algebra (a one-parameter deformation of ω_∞)*

$$[\tilde{u}(\alpha,\beta,t), \tilde{u}(\alpha',\beta',t)] = \frac{i}{\pi}\sin\frac{\hbar}{2}(\alpha\beta' - \beta\alpha')\tilde{u}(\alpha+\alpha', \beta+\beta', t) \tag{4}$$

An exact boson representation of the fermion field theory that reflects the W_∞ symmetry can be acheived in terms of the 3-dim. field $u(p,q,t)$, provided we impose the constraints that follow from its microscopic definition

$$\int \frac{dp\ dq}{2\pi\hbar} u(p,q,t) = N \tag{5}$$

$$\cos\frac{\hbar}{2}\left(\partial_q\partial_{p'} - \partial_{q'}\partial_p\right) u(p,q,t)u(p',q',t)\Big|_{\substack{p'=p \\ q'=q}} = u(p,q,t) \tag{6}$$

where N is the total number of fermions. Also the equation of motion that follows from the definition (1) is

$$(\partial_t + p\partial_q + q\partial_p)u(p,q,t) = 0 \tag{7}$$

The constraints (5) and (6) in fact specify a co-adjoint orbit of W_∞, and the

* We have in our previous works also used the notation $W(\alpha,\beta,t)$ for $\tilde{u}(\alpha,\beta,t)$.

classical action is constructed using the method of Kirillov

$$
\begin{aligned}
S[u,h] = \int ds\ dt \int \frac{dp\ dq}{2\pi\hbar} u(p,q,t,s)\hbar^2 \left\{\partial_s u(p,q,t,s), \partial_t u(p,q,t,s)\right\}_{MB} \\
+ \int dt \int \frac{dp\ dq}{2\pi\hbar} u(p,q,t) h(p,q).
\end{aligned}
\tag{8}
$$

where $\{\ ,\ \}_{MB}$ is the Moyal bracket (for details see [4]).

We wish to emphasize that the action (8), together with a measure in the functional integral that incorporates the constraints (5) and (6) can be **derived**, starting from the fermion field theory, by the standard procedure of time-slicing and inserting complete sets of W_∞-coherent states. Thus, one can derive the following identity for the n-point function of the bilocal fermion operator $u(p,q,t)$:

$$
\langle\mu|T(u(p_1,q_1,t_1)\cdots u(p_n,q_n,t_n))|\mu\rangle = \int \mathcal{D}u\, u(p_1,q_1,t_1)\cdots u(p_n,q_n,t_n) \exp(\frac{i}{\hbar} S[u,h])
\tag{9}
$$

where $S[u,h]$ is the action (8) and the measure $\mathcal{D}u$ includes δ-functions incorporating the constraints (5) and (6). The state $|\mu\rangle$ on the left hand side refers to the fermi ground state.

Let us now briefly indicate the classical limit of the string theory ($\hbar \to 0$). In this limit the constraint (6) implies that $u(p,q,t)$ is a characteristic function of a region of phase space specified by a boundary [11, 12, 13, 14]. For example the ground state corresponds to the static solution $u(p,q) = \theta(\mu - h(p,q))$, $\mu \sim -\frac{1}{\hbar}$. The massless excitation (tachyon) [15, 16] corresponds to a curve that is a small deviation from the fermi surface $h(p,q) = \frac{1}{2}(p^2 - q^2) = \mu$.

3. Time-dependent Classical Solution For Single-fermion Tunnelling:

We shall now describe a time-dependent classical solution of the $u(p,q,t)$ theory (5)-(8) that describes the phenomenon of *quantum mechanical* tunnelling of a single fermion through the potential barrier.

We wish to emphasize at the outset that **an effect which is genuinely quantum mechanical in terms of a single fermion can be described entirely by a classical solution of the $u(p,q,t)$ theory in real time**. Such a phenomenon is not unfamiliar: the classical Euler-Lagrange equation of a Schrödinger field theory is identical to the Schrödinger equation of single-particle quantum mechanics. The fact that the classical theory of $u(p,q,t)$ describes the single-particle quantum mechanics exactly is indicated by the appearance of explicit factors of $\hbar$ in the classical action and the constraints, as well as by the fact that the action is derived from coadjoint orbit of W_∞ rather than w_∞. The latter is the group of canonical transformations in the classical single-particle phase space whereas the former is the group of unitary transformations in the single-particle Hilbert space. This, indeed, is the main difference between our formalism and standard collective field theory [16]— classical solutions of the latter describe only classical motion of the fermions and do not accomodate their quantum fluctuations. We shall see this difference quantitatively in Sec. 6.

Single-fermion wave-packet in phase space:

We shall first describe a solution $u_1(p,q,t)$ of (5)-(8) with N in (5) put equal to one. This phase space density corresponds to a single isolated fermion tunnnelling across the potential barrier. Later we will combine this solution with a stationary solution corresponding to a fermi sea built out of $N-1$ fermions to construct a solution of the full N-particle system.

It is easy to verify that

$$u_1(p,q,t) = 2\exp\frac{-1}{\hbar}[(p\cosh t - q\sinh t - p_0)^2 + (-p\sinh t + q\cosh t - q_0)^2] \quad (10)$$

satisfies the equation of motion

$$\partial_t u_1(p,q,t) = \{h, u_1\}_{MB} = -(p\partial_q + q\partial_p)u_1 \tag{11}$$

and the constraints

$$\int \frac{dpdq}{2\pi\hbar} u_1(p,q) = 1 \tag{12}$$

$$\cos\frac{\hbar}{2}(\partial_q\partial_{p'} - \partial_{q'}\partial_p)[u_1(p,q)u_1(p',q')]_{p'=p,q'=q} = u_1(p,q) \tag{13}$$

It is clear that $u_1(p,q,t)$ is a configuration that describes the phase space density of a single fermion. In the next section we shall see that $u_1(p,q,t)$ corresponds to the phase space density of a fermion in a minimum uncertainty wavepacket (Eqn. (39)). Note that the peak of the phase space density at time t is given by

$$p\cosh t - q\sinh t - p_0 = 0 = -p\sinh t + q\cosh t - q_0 \tag{14}$$

The above equations give the position of the peak at time t as

$$\bar{p}(t) = p_0\cosh t + q_0\sinh t, \quad \bar{q}(t) = p_0\sinh t + q_0\cosh t \tag{15}$$

Let us choose $p_0 > 0, q_0 < 0$ and $p_0 < |q_0|$ for definiteness, so that the mean trajectory (15) describes a hyperbola in the left half space corresponding to negative energy (negative value of $(p_0^2 - q_0^2)/2$). We shall equivalently use a parametrization

$$p_0 = \sqrt{2|E_0|}\sinh\theta_0, \qquad q_0 = -\sqrt{2|E_0|}\cosh\theta_0 \tag{16}$$

where $E_0 = -|E_0| = (p_0^2 - q_0^2)/2$ denotes the energy of the trajectory.

The trajectory of the peak, (15), suggests that classically the fermion is completely reflected off the barrier. To see this more quantitatively, note that in the $\hbar \to 0$ limit we get

$$\begin{aligned} \frac{1}{2\pi\hbar} u_1(p,q,t) \to & \delta(p\cosh t - q\sinh t - p_0)\delta(-p\sinh t + q\cosh t - q_0) \\ = & \delta(p - \bar{p}(t))\delta(q - \bar{q}(t)) \end{aligned} \tag{17}$$

where $\bar{p}(t)$ and $\bar{q}(t)$ are given by (15). For finite $\hbar$, however, the phase space density has a finite spread and a finite amount of the phase space density trickles across to the other side of the potential barrier. The easiest way to see this is to look at the fermion density $\rho(q,t)$:

$$\begin{aligned} \rho(q,t) \equiv & \int \frac{dp}{2\pi\hbar} u_1(p,q,t) \\ = & (\pi\hbar\cosh 2t)^{-1/2} \exp[-\frac{(q - \bar{q}(t))^2}{\hbar\cosh 2t}] \end{aligned} \tag{18}$$

where $\bar{q}(t)$ has been defined in (15).

There are several interesting facts about (18). First of all, it is defined for all q, positive and negative. Therefore, there is a non-zero probability density of the fermion in the right half of the world ($q > 0$) at any time. Moreover, although the mean position $\bar{q}(t)$ again shows the classically reflected trajectory, the dispersion

$$\Delta q(t) = \sqrt{\frac{\hbar}{2}\cosh 2t}$$

increases exponentially rapidly at large times (positive as well as negative). This means that the density (18) is reasonably peaked at finite times around its mean but at large negative and positive times it gets very spread out. How does one find out if there is a finite amount of probability that actually moves over from the left side of the barrier to the right?

Let us consider the total probability, at any given time, of the fermion to be in the right (or left) half of q-space. In other words, we define

$$N_+(t) = \int_0^\infty dq\, \rho(q,t) \tag{19}$$

and

$$N_-(t) = \int_{-\infty}^0 dq\, \rho(q,t) \tag{20}$$

By (12), $N_+(t) + N_-(t) = 1$; hence only one of them is independent. We shall focus on $N_+(t)$. Using (18), we find that

$$N_+(t) = \frac{1}{2}[1 - \mathrm{erf}(\bar{x}(t))] \tag{21}$$

where

$$\bar{x}(t) = -\frac{\bar{q}(t)}{\sqrt{\hbar \cosh 2t}}$$

and the error function is defined by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2}$$

Note that with our choice of the classical trajectory (15)-(16), $\bar{x}(t)$ is positive for all t. It is easy to calculate the $t \to \pm\infty$ limits of (21):

$$\begin{aligned} N_+(\pm\infty) =& \frac{1}{2}[1 - \mathrm{erf}(\bar{x}(\pm\infty))] \\ \bar{x}(\pm\infty) =& \sqrt{\frac{|E_0|}{\hbar}} \exp(\mp\theta_0) \end{aligned} \tag{22}$$

where we have used the parametrization (16) of p_0, q_0 in terms of $|E_0|, \theta_0$. For $\theta_0 > 0$, *i.e.* $p_0 > 0$, we see that $N_+(\infty) > N_+(-\infty)$, showing that a finite amount of "trickling" has taken place, the amount being

$$\mathcal{T} \equiv N_+(\infty) - N_+(-\infty) = \frac{1}{\sqrt{\pi}} \int\limits_{\bar{x}(-\infty)}^{\bar{x}(\infty)} dt\, e^{-t^2} \tag{23}$$

Note that we find a positive "trickle" from the left to the right when $p_0 > 0$. This is understandable because $p_0 > 0$ means that the mean momentum of the wave-packet is also directed from the left to the right (in the direction of increasing q). For negative θ_0, or equivalently negative p_0, we find a negative value for $\mathcal{T}$, while $\mathcal{T}$ vanishes for $\theta_0 = 0 = p_0$.

For small θ_0 we get

$$\mathcal{T} = 2\theta_0 \sqrt{\frac{|E_0|}{2\pi\hbar}} \exp[-\frac{|E_0|}{2\hbar}] + o(\theta_0^2) \tag{24}$$

This can be compared with the leading WKB result for the tunnelling amplitude, which is given by

$$\exp[-\frac{1}{\hbar} \int\limits_{-a}^{a} dx' \sqrt{V(x') - E_0}] \propto \exp[-|E_0|\pi/\hbar\sqrt{2}] \tag{25}$$

Here $\pm a$ are the classical turning points, satisfying $V(\pm a) = E_0$.

4. Time-dependent Classical Solution of $c = 1$ String Field Theory and $\exp(-C/g_{\rm str})$ Effects:

In the last section we constructed an exact solution of the $u(p, q, t)$ theory which corresponds to fermion number equal to one. It describes the phase space density of a single-fermion wave packet, part of which tunnels through. To use this solution in constructing a solution of the N-fermion problem we proceed as follows. We consider the fermi sea of the N-fermion system and imagine removing one fermion from the fermi level to a 'wave-packet state' of the kind described above, with a mean energy that is much higher than the fermi energy but still far lower than the top of the potential barrier (we will presently make these statements more exact). The fermi sea of $N-1$ fermions corresponds to a phase space density

$$\begin{aligned} u_0(p,q) =& \langle F| \int dx\, e^{-ipx} \psi^\dagger(q-\hbar x/2,t)\psi(q+\hbar x/2,t)|F\rangle \\ =& \int^{\mu} d\nu \int dx\, e^{-ipx} \phi^*_\nu(q-\hbar x/2)\phi_\nu(q+\hbar x/2) \end{aligned} \tag{26}$$

Here $|F\rangle$ is the ground state of $(N-1)$ fermions, $\phi_\nu(x)$ is the eigenstate of the single-particle hamiltonian

$$\hat{h} = \frac{1}{2}(\hat{p}^2 - \hat{x}^2 + \frac{g_3}{\sqrt{N}}\hat{x}^3 + \cdots) \tag{27}$$

with energy ν and μ is the fermi level for $(N-1)$ fermions. In the limit $N \to \infty$, the right hand side of (26) can be evaluated explicitly. Denoting this limiting value by $\bar{u}_0(p,q)$, we have

$$\bar{u}_0(p,q) = \frac{1}{2\pi} \int_{-\infty}^{\mu} d\nu \int_{-\infty}^{\infty} \frac{d\lambda}{\cosh \lambda/2} \exp i[\nu\lambda - \frac{1}{\hbar}(p^2-q^2)\tanh\frac{\lambda}{2}] \tag{28}$$

By construction, the $u_0(p,q)$ in (26) satisfies the constraints

$$\int \frac{dp\, dq}{2\pi\hbar} u_0(p,q) = N-1 \tag{29}$$

$$\cos\frac{\hbar}{2}\left(\partial_q\partial_{p'} - \partial_{q'}\partial_p\right) u_0(p,q)u_0(p',q')\Big|_{\substack{p'=p\\ q'=q}} = u_0(p,q) \tag{30}$$

The equation of motion is also satisfied since $u_0(p,q)$ is time independent and can be shown to depend on p and q only through the classical hamiltonian $h(p,q) = \frac{1}{2}(p^2 - q^2 + \frac{g_3}{\sqrt{N}}q^3 + \cdots)$.

There are two ways one can approach the problem of constructing the full $u(p,q,t)$ that combines the phase space densities $u_0(p,q)$ and $u_1(p,q,t)$. The first is to try to see if

$$u(p,q,t) = u_0(p,q) + u_1(p,q,t) \tag{31}$$

is a solution of the equations of motion and the constraints. It is easy to see that in the large N limit the equation of motion and the total fermion number constraint is satisfied since the corresponding equations are linear. The quadratic constraint, however, is not satisfied because of the cross term

$$C_{01} \equiv \cos\frac{\hbar}{2}\left(\partial_q\partial_{p'} - \partial_{q'}\partial_p\right) u_0(p,q)u_1(p',q',t)\Big|_{\substack{p'=p\\ q'=q}} \tag{32}$$

Note, however, that in the classical limit $\hbar \to 0$,

$$u_0(p,q) \to \theta(\mu - \frac{p^2-q^2}{2}), \quad \frac{1}{2\pi\hbar}u_1(p,q,t) \to \delta(p-\bar{p}(t))\delta(q-\bar{q}(t)) \tag{33}$$

where $\bar{q}(t)$ and $\bar{p}(t)$ are given by (15). Clearly if in this limit we choose (p_0, q_0), the initial $(t=0)$ position of the peak, to be outside the support of u_0, or alternatively the energy $E_0 \equiv (p_0^2 - q_0^2)/2$ to be greater than the fermi energy μ, then the cross term (32) vanishes. In other words,

$$u(p,q,t) = \theta(\mu - \frac{p^2-q^2}{2}) + 2\pi\hbar\delta(p-\bar{p}(t))\delta(q-\bar{q}(t) \tag{34}$$

is a solution of the equation of motion and the constraints in the limit $\hbar \to 0$.

How about $\hbar \neq 0$? After all, if we put $\hbar = 0$ the physical effect that we are after, the "trickle", vanishes. Let us assume that $\hbar$ is non-zero but small. It is easy to see, by using explicit expressions for $u_0(p,q)$ and $u_1(p,q,t)$, that they develop exponential tails away from the support of the θ-function and δ-function respectively. Thus, by choosing the energy E_0 of the wave-packet to be sufficiently far away from the fermi energy μ, we can make the cross term C_{01} in (32) exponentially small. The region where the cross term is the strongest is given by $(p, q) \approx (p_0, q_0)$ where u_1 is of order 1 and u_0 is of order $\exp[-(a|\mu| - b\sqrt{|\mu E_0|})/\hbar]$ (a, b positive numbers of order 1). If we choose $|\mu| >> |E_0| >> 0$ then $C_{01} \sim \exp[-a|\mu|/\hbar]$. This implies that the solution $u(p, q, t) = u_0 + u_1$ is off from the exact solution by terms of the order $\exp[-(a|\mu|)/\hbar]^\star$. Note that we cannot outright ignore such terms because the "trickle" that we are looking for is also exponentially small as $\hbar \to 0$. The implication of this is the following. As in the previous section, let us define the quantities $N_+(t)$ and $\mathcal{T}$ as

$$N_+(t) = \int_0^\infty dq \int_{-\infty}^\infty dp \, u(p, q, t) \tag{35}$$

$$\mathcal{T} = N_+(\infty) - N_+(-\infty) \tag{36}$$

If $u = u_0 + u_1$ were the exact solution, then $\mathcal{T}$ would again be given by (23)-(24), since u_0 is time-independent and does not contribute to the trickle. The question is, if u has additional terms of order $\exp[-(a|\mu|)/\hbar]$ (which are also clearly time-dependent) then how does the estimate for the "trickle" modify? For instance, can the new contribution cancel the trickle by contributing an equal amount with an opposite sign? Fortunately such bizarre things do not happen. The basic reason is that expressions like (23) or (24) do not depend on μ, and in the domain of parameters $|\mu| >> |E_0| >> 0$ we can claim that, to leading order, the trickle is again given by

$$\mathcal{T} = 2\theta_0 \sqrt{\frac{|E_0|}{\pi\hbar}} \exp[-\frac{|E_0|}{\hbar}] + o(\theta_0^2) \tag{37}$$

There is a more precise way of seeing the above result by going back to fermions and constructing an exact classical solution $u(p, q, t)$ that satisfies all the con-

$\star$ We shall soon verify this statement explicitly by presenting the exact solution.

straints and the equation of motion exactly. It is given by

$$u(p,q) = \langle \Phi | \int dx \; e^{-ipx} \psi^\dagger(q - \hbar x/2, t)\psi(q + \hbar x/2, t) | \Phi \rangle \tag{38}$$

Here $|\Phi\rangle$ is the N-fermion state in which $N-1$ fermions occupy the $N-1$ lowest energy eigenstates $\phi_\nu(x)$, $\nu = -M, \cdots, \mu$ ($-M$ is a large negative number denoting the ground state energy of the single-particle hamiltonian; in the limit of $N \to \infty$, $-M \to -\infty$ and the energy levels become continuous) and one fermion belongs to a wave-packet state, which is given, in the limit $N \to \infty$ in which the single-particle hamiltonian is $\hat{h} = \frac{1}{2}(\hat{p}^2 - \hat{x}^2)$, by

$$\begin{aligned} \psi_1(x,t) &= \exp(\frac{it}{2}[\hbar \partial_x^2 + x^2/\hbar])\{(\pi\hbar)^{-1/4} \exp(-\frac{1}{2\hbar}[(x-q_0)^2 - 2ip_0 x])\} \\ &= \frac{(\pi\hbar)^{-1/4}\exp(ip_0 q_0/2\hbar)}{\sqrt{f(t)}} \exp[-\frac{1}{2\hbar \cosh 2t}\{(xf^*(t) - z_0)^2 + \frac{1}{2}(|z_0|^2 \cosh 2t - z_0^2 f(t))\}] \end{aligned} \tag{39}$$

where $z_0 \equiv q_0 + ip_0$ and $f(t) = \cosh t + i \sinh t$. $\psi_1(x,t)$ is the wave-function whose phase space density is $u_1(p,q,t)$. The state $|\Phi\rangle$ is explicitly given by the Slater determinant of the single-particle states $\{\phi_\nu(x),\, \nu = -M, \cdots, \mu;\, \psi_1(x,t)\}$. To see that (38) satisfies the quadratic constraint (6) it is convenient to define the first-order density matrix [12] $\gamma_\Phi(x,y,t) = \langle \Phi | \psi^\dagger(x,t)\psi(y,t) | \Phi \rangle$ in terms of which the constraint (6) reads as $\int dy \, \gamma_\Phi(x,y,t)\gamma_\Phi(y,z,t) = \gamma_\Phi(x,z,t)$. The last statement is true for *any* state Φ which can be written as a single Slater determinant of any arbitrary one-particle wavefunctions. Indeed, this is the easiest way to check the validity of the quadratic constraint for u_0 and u_1 also.

Let us expand $\psi_1(x,t)$ in terms of the energy eigenfunctions $\phi_\nu(x,t) \equiv \phi_\nu(x)\exp(-i\nu t)$:

$$\psi_1(x,t) = \sum_\nu A(\nu)\phi_\nu(x)\exp(-i\nu t) \tag{40}$$

Using these one can evaluate (38):

$$u(p,q,t) = \sum_{\nu(\text{sea})} u_\nu(p,q) + \frac{1}{C^2} B \tag{41}$$

where

$$C^2 = \sum_{\nu(\text{above})} |A(\nu)|^2 = 1 - \sum_{\nu(\text{sea})} |A(\nu)|^2 \tag{42}$$

and

$$B = \sum_{\nu(\text{above})} |A(\nu)|^2 u_\nu(p,q) + \sum_{\nu \neq \nu'(\text{above})} A^*(\nu) A(\nu') u_{\nu\nu'}(p,q,t) \tag{43}$$

In the above, sum over energy levels that belong to the fermi *sea* or *above* have been denoted appropriately. An unspecified sum means sum over all states. The notation $u_{\nu\nu'}(p,q,t)$ stands for $\int dx \, \exp(-ipx)\phi^*_\nu(q-\hbar x/2,t)\phi_{\nu'}(q+\hbar x/2,t)$.

It is clear that the first term in (41) is simply $u_0(p,q)$ of Eqn. (26). Let us compare the second term B/C^2 with $u_1(p,q,t)$; the latter (Eqn. (10)) looks in the present notation as

$$u_1(p,q,t) = \sum_\nu |A(\nu)|^2 u_\nu(p,q) + \sum_{\nu \neq \nu'} A^*(\nu) A(\nu') u_{\nu\nu'}(p,q,t) \tag{44}$$

We see that, in the limit $N \to \infty$, B/C^2 differs from $u_1(p,q,t)$ by terms of the order of $|A(\mu)|^2$. Now, $A(\mu)$ is simply the scalar product between the Gaussian wavefunction ψ_1 and the fermi level wave-function ϕ_μ and can be shown to be $\sim \exp[-a|\mu|/\hbar], a > 0$ if we have $|\mu| >> |E_0| >> 0$. This verifies our earlier conclusion that $u = u_0 + u_1 + o(\exp[-a|\mu|/\hbar]),\ a > 0$.

<u>Stringy Non-perturbative effect:</u>

Since we are working in the weak coupling limit ($\hbar = g_{\text{str}} \to 0$), the expression for the "trickle" that we have calculated using a classical solution can be regarded as the leading result for the following field theory amplitude

$$\mathcal{A} \equiv \frac{\int \mathcal{D}u \exp(iS[u]) \mathcal{T}[u]}{\int \mathcal{D}u \exp(iS[u])} \tag{45}$$

where

$$\mathcal{T}[u] = \int_0^\infty dq \int_{-\infty}^\infty dp [u(p,q,t=+\infty) - u(p,q,t=-\infty)] \tag{46}$$

In the above, $S[u]$ is the classical action described in Sec. 2 and the measure $\mathcal{D}u$ incorporates the constraints on the $u(p,q,t)$ field. To pick out the classical solution described above, we of course need to specify boundary conditions in the functional integral appropriately so that they match the behaviour of the desired classical solution at large initial and final times. By the results described above, we find that

$$\mathcal{A} \sim \exp[-|E_0|/g_{\rm str}] \tag{47}$$

where $E_0 = (p_0^2 - q_0^2)/2$ is a parameter of the classical solution specifying the mean energy of the wave packet. As already mentioned, the physics of this amplitude is the tunnelling of a single fermion.

For treatment of non-perturbative effects within the framework of collective field theory, see [17-19]. Stringy non-perturbative effects arising from the motion of a single eigenvalue in an effective potential have been discussed previously in $c < 1$ models in [20].

5. Marinari-Parisi Model:

In this section we briefly outline how the Marinari-Parisi model [3] can be treated in our formalism of $u(p,q,t)$-theory so that non-perturbative effects may be calculated in a field theory framework. The essential point is that the bosonic sector of the model corresponds to a non-relativistic fermi gas in one space dimension. The basic difference with the $c = 1$ model is that the classical single-particle hamiltonian is given by

$$h(p,q) = \frac{p^2}{2} + V(q), \quad V(q) = q^3 - \alpha q \tag{48}$$

Thus, except for the equation of motion for the u-field, which becomes

$$\partial_t u = \{h, u\}_{MB} = \{h, u\}_{PB} - \frac{\hbar^2}{4}\partial_p^3 u, \tag{49}$$

everything else (like the constraints, the classical action etc.) remains unchanged.

The interesting physical effect in this model is assoicated with the tunnelling of a single fermion. For $\alpha > 0$, the potential $V(q)$ has two minima at $q = q_m \equiv +\sqrt{\alpha/3}$ and $q = -\infty$, separated by a maximum at $q = -q_m$. It has been shown in [3] that at the critical point $\alpha = 0$, where the secondary well disappears, the ground state of the fermi system is given by a fermi sea which reaches upto the position of the point of inflexion. This means that as one decreases α from the positive side towards zero, more and more fermions escape out of the secondary well. In the limiting situation $\alpha \to 0^+$ only one fermion remains and criticality is characterized by the tunnelling of this fermion. This causes non-perturbative supersymmetry breaking, leading to amplitudes that go as $\exp(-C/g_{\rm str})$. Since the tunnelling involves a single fermion, the interpolating configuration $u(p, q, t)$ again consists of a "large" piece $u_0(p, q)$ describing $N - 1$ stationary fermions and a "small" piece $u_1(p, q, t)$. We can explicitly construct $u_1(p, q, t)$ as follows. Let u_1 at time $t = 0$ be given by

$$u_1(p, q, 0) = 2 \exp[-\frac{1}{\hbar}\{(p - p_0)^2 + (q - q_0)^2\}]. \tag{50}$$

It is easy to verify that (50) satisfies the constraints (12) and (13) at $t = 0$. Now a useful fact about the time-evolution $\partial_t u = \{h, u\}_{MB}$ is that if we ensure that $u(p, q, 0)$ satisfies the two constraints (12) and (13), then the time-evolved $u(p, q, t)$ automatically satisfies them for any *arbitrary* hamiltonian $h(p, q)$ (this is of course required for consistency between equations of motion and constraints). It is not difficult to show that the solution to (49), with the initial condition (50), is

$$u_1(p, q, t) = 2 \exp[-\frac{1}{\hbar}\{(P(p, q, t) - p_0)^2 + (Q(p, q, t) - q_0)^2\}] + o(\hbar^2). \tag{51}$$

where $P(p, q, t), Q(p, q, t)$ describe a classical trajectory for the hamilton (48), with the initial condition $P(p, q, 0) = p, Q(p, q, 0) = q$.

In order that (51) describes the appropriate tunnelling configuration, we should take q_0 to be inside the secondary well ($q_0 \approx q_m \equiv \sqrt{\alpha/3}$) and p_0 to be negative such that the wave-packet is directed towards the other well (the other well is strictly speaking bottomless in the double scaled limit, but for any finite N one has a regulated potential with a finite depth just as in the standard $c = 1$ model). The calculation of the "trickle" can again be performed in a similar fashion to the earlier sections. We shall present the details elsewhere.

6. **Comparison with Collective Field Theory:**

In this section we ask whether our classical solution $u(p, q, t)$ could be derived from the equations of motion of the standard collective field theory [16]. The answer will turn out to be negative. But before going to that, let us first see how one might make a comparison between the $u(p, q, t)$-theory and the standard collective field theory.

It is convenient to define the following moments of the phase space density $u(p, q, t)$:

$$\begin{aligned} \rho(q,t) \equiv \frac{\tilde{\rho}(q,t)}{2\pi\hbar} &= \int \frac{dp}{2\pi\hbar} u(p,q,t) \\ \Pi(q,t)\rho(q,t) &= \int \frac{dp}{2\pi\hbar} p u(p,q,t) \\ \Pi_2(q,t)\rho(q,t) &= \int \frac{dp}{2\pi\hbar} p^2 u(p,q,t) \\ \cdots &= \cdots \end{aligned} \tag{52}$$

In the following we shall assume that the $N \to \infty$ limit has been taken. In this limit the classical hamiltonian is $h(p, q) = \frac{1}{2}(p^2 - q^2)$ and therefore the equation of motion is

$$(\partial_t + p\partial_q + q\partial_p)u(p, q, t) = 0 \tag{53}$$

This equation implies equations of motion for the moments. One can obtain them by taking moments of (53). Let us write down the first two equations obtained this way:

$$\partial_t \tilde{\rho}(q,t) = -\partial_q(\Pi\tilde{\rho}) \tag{54}$$

$$\partial_t \Pi(q,t) = q + \partial_q(\frac{\Pi^2}{2} - \Pi_2) + \frac{\partial_q \rho}{\rho}(\Pi^2 - \Pi_2) \tag{55}$$

It is clear that one does not obtain a closed set of equations for ρ, Π— their equations of motion involve the next higher moment. In fact this pattern continues *ad infinitum*.

Let us compare (54)-(55) with collective field theory equations

$$\begin{aligned} \partial_t \tilde{\rho}(q,t) &= -\,\partial_q(\Pi\tilde{\rho}) \\ \partial_t \Pi(q,t) &= q - \partial_q(\frac{\Pi^2}{2} + \frac{\tilde{\rho}^2}{8}) \end{aligned} \tag{56}$$

How does one understand getting a closed set of equations for ρ, Π from our viewpoint? To see this, we turn again to quadratic profiles (for details see [4, 12]) for which the classical solution $u(p,q,t)$, in the limit $\hbar \to 0$, looks like

$$u(p,q,t) = \theta[(p_+(q,t) - p)(p - p_-(q,t))] + \hbar \,\text{corrections} \tag{57}$$

Remarkably, for these kinds of solutions we can show that the moment Π_2 can be determined in terms of Π, ρ as

$$\Pi_2 = \Pi^2 + \frac{1}{12}\tilde{\rho}^2 + \hbar \,\text{corrections} \tag{58}$$

If one puts this in (55), one recovers the second equation of (56) upto $\hbar$-corrections (the first equation already agreed with (54)).

There are two lessons to be learnt from the above exercise: (a) collective field theory equations can be recovered from the equation of motion of $u(p,q,t)$-theory under the assumption (57), in the limit $\hbar \to 0$; (b) even under the "quadratic profile" assumption, **classical** equations of the collective field theory are violated by **classical** solutions of u-theory (like (57)) by $\hbar$-corrections. The last observation reflects the fact that the classical solutions of the u-theory incorporate the single-particle quantum mechanics exactly, as was remarked in Sec. 3. To give a more explicit example of this point, consider $u = u_0(p, q)$ of the Sec. 4, and regard it for the moment as the fermi sea of the N-body problem (rather than $N-1$). This classical solution satisfies the ansatz (57) with non-trivial non-perturbative corrections in $\hbar$. It is easy to see that the second equation of (56) is violated by non-perturbative $\hbar$-corrections.

Let us now see if the full time-dependent $u(p, q, t)$ including u_0 and u_1 of the previous section satisfies equations (56). To a first approximation, let us ignore the overlap regions between (ρ_0, Π_0) and (ρ_1, Π_1) and try to see if each of these pairs satisfies the collective field equations independently. This amounts to ignoring non-perturbative terms in $\hbar$ (recall that both ρ_0 and ρ_1 have exponential tails in the intermediate region between them). This attitude is similar to the one that we had adopted while calculating the "trickle". In addition, since we have established non-perturbative violations of the collective field equations already in the last paragraph, one may be interested in looking for new violations this time which persist even when one ignores terms of order $\exp[-1/\hbar]$. It turns out that ρ_1, Π_1, taken by themselves, indeed violate the second equation of (56) by the amount

$$\begin{aligned}\partial_t \Pi_1(q,t) - &\{q - \partial_q(\frac{\Pi_1^2}{2} + \frac{\tilde{\rho}_1^2}{8})\} \\ &= \frac{q - \bar{q}(t)}{\cosh^2 2t} - \frac{\pi}{2} \frac{q - \bar{q}(t)}{\cosh^2 2t} \exp[-\frac{2(q - \bar{q}(t))^2}{\hbar \cosh 2t}]\end{aligned} \tag{59}$$

The non-perturbative term is already expected from earlier considerations; its magnitude may change when one takes into account the overlap terms between ρ_0 and ρ_1, though following the logic of previous sections, the modifications are smaller than the original term. The first term is more of a surprise because it is non-zero even in the $\hbar \to 0$ limit. The way to understand this is to note that the wave-packet solution $u_1(p, q, t)$ does not satisfy the criterion (57) appropriate for quadratic profiles. As a result it does not lead to (58)(which one may also verify directly). This is the reason why there is a **classical violation** of the collective field equations by the wave packet solution. Indeed, since the relation (58) is rather crucial in deriving the classical equations of collective field theory, and this in turn crucially depends on the assumption (57) of quadratic profiles, it is trivial to generate other examples of $u(p,q,t)$ which in the limit $\hbar \to 0$ go over to something other than quadratic profiles and thus end up satisfying different collective field equations!

7. Interpretation of the Time-dependent solution in the Black Hole Context:

In [6] we found a correspondence between the weak coupling regime of $c = 1$ string field theory and the black hole of two-dimensional string theory. One feature of the correspondence is that if one considers $\langle R|u(p,q,t)|R\rangle \equiv u_R(p,q,t)$ in states $|R\rangle$ which are 'small fluctuations' on the ground state $|R_0\rangle$ then the "hyperbolic transform"(HT) of the fluctuation $u_R - u_{R_0} \equiv \eta$ satisfies, in the classical limit, the differential equation of a massless scalar field in black hole background*. The 'small fluctuation' condition above means that the support of η must be in a small neighbourhood of the fermi surface, satisfying $|(h(p,q)-\mu)/\mu| << 1$ whenever $\eta(p,q) \neq 0$. Let us now see how we can find such solutions from the "tunnelling" state that we have constructed and used in the preceding sections to see somewhat different physical effects.

Consistent with the approximations that have been made in the earlier sections, we shall regard the full solution for u as

$$u(p,q,t) = u_{N-1}(p,q) + u_1(p,q,t) \tag{60}$$

where we have used the notation u_{N-1} in place of u_0 to emphasize that it corresponds to the fermi sea of an $(N-1)$-fermion system. The fluctuation η is the

$\star$ In equation (19) of [6] we made this claim for $\partial_\mu \eta$. By going through the steps that led to (19), we can show that the equation is equally valid for η itself.

difference between (60) and the expectation value of the $u(p,q,t)$-operator in the state $|R_0\rangle$ which describes the N-particle fermi sea. We have

$$\begin{aligned} \eta(p,q,t) \equiv u(p,q,t) - u_N(p,q) &= -\delta_0 u(p,q) + u_1(p,q,t) \\ \delta_0 u(p,q) &= u_N(p,q) - u_{N-1}(p,q) \end{aligned} \tag{61}$$

$\delta_0 u(p,q)$ is simply the phase space density corresponding to the fermion at the top of the N-particle fermi sea and $u_1(p,q,t)$ is the phase space density of the wave-packet. Now, by the arguments given in [6], the HT ('hyperbolic transform') of $\delta_0 u(p,q)$, denoted by $\delta_0 T(u,v)$, satisfies the differential equation

$$[4(uv - \frac{\mu}{2})\partial_u\partial_v + 2(u\partial_u + v\partial_v) + 1]\delta_0 T(u,v) = 0 + \hbar\,\text{corrections} \tag{62}$$

where u,v are defined by $u = \frac{1}{2}(p+q)e^{-t}$, $v = \frac{1}{2}(p-q)e^{t}$. This is because in the $\hbar \to 0$ limit the support of $\delta_0 u$ in the phase space is confined to a small strip near the fermi surface. Now if we can ensure that in the $\hbar \to 0$ limit u_1 satisfies the 'small flucutation' condition mentioned in the last paragraph, then η will also satisfy this condition and as a result the HT of η will satisfy Eqn. (62). This would imply that the HT of $u_1(p,q,t)$, would also satisfy Eqn. (62). The 'small fluctuation' condition on $u_1(p,q,t)$ implies that we must have $|(E_0-\mu)/\mu| << 1$. If we go back to the arguments in Sec. 4, we can see that in such a region of parameters p_0, q_0, (60) is a solution only in the classical limit $\hbar \to 0$, since otherwise the cross terms discussed in (32) are important. In the limit $\hbar \to 0$,

$$u_1(p,q,t) \to 2\pi\hbar\delta(q-\bar{q}(t))\delta(p-\bar{p}(t))$$

The HT of this limiting δ-function is easy to calculate and turns out to be proportional to

$$T_1(u,v) = |(u-u_0)(v-v_0)|^{-1/2} \tag{63}$$

where $u_0 = \frac{1}{2}(p_0+q_0)$ and $v_0 = \frac{1}{2}(p_0-q_0)$. It can be directly verfied that $T_1(u,v)$ satisfies the differential equation (62) for $|(u_0v_0 - \mu/2)/\mu| << 1$.

Note that in the limit that we are working with in this section, our solution $u_1(p,q,t)$ does not exhibit any "trickling" and is not linked with any non-perturbative effect. However, the solution $T_1(u,v)$ that it gives rise to is rather interesting from the black hole point of view. Let us emphasize that if one chooses parameters u_0, v_0 such that $u_0 v_0 = \mu/2$ then (63) provides an **exact solution** of the **differential equation for propagation of massless scalar fields in the black hole geometry**:

$$[4(uv - \frac{\mu}{2})\partial_u \partial_v + 2(u\partial_u + v\partial_v) + 1]T_1(u,v) = 0 \tag{64}$$

This solution has the intriguing feature that it has singularities along the two lines $u = u_0$ and $v = v_0$. Since μ in our convention is negative, $u_0 v_0 = \mu/2$ is satisfied by a family of values

$$u_0 = \alpha\sqrt{-\mu/2},\ v_0 = -\alpha^{-1}\sqrt{-\mu/2} \tag{65}$$

where α is a non-zero real number.

Let us try to understand this solution as a tachyon wave in the black hole geometry. In the following we shall concentrate on the $v > 0$ half of the Kruskal diagram and use "space" and "time" coordinates ξ and T, related to u, v by

$$u = \epsilon \exp(\xi + T), \quad v = \exp(\xi - T) \tag{66}$$

where $\epsilon = \pm 1$ depending on whether we are in the region $uv > 0$ or $uv < 0$. Since $uv = \mu/2 = -|\mu|/2$ denotes the position of the black hole singularity in our convention, $uv > 0$ or $\epsilon = +1$ denotes spacetime regions outside the event horizon. The ξ and T coordinates introduced above are simple functions of the Schwarzschild space and time coordinates, respectively. The solution (63) then looks like

$$T_1(\xi,T) = |(\exp(\xi+T) - \epsilon\alpha\sqrt{|\mu|/2})(\exp(\xi-T) + \alpha^{-1}\sqrt{|\mu|/2})|^{-1/2} \qquad (67)$$

Let us consider first the case $\alpha > 0$. In this case, (67) has singularities only outside the horizon $\epsilon > 0$. At a fixed ξ this singularity occurs at

$$T = -\xi + \frac{1}{2}\log(|\mu|/2) \qquad (68)$$

This singularity can be interpreted in two ways. If we think of (67) as a propagating tachyon wave, then it implies that the initial data has a singularity irrespective of the choice of the initial spacelike surface. A more interesting interpretation comes about if we think of additional observers who can couple to tachyon backgrounds. For such an observer at fixed ξ, the tachyon solution (67) will appear as a singularity at the instant of time (68) and will be well-behaved before and after. Since the lines of singularity of our solution $u = u_0$, $v = v_0$ are light-like, these can perhaps be interpreted as light-like "thunderbolts" [21]. The case $\alpha < 0$ has the property that here one encounters singularities only inside the event horizon. Note that the symmetry between positive and negative values of α can be restored if one includes in the discussion the other half of the Kruskal diagram $v < 0$. For further discussion of this and other solutions to (64), see [22].

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