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On the equisummability of Hermite and Fourier expansions

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Abstract. We prove an equisummability result for the Fourier expansions and Hermite expansions as well as special Hermite expansions. We also prove the uniform boundedness of the Bochner-Riesz means associated to the Hermite expansions for polyradial functions.

Keywords. Hermite functions; special Hermite expansions; equisummability.

1. Introduction

This paper is concerned with a comparative study of the Bochner-Riesz means associated to the Hermite and Fourier expansions. Recall that the Bochner-Riesz means associated to the Fourier transform on $\mathbb{R}^n$ are defined by

$$S_t^\delta f(x) = (2\pi)^{-n/2} \int_{|y|\le t} \mathrm{e}^{ix\cdot y} \left(1 - \frac{|y|^2}{t^2}\right)^\delta \hat{f}(y)\mathrm{d}y,$$

where

$$\hat{f}(y) = (2\pi)^{-n/2} \int \mathrm{e}^{-ix.y} f(x)\mathrm{d}x$$

is the Fourier transform on $\mathbb{R}^n$. Let $\Phi_\alpha, \alpha \in N^n$ be the n-dimensional Hermite functions which are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$ with the eigenvalue $(2|\alpha| + n)$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let P_k be the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the kth eigenspace spanned by $\Phi_\alpha, |\alpha| = k$. More precisely,

$$P_k f(x) = \sum_{|\alpha|=k} \left(\int f(y)\Phi_\alpha(y)\mathrm{d}y\right) \Phi_\alpha(x).$$

Then the Bochner-Riesz means associated to the Hermite expansions are defined by

$$S_R^\delta f(x) = \sum \left(1 - \frac{2k+n}{R}\right)_+^\delta P_k f(x).$$

For the properties of Hermite functions and related results, see [6].

In our study of the Bochner-Riesz means associated to Hermite and special Hermite expansions we make use of a transplantation theorem of Kenig-Stanton-Tomas [2]. Let us briefly recall their result. Let P be a differential operator acting on $C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ which is self adjoint. Let

$$Pf = \int \lambda \mathrm{d}E_\lambda$$

be the spectral resolution of P. Let m be a bounded function on $\mathbb{R}$ and define

$$m_R(P) = \int m\left(\frac{\lambda}{R}\right)\mathrm{d}E_\lambda.$$

Let K be a subset of $\mathbb{R}^n$ with positive measure and define the projection operator Q_k on $L^2(\mathbb{R}^n)$ by

$$Q_k f(x) = \chi_K(x) f(x),$$

where $\chi_K(x)$ is the characteristic function of K. Let $p(x,\xi)$ be the principal symbol of P. Since P is symmetric p is real valued. Then we have the following theorem.

Theorem 1.1. *Assume $1 \leq p \leq \infty$ and that there is a set of positive measure K_0 for which the operators $Q_{K_0} m_R(P) Q_{K_0}$ are uniformly bounded on $L^p(\mathbb{R}^n)$. If x_0 in K_0 is any point of density, then $m(p(x_0,\xi))$ is a Fourier multiplier of $L^p(\mathbb{R}^n)$.*

Let B be any compact set in $\mathbb{R}^n$ containing origin as a point of density and let χ_B be the operator

$$\chi_B f(x) = \chi_B(x) f(x).$$

Then from Theorem 1.1 it follows that the uniform boundedness of $\chi_B S_R^\delta \chi_B$ on $L^p(\mathbb{R}^n)$ implies the uniform boundedness of S_t^δ on $L^p(\mathbb{R}^n)$. Thus once we have the local summability theorem for Hermite expansions then a global result is true for the Fourier transform. At this point a natural question arises, to what extend the converse is true? In this paper we answer this question in the affirmative in dimensions one and two and partially in higher dimensions. We also study the equisummability of the special Hermite expansions, namely the eigenfunction expansion associated to the operator

$$L = -\Delta + \frac{1}{4}|z|^2 - i\sum\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

on $\mathbb{C}^n$. In this case we show that the local uniform boundedness of the Bochner-Riesz means for the special Hermite operator is equivalent to the uniform boundedness of S_t^δ on $\mathbb{R}^{2n}$. Using a recent result of Stempak and Zienkiewicz [4], on the restriction theorem we study the Bochner-Riesz means associated to the Hermite expansions on $\mathbb{R}^{2n}$ for functions having some homogeneity. We also prove a weighted version for the Hermite expansions which slightly improves the local estimates proved in [5]. Eigenfunction expansions associated to special Hermite operator L has been studied by Thangavelu [6].

2. Hermite expansions on $\mathbb{R}^n$

The Hermite functions h_k on $\mathbb{R}$ are defined by

$$h_k(x) = (2^k k!\sqrt{\pi})^{-\frac{1}{2}}(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}(\mathrm{e}^{-x^2})\mathrm{e}^{\frac{1}{2}x^2}.$$

In the higher dimensions the Hermite functions are defined by taking tensor products:

$$\Phi_\alpha(x) = h_{\alpha_1}(x_1)\ldots h_{\alpha_n}(x_n).$$

Given $f \in L^p(\mathbb{R})$ consider the Hermite expansion

$$f(x) = \sum_{k=0}^{\infty}(f,h_k)h_k(x),$$

where $(f,h_k) = \int f(x)h_k(x)\mathrm{d}x$.

Let $S_N f(x) = \sum_{k=0}^{N}(f,h_k)h_k(x)$ be the partial sums associated to the above series. In 1965, Askey-Wainger [1] proved the following celebrated theorem.

Theorem 2.1. *$S_N f \to f$ in the L^p norm iff $\frac{4}{3} < p < 4$.*

Let S_t be the partial sum operator associated to the Fourier transform on $\mathbb{R}$. Then it is well known that $S_t f \to f$ in L^p norm for all $1 < p < \infty$. In this section we show that on a subclass of $L^p(\mathbb{R})$ the same is true for the Hermite expansions.

In the higher dimensions it is convenient to work with Cesaro means rather than Riesz means. These are defined by

$$\sigma_N^\delta f(x) = \frac{1}{A_N^\delta}\sum_{k=0}^{N} A_{N-k}^\delta P_k f(x),$$

where A_k^δ are the binomial coefficients defined by $A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)}$. It is well known that σ_N^δ are uniformly bounded on $L^p(\mathbb{R}^n)$ iff S_R^δ are uniformly bounded. We have the following equisummability result. Let E stand for the operator $Ef(x) = \mathrm{e}^{-\frac{1}{2}|x|^2}f(x)$.

Theorem 2.2. *$E\sigma_N^\delta E$ are uniformly bounded on $L^p(R^n)$ iff S_t^δ are uniformly bounded, provided $\delta \geq \max\{0, \frac{n}{2}-1\}$.*

As a corollary we have the following.

COROLLARY 2.3

Let $1 < p < \infty$. Then for the partial sum operators associated to the one dimensional Hermite expansion we have the uniform estimate

$$\int |S_N f(x)|^p \mathrm{e}^{-\frac{p}{2}x^2}\mathrm{d}x \leq C\int |f(y)|^p \mathrm{e}^{\frac{p}{2}y^2}\mathrm{d}y.$$

Thus for $f \in L^p(\mathrm{e}^{\frac{p}{2}y^2}\mathrm{d}y), 1 < p < \infty$ the partial sums converge to f in $L^p(\mathrm{e}^{-\frac{p}{2}x^2}\mathrm{d}x)$.

For a general weighted norm inequality for Hermite expansions, see Muckenhoupt's paper [3].

The celebrated theorem of Carleson-Sjolin for the Fourier expansion on $\mathbb{R}^2$ says that if $\delta > 2(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$, $1 \leq p < \frac{4}{3}$ then S_t^δ are uniformly bounded on $L^p(\mathbb{R}^2)$. As a corollary to this we obtain the following result for the Cesaro means σ_N^δ on $\mathbb{R}^2$.

COROLLARY 2.4

Let $n = 2$, $1 \leq p < \frac{4}{3}$ and $\delta > 2(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$. Then for $f \in L^p(\mathbb{R}^2)$

$$\int |\sigma_N^\delta f(x)|^p \mathrm{e}^{-\frac{p}{2}|x|^2}\mathrm{d}x \leq C\int |f(y)|^p \mathrm{e}^{\frac{p}{2}|y|^2}\mathrm{d}y.$$

It is an interesting and more difficult problem to establish the above without the exponential factors.

We now proceed to prove Theorem 2.2. It is a trivial matter to see that uniform boundedness of $E\sigma_N^\delta E$ implies the same for $\chi_B\sigma_N^\delta\chi_B$ for any compact subset B of $\mathbb{R}^n$. In fact, if $E\sigma_N^\delta E$ are uniformly bounded then

$$\begin{aligned}
&\int |\chi_B\sigma_N^\delta\chi_B f|^p \mathrm{d}x \\
&\quad = \int_B \mathrm{e}^{-\frac{p}{2}|x|^2}\mathrm{e}^{\frac{p}{2}|x|^2}|\sigma_N^\delta(\mathrm{e}^{-\frac{1}{2}|y|^2}(\chi_B f(y)\mathrm{e}^{\frac{1}{2}|y|^2}))|^p \mathrm{d}x \\
&\quad \leq C\int |E\sigma_N^\delta E(\chi_B f(y)\mathrm{e}^{\frac{1}{2}|y|^2})|^p \mathrm{d}x \\
&\quad \leq C\int |f(x)|^p \mathrm{d}x
\end{aligned}$$

which proves the one way implication, by the transplantation theorem [2]. To prove the converse we proceed as follows. Let

$$\Phi_k(x,y) = \sum_{|\alpha|=k}\Phi_\alpha(x)\Phi_\alpha(y)$$

be the kernel of the projection operator P_k. Then the kernel $\sigma_N^\delta(x,y)$ of the Cesaro means is given by

$$\sigma_N^\delta(x,y) = \frac{1}{A_N^\delta}\sum_{k=0}^N A_{N-k}^\delta\Phi_k(x,y).$$

We first obtain a usable expression for this kernel in terms of certain Laguerre functions. Let $L_k^\alpha(t)$ be the Laguerre polynomials of the type $\alpha > -1$ defined by

$$\mathrm{e}^{-t}t^\alpha L_k^\alpha(t) = (-1)^k\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\mathrm{e}^{-t}t^{k+\alpha}), \quad t > 0.$$

We have the following expression.

PROPOSITION 2.5

$$\sigma_N^\delta(x,y) = \frac{1}{A_N^\delta}\sum_{k=0}^N(-1)^k L_{N-k}^{\delta+\frac{n}{2}}\left(\frac{1}{2}|x-y|^2\right)\mathrm{e}^{-\frac{1}{4}|x-y|^2}L_k^{\frac{n}{2}-1}\left(\frac{1}{2}|x+y|^2\right)\mathrm{e}^{-\frac{1}{4}|x+y|^2}.$$

Proof. The generating function identity for the projection kernels $\Phi_k(x,y)$ reads

$$\sum_{k=0}^\infty r^k\Phi_k(x,y) = \pi^{-\frac{n}{2}}(1-r^2)^{-\frac{n}{2}}\,\mathrm{e}^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2rx.y}{1-r^2}}.$$

Since

$$(1-r)^{-\delta-1} = \sum_{k=0}^\infty A_k^\delta r^k$$

the generating function for $\sigma_k^\delta(x,y)$ is given by

$$\sum_{k=0}^{\infty} r^k A_k^\delta \sigma_k^\delta(x,y) = (1-r)^{-\delta-\frac{n}{2}-1}(1+r)^{-\frac{n}{2}}\, \mathrm{e}^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2rx.y}{1-r^2}}.$$

The right hand side of the above expression can be written as

$$(1-r)^{-\delta-\frac{n}{2}-1}\mathrm{e}^{-\frac{1}{4}\frac{1+r}{1-r}|x-y|^2}(1+r)^{-\frac{n}{2}}\, \mathrm{e}^{-\frac{1}{4}\frac{1-r}{1+r}|x+y|^2}.$$

Now the generating function for the Laguerre polynomials L_k^α is

$$\sum_{k=0}^{\infty} r^k L_k^\alpha\left(\frac{1}{2}t^2\right)\mathrm{e}^{-\frac{1}{4}t^2} = (1-r)^{-\alpha-1}\mathrm{e}^{-\frac{1}{4}\frac{1+r}{1-r}t^2}.$$

Therefore, we have

$$\sum_{k=0}^{\infty} r^k A_k^\delta \sigma_k^\delta(x,y) = \left(\sum_{j=0}^{\infty} r^j L_j^{\delta+\frac{n}{2}}\left(\frac{1}{2}|x-y|^2\right)\mathrm{e}^{-\frac{1}{4}|x-y|^2}\right)$$
$$\left(\sum_{i=0}^{\infty}(-r)^i L_i^{\frac{n}{2}-1}\left(\frac{1}{2}|x+y|^2\right)\mathrm{e}^{-\frac{1}{4}|x+y|^2}\right).$$

Equating the coefficients of r^k on both sides we obtain the proposition.

The Laguerre functions L_k^α are expressible in terms of Bessel functions J_α. More precisely, we have the formula

$$\mathrm{e}^{-x}x^{\frac{\alpha}{2}}L_k^\alpha(x) = \frac{1}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-t}t^{k+\frac{\alpha}{2}}J_\alpha(2\sqrt{tx})\mathrm{d}t.$$

Using this, the kernel $\mathrm{e}^{-\frac{1}{2}|x|^2}\sigma_N^\delta(x,y)\,\mathrm{e}^{-\frac{1}{2}|y|^2}$ of the operator $E\sigma_N^\delta E$ is given by.

$$\mathrm{e}^{-\frac{1}{2}|x|^2}\sigma_N^\delta(x,y)\mathrm{e}^{-\frac{1}{2}|y|^2} =$$
$$\frac{C}{A_N^\delta}\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}\frac{(t-s)^N}{N!}t^\delta s^{\frac{n}{2}-1}t^{\frac{n}{2}}\frac{J_{\delta+\frac{n}{2}}(\sqrt{2t}|x-y|)}{(\sqrt{2t}|x-y|)^{\delta+\frac{n}{2}}}\frac{J_{\frac{n}{2}-1}(\sqrt{2s}|x+y|)}{(\sqrt{2s}|x+y|)^{\frac{n}{2}-1}}\mathrm{d}t\mathrm{d}s,$$

where C depends only on δ. Now the kernel of the Bochner-Riesz means S_t^δ on $\mathbb{R}^n$ is given by

$$S_t^\delta(x,y) = t^n\frac{J_{\delta+\frac{n}{2}}(t|x-y|)}{(t|x-y|)^{\delta+\frac{n}{2}}}.$$

When $n=1$,

$$\frac{J_{-\frac{1}{2}}(t)}{t^{-\frac{1}{2}}} = \left(\frac{2}{\pi}\right)^{1/2}\cos t$$

and hence

$$E\sigma_N^\delta Ef(x) = C\frac{1}{A_N^\delta}\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}\frac{(t-s)^N}{N!}t^\delta s^{-\frac{1}{2}}T_t^\delta f(x)\mathrm{d}t\mathrm{d}s,$$

where

$$T_t^\delta f(x) = \int_{\mathbb{R}} S^\delta_{\sqrt{2t}}(x,y)\cos(\sqrt{2s}|x+y|)f(y)\mathrm{d}y$$

and C an absolute constant.

By Minkowski's integral inequality we get

$$\begin{aligned}||E\sigma_N^\delta Ef||_p &\le C\frac{1}{N^\delta}\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}\frac{|t-s|^N}{N!}t^\delta s^{-\frac{1}{2}}||T_t^\delta f||_p\mathrm{d}t\mathrm{d}s\\ &\le C||f||_p\end{aligned}$$

since

$$\begin{aligned}&\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}|t-s|^N t^\delta s^{-\frac{1}{2}}\mathrm{d}t\,\mathrm{d}s\\ &\quad\le \int_0^\infty \mathrm{e}^{-t}t^\delta\left(\int_0^t \mathrm{e}^{-s}t^N s^{-\frac{1}{2}}\mathrm{d}s + \int_t^\infty \mathrm{e}^{-s}s^N s^{-\frac{1}{2}}\,\mathrm{d}s\right)\mathrm{d}t\\ &\quad\le C\int_0^\infty \mathrm{e}^{-t}t^\delta t^N\mathrm{d}t + \Gamma\left(N+\frac{1}{2}\right)\int_0^\infty \mathrm{e}^{-t}t^\delta\,\mathrm{d}t\\ &\quad\le CN!N^\delta\end{aligned}$$

which proves the theorem in one dimension.

When $n \ge 2$ we have the Bessel functions $J_{\frac{n}{2}-1}$ inside the integral. If $\mathrm{d}\mu$ is the surface measure on the unit circle $|x| = 1$ in $\mathbb{R}^n$ then we have

$$C\frac{J_{\frac{n}{2}-1}(|x|)}{|x|^{\frac{n}{2}-1}} = \int_{|y|=1}\mathrm{e}^{ix.y}\mathrm{d}\mu(y),$$

where C is an absolute constant. If we use this in the above we get $E\sigma_N^\delta Ef(x)$ equals

$$\frac{C}{A_N^\delta}\int_{|\xi|=1}\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}\frac{(t-s)^N}{N!}t^\delta s^{\frac{n}{2}-1}S^\delta_{\sqrt{2t}}(f(y)\mathrm{e}^{i\sqrt{2s}y.\xi})(x)\mathrm{e}^{i\sqrt{2s}x.\xi}\mathrm{d}t\mathrm{d}s\mathrm{d}\mu(\xi).$$

As before, using Minkowski's inequality we get

$$||E\sigma_N^\delta Ef||_p \le C||f||_p$$

since

$$\begin{aligned}&\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}|t-s|^N t^\delta s^{\frac{n}{2}-1}\mathrm{d}t\mathrm{d}s\\ &\quad\le \int_0^\infty \mathrm{e}^{-t}t^\delta\left(\int_0^t \mathrm{e}^{-s}t^N s^{\frac{n}{2}-1}\mathrm{d}s + \int_t^\infty \mathrm{e}^{-s}s^{N+\frac{n}{2}-1}\mathrm{d}s\right)\mathrm{d}t\\ &\quad\le C\int_0^\infty \mathrm{e}^{-t}t^\delta t^N\mathrm{d}t + \Gamma\left(N+\frac{n}{2}\right)\int_0^\infty \mathrm{e}^{-t}t^\delta\mathrm{d}t\\ &\quad\le C\Gamma(N+\delta+1)\end{aligned}$$

provided $\delta \ge \frac{n}{2} - 1$. This completes the proof.

3. Special Hermite expansions

Let $\Phi_{\alpha\beta}$, $\alpha, \beta \in N^n$, be the special Hermite functions on $\mathbb{C}^n$ which form an orthonormal basis for $L^2(\mathbb{C}^n)$. The special Hermite expansion of a function f in $L^p(\mathbb{C}^n)$ is given by

$$f = \sum \sum (f, \Phi_{\alpha\beta})\Phi_{\alpha\beta}.$$

The functions $\Phi_{\alpha\beta}$ are the eigenfunctions of the operator L with eigenvalues $(2|\beta| + n)$. Let

$$Q_k f = \sum_{|\alpha|=k} \sum_{\beta} (f, \Phi_{\alpha\beta})\Phi_{\alpha\beta}$$

be the projection onto the kth eigenspace. Then we have

$$Q_k f(z) = (2\pi)^{-n} f \times \varphi_k(z),$$

where $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)\mathrm{e}^{-\frac{1}{4}|z|^2}$ are the Laguerre functions and $f \times g$ is the twisted convolution

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)\mathrm{e}^{\frac{i}{2}\operatorname{Im} z.\overline{w}}\mathrm{d}w.$$

The special Hermite expansion then takes the compact form

$$f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k.$$

The Cesaro means are then defined by

$$\sigma_N^\delta f(z) = \frac{(2\pi)^{-n}}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta f \times \varphi_k(z).$$

In this section we prove the following theorem.

Let S_t^δ be the Bochner-Riesz means for the Fourier transform on $\mathbb{R}^{2n} = \mathbb{C}^n$.

Theorem 3.1. *Let B be any compact subset of $\mathbb{C}^n$ containing the origin. Then $\chi_B \sigma_N^\delta \chi_B$ are uniformly bounded on L^p, $1 \le p \le \infty$ if and only if S_t^δ are uniformly bounded on the same L^p.*

Proof. The kernel $\sigma_N^\delta(z)$ of σ_N^δ is given by

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta \varphi_k(z).$$

Using the formula

$$\sum_{k=0}^{N} A_{N-k}^\delta L_k^\alpha(t) = L_N^{\alpha+\delta+1}(t)$$

we have

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta} L_N^{\delta+n}\left(\frac{1}{2}|z|^2\right)\mathrm{e}^{-\frac{1}{4}|z|^2}.$$

As in the previous section we can express the Laguerre function in terms of the Bessel functions, thus getting

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta} \frac{1}{\Gamma(N+1)} \mathrm{e}^{\frac{1}{4}|z|^2} \int_0^\infty \mathrm{e}^{-t} t^{\delta+N+n} \frac{J_{\delta+n}(\sqrt{2t}|z|)}{(\sqrt{2t}|z|)^{\delta+n}} \mathrm{d}t.$$

Now, $\sigma_N^\delta f = f \times \sigma_N^\delta$ so that

$$\sigma_N^\delta f(z) = \int \sigma_N^\delta(z,w) f(w) \mathrm{d}w,$$

where

$$\sigma_N^\delta(z,w) = \mathrm{e}^{\frac{i}{2}\operatorname{Im} z.\overline{w}} \sigma_N^\delta(z-w).$$

Writing $|z-w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re} z\cdot\overline{w}$ we have

$$\begin{aligned}\sigma_N^\delta(z,w) &= \mathrm{e}^{\frac{1}{4}|z|^2} \mathrm{e}^{-\frac{1}{2}z.\overline{w}} \frac{1}{A_N^\delta \Gamma(N+1)} \int_0^\infty \mathrm{e}^{-t} t^{\delta+n+N} \frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}} \mathrm{e}^{\frac{1}{4}|w|^2} \mathrm{d}t \\ &= \left(\sum_\alpha \left(-\frac{1}{2}\right)^{|\alpha|} \frac{(z.\overline{w})^\alpha}{\alpha!} \right) \mathrm{e}^{\frac{1}{4}|z|^2} \frac{1}{A_N^\delta \Gamma(N+1)}{}_t \\ &\quad \times \int_0^\infty \mathrm{e}^{-t} t^{\delta+n+N} \frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}} \mathrm{e}^{\frac{1}{4}|w|^2} \mathrm{d}t,\end{aligned}$$

where $(z\cdot\overline{w})^\alpha = (z_1\overline{w}_1)^{\alpha_1} \cdots (z_n\overline{w}_n)^{\alpha_n}$. Therefore,

$$\chi_B \sigma_N^\delta \chi_B f(z) = \frac{1}{A_N^\delta \Gamma(N+1)} \sum_\alpha \left(-\frac{1}{2}\right)^{|\alpha|} \frac{1}{\alpha!} \int_0^\infty \mathrm{e}^{-t} t^{\delta+N} T_{\alpha,\delta}^t f(z) \mathrm{d}t,$$

where

$$T_{\alpha,\delta}^t f(z) = \chi_B(z) z^\alpha \mathrm{e}^{\frac{1}{4}|z|^2} \int_{\mathbb{C}^n} t^n \frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}} \chi_B(w) \overline{w}^\alpha \mathrm{e}^{\frac{1}{4}|w|^2} f(w) \mathrm{d}w.$$

If we assume that S_t^δ are uniformly bounded we get

$$||T_{\alpha,\delta}^t f||_p \leq C R^{2|\alpha|} ||f||_p,$$

when B is contained in the ball $\{z : |z| \leq R\}$. Using this in the above equation we get

$$||\chi_B \sigma_N^\delta \chi_B f||_p \leq C_B ||f||_p.$$

The converse is the transplantation theorem of Kenig-Stanton-Tomas.

In [5], Thangavelu has established the following local estimates for the Cesaro means.

Theorem 3.2. *Let* $\frac{2(2n+1)}{2n-1} \leq p \leq \infty$ *and* $\delta > \delta(p) = 2n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ *then for any compact subset* B *of* $\mathbb{C}^n$

$$\int_B |\sigma_N^\delta f(z)|^p \mathrm{d}z \leq C_B \int |f(z)|^p \mathrm{d}z.$$

Recently Stempak and Zienkiewicz have proved the global estimate

$$\int_{\mathbb{C}^n} |\sigma_N^\delta f(z)|^p \mathrm{d}z \leq C \int_{\mathbb{C}^n} |f(z)|^p \mathrm{d}z$$

for the above range. The key point is the restriction theorem namely, the estimate

$$||f \times \varphi_k||_2 \leq Ck^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}||f||_p$$

which they established in the range $1 \leq p \leq \frac{2(2n+1)}{2n+3}$. In the next section we use this restriction theorem in order to prove a positive result for the Hermite expansions on $\mathbb{R}^{2n}$.

4. Hermite expansions on $\mathbb{R}^{2n}$

In this section we consider the operator $-\Delta + \frac{1}{4}|z|^2$ rather than the operator $-\Delta + |z|^2$. If $\Phi_\mu(x,y), \mu \in N^{2n}$ are the eigenfunctions of the operator $-\Delta + |z|^2$ then $\Psi_\mu(z) = \Phi_\mu(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}})$ are the eigenfunctions of $-\Delta + \frac{1}{4}|z|^2$ with eigenvalues $(|\mu| + n)$. The operator $-\Delta + \frac{1}{4}|z|^2$ has another family of eigenfunctions namely the special Hermite functions. In fact, $\Phi_{\alpha\beta}$ are eigenfunctions of the operator $-\Delta + \frac{1}{4}|z|^2$ with eigenvalue $(|\alpha|+|\beta| + n)$; here $\alpha, \beta \in N^n$.

In this section we study the expansion in terms of Ψ_μ for functions having some homogeneity. The torus $T(n) = \{(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}, \ldots, \mathrm{e}^{i\theta_n}) : \theta \in \mathbb{R}^n\}$ acts on functions on $\mathbb{C}^n$ by $\tau_\theta f(z) = f(\mathrm{e}^{i\theta}z)$ where $\mathrm{e}^{i\theta}z = (\mathrm{e}^{i\theta_1}z_1, \mathrm{e}^{i\theta_2}z_2, \ldots, \mathrm{e}^{i\theta_n}z_n)$. We say that a function is m-homogeneous if $\tau_\theta f(z) = \mathrm{e}^{im.\theta} f(z)$, here $m \in Z^n$ and $m.\theta = m_1.\theta_1 + \cdots + m_n{\cdot}\theta_n$. It is a fact that $\Phi_{\alpha\beta}$ is $(\beta - \alpha)$ homogeneous. 0-homogeneous functions are also called polyradial.

The operator $-\Delta + \frac{1}{4}|z|^2$ commutes with τ_θ for all θ, therefore $P_k\tau_\theta f = \tau_\theta P_k f$ which shows that $P_k f$ is m-homogeneous if f is . In particular, $P_k f$ is polyradial if f is. Therefore, for such functions $L(P_k f) = (-\Delta + \frac{1}{4}|z|^2)P_k f = (k+n)P_k f$. This shows that $P_k f$ is an eigenfunction of L with eigenvalue $k+n$. But the spectrum of L is $\{2k+n : k = 0, 1, \ldots\}$ which forces $P_k f = 0$ when k is odd.

PROPOSITION 4.1

Let f be polyradial on $\mathbb{C}^n$. Then $P_{2k+1}f = 0$ and $P_{2k}f = f \times \varphi_k$.

Proof. We show that when f is polyradial the operators $P_{2k}f$ and $f \times \varphi_k$ have the same kernel. Let

$$\Psi_k(z,w) = \sum_{|\mu|=k} \Psi_\mu(z)\Psi_\mu(w)$$

be the kernel of P_k. Then by Mehler's formula

$$\sum_{k=0}^{\infty} t^k \Psi_k(z,w) = \pi^{-n}(1-t^2)^{-n} \mathrm{e}^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(|z|^2+|w|^2)+\frac{t}{1-t^2}\mathrm{Re}(z.\overline{w})}$$

so that

$$\sum_{k=0}^{\infty} t^k P_k f(z) = \pi^{-n}(1-t^2)^{-n} \int \mathrm{e}^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(|z|^2+|w|^2)+\frac{t}{1-t^2}\mathrm{Re}(z.\overline{w})} f(w)\mathrm{d}w.$$

Let $w_j = u_j + iv_j = r_j e^{i\theta_j}$. When f is polyradial $f(w) = f_0(r_1, r_2, \ldots, r_n)$ and so we have

$$\sum_{k=0}^{\infty} t^k P_k f(z) = \int_0^{\infty} \cdots \int_0^{\infty} \Psi(s,r) f_0(r_1, \ldots, r_n) r_1 r_2, \ldots, r_n \mathrm{d}r_1 \mathrm{d}r_2 \ldots \mathrm{d}r_n,$$

where $s = (s_1, s_2, \ldots, s_n), s_j = |z_j|$ and Ψ is given by

$$\Psi(s,r) = (1-t^2)^{-n} \int_{[0,2\pi]^n} \mathrm{e}^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(r^2+s^2)} \mathrm{e}^{\frac{t}{1-t^2}\mathrm{Re}\, z.\overline{w}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_n.$$

Now Re $z_j \cdot \overline{w}_j = r_j s_j \cos(\theta_j - \varphi_j)$ where $z_j = s_j \mathrm{e}^{i\varphi_j}, w_j = r_j \mathrm{e}^{i\theta_j}$. Consider the integral

$$\int_0^{2\pi} \mathrm{e}^{\frac{t}{1-t^2} r_j s_j \cos(\theta_j - \varphi_j)} \mathrm{d}\theta_j$$

which equals, if we recall the definition of the Bessel functions, $J_0(\frac{it}{1-t^2} r_j s_j)$. Thus we have proved

$$\Psi(s,r) = (1-t^2)^{-n} \mathrm{e}^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(r^2+s^2)} \Pi_{j=1}^n J_0\left(\frac{it}{1-t^2} r_j s_j\right).$$

On the other hand when f is polyradial $f \times \varphi_k$ reduces to the finite sum

$$\begin{aligned} f \times \varphi_k &= \sum_{|\alpha|=k} (f, \Phi_{\alpha\alpha}) \Phi_{\alpha\alpha}(z) \\ &= \sum_{|\alpha|=k} \left(\int_0^{\infty} \cdots \int_0^{\infty} f_0(r_1, \ldots, r_n) \Phi_{\alpha\alpha}(r_1, \ldots, r_n) r_1, \ldots, r_n \mathrm{d}r_1, \ldots, \mathrm{d}r_n \right) \\ &\quad \times \Phi_{\alpha\alpha}(s_1, \ldots, s_n) \end{aligned}$$

where we have written

$$\Phi_{\alpha\alpha}(z) = \Phi_{\alpha\alpha}(r_1, \ldots, r_n)$$

as it is polyradial. Then $f \times \varphi_k$ is given by the integral operator

$$\begin{aligned} f \times \varphi_k(z) = \int_0^{\infty} \cdots \int_0^{\infty} &\left(\sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(r_1, \ldots, r_n) \Phi_{\alpha,\alpha}(s_1, \ldots, s_n) \right) \\ &f_0(r_1, \ldots, r_n) r_1, \ldots, r_n \mathrm{d}r_1, \ldots, \mathrm{d}r_n. \end{aligned}$$

We have the formula (see [6])

$$\Phi_{\mu\mu}(z) = (2\pi)^{-\frac{n}{2}} \Pi_{j=1}^n L_{\mu_j}\left(\frac{1}{2}|z_j|^2\right) \mathrm{e}^{-\frac{1}{4}|z_j|^2}.$$

Recalling the generating function identity for the Laguerre polynomials of type 0,

$$\sum_{k=0}^{\infty} L_k(x) L_k(y) w^k = (1-w)^{-1} \mathrm{e}^{-\frac{w}{1-w}(x+y)} J_0\left(\frac{2(-xyw)^{\frac{1}{2}}}{1-w}\right)$$

we get, if $S_k(r,s)$ is the kernel for $f \times \varphi_k$

$$\sum_{k=0}^{\infty} t^k S_k(r,s) = (1-t)^{-n} \mathrm{e}^{-\frac{1}{4}\frac{1+t}{1-t}(r^2+s^2)} \Pi_{j=1}^{n} J_0\left(\frac{i\sqrt{t}}{1-t} r_j s_j\right).$$

Comparing the two generating functions we see that

$$\sum_{k=0}^{\infty} t^{2k} S_k(r,s) = \sum_{k=0}^{\infty} t^k \Psi_k(r,s)$$

from which follows $\Psi_{2k}(r,s) = S_k(r,s)$ and this proves the proposition.

Consider now the Bochner-Riesz means associated to the expansions in terms of $\Psi_\mu(z)$ defined by

$$S_R^\delta f(z) = \sum_{\mu} \left(1 - \frac{(|\mu| + n)}{R}\right)_+^\delta (f, \Psi_\mu)\Psi_\mu(z).$$

For these means we have the following result.

Theorem 4.2. *Let* $1 \leq p \leq 2(\frac{2n+1}{2n+3})$, $\delta > \delta(p) = 2n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ *and let* $f \in L^p(\mathbb{C}^n)$ *be polyradial. Then*

$$||S_R^\delta f||_p \leq C||f||_p,$$

where C *is independent of* f *and* R.

The key ingredient in proving the above theorem is the $L^p - L^2$ estimates

$$||P_k f||_2 \leq C k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} ||f||_p$$

which now follows from the corresponding estimates for $f \times \varphi_k$. We omit the details.

We conclude this section with the following remarks. As we have observed, $P_k f$ is m-homogeneous whenever f is and so $P_k f$ can be obtained in terms of $f \times \varphi_k$ when f is m-homogeneous. So an analogue of the above theorem is true for all m-homogeneous functions. More generally, let us call a function f of type N if it has the Fourier expansion

$$f(z) = \sum_{|m| \leq N} f_m(z),$$

where

$$f_m(z) = \int f(\mathrm{e}^{i\theta} z) \mathrm{e}^{-im.\theta} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

Note that f_m is m-homogeneous. We can show that when f is of type N then

$$||S_R^\delta f||_p \leq C_N ||f||_p,$$

under the conditions of the above theorem on p and δ where now C_N depends on N. We leave the details to the interested reader. It is an interesting problem to see if the theorem is true for all functions.

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