A Solvable Model of Interacting Fermions in Two Dimensions

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We introduce and study an exactly solvable model of several species of fermions in which particles interact pairwise through a mutual magnetic field; the interaction operates only between particles belonging to different species. After an unitary transformation, the model reduces to one in which each particle sees a magnetic field which depends on the total numbers of particles of all the other species; this may be viewed as the mean-field model for a class of anyonic theories. Our model is invariant under charge conjugation C and the product PT (parity and time reversal). For the special case of two species, we examine various properties of this system, such as the Hall conductivity, the wave function overlap arising from the transfer of one particle from one species to another, and the one-particle off-diagonal density matrix. Our model is a generalization of a recently introduced solvable model in one dimension.

PACS number: 71.10.Pm, 71.27.+a

Exactly solvable models of interacting particles have often been very useful in illustrating some general concepts in many-body physics. While there is a large variety of such models available in one dimension, many of which fall into the class of Tomonaga-Luttinger liquids[1], there are few models known in two dimensions which are completely solvable. In this paper, we introduce and study a model of several species of fermions which interact with each other through a magnetic field term which depends on the coordinates of pairs of particles belonging to two different species. The model can be solved by a unitary transformation which reduces it to a model of fermions in a magnetic field which depends on the total numbers of fermions belonging to the other species. Our model is a direct generalization of the recent reinterpretation of the well-known model of Luttinger in one dimension[2]. The one-dimensional model also has pairwise "gauge" interactions depending on the coordinates of the particles; the model is exactly solvable because the interactions can be unitarily gauged away at the cost of modifying the boundary conditions in a non-trivial way. As we will see, in our two-dimensional model the interactions cannot be gauged away in the bulk of the system; the unitary transformation leaves behind a static magnetic field.

Let us consider ν species of fermions in two dimensions (say, the $\hat{x}-\hat{y}$ plane), with the charge and number of fermions of type α being denoted by q_α and N_α respectively. The coordinates of the particles will be denoted by $\vec{r}_{i,\alpha}$, where $1 \le i \le N_\alpha$ and $1 \le \alpha \le \nu$. We will consider the Hamiltonian

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$$
\begin{aligned}
\mathcal{H} \;&=\; \sum_{i,\alpha} \; \frac{1}{2m_\alpha} \; \Big(\; \vec{p}_{i,\alpha} \; - \; \frac{q_\alpha}{c} \vec{A}_{i,\alpha} \; - \; \frac{q_\alpha}{c} \vec{\mathcal{A}}_{i,\alpha} \; \Big)^2 \; , \\
\vec{A}_{i,\alpha} \;&=\; \frac{1}{2} \; B_0 \; \hat{z} \; \times \; \vec{r}_{i,\alpha} \; ,
\end{aligned}
\tag{1}
$$

where c is the velocity of light. $B_0 \hat{z}$ is an external magnetic field pointing in a direction perpendicular to the two-dimensional plane; we have chosen the symmetric gauge for its vector potential $\vec{A}_{i,\alpha}$ in order to explicitly maintain invariance under rotations of the plane. The other vector potential $\vec{\mathcal{A}}_{i,\alpha}$ arises from two-body interactions; it will be taken to have the following form which is natural in two dimensions[3],

$$
\vec{\mathcal{A}}_{i,\alpha} \;=\; \eta_\alpha \; \sum_{j\beta} \; \xi_{\alpha\beta} \; \hat{z} \; \times \; (\; \vec{r}_{i,\alpha} \; - \; \vec{r}_{j,\beta} \;) \; ,
\tag{2}
$$

where η_α and $\xi_{\alpha\beta}$ are some constants to be fixed below. Note that the Hamiltonian (1) is invariant under translations in the plane.

We may now perform an unitary transformation on the Hamiltonian of the form

$$
\begin{aligned}
\tilde{\mathcal{H}} \;&=\; U \; \mathcal{H} \; U^{-1} \; , \\
U \;&=\; \exp \; \Big[\; - \; \frac{iq}{\hbar c} \; \sum_{\alpha<\beta} \sum_{i,j} \; \xi_{\alpha\beta} \; \hat{z} \; \cdot \; \vec{r}_{i,\alpha} \; \times \vec{r}_{j,\beta} \; \Big] \; .
\end{aligned}
\tag{3}
$$

where q is the charge of an electron. (We note that the phase factor in U only depends on the total coordinates $\vec{R}_\alpha = \sum_i \vec{r}_{i,\alpha}$ of the various species of fermions). This gives the transformed Hamiltonian

$$
\begin{aligned}
\tilde{\mathcal{H}} \;&=\; \sum_{i,\alpha} \; \frac{1}{2m_\alpha} \; \Big(\; \vec{p}_{i,\alpha} \; - \; \frac{q_\alpha}{c} \vec{A}_{i,\alpha} \; - \; \frac{q_\alpha}{c} \vec{a}_{i,\alpha} \; \Big)^2 \; , \\
\vec{a}_{i,\alpha} \;&=\; \frac{1}{2} \; \Big(\; \eta_\alpha \; \sum_{\beta\neq\alpha} \; \xi_{\alpha\beta} \; N_\beta \; \Big) \; \hat{z} \; \times \; \vec{r}_{i,\alpha} \; ,
\end{aligned}
\tag{4}
$$

provided that

$$
\begin{aligned}
\xi_{\alpha\beta} \;&=\; - \; \xi_{\beta\alpha} \; , \\
\text{and} \quad q_\alpha \eta_\alpha \;&=\; q \quad \text{for all} \quad \alpha \; .
\end{aligned}
\tag{5}
$$

The antisymmetry of $\xi_{\alpha\beta}$ implies that the two-particle magnetic interaction can only act between particles belonging to two different species.

It is interesting to consider the effects of some discrete symmetries such as time reversal (T), parity (P) and charge conjugation (C). Let us first set the external magnetic field $B_0 = 0$. Under T, the wave functions and factors of i are complex conjugated (thus, the momentum operators $\vec{p}_{i,\alpha} \to -\vec{p}_{i,\alpha}$) and the time coordinate $t \to -t$; the space coordinates x, y and the various parameters q_α, η_α and $\xi_{\alpha\beta}$ remain unchanged. Under P, one of the space coordinates, say, $x \to -x$, while y, t and all the parameters remain unchanged. We therefore see that the model is not invariant under P and T separately, but it is invariant under the combined operation PT. Under charge conjugation, we demand that $q_\alpha \to -q_\alpha$ and $\xi_{\alpha\beta} \to -\xi_{\alpha\beta}$, while η_α and the space-time coordinates remain unchanged; thus the model is invariant under C and therefore under CPT. Finally, if the external magnetic field B_0 is nonzero, the model is again invariant under C and PT, but not under P and T separately; this is because a magnetic field (which must be produced by some external currents) changes sign under C, P and T separately.

It may be useful to point out here that our model has some resemblance to the mean field theory of several species of anyons. In the usual theories of anyons, the wave function is assumed to pick up a phase

θ_{ij} whenever particle i is taken in an anticlockwise loop around particle j, no matter what the size and shape of the loop is[4]. This is often modeled by treating each particle as a point-like composite of charge and magnetic flux; when one particles encircles another, the wave function picks up an Aharonov-Bohm phase. In understanding the many-body properties of such a system, a fruitful approach has been to begin with a mean field theory in which the magnetic flux of each anyon is smeared out over the entire plane[5,6]. Thus each particle sees a magnetic field proportional to the average density of particles, which is similar to our situation. Of course, the analysis of anyons then goes beyond mean field theory to study the fluctuations about the average magnetic field, while our simplified model has no fluctuations. It is worth remarking that our model has no counterpart for the most popular anyon model which has only one species; we need a a minimum of two species.

To continue, the total magnetic field seen by a particle of type α in our model is given by $B_\alpha \hat{z}$, where

$$B_\alpha \ = \ B_0 \ + \ \eta_\alpha \sum_{\beta \neq \alpha} \xi_{\alpha\beta} \ N_\beta \ . \tag{6}$$

In order to have a well-defined thermodynamic limit $N_\alpha \to \infty$, the $\xi_{\alpha\beta}$ must be taken to scale as $1/A$, where A is the area of the system; thus the magnetic field strengths B_α in (6) remain of order 1 as $A \to \infty$ with the densities $\rho_\alpha = N_\alpha / A$ held fixed. We then expect Landau levels to form for each species[7]. It is well-known that each Landau level has a macroscopic degeneracy equal to $A|q_\alpha B_\alpha|/(2\pi\hbar c)$. The filling fraction of fermions of type α is given by

$$f_\alpha \ = \ \rho_\alpha \ \frac{2\pi\hbar c}{|q_\alpha B_\alpha|} \ . \tag{7}$$

If f_α is not equal to an integer for one or more values of α, the ground state of the system is highly degenerate.

For computational purposes, it is convenient to break this degeneracy in one of two ways. We can either add a simple harmonic confining potential to the Hamiltonians (1) and (4) of the form

$$\mathcal{H}_{sh} \ = \ \frac{k}{2} \sum_{i,\alpha} \vec{r}_{i,\alpha}^{\,2} \ , \tag{8}$$

and take the limit $k \to 0$ at the end of the calculation, or we can simply impose a hard wall boundary condition at some large radius R. Analytically, it is easier to work with the first method since the problem of free particles in a combination of an uniform magnetic field and a simple harmonic confinement is exactly solvable as we will now discuss. (Let us drop the species label α in the rest of this paragraph and in the next). Since the problem has rotational symmetry, the energies and wave functions are specified by two quantum numbers, a radial quantum number $n = 0, 1, 2, ...$ and the angular momentum $l = 0, \pm 1, \pm 2, ...$ If only a magnetic field is present (with, say, the product qB being positive), the single-particle states have energies which only depend on the integer n which counts the number of nodes in the radial direction; thus

$$\begin{aligned} E_{n,l} \ &= \ \hbar\omega_c \ (\ n \ + \ \frac{1}{2} \ ) \ , \\ \omega_c \ &= \ \frac{qB}{mc} \ . \end{aligned} \tag{9}$$

In the lowest Landau level (LLL), $n = 0$ while l can only take non-negative values; all states have the energy $E_{0,l} = \hbar\omega_c/2$ independent of l. The normalized wave functions in the LLL are given in terms of the complex coordinates $z = x + iy$ and $z^\star = x - iy$ as

$$\psi_{0,l}(z, z^\star) \ = \ \left(\ \frac{qB}{2\hbar c} \ \right)^{(l+1)/2} \ \frac{z^l}{\sqrt{l! \ \pi}} \ \exp\Big[\ - \ \frac{qB}{4\hbar c} z z^\star \ \Big] \ , \tag{10}$$

where $l = 0, 1, 2, ...$ The amplitudes of these wave functions are peaked on circles of various radii centered about the origin $\vec{r} = \vec{0}$; the radii of these "ring" states are given by $r_l = \sqrt{2l\hbar c/(qB)}$. (If qB is negative, the LLL wave functions are given by Eq. (10) with z replaced by $z^\star$. Then the angular momentum only takes non-positive values).

If we now add a weak simple harmonic potential $m\omega^2 \vec{r}^2/2$ for all the particles, the energies of the ring states in the LLL become

$$E_l \;=\; \frac{\hbar}{2} \left[\sqrt{\omega_c^2 \;+\; 4\omega^2} \;+\; (\sqrt{\omega_c^2 \;+\; 4\omega^2} \;-\; \omega_c \;) \;|l| \right] , \tag{11}$$

which increase from the origin outwards as $|l|$ increases from zero. In the many-particle ground state, therefore, the fermions fill up the individual ring states from the origin outwards. In the following discussion, we will assume this order of filling in the LLL, without explicitly mentioning the simple harmonic confinement which justifies it.

We will now specialize to the case of two species of fermions to illustrate some properties of our model. Let us take the masses equal to m for both species, the charges equal to $q_1 = q_2 = q$ (thus, $\eta_1 = \eta_2 = 1$), and the numbers of particles equal to N_1 and N_2 for the two species respectively. We will also set

$$\xi_{12} \;=\; -\; \xi_{21} \;=\; \frac{\gamma}{A} \;, \tag{12}$$

where γ is a number of order 1. After the unitary transformation in (3), the two species see uniform magnetic fields equal to

$$\begin{aligned} B_1 \;&=\; B_0 \;+\; \gamma \; \frac{N_2}{A} \\ \text{and} \quad B_2 \;&=\; B_0 \;-\; \gamma \; \frac{N_1}{A} \end{aligned} \tag{13}$$

respectively. If the number of particles $N_1 = N_2$, the model is invariant under the exchange of the species labels $1 \leftrightarrow 2$ and $\gamma \to -\gamma$; this is in addition to the discrete symmetries C and PT discussed in general before.

One of the properties of interest for such a model is the Hall conductivity. In the absence of impurities and any other interactions (such as Coulomb repulsion), what is the Hall conductivity of this system if the filling fractions f_1 and f_2 are both integers? It is fairly easy to see that the answer is

$$\sigma_{xy} \;=\; [\; f_1 \;\text{sign}(B_1) \;+\; f_2 \;\text{sign}(B_2) \;] \; \frac{q^2}{2\pi\hbar} \;. \tag{14}$$

This can be derived from the usual formula for the frequency-dependent conductivity

$$\sigma_{xy} \;=\; \frac{i}{\omega} \sum_{a \neq 0} \left[\frac{\langle 0|J_x|a\rangle\langle a|J_y|0\rangle}{\omega - E_a + E_0 + i\eta} \;-\; \frac{\langle 0|J_y|a\rangle\langle a|J_x|0\rangle}{\omega + E_a - E_0 + i\eta} \right] , \tag{15}$$

where $|0\rangle$ is the ground state of the many-body system, and the sum over $|a\rangle$ runs over all the excited states; η is an infinitesimal positive number. The current $\vec{J}$ is given by the second-quantized expression

$$\vec{J} \;=\; -\; c \; \frac{\delta \mathcal{H}}{\delta \vec{A}} \;=\; \frac{q}{2mi} \sum_{\alpha=1}^{2} \int d^2\vec{r} \left[\Psi_\alpha^\dagger \left(\vec{p} - \frac{q_\alpha}{c}\vec{A} \;-\; \frac{q_\alpha}{c}\vec{\mathcal{A}}_\alpha \;\right) \Psi_\alpha \;-\; \text{hermitian conjugate} \right] . \tag{16}$$

If we now perform the unitary transformation in (3) on both the current and the states, then (16) reduces to the conventional expression for the current operator of two species of fermions placed in the magnetic fields given by (13). Eq. (15) can then be evaluated in the usual way[5,6]; in the zero frequency limit, we obtain the expression given in (14). The Hall conductivity will remain unchanged if we make our model more realistic by including Coulomb repulsion between the particles.

Another object of interest in this model is the matrix element of the “hopping” operator

$$M(\vec{r}) \ = \ c_1^\dagger(\vec{r}) \ c_2(\vec{r}) \tag{17}$$

between the ground state of the system with (N_1, N_2) particles and all possible states of the system with (N_1+1, N_2-1) particles. [The calculation of this overlap is of interest in connection with the “orthogonality” catastrophe which is known to occur in Luttinger liquids in one dimension. It may also be useful in the context of a two-layer quantum Hall system in which electrons can hop from one layer to the other]. Since our original Hamiltonian (1) is translation invariant, it is sufficient to compute the matrix element of $M(\vec{0})$ located at the origin. This simplifies the computation for the following reason. In a second quantized form, the annihilation operator for any species is given by

$$c(\vec{r}) \ = \ \sum_{n,l} \ \psi_{n,l}(\vec{r}) \ c_{n,l} \ , \tag{18}$$

where the sum runs over all one-particle states (n, l) with wave functions $\psi_{n,l}$, and $c_{n,l}$ annihilates a fermion in the state (n, l). Since only the zero angular momentum states have non-vanishing wave functions at the origin, $c(\vec{0})$ gets a contribution from only $l = 0$ but all possible radial quantum numbers n. Thus

$$c(\vec{0}) \ = \ \sum_{n} \ \psi_{n,0}(\vec{0}) \ c_{n,0} \ , \tag{19}$$

where

$$|\psi_{n,0}(\vec{0})|^2 \ = \ \frac{qB}{2\pi\hbar c} \tag{20}$$

for *all* n. (This follows from the normalization of the Laguerre polynomials given in Refs. 7 and 8). We can now compute the frequency-dependent hopping function

$$\mathcal{M}(\omega) \ = \ \sum_a \ |\langle a; N_1+1, N_2-1| \ M(\vec{0}) \ |0; N_1, N_2\rangle|^2 \ 2\pi \ \delta(\hbar\omega \ - \ E_a \ + \ E_0) \ , \tag{21}$$

where $|0; N_1, N_2\rangle$ denotes the ground state of the system with (N_1, N_2) particles, while $|a; N_1+1, N_2-1\rangle$ denotes all possible states of the system with (N_1+1, N_2-1) particles.

For simplicity, let us consider the case in which the filling fractions f_1 and f_2 are both less than 1, and $N_1 = N_2 = N$. Then the ground state $|0; N, N\rangle$ is one which the both the type 1 and type 2 particles occupy the LLL states with $n = 0$ and angular momentum $l = 0, 1, 2, ..., N-1$. Upon acting on this state with the operator $M(\vec{0})$ in (17), we get a state $|a; N+1, N-1\rangle$ in which a type 2 particle has been removed from the state $(0, 0)$, and a type 1 particle has been added to the state $(n, 0)$, where $n \neq 0$ due to the Pauli exclusion principle. Hence the energy difference is

$$E_a - E_0 \ = \ \frac{\hbar q}{mc} \left[\ (\ n \ + \ \frac{1}{2} \ ) \ |B_1| \ - \ \frac{1}{2} \ |B_2| \ \right] \ , \tag{22}$$

where $n = 1, 2, 3, ...$ This gives the locations of the δ-functions on the right hand side of (21). We now have to find the weights. We can use (20) to show that for species 2,

$$|\langle a_2; N-1|\ c_2(\vec{0})\ |0;N\rangle|^2 \ = \ \frac{q|B_2|}{2\pi\hbar c} \ , \tag{23}$$

where $|a_2; N-1\rangle$ represents the state in which a particle of type 2 has been removed from the state $(0,0)$. Similarly, for species 1, we have

$$|\langle a_1; N+1|\ c_1^\dagger(\vec{0})\ |0;N\rangle|^2 \ = \ \frac{q|B_1|}{2\pi\hbar c} \ , \tag{24}$$

where $|a_1; N+1\rangle$ represents the state in which a particle of type 1 has been added to the state $(n,0)$ where $n \geq 1$. To use the results (23) and (24) for evaluating the matrix elements in (21), we now perform the unitary transformation in (3). At this point, we have to worry about two things. Firstly, the $N+1$ particles of type 1 in the states $|a\rangle$ see a slightly different magnetic field than the N particles of type 1 in the state $|0\rangle$, since the number of type 2 particles differ by one in the two cases. However, the difference in the two magnetic fields is of order $1/A$ which vanishes in the thermodynamic limit. Thus the wave functions in the two cases look almost the same since the magnetic field B appearing in (10) differs only slightly in the two cases; so when we use Eq. (24), it does not matter much if we set N_2 equal to N or $N-1$ to determine the value of B_1 given by Eq. (13). Secondly, we have to worry about the phase factor appearing in U which depends on the total coordinates $\vec{R}_{N,\alpha} = \sum_i \vec{r}_{i,\alpha}$; recall Eq. (3). At this point, another advantage of locating the hopping operator at the origin $\vec{r} = \vec{0}$ becomes apparent. Namely, we see that the phase factors $\hat{z} \cdot \vec{R}_{N+1,1} \times \vec{R}_{N-1,2}$ appearing in the states $|a\rangle$ cancels with the phase factor $\hat{z} \cdot \vec{R}_{N,1} \times \vec{R}_{N,2}$ appearing in the state $|0\rangle$, since the two states only differ by the addition or removal of particles at the origin; this does not change the total coordinate $\vec{R}_\alpha$ of either species. Putting Eqs. (23-24) and (22) together, we see that the hopping function is given by

$$\mathcal{M}(\omega) \ = \ \frac{q^2|B_1 B_2|}{(2\pi\hbar c)^2} \ \sum_{n=1}^{\infty} \ 2\pi \ \delta\Big(\hbar\omega \ - \ \frac{\hbar q}{mc}(n \ + \ \frac{1}{2}) \ |B_1| \ + \ \frac{\hbar q}{2mc} \ |B_2| \ \Big) \ . \tag{25}$$

We thus get an infinite sequence of δ-functions with equal weight.

Finally, let us compute the one-particle off-diagonal density matrix for, say, species 1. We assume again that $N_1 = N_2 = N$ and both the filling fractions f_α are less than 1. We have to evaluate

$$\rho(\vec{r},\vec{r}^{\,\prime}) = \int \ \prod_{i=2}^{N} d^2\vec{r}_{i,1} \prod_{j=1}^{N} d^2\vec{r}_{j,2} \ \psi^\star(\vec{r},\vec{r}_{2,1},...,\vec{r}_{N,1};\vec{r}_{1,2},\vec{r}_{2,2},...,\vec{r}_{N,2}) \ \psi(\vec{r}^{\,\prime},\vec{r}_{2,1},...,\vec{r}_{N,1};\vec{r}_{1,2},\vec{r}_{2,2},...,\vec{r}_{N,2}), \tag{26}$$

where we assume that the particles fill up the states $l = 0,1,2,...,N$ in the LLL. We again perform the unitary transformation (3). The integrand in (26) then becomes the product of a phase factor

$$\exp \ [\ \frac{iq\hbar\gamma}{cA} \hat{z} \cdot (\vec{r} - \vec{r}^{\,\prime}) \times \vec{R}_2 \ ] \tag{27}$$

(where $\vec{R}_2 = \sum_j \vec{r}_{j,2}$ and we have used Eq. (12)), four Van der Monde determinants which typically look like $\prod_{k<l}(z_{k,\alpha} - z_{l,\alpha})$ and its complex conjugate for both the species, and the Gaussian factor

$$\exp \ [\ - \frac{q|B_1|}{4\hbar c}(\vec{r}^{\,2} + \vec{r}^{\,\prime 2} + 2\sum_{i=2}^{N} \vec{r}_{i,1}^{\,2}) \ - \ \frac{q|B_2|}{2\hbar c} \sum_{j=1}^{N} \vec{r}_{j,2}^{\,2} \ ] \ . \tag{28}$$

Since the Van der Monde determinants are invariant under translations, we can immediately integrate over the $N-1$ independent relative coordinates (i.e., $\vec{r}_{k,2} - \vec{r}_{l,2}$) of the type 2 particles. The total coordinate $\vec{R}_2$ of species 2 then remains in the form

$$\exp\left[\frac{iq\hbar\gamma}{cA}\hat{z}\cdot(\vec{r}-\vec{r}')\times\vec{R}_2 - \frac{q|B_2|}{2\hbar c}\frac{\vec{R}_2^2}{N}\right] . \tag{29}$$

When we integrate over $\vec{R}_2$, we get a Gaussian of the form $\exp[-d(\vec{r}-\vec{r}')^2/A]$, where d is a number of order 1. In the thermodynamic limit, we can set this equal to 1 since we can assume that the separation $|\vec{r}-\vec{r}'|$ is much smaller than the size of the system. We are now left with only the coordinates $\vec{r}_{i,1}$, with $i = 2, 3, ..., N$, to integrate over. We finally get

$$\rho(\vec{r},\vec{r}') = \frac{1}{N}\sum_{l=0}^{N}\left(\frac{qB_1}{2\hbar c}\right)^{l+1}\frac{(zz'^{\star})^l}{l!\,\pi}\exp\left[-\frac{qB_1}{4\hbar c}(zz^{\star} + z'z'^{\star})\right] . \tag{30}$$

where we have assumed that qB_1 is positive. If we now take the limit $N\to\infty$, we find that the off-diagonal density matrix is the product of a Gaussian times a phase,

$$\rho(\vec{r},\vec{r}') \sim \frac{qB_1}{2\pi\hbar c}\exp\left[-\frac{qB_1}{4\hbar c}|z - z'|^2 - \frac{qB_1}{4\hbar c}(z^{\star}z' - zz'^{\star})\right] , \tag{31}$$

which is the usual result for a single species of particles in the LLL.

To summarize, we have introduced and solved a two-dimensional multi-species fermi system with mutual interactions of a particular type. The interaction can be converted via an unitary transformation into a static magnetic field whose strength depends on the density of particles. The model is quite simple; after all, the exact solvability of a Hamiltonian which has a quadratic form should surprise no one. Yet the physics of the model is quite interesting. We end up with a strongly non-fermi liquid system; further, the elements of the orthogonality catastrophe, i.e., a readjusting of *all* states in response to the addition of a single particle, also carry over from the one-dimensional physics of the Luttinger model. Our model may thus serve some purpose in understanding the physics of non-fermi liquids in higher dimensions.

We would like to dedicate this paper to the memory of Heinz Schulz.

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