Lorentz Invariance, Local Field Theory, and Faster-than-Light Particles*

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The quantum theory of faster-than-light particles is studied following the earlier classical theory of Bilanuik, Deshpande, and Sudarshan and of Terletski. The ingenious scheme of quantization formulated by Feinberg is seen, on closer examination, to violate Lorentz invariance. Another scheme of quantization involving a new physical postulate is formulated. The consistency and novel features of this formulation are discussed in some detail.

I. INTRODUCTION

IN recent years, there has been some discussion of particles with velocities greater than the speed of light, particularly in relation to the special theory of relativity. Bilaniuk, Deshpande, and Sudarshan (BDS) have treated the question of the existence of such particles within the framework of classical (nonquantum) relativity theory.[1] More recently, Feinberg (GF) has proposed a quantum field theory of noninteracting, spinless, faster-than-light particles.[2] The question is also discussed by Terletski.[3]

In this paper, we wish to continue this discussion with particular attention to Lorentz invariance in the context of local quantum field theory. For a general discussion of some of the interpretational questions and answers to some of the standard objections to faster-than-light particles, the reader is referred to BDS and GF.

We only mention here the crucial point of immediate interest. Since we are dealing with spacelike four-momenta, the mass hyperboloid is single-sheeted. This implies that a proper Lorentz transformation can change the sign of the energy in contradistinction to the usual case. However, one can maintain the usual interpretation of negative-energy particles as moving backward in time, only because the same Lorentz transformation that reverses the sign of the energy also reverses the time ordering of any two points in the path of the particle.[1,2] As Feinberg points out, although with this interpretation the qualitative features of faster-than-light particles can be treated, one must proceed to a detailed mathematical description in order to determine if a consistent theory can be constructed.

Since we are concerned with a Lorentz-invariant local *field* theory, let us define what we mean more explicitly. We insist on the existence of a set of unitary operators providing a representation of the inhomogeneous Lorentz group $U(\Lambda,a)$ that implements the automorphisms induced by such transformation on a local scalar field operator $[\phi(x)\rightarrow\phi(\Lambda x+a)]$. That is, we demand the existence of unitary $U(\Lambda,a)$ such that

$$U(\Lambda,a)\phi(x)U^{-1}(\Lambda,a)=\phi(\Lambda x+a). \qquad (1.1)$$

We propose to show in Sec. II that with this criterion, the scalar field theory proposed by Feinberg fails to be Lorentz-invariant.[4] We wish to emphasize that it was essential to explore the Feinberg scheme as a reasonable first attempt and that it was the natural avenue to quantization that was suggested by previous work in dealing with this problem. One should attempt to construct a theory as close to the conventional framework as possible, carry it through until it encounters difficulty, and then modify it as minimally as necessary to resolve the difficulty. In this spirit, in Sec. III, we propose an alternative quantization that deviates further from the conventional theory and requires an additional postulate for a physically meaningful interpretation. We will then remark on some of the characteristics of such a theory. A corollary to this treatment is that Feinberg's restriction to Fermi statistics is no

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[1] O. M. P. Bilaniuk, V. K. Deshpande, and E. C. G. Sudarshan, Am. J. Phys. **30**, 718 (1962); this paper is referred to as BDS.

[2] G. Feinberg, Phys. Rev. **159**, 1089 (1967); this paper is referred to as GF.

[3] Ya. P. Terletsky, *Paradoksy Teorii Otnositel'nosti* (Academy of Sciences, U.S.S.R., NAUKA Press, Moscow, 1966). (An English translation is to be published by Plenum Press.)

[4] A remark on this criterion for Lorentz invariance might be in order. It is true that the only really necessary requirement is that the predictions of the theory should not contradict physical reality, a reality that appears consistent with special relativity. Since any theory is a mathematically constructed model, one must translate the "physical requirement" of relativistic invariance into precise mathematical conditions on the theory. Since it is well known in quantum field theory, particularly for free fields, that our criterion does indeed insure that this requirement is met, we consider it reasonable to impose this requirement in the case under discussion. [In fact, it is just the condition stated in equation (4.2) of Feinberg's paper.] We note here that the criterion of (1.1) makes sense only in an infinite space, since, for example, we could not define the translation operator in a "finite box" volume. If a theory fails to obey our criterion, one may choose to proceed and argue that perhaps its physical predictions will not contradict reality, but then it cannot be called a Lorentz-invariant *field* theory in the sense generally understood, and in fact cannot be properly called a Lorentz-invariant theory at all without stating precisely what the condition is for Lorentz invariance and showing that the condition leads to relativistically invariant predictions.

longer relevant. We would have either Fermi or Bose statistics for the faster-than-light particles.

We do not treat here the still unsolved problem of constructing a consistent, Lorentz-invariant interacting-field theory. It is clear that until this is done, we cannot be sure that such a theory is even possible. However, in spite of our objections to the particular scheme proposed by Feinberg, we share with him the feeling that no compelling theoretical argument has yet been presented for the impossibility of faster-than-light particles.

Not much experimental effort has gone into the problem of searching for faster-than-light particles. One experiment is mentioned in Sec. IV, where the question of such particles being identified in a high-energy collision is also briefly discussed.

In connection with the recent interest in multimass relativistic equations, we note that all the known ones include a spacelike spectrum so that a quantum theory of infinite component wave functions would contain faster-than-light quantum-mechanical particles. From the point of view of the foundations of relativistic quantum mechanics, it is also important to ask whether the scattering states which obey the superposition principle, etc., can be covariantly defined. These questions are discussed in Sec. IV. Section II reviews Feinberg's quantization scheme and demonstrates its lack of relativistic invariance. Section III deals with an alternative method of quantization that is covariant.

II. FEINBERG'S SCALAR FREE-FIELD THEORY AND THE QUESTION OF LORENTZ INVARIANCE

In this section, we will examine in detail the model proposed by Feinberg. Because we believe that it is the presence of certain infinite quantities that destroys the Lorentz invariance of the theory, we will redo his quantization taking special care to use normalizable rather than plane-wave solutions of the wave equation. This will avoid any possible confusion that the infinities appearing might be due to the use of non-normalizable solutions.

We begin by writing down a few kinematic formulas for convenience, referring the reader to BDS and GF for greater detail. We are concerned with particles whose momenta are spacelike, that is, we have

$$p^2=p_0{}^2-\mathbf{p}^2=-\mu^2 \tag{2.1}$$

or, in terms of velocity v ($c=1$),

$$E=p_0=\mu/(v^2-1)^{1/2},\quad \mathbf{p}=\mu\mathbf{v}/(v^2-1)^{1/2}, \tag{2.2}$$

where $|E|$ ranges from 0 to ∞, $|\mathbf{p}|$ from μ to ∞ as $v\to\infty$ to $1(c)$. Note also that $|\mathbf{p}|/p_0=v$ which is always ≥ 1, so $|\mathbf{p}|$ is always greater than E.

Under the "boost" Λ characterized by boost velocity u, we are interested in the transformation of the energy. As $x\to x'=\Lambda x$, $p\to p'=\Lambda p$, where

$$p_0'=\gamma(p_0-\mathbf{p}\cdot\mathbf{u}),\quad \gamma=(1-u^2)^{-1/2}. \tag{2.3}$$

Since $|\mathbf{p}|>p_0$, it is clear that a $\mathbf{u}$ can be chosen to change the sign of the energy. With these kinematic preliminaries, we now proceed to construct the free-field theory, following Feinberg except for the use of normalizable wave functions.

We want local scalar field operators $\phi(x)$ that satisfy the equation

$$(\Box^2-\mu^2)\phi(x)=\left(\frac{\partial^2}{\partial t^2}-\nabla^2-\mu^2\right)\phi(x)=0. \tag{2.4}$$

For this we need a complete set of one-particle positive-energy wave functions $f_\alpha(x)$ satisfying (2.4). It is easiest to write these in terms of a Fourier transform of a complete set of functions in three-dimensional k space in the domain $|\mathbf{k}|\geq\mu$, $\Phi_\alpha(\mathbf{k})$ which satisfy the orthogonality and completeness relations

$$\int\frac{d^3k}{2\omega_k}\theta(|\mathbf{k}|-\mu)\Phi_\alpha{}^*(\mathbf{k})\Phi_\beta(\mathbf{k})=\delta_{\alpha\beta}, \tag{2.5}$$

$$\sum_\alpha\Phi_\alpha{}^*(\mathbf{k})\Phi_\alpha(\mathbf{k}')=2\omega_k\delta(\mathbf{k}-\mathbf{k}')\theta(|\mathbf{k}|-\mu), \tag{2.6}$$

where $\omega_k=+(|\mathbf{k}|^2-\mu^2)^{1/2}$ and the δ function in (2.6) will always be understood to be for functions whose support lies in the domain $|\mathbf{k}|\geq\mu$. A specific set of $\{\Phi_\alpha(\mathbf{k})\}$ for this domain is displayed in the Appendix. If we then define

$$f_\alpha(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2\omega_k}e^{-i\omega_kt}e^{i\mathbf{k}\cdot\mathbf{x}}\theta(|\mathbf{k}|-\mu)\Phi_\alpha(\mathbf{k}), \tag{2.7}$$

then the f's are an orthonormal set satisfying

$$i\int d^3x\, f_\alpha{}^*(x)\overset{\leftrightarrow}{\partial^0}f_\beta(x)=\delta_{\alpha\beta},$$
$$i\int d^3x\, f_\alpha(x)\overset{\leftrightarrow}{\partial^0}f_\beta(x)=0. \tag{2.8}$$

With these one-particle wave functions, the scalar field operator can then be written, following the conventional fashion,

$$\phi(x)=\sum_\alpha[f_\alpha(x)a_\alpha+f_\alpha{}^*(x)a_\alpha{}^\dagger]. \tag{2.9}$$

We are now particularly interested in the transformation properties of the a_α under the operators $U(\Lambda,a)$ implied by (1.1). Choosing first a boost $U(\Lambda)$ characterized by boost velocity u, (1.1) implies

$$\sum_\alpha[f_\alpha(x)U(\Lambda)a_\alpha U^{-1}(\Lambda)+f_\alpha{}^*(x)U(\Lambda)a_\alpha{}^\dagger U^{-1}(\Lambda)]=\sum_\alpha[f_\alpha(\Lambda x)a_\alpha+f_\alpha{}^*(\Lambda x)a_\alpha{}^\dagger]. \tag{2.10}$$

So using (2.8), we have

$$U(\Lambda)a_\alpha U^{-1}(\Lambda)=i\int d^3x\, f_\alpha{}^*(x)\overset{\leftrightarrow}{\partial^0}\phi(\Lambda x). \tag{2.11}$$

Now

$$f_\alpha(\Lambda x)=\frac{1}{(2\pi)^{3/2}}\int_{k_0>0}\frac{d^3k}{2\omega_k}e^{-ik\cdot\Lambda x}\theta(|\mathbf{k}|-\mu)\Phi_\alpha(\mathbf{k})=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k''}{2\omega_{k''}}e^{-ik''\cdot x}\theta(|\mathbf{k}''|-\mu)\Phi_\alpha(\Lambda\mathbf{k}''),\quad(2.12)$$

where

$$k''=\Lambda^{-1}k.$$

We can now separate the latter integral into positive and negative energy parts. Since $k_0''=\gamma(\omega_k+\mathbf{u}\cdot\mathbf{k})$, we can characterize the region of $\mathbf{k}''$ where $k_0''>0$ by all $\mathbf{k}''$ such that

$$\omega_{k''}-\mathbf{u}\cdot\mathbf{k}''>0$$

and the region where $k_0''<0$ by all $\mathbf{k}''$ such that

$$-\omega_{k''}-\mathbf{u}\cdot\mathbf{k}''>0.$$

Then we find that (2.12) can be written

$$f_\alpha(\Lambda x)=\frac{1}{(2\pi)^{3/2}}\int_{(\omega_k-\mathbf{u}\cdot\mathbf{k})>0}\frac{d^3k}{2\omega_k}e^{-i\omega_kt}e^{i\mathbf{k}\cdot\mathbf{x}}\theta(|\mathbf{k}|-\mu)\Phi_\alpha(\Lambda\mathbf{k})+\frac{1}{(2\pi)^{3/2}}\int_{(\omega_k-\mathbf{u}\cdot\mathbf{k})<0}\frac{d^3k}{2\omega_k}e^{i\omega_kt}e^{-i\mathbf{k}\cdot\mathbf{x}}\times\theta(|\mathbf{k}|-\mu)\Phi_\alpha(-\Lambda\mathbf{k}).\quad(2.13)$$

In the second term on the right, we have replaced $\mathbf{k}''$ by $-\mathbf{k}$. Doing the same for $f_\alpha^*(\Lambda x)$ and substituting in (2.11), we get

$$U(\Lambda)a_\alpha U^{-1}(\Lambda)=\sum_\beta a_\beta\int_{(\omega_k-\mathbf{u}\cdot\mathbf{k})>0}\frac{d^3k}{2\omega_k}\theta(|\mathbf{k}|-\mu)\Phi_\alpha^*(\mathbf{k})\Phi_\beta(\Lambda\mathbf{k})+\sum_\beta a_\beta^\dagger\int_{(\omega_k-\mathbf{u}\cdot\mathbf{k})<0}\frac{d^3k}{2\omega_k}\theta(|\mathbf{k}|-\mu)\Phi_\alpha^*(\mathbf{k})\Phi_\beta(-\Lambda\mathbf{k})\quad(2.14)$$

with a similar expression for $Ua_\alpha^\dagger U^{-1}$. This result exhibits the crucial property that a Lorentz transformation does not just take creation operators into creation operators but mixes creation and destruction operators. Thus, as Feinberg remarks, (2.14) then precludes commutation relations between a_α and $a_\beta^\dagger$ and thus forces anticommutation relations. Thus we will take

$$\{a_\alpha,a_\beta^\dagger\}=\delta_{\alpha\beta}\quad(2.15)$$

for the remainder of this section. Now we will examine the energy-momentum operator P_μ which is the generator for the unitary transformations that induce space-time translations. That is, we want a P_μ such that [compare (1.1)]

$$e^{ia^\mu P_\mu}\phi(x)e^{-ia^\mu P_\mu}=\phi(x+a).\quad(2.16)$$

We may construct such an operator in a straightforward fashion by defining an operator $\hat{P}_\mu$ on the one-particle wave functions such that

$$\hat{P}_\mu f_\alpha(x)\equiv\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2\omega_k}k_\mu e^{-i\omega_kt}e^{i\mathbf{k}\cdot\mathbf{x}}\times\Phi_\alpha(\mathbf{k})\theta(|\mathbf{k}|-\mu).\quad(2.17)$$

Then a P_μ satisfying (2.16) can be written as

$$P_\mu=\sum_{\alpha\beta}\hat{P}_{\mu\alpha\beta}a_\alpha^\dagger a_\beta,\quad(2.18)$$

where

$$P_{\mu\alpha\beta}\equiv i\int d^3x\,f_\alpha^*(x)\overleftrightarrow{\partial_0}\hat{P}_\mu f_\beta(x).\quad(2.19)$$

Carrying this out, we get

$$P_\mu=\sum_{\alpha,\beta}\int\frac{d^3k}{\omega_k}\theta(|\mathbf{k}|-\mu)k_\mu\Phi_\alpha^*(\mathbf{k})\Phi_\beta(\mathbf{k})a_\alpha^\dagger a_\beta.\quad(2.20)$$

With this expression, and our transformation law for the a_α, we can calculate how P_μ transforms under a Lorentz transformation and compare it with the known properties of the inhomogeneous Lorentz group. Using (2.14) and (2.20), we get

$$\begin{aligned}U(\Lambda)P_\mu U^{-1}(\Lambda)=\sum_{\alpha,\beta,\beta',\beta''}\Bigg\{\int\frac{d^3k}{2\omega_k}k_\mu\Phi_\alpha^*(\mathbf{k})\Phi_\beta(\mathbf{k})\\
\times\Bigg[\int_{(\omega_{k'}-\mathbf{u}\cdot\mathbf{k}')>0}\frac{d^3k'}{2\omega_{k'}}\int_{(\omega_{k''}-\mathbf{u}\cdot\mathbf{k}'')>0}\frac{d^3k''}{2\omega_{k''}}\Phi_\alpha(\mathbf{k}')\Phi_{\beta'}^*(\Lambda\mathbf{k}')\Phi_\beta^*(\mathbf{k}'')\Phi_{\beta''}(\Lambda\mathbf{k}'')a_{\beta'}^\dagger a_{\beta''}\\
\times\int_{(\omega_{k'}-\mathbf{u}\cdot\mathbf{k}')>0}\frac{d^3k'}{2\omega_{k'}}\int_{(\omega_{k''}-\mathbf{u}\cdot\mathbf{k}'')<0}\frac{d^3k''}{2\omega_{k''}}\Phi_\alpha(\mathbf{k}')\Phi_{\beta'}^*(\Lambda\mathbf{k}')\Phi_\beta^*(\mathbf{k}'')\Phi_{\beta''}(-\Lambda\mathbf{k}'')a_{\beta'}^\dagger a_{\beta''}^\dagger\\
+\int_{(\omega_{k'}-\mathbf{u}\cdot\mathbf{k}')<0}\frac{d^3k'}{2\omega_{k'}}\int_{(\omega_{k''}-\mathbf{u}\cdot\mathbf{k}'')>0}\frac{d^3k''}{2\omega_{k''}}\Phi_\alpha(\mathbf{k}')\Phi_{\beta'}^*(-\Lambda\mathbf{k}')\Phi_\beta^*(\mathbf{k}'')\Phi_{\beta''}(+\Lambda\mathbf{k}'')a_{\beta'}a_{\beta''}\\
+\int_{(\omega_{k'}-\mathbf{u}\cdot\mathbf{k}')<0}\frac{d^3k'}{2\omega_{k'}}\int_{(\omega_{k''}-\mathbf{u}\cdot\mathbf{k}'')<0}\frac{d^3k''}{2\omega_{k''}}\Phi_\alpha(\mathbf{k}')\Phi_{\beta'}^*(-\Lambda\mathbf{k}')\Phi_\beta^*(\mathbf{k}'')\Phi_{\beta''}(-\Lambda\mathbf{k}'')a_{\beta'}a_{\beta''}^\dagger\Bigg]\Bigg\}.\quad(2.21)\end{aligned}$$

[We have not explicitly indicated the factor $\theta(|\mathbf{k}|-\mu)$ since all integrations are restricted to this range.]

To evaluate this horrendous expression, we see that the summations of α and β give $2\omega_k\delta(\mathbf{k}-\mathbf{k}')$ times $2\omega_k\delta(\mathbf{k}''-\mathbf{k})$ in all four terms on the right. Performing the integration over $d^3k/2\omega_k$ gives $2\omega_{k'}\delta(\mathbf{k}'-\mathbf{k}'')$. But the second and third terms have $\mathbf{k}'$ and $\mathbf{k}''$ in nonoverlapping regions, so they vanish; and (2.21) becomes

$$U(\Lambda)P_\mu U^{-1}(\Lambda) = \sum_{\alpha,\beta}\left\{\int_{(\omega_k-\mathbf{u}\cdot\mathbf{k})>0}\frac{d^3k}{2\omega_k}k_\mu\Phi_\alpha^*(\Lambda\mathbf{k})\Phi_\beta(\Lambda\mathbf{k})a_\alpha{}^\dagger a_\beta + \int_{(\omega_k-\mathbf{u}\cdot\mathbf{k})<0}\frac{d^3k}{2\omega_k}k_\mu\Phi_\alpha(-\Lambda\mathbf{k})\Phi_\beta^*(-\Lambda\mathbf{k})a_\alpha a_\beta{}^\dagger\right\}. \tag{2.22}$$

Changing variables to $k'=\Lambda k$ in the first term leaves as the domain of integration all k' such that $(\omega_{k'}+\mathbf{u}\cdot\mathbf{k}')>0$. Changing to $k'=-\Lambda k$ in the second term we have the domain $(\omega_{k'}+\mathbf{u}\cdot\mathbf{k}')<0$, so using $a_\alpha a_\beta{}^\dagger=-a_\beta{}^\dagger a_\alpha+\delta_{\alpha\beta}$ we get

$$U(\Lambda)P_\mu U^{-1}(\Lambda)=\sum_{\alpha,\beta}\left\{\Lambda^{-1}{}_\mu{}^\nu\int\frac{d^3k}{2\omega_k}k_\nu\Phi_\alpha^*(\mathbf{k})\Phi_\beta(\mathbf{k})a_\alpha{}^\dagger a_\beta + \int_{(\omega_k+\mathbf{u}\cdot\mathbf{k})<0}\frac{d^3k}{2\omega_k}\Lambda^{-1}{}_\mu{}^\nu k_\nu\Phi_\alpha^*(\mathbf{k})\Phi_\beta(\mathbf{k})\delta_{\alpha\beta}\right\} \tag{2.23}$$

or

$$U(\Lambda)P_\mu U^{-1}(\Lambda)=\Lambda^{-1}{}_\mu{}^\nu P_\nu+C, \tag{2.24}$$

where the constant C is given as

$$C=\int_{(\omega_k+\mathbf{u}\cdot\mathbf{k})<0}\frac{d^3k}{2\omega_k}\Lambda^{-1}k\sum_\alpha\Phi_\alpha^*(\mathbf{k})\Phi_\alpha(\mathbf{k}) = \delta^3(0)\int_{(\omega_k+\mathbf{u}\cdot\mathbf{k})<0}d^3k\,\Lambda^{-1}k = \infty. \tag{2.25}$$

The fact that P_μ transforms differently than it should under the Lorentz transformation by an infinite constant is the first indication of the lack of Lorentz invariance of the theory. While we can adjust the generators by finite c-number constants without changing the global operator relations of the group, an *infinite* constant strongly implies the nonexistence of the unitary operators $U(\Lambda,a)$. An even clearer demonstration awaits us as we go on to examine the Fock space constructed from these operators. In particular, let us construct a Fock space from a vacuum $|0\rangle$ defined so that

$$a_\alpha|0\rangle=0,\quad \text{for all } \alpha. \tag{2.26}$$

Then the many-particle states are

$$|\alpha,\beta,\cdots\rangle=a_\alpha{}^\dagger a_\beta{}^\dagger\cdots|0\rangle. \tag{2.27}$$

As Feinberg shows, this vacuum is not Lorentz-invariant, which is not necessarily a major problem. Let us, however, as he does, calculate the number of particles in the transformed state, using the number operator

$$N=\sum_\alpha a_\alpha{}^\dagger a_\alpha. \tag{2.28}$$

Then if we define

$$|0\rangle_\Lambda\equiv U(\Lambda)|0\rangle, \tag{2.29}$$

we can calculate

$$N|0\rangle_\Lambda=U(\Lambda)\sum_\alpha U^{-1}(\Lambda)\times a_\alpha{}^\dagger U(\Lambda)U^{-1}(\Lambda)a_\alpha U(\Lambda)|0\rangle. \tag{2.30}$$

Modifying (2.14) for $U^{-1}(\Lambda)$, and substituting similarly to the way used for P_μ, we get

$$\sum_\alpha U^{-1}(\Lambda)a_\alpha{}^\dagger U(\Lambda)U^{-1}(\Lambda)a_\alpha U(\Lambda) = \sum_{\alpha\beta}\int_{(\omega_k+\mathbf{u}\cdot\mathbf{k})>0}\frac{d^3k}{2\omega_k}\Phi_\alpha(\Lambda^{-1}\mathbf{k})\Phi_\beta^*(\Lambda^{-1}\mathbf{k})a_\alpha{}^\dagger a_\beta + \sum_{\alpha\beta}\int_{(\omega_k+\mathbf{u}\cdot\mathbf{k})<0}\frac{d^3k}{2\omega_k}\Phi_\beta^*(-\Lambda^{-1}\mathbf{k})\Phi_\alpha(-\Lambda^{-1}\mathbf{k})a_\beta a_\alpha{}^\dagger. \tag{2.31}$$

Then putting this in (2.30), we get

$$N|0\rangle_\Lambda=n|0\rangle_\Lambda, \tag{2.32}$$

where

$$n=\int_{(\omega_k+\mathbf{u}\cdot\mathbf{k})<0}\frac{d^3k}{2\omega_k}\sum_\alpha\Phi_\alpha^*(-\Lambda^{-1}\mathbf{k})\Phi_\alpha(-\Lambda^{-1}\mathbf{k}) = \delta^3(0)\int_{(\omega_k+\mathbf{u}\cdot\mathbf{k})<0}d^3k = \tfrac{2}{3}\pi\mu^3(\gamma-1)\delta^3(0) = \infty. \tag{2.33}$$

Thus, performing the calculating carefully, we find that there are an infinite number of particles in the Lorentz-transformed vacuum. But since such states, differing by an infinite number of particles, are known to belong to *inequivalent* representations of the commutation relations,[5] they cannot be connected by a unitary operator.[6]

[5] See, for example, S. S. Schweber, and A. S. Wightman, Phys. Rev. **98**, 812, (1955); E. C. G. Sudarshan, J. Math. Phys. **4**, 1029 (1963).

[6] The quantum field theory of faster-than-light particles was first attempted by S. Tanaka, Prog. Theoret. Phys. (Kyoto) **24**, 171 (1960). His theory is also unfortunately not relativistically invariant for the same reasons that Feinberg's work fails to satisfy relativistic invariance. [Compare his equation (3.22).] The problem has been studied more recently by D. Kroff and Z. Fried, Nuovo Cimento **52**, 173 (1967), who seem unaware of the work cited in Ref. 1. In their approach, however, Kroff and Fried do not even attempt to construct a four-vector energy-momentum operator.

Thus the unitary operator $U(\Lambda)$ does not exist, violating the criterion for Lorentz invariance of the theory.[7]

III. ALTERNATIVE QUANTIZATION

The next question to consider is: How may one surmount this difficulty? It should be clear that the noninvariance has come from the attempt to retain the standard positive-energy particle interpretation by associating creation operators with the negative-frequency part of the scalar field. It is because of the well-known fact that this separation is not covariant for spacelike momenta that the unitary operators do not exist.[8] We will avoid this problem by not making that separation. The result will be a Fock space that explicitly allows negative-energy states. We will achieve a physically "sensible" theory by insisting that the only physical quantities are transition amplitudes and a negative-energy in (out) state is physically understood to be a positive-energy out (in) state. This will be now made explicit. (For simplicity, since we are not concerned with proving any rigorous result here, we shall go back to the planewave limits. It is straightforward but tedious to transcribe what follows in terms of the normalizable wave functions used in the previous section.)

We write the non-Hermitian scalar field satisfying (2.4) in terms of destruction operators,

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int d^4k \times\delta(k^2+\mu^2)e^{-ik\cdot x}a(k)\theta(|\mathbf{k}|-\mu),$$

$$\phi^\dagger(x)=\frac{1}{(2\pi)^{3/2}}\int d^4k \times\delta(k^2+\mu^2)e^{ik\cdot x}a^\dagger(k)\theta(|\mathbf{k}|-\mu). \quad (3.1)$$

As before, we wish the existence of $U(\Lambda,a)$ so as to satisfy Eq. (1.1):

$$\begin{aligned}U(\Lambda,a)\phi(x)U^{-1}(\Lambda,a) &=\phi(\Lambda x+a)\\ &=\frac{1}{(2\pi)^{2/3}}\int d^4k\ \delta(k^2+\mu^2)e^{-ik\cdot a}e^{ik\cdot\Lambda x}a(k)\theta(|\mathbf{k}|-\mu)\\ &=\frac{1}{(2\pi)^{3/2}}\int d^4k\ \delta(k^2+\mu^2)e^{-i\Lambda k\cdot a}e^{-ik\cdot x}a(\Lambda k)\theta(|\mathbf{k}|-\mu),\end{aligned} \quad (3.2)$$

which implies that the a transforms as follows:

$$U(\Lambda,a)a(k)U^{-1}(\Lambda,a)=e^{-i\Lambda k\cdot a}a(\Lambda k). \quad (3.3)$$

[7] The noninvariance can already be seen in the work of Feinberg. He indicates that as long as you quantize "in a box", the constants occurring are finite [contrast our (2.24), (2.25), and (2.33)]. But we insist that only in the infinite-volume limit is the requirement for Lorentz invariance mathematically precise, and then the infinities are unavoidable.

Since now the boosts do not mix creation and destruction operators, we are not forced to use anticommutation relations as in Sec. II. In fact, we are free to use either commutation or anticommutation relations. For convenience we will choose the former. Then (3.3) is consistent with[9]

$$\delta(k^2+\mu^2)\delta(k'^2+\mu^2)[a(k),a^\dagger(k')] =\delta(k^2+\mu^2)\delta^4(k-k'), \quad (3.4)$$

where, for example, the energy-momentum operator is then explicitly

$$P_\mu=\int d^4k\, k_\mu a^\dagger(k)a(k)\delta(k^2+\mu^2)\theta(|\mathbf{k}|-\mu). \quad (3.5)$$

Similarly, the generator for homogeneous Lorentz transformations is then

$$M_{\mu\nu}=-\frac{1}{2}\int d^4k\ \delta(k^2+\mu^2) \times\theta(|\mathbf{k}|-\mu)\left(k_\mu\frac{\partial}{\partial k^\nu}-k_\nu\frac{\partial}{\partial k^\mu}\right)a^\dagger(k)a(k). \quad (3.6)$$

We can then proceed to calculate the commutation relations of the ϕ's

$$[\phi(x),\phi^\dagger(x')]=\frac{1}{(2\pi)^3}\int\frac{d^3k}{\omega_k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\cos\omega_k(t-t'), \quad (3.7)$$

which gives at equal times

$$[\phi(x),\phi^\dagger(x')]_{x_0=x_0'}=\frac{1}{(2\pi)^3}\int\frac{d^3k}{\omega_k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \quad (3.8)$$

This shows clearly that for this theory, ϕ and $\phi^\dagger$ do not commute for spacelike separations, a major difference from the usual field theory. We then construct a Fock space with an invariant vacuum $|0\rangle$ in the usual manner:

$$a(k)|0\rangle=0,\quad \text{for all } k$$
$$|k_1,k_2,\cdots\rangle\equiv a^\dagger(k_1)a^\dagger(k_2)\cdots|0\rangle. \quad (3.9)$$

Since the $a^\dagger(k)$ can create negative-energy states, we need a physical *postulate* which is stated as follows.

[8] I. E. Segal, in considering representations of the proper Lorentz group on a complex Hilbert space for the scalar field, where the complex structure is connected to the separation of positive and negative frequencies, states: "When m is not real, the transformations are definitely not unitary, a situation closely related to what is referred to in the theoretical physics literature as the impossibility of making a covariant separation of a field with imaginary mass into positive- and negative-frequency parts". Irving E. Segal, *Mathematical Problems of Relativistic Physics* (American Mathematical Society, Providence, R. I., 1963), p. 34.

[9] We could rewrite everything in terms of $a_+(\mathbf{k})\equiv a(k)/(2\omega_k)^{1/2}$, $k_0=+\omega_k$ and $a_-(\mathbf{k})\equiv a(k)/(2\omega_k)^{1/2}$, $k_0=-\omega_k$, in which case, $[a_\pm(\mathbf{k}),a_\pm^\dagger(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}')$, which are the more familiar relations. For simplicity we retain the covariant notation.

Physical postulate: The only physically relevant quantities are the transition amplitudes. Any transition amplitude is to be interpreted as the amplitude for transition between positive-energy particles where all negative-energy particles (of momentum p) in the initial state are interpreted as outgoing positive-energy particles (of momentum $-p$); similarly, for negative-energy particles in the final state.

In other words, the calculated amplitude

$$T=\langle p_1'\cdots p_n'|T|p_1\cdots p_n\rangle$$

is to be interpreted as the amplitude for $\{\bar{p}_i\}\rightarrow\{\bar{p}_i'\}$, where the bars mean take all negative-energy p's and put them in $\{\bar{p}_i'\}$ as $-p$ and vice versa.[10]

Remarks

(1) This physical interpretation has allowed us to keep the understanding of negative-energy particles as positive-energy particles moving backward in time without requiring the noncovariant separation for the underlying field theory that destroyed Lorentz invariance in the Feinberg scheme.

(2) Although the field $\phi(x)$ has the property that it annihilates the vacuum, the usual theorems that would require it to vanish do not apply because of the lack of spacelike commutativity.[11]

(3) The physical postulate gives us s-u crossing symmetry as a consequence of Lorentz invariance. Consider the elastic scattering of a particle of type I ($v<c$) (momenta p_1, p_2) with a particle of type III ($v>c$) (q_1, q_2), $p_1+q_1\rightarrow p_2+q_2$ with all energies positive.

Now we consider the amplitude in a new frame where $q_{10}'<0$, $q_{20}'<0$. These amplitudes are equal by Lorentz invariance, yet our physical postulate gives us

$$T^{\rm phys}(p_1,q_1;p_2q_2)=T^{\rm phys}(p_1',-q_2';p_2',-q_1'), \quad (3.10)$$

but in terms of the invariant variables,

$$s=(p_1+q_1)^2=u',$$
$$t=(q_1-q_2)^2=t',$$
$$u=(p_1-q_2)^2=s',$$

so we have

$$T(s,t,u)=T(u,t,s) \quad (3.11)$$

as a consequence of Lorentz invariance.

(4) Because we do not know how to get the commutation relations from a Lagrangian and because the lack of spacelike commutativity makes the time-ordered (or retarded) product noninvariant, the problem of proceeding to a consistent interacting theory seems far from easy to solve.

IV. DISCUSSION

The study presented in this paper has two aims: first, to point out that the quantum theory of faster-than-light particles according to the scheme of Feinberg is not compatible with Lorentz invariance [in the sense of (1.1)]; and to outline a new quantum-theory scheme which realizes the physical reinterpretation discussed in BDS. The construction of a theory of interacting fields associated with such particles is a considerably harder problem. It is not attempted in this paper. We are content to draw attention to the novelty and consistency of the quantum theory of faster-than-light particles.

While we have confined our attention to "single mass" fields, similar considerations apply to the quantization of most "infinite component" wave equations with a mass spectrum. These spectra usually include spacelike solutions and the quantum theory of these fields would exhibit the novel features encountered in the single-mass case. It is known that nontrivial local fields which annihilate the vacuum exist.[12] Todorov has discussed[12] some of the related problems connected with the quantization of infinite-component wave fields.

It is our belief that the *usual* objections to the possibility of faster-than-light particles in relativistic quantum theory can be overcome. There may be hitherto unforeseen difficulties of a fundamental nature which may make it impossible to entertain such entities. However, no reason really exists for not investigating the possible existence of them experimentally. The only experiment of this kind known to the authors is due to a group at the Nobel Institute, who searched for such particles in the emissions of an intense β emitter.[13] They made the arbitrary but plausible assumption that such

[10] An exactly soluble static theory (the "charged scalar theory") in which crossing symmetry was obeyed and which involved both negative- and positive-energy particles was constructed by E. C. G. Sudarshan, in *Brandeis University Lectures in Theoretical Physics* (W. A. Benjamin, Inc., New York, 1961). The scattering amplitudes in that theory were to be reinterpreted using this physical postulate. In the present work, it is to be noted that the reinterpretation contained in our physical postulate is to be made *only* on the transition amplitude. If it were to be extended to the S matrix, contradictions would follow. This is because of the fact that $S=1+2\pi iT$, and the reinterpretation should not be applied to the unit operator. To see this, consider, for example, the matrix element of the creation operator for a positive-energy particle ($k_0=+\omega,\mathbf{k}$) between a one-particle state and the vacuum [$\omega=+(\mathbf{k}^2-\mu^2)^{1/2}$]:

$$\langle+\omega',\mathbf{k}'|a_{\rm in}^\dagger(\omega,\mathbf{k})|0\rangle=\delta(\mathbf{k}-\mathbf{k}').$$

(Because of the stability of the physical one-particle state, we need no "in" or "out" designations on the one-particle states.) Yet had we attempted to apply the reinterpretation to the whole S matrix, we would require that the above matrix element be equal to the matrix element of the destruction operator for negative energy ($k_0=-\omega$, $-\mathbf{k}$) between the same states:

$$\langle\omega',\mathbf{k}'|a_{\rm in}(-\omega,-\mathbf{k})|0\rangle.$$

However, the later matrix element is equal to zero, not $\delta(\mathbf{k}-\mathbf{k}')$. We avoid this type of contradiction by applying our reinterpretation only to the transition part of the S matrix.

[11] R. F. Streater and A. S. Wightman, *TCP, Spin, Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964).

[12] D. Tz. Stoyanov and I. T. Todorov, Trieste Report No. IC/67/58, 1967 (unpublished); I. T. Todorov, in *Proceedings of the International Conference on Particles and Fields, Rochester, 1967* (Interscience Publishers, Inc., New York, 1967).

[13] P. Erman (private communication).

particles have a mass whose absolute value is equal to the electron mass. They could place an upper limit of 10^{-4} for the frequency of faster-than-light particles *of this (imaginary) mass*. More systematic experiments of such a nature would be very desirable.

In a high-energy reaction as recorded, say, in a bubble chamber, such an event would look like an apparent instability of a known stable particle (like the proton). As discussed in detail in BDS, the elastic scattering of a faster-than-light particle by an ordinary (slower-than-light) particle will appear, in a suitable reference frame, as the instability of the ordinary particle of a certain four-momentum into the same particle with another four-momentum and a pair of faster-than-light particles. In the normal course of events, such an event would be discarded since it would be said not to satisfy usual kinematic criteria.

From the theoretical point of view, an interesting possibility about the structure of relativistic quantum theory is revealed by the quantization scheme that we have discussed. In quantum theory the "principle of superposition of states" must hold, and hence the states must form a linear manifold.[14] Consequently, the scattering amplitudes must factorize; it is the scalar product of an initial "in" state and a final "out" state. On the other hand, by definition, in a relativistic quantum theory the scattering amplitudes must be relativistically invariant. It is natural to demand, in the normal course of things, that the factorization of the scattering amplitude into the "in" and "out" states be relativistically invariant. It is also natural to demand that the particles that we observe, must have positive energies only. We find that these requirements are not all compatible; the notion of only positive-energy particles is not compatible with relativistic invariance. We could now ask whether the following question is relevant for relativistic quantum field theory in general. Is it possible that the notion of a physical state itself is not expressible in a relativistically invariant fashion, though the physical predictions like mass levels and transition probabilities are all relativistically invariant? We do not know the answer. But we remark that this is akin to the question: Is it possible that relativistic formulation of a quantum field theory is not expressible in a local form in a theory with positive definite metric?

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[14] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, London, 1958), p. 14.

APPENDIX: A COMPLETE SET OF FUNCTIONS $\Phi_\alpha(\mathbf{k})$ IN THE REGION $|\mathbf{k}|>\mu$

We wish to have a set $\{\Phi_\alpha(\mathbf{k})\}$ such that

$$\int \frac{d^3k}{2\omega_k}\theta(|\mathbf{k}|-\mu)\Phi_\alpha{}^*(\mathbf{k})\Phi_\beta(\mathbf{k})=\delta_{\alpha\beta}, \qquad \text{(A1)}$$

and

$$\theta(|\mathbf{k}|-\mu)\theta(|\mathbf{k}'|-\mu)\sum_\alpha \Phi_\alpha{}^*(\mathbf{k})\Phi_\alpha(\mathbf{k}') =2\omega_k\delta(\mathbf{k}-\mathbf{k}')\theta(|\mathbf{k}|-\mu). \qquad \text{(A2)}$$

Let us use spherical coordinates: Let $k=|\mathbf{k}|$ and Ω be the solid angle. Consider

$$\psi_{nlm}(\mathbf{k})=f_n(k)Y_m{}^l(\Omega), \qquad \text{(A3)}$$

where the $Y_m{}^l$ satisfy

$$\int d\Omega\, Y_{m'}{}^{l'*}(\Omega)Y_m{}^l(\Omega)=\delta_{ll'}\delta_{mm'}. \qquad \text{(A4)}$$

Putting (A3) and (A4) into (A1) shows that we need $f_n(k)$ satisfying

$$\int f_n{}^*(k)f_{n'}(k)\frac{k^2dk}{\omega_k}\theta(k-\mu)=\delta_{nn'}, \qquad \text{(A5)}$$

where $\omega_k=(k^2-\mu^2)^{1/2}$.

Let us consider a complete orthonormal set on the real line from 0 to ∞, e.g., the Laguerre polynomials $Q_\nu(x)$[15] such that

$$\int_0^\infty dx\, e^{-x}Q_\nu(x)Q_{\nu'}(x)=\delta_{\nu\nu'}. \qquad \text{(A6)}$$

Changing variables to

$$k=\mu e^x, \quad x=\ln(k/\mu),$$

we get

$$\int_\mu^\infty \mu\frac{dk}{k^2}Q_\nu\left(\ln\frac{k}{\mu}\right)Q_{\nu'}\left(\ln\frac{k}{\mu}\right)=\delta_{\nu\nu'}. \qquad \text{(A7)}$$

If we define

$$f_n(k)=(2\mu\omega_k)^{1/2}Q_n[\ln(k/\mu)]/k^2,$$

(A7) becomes

$$\int_0^\infty \frac{k^2dk}{2\omega_k}f_n(k)f_{n'}(k)\theta(k-\mu)=\delta_{nn'},$$

which is the required relation (A5). Thus, if we write

$$\Phi_\alpha(\mathbf{k})=\sqrt{2}\mu\omega_kQ_n(\ln(k/\mu))Y_m{}^l(\hat{k}), \quad |\mathbf{k}|\geq\mu$$

we have the desired orthonormal set.

[15] R. Courant and P. Hilbert, *Methoden der Mathematischen Physik* (Springer, Berlin, 1931), Vol. 1, p. 80.