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f-Oscillators and Nonlinear Coherent States

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Abstract

The notion of *f*-oscillators generalizing *q*-oscillators is introduced. For classical and quantum cases, an interpretation of the *f*-oscillator is provided as corresponding to a special nonlinearity of vibration for which the frequency of oscillation depends on the energy. The *f*-coherent states (nonlinear coherent states) generalizing *q*-coherent states are constructed. Applied to quantum optics, photon distribution function, photon number means, and dispersions are calculated for the *f*-coherent states as well as the Wigner function and *Q*-function. As an example, it is shown how this nonlinearity may affect the Planck distribution formula.

1. Introduction

In quantum physics, harmonic oscillators are synonymous with creation and annihilation operators. For this reason, in the first attempts to realize Hopf algebras (quantum groups) in terms of creation and annihilation operators (a generalization of the Jordan–Schwinger map) the resulting oscillators were named *q*-oscillators. This pervading property of the oscillator formalism in many physical situations has induced a lot of interest in looking for physical consequences, where honest oscillators are replaced by *q*-deformed ones (partition functions, field theories, nonlinear optics, etc.).

Coherent states, defined through creation and annihilation operators, provide us with a beautiful connection between quantum and classical oscillators. The notion of coherent states [1–3] permitted the use of language and intuition developed from the study of the classical mechanics of harmonic oscillators in order to treat their quantum counterpart, because the trajectory of the center of quantum coherent packet is the same as the classical trajectory and the width of the packet is the minimal possible one in the frame of Heisenberg uncertainty relation [4]. The notion of coherent state turned out to be appropriate also to describe simple quantum systems like spin [5] and cyclotron motion of a charge in magnetic field [6]. On the other hand, the notion of the quantum *q*-oscillator [7, 8] was interpreted [9, 10] as a nonlinear oscillator with a very specific type of nonlinearity, in which the frequency of vibration depends on the energy of these vibrations through the hyperbolic cosine function containing a parameter of nonlinearity.

This interpretation of *q*-oscillators becomes obvious if one used the classical counterpart of the original quantum *q*-oscillators. This observation suggests that there might exist other types of nonlinearity for which the frequency of oscillation varies with the amplitude in a manner different from the cosh-dependence; we will label this dependence by a function *f*. Such classical oscillators (and their quantum

partners) may be called *f*-oscillators [9]. It is interesting to consider the statistical mechanics of a gas of deformed oscillators (free energy, partition function behaviour) and to compare it with the one associated with standard oscillators.

The problems have obvious counterparts in quantum mechanics. Here, the role of the phase diagram is played by the eigenstates of the Hamiltonian. For stationary systems, one could consider such changes of the Hamiltonian, which is an integral of motion, that produce new Hamiltonian which is some function of the initial one. Then if there are no degeneracies in the spectra for the initial Hamiltonian, the eigenstates of the new Hamiltonian coincide with the old ones. But for the new system the energy spectra are different. This produces time evolution of the phase factors of the eigenstates such that these vary with different velocities in complete analogy to the classical motion of the corresponding deformed classical systems, moving along their trajectories in phase space with reparametrized velocities.

In fact, in more general situations the new quantum systems having the same stationary eigenstates as the initial ones possess the new Hamiltonian which is a function of the usually commuting time independent integrals of motion (complete set of observables). This is the analogue of the new classical Hamiltonian which was deformed using reparametrizations depending on all available classical invariants which are in involution.

The aim of our paper is to study the behaviour of classical and quantum systems belonging to the subclass described above, and to clarify the role of nonlinearities corresponding to these systems. This goal is motivated by the fact that *q*-oscillator belongs to the system of the subclass corresponding to the specific *q*-nonlinearity [9].

In classical case, we consider simple linear systems (oscillators) and their deformations producing nonlinear integrable systems as well as symmetries based on the nonlinear noncanonical transform of the conjugate variables preserving the vectorfield. In the quantum counterpart, we study such systems which differ in Hamiltonian but have the same set of eigenstates. In these cases, we analyze the possibility of extending the notion of coherent states of usual harmonic oscillator to the case of *f*-oscillators. Algebraic extensions of the notion of *q*-oscillator coherent states have been discussed in [11, 12] and applications in [13]. The particular case of *f*-coherent states called also as nonlinear coherent states for the function *f* expressed in terms of Laguerre polynomials was shown to be created for trapped ion in [14]. Shortly *f*-oscillators were discussed in [15].

We will study some physical consequences of the existence of *f*-coherent states like the change of the particle distribution function, the possibility of having super- or sub-Poissonian statistics, influence of *f*-nonlinearity in the black body radiation formula with the particular example of the *q*-oscillators. As a particular example we will apply these results to *q*-nonlinear systems (*q*-oscillators) and will show that for *q*-coherent states there exists sub-Poissonian statistics which mean that *q*-nonlinearity of their fields decreases the fluctuation of the particle number in a *q*-coherent state.

The next three Sections (2, 3 and 4) illustrate in detail the situation in classical mechanics with the examples of one-dimensional, two-dimensional and three-dimensional oscillators. The introduction of the one-dimensional quantum analogues (the quantum *f*-oscillator) is given in Section 5 and in the following one (Section 6) some algebraic relations are shown for the operators describing them. Then the eigenstates of a one-mode *f*-annihilation operator (*f*-coherent states) are considered in the Fock space (Section 7) as well as in different representations (Wigner and Husimi) in Section 8. Properties of such states are studied in Sections 9, 10 and 11, namely, their evolution and completeness relations with a remark on the Stone-von Neumann theorem, with the main end of underlining that they are not always coherent in the ordinary sense [16]. Extensions to many modes are shown to be possible not only when they are independent of each other but also when there is a nonlinear coupling among them (Section 12). The current setup of quantum mechanics having been maintained, an application of the above objects in quantum optics is given in the last Sections 13, 14 and 15 obtaining new photon distributions, squeezing in the nonlinear coherent states, correlation of quadratures, and Planck formula deformation.

2. Construction of Nonlinear Systems from Linear Systems

It was shown, that in the classical limit the one-dimensional *q*-oscillator is represented by a reparametrized oscillator [9]. The reparametrization is provided by a constant of motion and the associated differential equations exhibit a special kind of nonlinearity. Here, we would like to consider these systems from a general viewpoint: nonlinear systems as “reparametrized” linear ones.

A linear dynamical system, say on \mathbf{R}^n , with coordinates $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is described for any $n \times n$ matrix A by the differential equation

$$\dot{x}_j = A_j^k x_k; \quad A_j^k \in \mathbf{R}, \quad (1)$$

with solutions

$$\mathbf{x}(t) = e^{tA} \mathbf{x}_0; \quad \mathbf{x}_0 = \mathbf{x}(t_0).$$

To obtain new (nonlinear) systems from the above, we replace the globally constant matrix A with a matrix valued function $B(\mathbf{x})$ and write the equation

$$\dot{x}_j = B_j^k(\mathbf{x}) x_k \quad (2)$$

with

$$\frac{d}{dt} B_j^k = x_n B_m^n(\mathbf{x}) \frac{\partial}{\partial x_m} B_j^k = 0.$$

It means that the matrix elements of the matrix B are integrals of motion for eq. (2). For this new system, we can write the solutions as

$$\mathbf{x}(t) = \exp [tB(\mathbf{x}_0)] \mathbf{x}_0.$$

For each initial condition, eq. (2) reduces to eq. (1) and a particular case is the orbit-dependent time reparametrization.

An example of such a deformed equation in field theory has been shown in [17] where a physical parameter was made dependent on Cauchy data. A particular family of such systems is when

$$B_j^i(\mathbf{x}) = f(\mathbf{x}) A_j^i,$$

with f any constant of motion of the system. For instance, let us consider the three-dimensional isotropic harmonic oscillators

$$\begin{aligned} \dot{\mathbf{x}} &= \omega \mathbf{y}; \\ \dot{\mathbf{y}} &= -\omega \mathbf{x}; \quad \omega \in \mathbf{R}. \end{aligned} \quad (3)$$

All constants of motion of this system are functions of $b_{ij}(x^i y^j - x^j y^i)$ and $a_{ij}(y^i y^j + x^i x^j)$. If we set new frequency $\Omega = \Omega[(b_{ij}(x^i y^j - x^j y^i), a_{ij}(x_i x_j + y_i y_j))]$ and make the replacement $\omega \rightarrow \Omega$ in eq. (3), we get a nonlinear system. Solution to this system is given by

$$\begin{pmatrix} x_j(t) \\ y_j(t) \end{pmatrix} = \begin{pmatrix} \cos t\Omega(x_0, y_0) & \sin t\Omega(x_0, y_0) \\ -\sin t\Omega(x_0, y_0) & \cos t\Omega(x_0, y_0) \end{pmatrix} \begin{pmatrix} x_j(0) \\ y_j(0) \end{pmatrix}. \quad (4)$$

For each choice of the functional dependence of Ω on the constants of motion, we get a foliation of the carrier space \mathbf{R}^6 with leaves given by

$$\Sigma_\lambda = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^6 : \Omega(\mathbf{x}, \mathbf{y}) = \lambda\}. \quad (5)$$

For each initial condition $\mathbf{x}_0, \mathbf{y}_0 \in \Sigma_\lambda$, the oscillatory motion has the same frequency for each one of the coordinates.

It is however possible to start more generally with

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{x}_3 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_1 & 0 & 0 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 & 0 & 0 \\ 0 & 0 & -\omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 \\ 0 & 0 & 0 & 0 & -\omega_3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix}, \quad (6)$$

and “nonlinearize” the system by making different choices of the constants of motion for the different frequencies. For instance, we could take

$$\begin{aligned} \Omega_1 &= f_1(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2); \\ \Omega_2 &= f_2(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2); \\ \Omega_3 &= f_3(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2). \end{aligned} \quad (7)$$

Now each mode will be “dynamically coupled” to the others in a different way.

If Γ is the dynamical vectorfield, it should be noticed that when we replaced Γ with $f\Gamma$, i.e., we reparametrize Γ by a constant of motion, we get the same phase portrait for both vectorfields. If we reparametrize different modes of vibration differently, we change the phase portrait of the new system with respect to the original one of Γ . Thus, our “nonlin-

earization" procedure is more than a parametrization of the original linear system.

3. Deformation of Linear Hamiltonian Systems and Symmetries

The above variety of choices has to be restricted if we start with a system admitting a Hamiltonian description and want to preserve the Hamiltonian character after we "reparametrize" it. The simplest choice to get a reparametrized Hamiltonian system is the following: Start with a linear Hamiltonian system

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \begin{pmatrix} \partial H / \partial y_i \\ -\partial H / \partial x_i \end{pmatrix}, \quad (8)$$

where

$$H = x_i A_j^i y^j + B_{ij} y^i y^j + C^{ij} x_i x_j$$

(the coordinates y_i are the momentum components) and consider the new system associated with the Hamiltonian $\tilde{H} = f(H)$. We get the new nonlinear equations of motion given by

$$\begin{aligned} \dot{x}_i &= f'(H) \frac{\partial H}{\partial y_i}; \\ \dot{y}_i &= -f'(H) \frac{\partial H}{\partial x_i} \end{aligned} \quad (9)$$

with $f'(H) = \partial f / \partial H$. This system can be explicitly integrated as any linear system can be. This has already been demonstrated in [9].

To make contact with [9], we also show, however, a different path to nonlinear Hamiltonian systems obtained from linear systems, from linear symmetries to nonlinear ones. If A is any $n \times n$ matrix, we denote by X_A the associated vectorfield setting $X_A = x_i A_j^i \partial / \partial x_j$. For a dynamical system, we prefer using the notation Γ instead of X_A . The Lie algebra of symmetries for Γ contains all linear vectorfields X_B such that $[B, A] = 0$; this follows trivially from $[X_A, X_B] = X_{[A, B]}$. In the case of the isotropic harmonic oscillator, we find that for the m -dimensional oscillator the symmetry algebra is $gl(m, C)$. Now we "reparametrize" any one of X_B by using constants of motion for Γ . Had we started with matrix \mathcal{B} generating symmetry transformations instead of infinitesimal ones, i.e., $\mathcal{B}^{-1} A \mathcal{B} = A$ (diffeomorphism in the differential geometric language), we could generate nonlinear changes of coordinates by following the same idea of "reparametrization". Generally, this procedure will turn canonical transformations into nonlinear noncanonical transformations.

We deal with the two-dimensional isotropic harmonic oscillator on \mathbf{R}^4

$$\dot{x}_i = y_i; \quad \dot{y}_i = -x_i; \quad i = 1, 2 \quad (10)$$

and define the change of coordinates

$$\begin{aligned} q_i &= f_i(H_i) x_i; \\ p_i &= f_i(H_i) y_i \end{aligned} \quad (11)$$

with $H_i = x_i^2 + y_i^2$; $f_i: \mathbf{R} \rightarrow \mathbf{R}$ and no summation on i . In these new coordinates, the equations of motion have still a

linear form given by

$$\begin{aligned} \dot{q}_i &= p_i; \\ p_i &= -q_i. \end{aligned} \quad (12)$$

Without loss of generality, we consider only one degree of freedom. In both coordinate systems, the dynamics admits a Hamiltonian description given respectively by

$$(a) \quad \{x, y\} = 1; \quad H = \frac{1}{2}(x^2 + y^2); \quad \frac{d}{dt} f = \{H, f\} \quad (13)$$

and

$$(b) \quad \{p, q\} = 1; \quad \tilde{H} = \frac{1}{2}(p^2 + q^2); \quad \frac{d}{dt} f = \{\tilde{H}, f\}. \quad (14)$$

However, using the Poisson bracket of (a) to compute the Poisson bracket in (b) for (p, q) as functions of (x, y) given by the system of eqs (11) we find

$$\{f(H)x, f(H)y\} = \frac{d}{dH} (Hf^2(H)) \neq 1,$$

i.e., the nonlinear transformation we have performed is non-canonical. The noncanonical property is there even if $F(H)$ is a constant f . To obtain the same right hand side of the Poisson bracket in (b), it is necessary to use for the independent variables (x, y) the new Poisson bracket

$$\{x, y\}' = \left[\frac{d}{dH} (Hf^2(H)) \right]^{-1}; \quad (15)$$

with it and the Hamiltonian function

$$H'(x, y) = \frac{1}{2}(x^2 + y^2) f^2(x^2 + y^2), \quad (16)$$

we have another Hamiltonian description in the coordinate (x, y) for Γ .

This can be written in symplectic terms as

$$i_r \omega = -dH; \quad \omega = dx \wedge dy \quad (17)$$

and

$$i_r \omega' = -dH'; \quad \omega' = \frac{d}{dH} (Hf^2(H)) dx \wedge dy. \quad (18)$$

Because both H and \tilde{H} are Hamiltonian functions for Γ , we can use H expressed as function of (p, q) through the inverse of (11) and the bracket $\{p, q\} = 1$ to introduce a new vectorfield. Otherwise, we can use

$$\{x, y\}' = \left[\frac{d}{dH} (Hf^2(H)) \right]^{-1}$$

and the Hamiltonian

$$H = \frac{1}{2}(x^2 + y^2).$$

In both cases, we get a "reparametrized" version of harmonic oscillator. In comparing the two procedures, it is clear that in one approach we have used the same Poisson Brackets ("commutation relations") as before with a new Hamiltonian which is a function of the standard quadratic Hamiltonian. In the second approach, we use the same Hamiltonian function but the "deformed" Poisson brackets

(“deformed commutation relations”). It is this second viewpoint which allows the connection with *q*-oscillators, while the first one has suggested the interpretation of the classical counterpart as a nonlinear *f*-oscillator. However, it should be clear that both views are legitimate and both of them give rise to a nonlinear dynamics out of a linear one by taking recourse to a “reparametrization.”

Having in mind the passage to quantum mechanics, we prefer turning now to complex variables in \mathbf{R}^4 and defining complex coordinates

$$\begin{aligned} \alpha_k &= 2^{-1/2}(x_k + iy_k); \\ \alpha_k^* &= 2^{-1/2}(x_k - iy_k); \quad k = 1, 2. \end{aligned} \tag{19}$$

The complex structure in \mathbf{R}^4 is given by the matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

satisfying $J^2 = -1$. This matrix defines a complex structure commuting with the dynamical evolution associated with the isotropic harmonic oscillator

$$\Gamma = i \left(\alpha_k \frac{\partial}{\partial \alpha_k} - \alpha_k^* \frac{\partial}{\partial \alpha_k^*} \right). \tag{20}$$

The equations of motion are

$$\begin{aligned} \dot{\alpha} &= -i\alpha; \\ \dot{\alpha}^* &= i\alpha^*. \end{aligned} \tag{21}$$

For the (p, q) coordinates, we have new complex coordinates

$$\begin{aligned} \xi_k &= 2^{-1/2}(q_k + ip_k); \\ \xi_k^* &= 2^{-1/2}(q_k - ip_k) \end{aligned} \tag{22}$$

and, setting

$$n_k = \alpha_k \alpha_k^*,$$

we find

$$\begin{aligned} \xi_k &= f_k(n_k)\alpha_k; \\ \xi_k^* &= f_k(n_k)\alpha_k^*. \end{aligned} \tag{23}$$

This transformation is not “analytic” (we notice that analyticity depends on the complex structure and here we have two alternative complex structures compatible with the dynamics).

The equations of motion for these new variables are the same as (21) as it was seen earlier, i.e., eqs (21) are form invariant under the transformation (23). In these complex coordinates, if we take the new point of view, stemming from quantum mechanics, which takes the Hamiltonian as a primitive concept for the dynamics, we are naturally led to consider the following two Hamiltonians

$$H_1 = \frac{1}{2} \sum_k \alpha_k \alpha_k^* \tag{24}$$

in the α -coordinates, and

$$H_2 = \frac{1}{2} \sum_k \xi_k \xi_k^* \tag{25}$$

in the ξ -coordinates. To compare, we express them in the same variables to find

$$H_1 = \frac{1}{2} \sum_k n_k \tag{26}$$

and

$$H_2 = \frac{1}{2} \sum_k f_k^2(n_k)n_k. \tag{27}$$

We now use the same bracket for them, say

$$\{\alpha_k, \alpha_j^*\} = -i\delta_{kj}. \tag{28}$$

We obtain two different dynamical systems, where the one associated with H_2 is not even necessarily isotropic. The evolution goes from a periodic orbit to an orbit whose closure is a two-dimensional torus, i.e., the associated systems are completely different. Of course, there is no contradiction. Indeed, to have the same dynamics using H_2 we should use different Poisson brackets, as it has been seen earlier.

For the nonlinear oscillator obtained by means of the deformation function f , the equations of motion are

$$\begin{aligned} \dot{\alpha} &= -i \frac{d}{dn} (nf^2(n))\alpha; \\ \dot{\alpha}^* &= i \frac{d}{dn} (nf^2(n))\alpha^* \end{aligned} \tag{29}$$

and are not invariant under the transformation (23), which indeed gives the equations of motion for another system of our class but with a different deformation function. They admit different Hamiltonian descriptions. For instance,

$$1. H = \alpha\alpha^*; \quad \{\alpha, \alpha^*\} = i \frac{d}{dn} (nf^2(n));$$

$$\omega = \left(\frac{d}{dn} (nf^2(n)) \right)^{-1} d\alpha \wedge d\alpha^*,$$

$$2. H = nf^2(n); \quad \{\alpha, \alpha^*\} = -i; \quad \omega = d\alpha \wedge \alpha^*.$$

It is now clear, that in the quantum picture these complex coordinates will be replaced by creation and annihilation operators. Therefore for the corresponding commutators, we can repeat what we have said for the Poisson bracket.

4. Some Examples

The first example we consider concerns the classical one-dimensional *q*-oscillator [9] where

$$\begin{aligned} \xi &= \sqrt{\frac{\sinh \lambda \alpha \alpha^*}{\alpha \alpha^* \sinh \lambda}} \alpha; \\ \xi^* &= \sqrt{\frac{\sinh \lambda \alpha \alpha^*}{\alpha \alpha^* \sinh \lambda}} \alpha^*. \quad \lambda \in \mathbf{R} \end{aligned} \tag{31}$$

The Poisson bracket for the new variables can be computed using (28) and expressed in terms of themselves, by the use of the map $(\xi, \xi^*) \rightarrow (\alpha, \alpha^*)$, the inverse of (23) (which in this case exists). One then obtains

$$\{\xi, \xi^*\} = -i \frac{\lambda}{\sinh \lambda} \sqrt{1 + |\xi|^4 (\sinh \lambda)^2}, \tag{32}$$

so that we can consider a new system described by such variables with Hamiltonian function

$$H(\xi, \xi^*) = \xi \xi^*. \quad (33)$$

The equations of motion then are

$$\dot{\xi} = -i \frac{\lambda}{\sinh \lambda} \sqrt{1 + |\xi|^4 (\sinh \lambda)^2} \xi \quad (34)$$

(and its complex conjugate) with solutions

$$\xi(t) = \xi(0) \exp \left[\frac{-it\lambda}{\sinh \lambda} \sqrt{1 + |\xi(0)|^4 (\sinh \lambda)^2} \right] \quad (35)$$

(and its conjugate). Such a system can be rewritten, of course, in terms of (α, α^*) variables: in these coordinates, the original Poisson bracket is unchanged while the Hamiltonian function is

$$H(\alpha, \alpha^*) = \frac{\sinh \lambda \alpha \alpha^*}{\sinh \lambda}. \quad (36)$$

It is clear, that this new dynamical system has a phase portrait which is the same as the usual linear harmonic oscillator. The only difference is in the frequencies, the new one being

$$\omega = \frac{\lambda}{\sinh \lambda} \cosh \lambda \alpha \alpha^*. \quad (37)$$

We notice that $\alpha \alpha^*$ is a constant of motion for both systems. The deformation function used has given both energy and frequency exponentially growing with $\alpha \alpha^*$. To have physically more acceptable functions, it is not difficult to consider a slightly different deformation function.

The other example we consider is the deformation which leads to the classical version of harmonious states to be discussed in next sections. Here the deformation function is taken to be

$$f(n) = \left(\frac{1}{\Gamma(n+1)} \right)^{1/2}; \quad n = \alpha \alpha^*, \quad (38)$$

so that

$$\xi = \left(\frac{1}{\Gamma(n+1)} \right)^{1/2} \alpha; \quad \xi^* = \left(\frac{1}{\Gamma(n+1)} \right)^{1/2} \alpha^*. \quad (39)$$

The Poisson bracket

$$\{\xi, \xi^*\} = \frac{d}{dn} \left(\frac{n}{\Gamma(n+1)} \right), \quad (40)$$

where the right hand side is the function of $\xi \xi^*$ which is its transform by the inverse of (23), remarking that the Γ function on the positive real line has well defined derivative as well as inverse.

5. Nonlinear Oscillators in Quantum Mechanics through Noncanonical Transformations

We wish now to deal with the quantum analogue systems: we will remain in the usual formulation where the Fock space is better to describe the quantum harmonic oscillator. To go along the same lines as in the previous sections, we

stay in the Heisenberg picture and write the equations of motion for the harmonic oscillator amplitude a

$$\dot{a} = -i\omega a$$

and for its conjugate

$$\dot{a}^\dagger = i\omega a^\dagger.$$

The transformation (23) is written here as $A = af(a^\dagger a)$; $A^\dagger = f(a^\dagger a)a^\dagger$. It is noncanonical since it does not preserve the commutation relations. The operators A and A^\dagger evolve with the same equations, i.e.,

$$\dot{A} = -i\omega A, \quad \dot{A}^\dagger = i\omega A^\dagger$$

in complete analogy with the classical case. We have in our Hilbert space the vacuum state $|0\rangle$ which satisfies

$$a|0\rangle = 0,$$

as well as

$$A|0\rangle = 0.$$

This allows us to construct two bases in the vector space having this vector in common. One is the standard (Fock) basis

$$|n\rangle = \frac{a^{+n}}{\sqrt{n!}} |0\rangle,$$

which is orthonormal in standard scalar product

$$\langle n|m\rangle = \delta_{nm}.$$

Another basis is constructed using the operator A^\dagger ,

$$|\tilde{n}\rangle = \frac{A^{+n}}{\sqrt{n!}} |0\rangle.$$

We define a new scalar product in the same vector space which gives

$$\langle n|\tilde{m}\rangle = \delta_{nm}.$$

Provided $f(a^\dagger a)$ to be nonsingular, we can then speak of two Hilbert space structures carried by the same vector space.

The adjoint with respect to this new scalar product does not coincide with the old one. We can then define the operators

$$b^*|\tilde{n}\rangle = \sqrt{n+1}|n+1\rangle; \quad b|\tilde{n}\rangle = \sqrt{n}|n-1\rangle;$$

where $*$ means the adjoint in the new scalar product. These operators satisfy the commutation relations $[b, b^*] = 1$. Taking the Hamiltonian $H = \omega b^* b$ we have for the operators b, b^* the equation of motion of the harmonic oscillator. This is actually the situation with A and A^\dagger .

In the second Hilbert space, the operators A and A^\dagger have an identical representation as a and a^\dagger have in the Fock space and so they satisfy the commutation relation $[A, A^\dagger] = 1$. Thus, for one and the same vector space, we have the possibility to introduce two Hilbert space structures. As for the dynamics of the harmonic oscillator, we have two different descriptions, which parallel the alternative Hamiltonian descriptions of the classical oscillator. Therefore, much as we did for the classical case, we can use the new Hamiltonian and the old commutation relations to get a

“deformed” dynamics, or vice versa. To keep the current physical interpretation of the operators a and a^\dagger , we choose to maintain the value of their commutator which is directly connected with measurements, while we will consider deformed Hamiltonian operators. Then operator of “energy” \tilde{H} is

$$\tilde{H} = \frac{\omega}{2} (A^\dagger A + A A^\dagger). \quad (41)$$

In the Fock space, its eigenvalues are

$$E_n = \frac{\omega}{2} [(n + 1)f(n + 1)f^*(n + 1) + nf(n)f^*(n)]. \quad (42)$$

We illustrate the situation with an example and consider the nonlinear noncanonical transformation such that

$$A = af_q(a^\dagger a) = a \sqrt{\frac{\sinh \lambda \hat{n}}{\hat{n} \sinh \lambda}}, \quad (43)$$

which is invertible,

$$a = A \left[\frac{\ln [\hat{N} \sinh \lambda + \sqrt{\hat{N}^2 \sinh^2 \lambda + 1}]}{\lambda \hat{N}} \right]^{1/2}, \quad (44)$$

where

$$\hat{N} = A^\dagger A. \quad (45)$$

The operators A , A^\dagger acting in the same Hilbert space as the operators a , a^\dagger (the original Fock space) satisfy in this case the commutation relations

$$[A, A^\dagger] = \hat{N}(\cosh \lambda - 1) + \sqrt{\hat{N}^2 \sinh^2 \lambda + 1}, \quad (46)$$

as can be seen expressing all the operators in terms of matrices.

If one has the linear dynamics for the operator a with frequency equal to unity, i.e.,

$$\dot{a} + ia = 0, \quad (47)$$

and boson commutation relation for the operators a and a^\dagger , the same dynamics exists for the operator A :

$$\dot{A} + iA = 0, \quad (48)$$

since

$$\dot{A} + iA = (\dot{a} + ia)f_q(a^\dagger a) \quad (49)$$

and the function f_q is an integral of motion. The Hamiltonian for this dynamics may be taken as

$$H = \frac{1}{\lambda} \ln [A^\dagger A \sinh \lambda + \sqrt{(A^\dagger A)^2 \sinh^2 \lambda + 1}] + \frac{1}{2}, \quad (50)$$

and one obtains

$$[A, H] = A. \quad (51)$$

For the same dynamics, it is possible however to have a different Hamiltonian formulation. In one case, it is related to the above Hamiltonian, while in another Hilbert space we define the Hamiltonian

$$H' = A^\dagger A + \frac{1}{2} \quad (52)$$

with commutation relation

$$[A, A^\dagger] = 1, \quad (53)$$

and again

$$[A, H'] = A. \quad (54)$$

Thus, we see that analogous to classical mechanics there are possible alternative descriptions of a quantum system. For the same equations of motion, for the operators we have two different Hamiltonians with corresponding different commutation relations.

The same situation is anyhow present also for the class of nonlinear oscillators we have considered. There, continuing with the example of the f_q deforming function, we start with the dynamics for the q -oscillator [9] given by the equation

$$\dot{a} + i \frac{1}{2 \sinh \lambda} [\sinh \lambda (a^\dagger a + 2) - \sinh \lambda a^\dagger a] a = 0. \quad (55)$$

It is worth remarking at this point, that if we multiply the last equation from the right hand side by the same function $f_q(a^\dagger a)$, for instance, we are led to a new f -oscillator. Since this function is an integral of motion for the above q -nonlinear equation, we have in fact for operator (43) the equation of motion

$$\dot{A} + i \frac{\sinh \lambda}{\lambda} (\cosh \lambda a^\dagger a) A = 0. \quad (56)$$

But after (44),

$$a^\dagger a = \frac{1}{\lambda} \ln [A^\dagger A \sinh \lambda + \sqrt{(A^\dagger A)^2 \sinh^2 \lambda + 1}], \quad (57)$$

the obtained dynamics is different from the initial one.

We return now to consider eq. (55) and the first Hamiltonian description is given by

$$H = \frac{1}{2 \sinh \lambda} [\sinh \lambda (a^\dagger a + 1) + \sinh \lambda a^\dagger a]; \quad [a, a^\dagger] = 1. \quad (58)$$

In another Hilbert space, let the operators B and B^\dagger evolve with the equation

$$\dot{B} + i \frac{1}{2 \sinh \lambda} [\sinh \lambda (B^\dagger B + 2) - \sinh \lambda B^\dagger B] B = 0. \quad (59)$$

If we take the commutation relation

$$[B, B^\dagger] = B^\dagger B (\cosh \lambda - 1) + \sqrt{(B^\dagger B)^2 \sinh^2 \lambda + 1}, \quad (60)$$

the form of the Hamiltonian for this system differs from the form of Hamiltonian (58).

The important physical consequence of the existence of the same dynamics for quadratures with different commutation relations is the possibility of existing identical harmonic vibrations of two kinds. One vibrational process respects the Heisenberg uncertainty relation since quadratures satisfy standard boson commutation relations. Another vibrational process is compatible with different uncertainty relation for its quadratures since they satisfy different commutation relations. Nevertheless, from the view point of dynamics (harmonic vibrations) both cases are undistinguishable.

6. *f*-oscillator Operators

The operators A and A^\dagger represent the dynamical variables to be associated with the quantum f -oscillators. The well

known q-oscillator operators belong to this class. In this section, after discussing some algebraic features useful for the f-generalization, we refer also to other examples of f-oscillators already taken into consideration.

We start by recalling some notions about the harmonic oscillator operators a and a^\dagger whose algebraic structure is $[a, a^\dagger] = 1$. In the Fock space with $a = (a^\dagger)^\dagger$; $\hat{n} = a^\dagger a$, the basis is given by the eigenfunctions of \hat{n}

$$\hat{n}|n\rangle = n|n\rangle; \quad n \in \mathbf{Z}^+. \quad (61)$$

We have also

$$\mathbf{1} = \sum_0^\infty |n\rangle\langle n|; \quad \langle n|m\rangle = \delta_{nm} \quad (62)$$

and $\forall f: \mathbf{Z}^+ \rightarrow \mathbf{C}$

$$f(\hat{n}) = \sum_{j=0}^\infty f(j)|j\rangle\langle j|. \quad (62)$$

Consider now a "distortion" of a and a^\dagger of the form

$$A = af(\hat{n}) = f(\hat{n} + 1)a; \quad (64)$$

$$A^\dagger = f^\dagger(\hat{n})a^\dagger = a^\dagger f^\dagger(\hat{n} + 1)$$

and note that

$$[A, \hat{n}] = A; \quad [A^\dagger, \hat{n}] = -A^\dagger. \quad (65)$$

The functions we are considering can be made dependent in general, also on continuous parameters, in such a way that for particular values of them the usual annihilation and creation operators are reconstructed. We will say then that we are in presence of continuous deformations. This was the case of q-deformations [18]. In principle, one may consider discontinuous deformations, too.

Since

$$a = \sum_{n=0}^\infty \sqrt{n}|n-1\rangle\langle n|;$$

$$a^\dagger = \sum_{n=0}^\infty \sqrt{n}|n\rangle\langle n-1|, \quad (66)$$

the same Fock space is a carrier space for A and A^\dagger , i.e.,

$$A = \sum_{n=0}^\infty \sqrt{n}f(n)|n-1\rangle\langle n|;$$

$$A^\dagger = \sum_{n=0}^\infty \sqrt{n}f^*(n)|n\rangle\langle n-1|. \quad (67)$$

This realization may or may not be irreducible depending on the assumed functional form of $f(n)$.

Following the choice of not deforming the commutators of the physical variables (a, a^\dagger), the commutator between A and A^\dagger can be easily computed and by using (67) reads

$$F(\hat{n}) \doteq [A, A^\dagger] = (\hat{n} + 1)f(\hat{n} + 1)(f(\hat{n} + 1))^\dagger - \hat{n}f(\hat{n})(f(\hat{n}))^\dagger, \quad (68)$$

while the "q-commutator" is

$$G(\hat{n}) \doteq [A, A^\dagger]_q \doteq AA^\dagger - qA^\dagger A$$

$$= (\hat{n} + 1)f(\hat{n} + 1)(f(\hat{n} + 1))^\dagger - q\hat{n}f(\hat{n})(f(\hat{n}))^\dagger; \quad q \in \mathbf{R}. \quad (69)$$

We have, also

$$F(\hat{n}) - G(\hat{n}) = (q - 1)\hat{n}f(\hat{n})f^\dagger(\hat{n}). \quad (70)$$

Since only ff^\dagger occurs, the phase of f is irrelevant and we may, without loss of generality, choose f to be real and non-negative:

$$f^\dagger(\hat{n}) = f(\hat{n}). \quad (71)$$

Alternatively, given the functions F or G , assumed hermitian, we obtain the following solution to eqs (68) and (69)

$$f(n) = \frac{1}{\sqrt{n}} \left(\sum_{j=0}^{n-1} F(j) \right)^{1/2}; \quad n \neq 0 \quad (72)$$

and

$$f(n) = \frac{1}{\sqrt{n}} \left(\sum_{j=0}^{n-1} q^j G(n-j-1) \right)^{1/2}; \quad n \neq 0, \quad (73)$$

respectively, with $f(0)$ arbitrary in both cases. Such solutions are unique, having been obtained by construction. Of course, we may use complex f to construct the functions F and G in (68) and (69). However, to obtain f from F and G , we remark that F has to be real as well as G . Then in (64), we have the freedom of choosing an n dependent phase of f , which corresponds to construction of generalized coherent states [19, 20]. These generalized coherent states were analyzed in [21] were different types of interesting new states were introduced.

As was mentioned earlier, in the case of the q-oscillator operators, the function f depends also on a continuous parameter in order to obtain the harmonic oscillator operators as a limiting case. Starting with the q-commutation relation [7, 8]

$$G(\hat{n}) = q^{-\hat{n}}; \quad \lambda = \log q; \quad \lambda \in \mathbf{R}, \quad (74)$$

using (73) one obtains

$$f(n) = \sqrt{\frac{1}{n} \frac{q^n - q^{-n}}{q - q^{-1}}} = \sqrt{\frac{\sinh \lambda n}{n \sinh \lambda}} \doteq f_q(n), \quad (75)$$

setting

$$f_q(0) = 1. \quad (76)$$

We see from (58) that the eigenvalues of the "energy" operator grow exponentially with increasing occupation number.

One can remark that the phase operators V, V^\dagger in [22] are actually deformations of the Bose operators of the kind we are studying and lead to the harmonious states [23] to be considered below. In this case,

$$f(n) = \frac{1}{\sqrt{n}} \doteq f_h, \quad (77)$$

so that

$$A|n\rangle = |n-1\rangle; \quad A^\dagger|n\rangle = |n+1\rangle; \quad n \neq 0; \quad A|0\rangle = 0. \quad (78)$$

When many degrees of freedom are involved, we have two possible choices. The first one defines q-oscillators which satisfy q-commutation relations among a single degree of freedom, but mutually commuting between different degrees of freedom, possibly with different functions for the various degrees of freedom. Another choice is to make f dependent

on the total occupation number operator

$$\hat{n}_{\text{tot}} = \sum_i \hat{n}_i. \tag{79}$$

For two degrees of freedom, this was discussed in [9].

7. Nonlinear Coherent States

Coherent states were originally introduced as eigenstates of the annihilation operator for the harmonic oscillator and then widely used in physics, particularly in quantum optics. This is therefore a concept of algebraic origin and having now constructed a similar annihilation operator it is natural, following the same procedure, to construct a new class of *f*-coherent states in the Fock space. This construction is in general different from other ones [24]. Further the *f*-coherent states may not be preserved under time evolution. Nevertheless, we are willing to call them *f*-coherent states for an easy identification, of the kind already proposed for the eigenstates of the *q*-annihilation operator which were named *q*-coherent states [7, 25].

Let us take for the one-mode case the operator *A* (64). Then one can consider the eigenfunctions $|\alpha, f\rangle$ of *A* in a Hilbert space. They therefore satisfy the equation

$$A|\alpha, f\rangle = \alpha|\alpha, f\rangle; \quad \alpha \in \mathbb{C}. \tag{80}$$

Looking for the decomposition of $|\alpha, f\rangle$ in the Fock space

$$|\alpha, f\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{81}$$

we obtain for the coefficients c_n the following recurrence relation

$$c_{n+1} \sqrt{n+1} f(n+1) = c_n \alpha. \tag{82}$$

This gives

$$c_n = c_0 \frac{\alpha^n}{\sqrt{n! [f(n)]!}}, \tag{83}$$

in which

$$[f(n)]! = f(0)f(1)\cdots f(n). \tag{84}$$

To fix c_0 , we use the condition

$$\langle \alpha, f | \alpha, f \rangle = 1 \tag{85}$$

and obtain

$$c_0 = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n! [f(n)]!^2} \right)^{-1/2}. \tag{86}$$

To emphasize the dependence of c_0 on *f* and $|\alpha|$, we will write

$$c_0 = N_{f, \alpha} \tag{87}$$

and in order to have states belonging to the Fock space it is required that

$$0 < N_{f, \alpha} < \infty; \tag{88}$$

therefore not any *f* and $|\alpha|$ are allowed. We will denote with $\bar{\rho}$ the positive number such that, given *f*, the above series converge $\forall |\alpha| \leq \bar{\rho}$. The scalar product is easily written

$$\langle \alpha | \beta \rangle = N_{f, \alpha} N_{f, \beta} \sum_{n=0}^{\infty} \frac{1}{n! [f(n)]!^2} (\alpha^* \beta)^n; \quad |\alpha|, |\beta| < \bar{\rho}. \tag{89}$$

No further constraints are then put on *f* and $\bar{\rho}$.

It should be remarked furthermore that, given $C(n) = C_n$ any real function on \mathbb{Z}^+ , the state $|\alpha, C\rangle$ defined by

$$|\alpha, C\rangle = \sum_{n=0}^{\infty} C_n \alpha^n |n\rangle \tag{90}$$

is an eigenfunction of some *A*. In fact, the corresponding function *f* is found to be

$$f(n) = \frac{1}{\sqrt{n}} \frac{C_{n-1}}{C_n}. \tag{91}$$

Such eigenstate can be normalized if the *f* so obtained satisfies (86). In the case

$$f(n) = 1, \tag{92}$$

$|\alpha, 1\rangle$ denotes the usual coherent state and

$$N_{1, \alpha} = \exp\left(-\frac{|\alpha|^2}{2}\right), \tag{93}$$

α can be any complex number.

As anticipated, the known *q*-coherent states [7, 25] turn out to be a particular case of *f*-coherent states, which we might also call f_q -coherent states. Normalization factor of such states is

$$N_{q, \alpha} = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!} \right)^{-1}, \tag{94}$$

in which

$$[n]! = \frac{\sinh \lambda n}{\sinh \lambda} \frac{\sinh \lambda(n-1)}{\sinh \lambda} \cdots 1 \tag{95}$$

Using the notation (84) we can also write

$$[n]! = [nf_q^2(n)]!. \tag{96}$$

It is seen that α can be any complex number. For the scalar product, we have

$$\langle \alpha | \beta \rangle = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!} \right)^{-1} \left(\sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{[n]!} \right)^{-1} \sum_{n=0}^{\infty} \frac{1}{[n]!} (\alpha^* \beta)^n. \tag{97}$$

Harmonious states [23] are eigenstates of the annihilation operator deformed by the factor f_h (77) to which corresponds the normalization

$$N_{h, \alpha} = (1 - |\alpha|^2)^{-1/2}, \tag{98}$$

and the acceptable α must have modulus less the 1. Following (80) they can be denoted $|\alpha, f_h\rangle$ and their scalar product is

$$\begin{aligned} \langle \alpha, f_h | \beta, f_h \rangle &= \left(\frac{1}{1 - |\alpha|^2} \right)^{-1/2} \\ &\times \left(\frac{1}{1 - |\beta|^2} \right)^{-1/2} (1 - \alpha^* \beta)^{-1}. \end{aligned} \tag{99}$$

8. Nonlinear Coherent States in Different Representations

Since the state $|\alpha, f\rangle$ is given as series of Fock states, we can

easily write the wave function of these states in different representations explicitly.

In coordinate representation, the wave function is

$$\psi_{\alpha, f}^{(x)} = \pi^{-1/4} N_{\alpha, f} e^{-x^2/2} \sum_{n=0}^{\infty} \left(\frac{\alpha}{\sqrt{2}} \right)^n \frac{1}{n! [f(n)]!} H_n(x), \quad (100)$$

where H_n is the Hermite polynomial of degree n .

For the momentum representation, the formula is the same.

For the Bargmann representation (the usual coherent states), the wave function $\langle z | \alpha, f \rangle$, where we use the basis $|z\rangle (z \in \mathbf{C})$ with $a|z\rangle = z|z\rangle$, takes the form

$$\psi_{\alpha, f}^{(z)} = N_{\alpha, f} e^{-|z|^2/2} \sum_{n=0}^{\infty} (z^* \alpha)^n \frac{1}{n! [f(n)]!}. \quad (101)$$

For a continuous parameter dependent f , in the limit $f \rightarrow 1$ the usual wave function is recovered

$$\psi_{\alpha, 1}^{(z)} = \exp \left(\frac{|z|^2 - |\alpha|^2}{2} + z^* \alpha \right). \quad (102)$$

In the Wigner–Moyal representation [26] the density matrix for the f -coherent state reads

$$W_f(x, p) = 2N_{\alpha, f}^2 e^{-(x^2+p^2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m! [f(m)]!} \frac{1}{[f(n)]!} \times (-\alpha)^n \alpha^{*m} [\sqrt{2}(x-ip)]^{m-n} L_n^{m-n}(2(x^2+p^2)), \quad (103)$$

where L_n^m denotes a generalized Laguerre polynomial. For the particular case of q -oscillator $f = f_q$,

$$W_{q, \alpha} = \left(\sum_0^{\infty} \frac{|\alpha|^{2n}}{[n]!} \right)^{-1} e^{-(x^2+p^2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{[m]!} \frac{n!}{[n]!} \times (-\alpha)^n \alpha^{*m} [\sqrt{2}(x-ip)]^{m-n} L_n^{m-n}(2(x^2+p^2)). \quad (104)$$

Finally, we consider Husimi–Kano [27] Q -function of f -coherent states. In the definition, $Q_{\psi}(z, z^*)$ is the diagonal matrix element of the density operator $|\psi\rangle\langle\psi|$ for the state $|\psi\rangle$ in the usual coherent state basis. For an f -coherent state $|\alpha, f\rangle$, we can write

$$Q_f(z, z^*) = e^{-|z|^2} N_{f, \alpha}^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(z^* \alpha)^m}{m! [f(m)]!} \frac{(z \alpha^*)^n}{n! [f(n)]!}. \quad (105)$$

For q -coherent state, the Q -function is

$$Q_q(z, z^*) = e^{-|z|^2} \left(\sum_0^{\infty} \frac{|\alpha|^{2n}}{[n]!} \right)^{-2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(z^* \alpha)^m}{\sqrt{m! [m]!}} \frac{(z \alpha^*)^n}{\sqrt{n! [n]!}}. \quad (106)$$

For the harmonious states, the Wigner–Moyal function is

$$W_h(x, p) = 2(1 - |\alpha|^2) e^{-(x^2+p^2)} \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sqrt{\frac{n!}{m!}} (-\alpha)^n \alpha^{*m} [\sqrt{2}(x-ip)]^{m-n} \times L_n^{m-n}(2(x^2+p^2)) \quad (107)$$

and the Husimi-Kano function

$$Q_h(z, z^*) = e^{-|z|^2} (1 - |\alpha|^2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(z^* \alpha)^m}{\sqrt{m!}} \frac{(z \alpha^*)^n}{\sqrt{n!}}. \quad (108)$$

9. f -Coherent State Evolution

Here we offer a few considerations with the aim of underlining the peculiarities of the new classes of states in the Fock space, cautioning against relying too much on the abuse of their name.

We consider our field mode evolution to be guided by the equation of motion with the quantum Hamiltonian

$$\tilde{H} = \frac{1}{2}(A^\dagger A + A A^\dagger), \quad (109)$$

i.e., in the variables (a, a^\dagger) it is the system with the self-interaction described by the Hamiltonian

$$\tilde{H} = \frac{1}{2}[\hat{n}f^2(\hat{n}) + (\hat{n} + 1)f^2(\hat{n} + 1)]. \quad (110)$$

Then the evolution operator

$$U(t) = \exp(-it\tilde{H}(\hat{n})) \quad (111)$$

for this quantum nonlinear oscillator gives the following solution to the Heisenberg equation of motion for the operator $a(t)$

$$a(t) = U^\dagger(t)aU(t) = a \exp[-i\omega(\hat{n})t], \quad (112)$$

where

$$\omega(\hat{n}) = \frac{1}{2}[(\hat{n} + 1)f^2(\hat{n} + 1) - (\hat{n} - 1)f^2(\hat{n} - 1)]. \quad (113)$$

Thus, we see then that also the quantum f -oscillator vibrates with a frequency depending on the amplitude.

Turning to the Schrödinger picture, we can remark that at time t the harmonic oscillator has become a deformed oscillator of the kind we discuss. The deformation function is actually complex of modulus 1 and we will denote it with

$$F_f(n, t) = \exp \left[it \frac{(n+1)f^2(n+1) - (n-1)f^2(n-1)}{2} \right]. \quad (114)$$

It is possible, in fact, to introduce the notion of f -coherent states also for complex deformation functions f as all formulae go through unaltered.

Then it can be seen, that if initially the state was the usual coherent state, i.e., in an eigenstate of the operator a , then it evolves becoming at time t an $F_f(t)$ -coherent state. Physically it means that f -nonlinearity creates the $F_f(t)$ -coherent states in the evolution of a usual coherent state. Due to this, the photon statistics of the initial coherent state, to be discussed later, is influenced by the f -nonlinearity of the field vibrations. It is evidently different from the usual coherent states. Interesting physical example of the f -nonlinear systems is quartic nonlinear oscillator usually used for modelling the Kerr medium.

10. Irreducibility and Deformation

The usual Stone–von Neumann theorem states that the operators q and p (or a and a^\dagger) have no invariant subspaces in the Hilbert space of the oscillator states. If $f(n)$ is chosen to have no zeroes in \mathbf{Z}^+ , the operators A and A^\dagger are irreducible over the Fock space. If there are one or more double zeroes, the Fock space breaks up into a countable number of irreducible representations (compare with Master Analytic representations [28]). If the zeroes are simple zeroes, some of the reduced pieces will not allow a unitary rerepresentation.

It is easy to prove, that if the function $f(n)$ has no zeroes at positive integers, the Stone-von Neumann theorem can be extended to the case of the operators A, A^\dagger . So for the q -oscillator, we are just in this case. Here the map $a \rightarrow A$ is invertible and the statement is obvious. More interesting situations arise when, for example, the function $f(n)$ has one double zero at the integer n_0 , i.e., $f(n_0) = f'(n_0) = 0$. Then the subspace of the states

$$|\psi\rangle = \sum_{n=0}^{n_0} s_n |n\rangle \tag{115}$$

is an invariant subspace for the operators A, A^\dagger constructed by means of such function $f(n)$. The subspace of the states

$$|\psi'\rangle = \sum_{n=n_0+1}^{\infty} s_n |n\rangle \tag{116}$$

is another invariant subspace for the above operators. Thus, the coherent states defined in this case do not contain the states with photon numbers less or equal to n_0 . In this case, the coherent state contains the states with photon numbers starting from $n_0 + 1$,

$$|\alpha, \tilde{f}\rangle = \tilde{N} \sum_{n=n_0+1}^{\infty} \frac{\alpha^n}{\sqrt{\tilde{n}! [f(n)]!}} |n\rangle, \tag{117}$$

where

$$\tilde{n}! = n(n-1)(n-2)\cdots(n_0+1), \tag{118}$$

$$[f(n)]! = f(n)f(n-1)\cdots f(n_0+1), \tag{119}$$

and

$$\tilde{N}_{\alpha, \tilde{f}} = \left(\sum_{n=n_0+1}^{\infty} \frac{|\alpha|^{2n}}{\tilde{n}! ([f(n)]!)^2} \right)^{-1/2}. \tag{120}$$

11. Completeness Relations

In this section, we will show how the f -coherent states form a complete system of states in the Hilbert space for non-singular $f(n)$, so that any state vector may be represented as a superposition of the f -coherent states. There may be different forms of completeness relations since the set of f -coherent states are over complete.

We introduce first an integral representation for the identity operator which uses the analyticity of the f -coherent states and the Cauchy theorem. By construction, the state $N_{f,\alpha}^{-1}|\alpha\rangle$ is an analytic vector valued function of the complex variable α . Hence, the following relation holds

$$|n\rangle = \frac{\sqrt{n!}}{2\pi i} [f(n)]! \oint d\alpha N_{f,\alpha}^{-1} |\alpha, f\rangle \alpha^{-n-1}. \tag{121}$$

Inserting this formula into the known resolution of identity (62) we obtain

$$1 = -\frac{1}{4\pi^2} \sum_{n=0}^{\infty} \oint \oint d\alpha d\beta^* |\alpha, f\rangle \times \langle \beta, f | N_{f,\alpha}^{-1} N_{f,\beta}^{-1} (\alpha\beta^*)^{-n-1} n! ([f(n)]!)^2. \tag{122}$$

These are line integrals along contours taken in a region of the complex planes where the convergence is guaranteed for the series considered in Section 7 to define the normalization constant $N_{f,\alpha}$. The introduced nondiagonal resolution of identity permits us to calculate the coefficients necessary to represent any vector as a superposition of f -

coherent states. We also point out that the components of the f -coherent state in any basis are the generating functions for components of the Fock states. It means that for any matrix representing an operator in Fock basis, the matrix elements of the same operator in f -coherent state basis are the generating functions.

In order that such states can be considered as coherent states in the usual definition [24], one should write a diagonal resolution of the identity

$$1 = \int d\mu(\alpha) |\alpha, f\rangle \langle \alpha, f|, \tag{123}$$

where $\mu(\alpha)$ is the weight function. Then the following relations have to be satisfied

$$2\pi \int_0^{\bar{\rho}} \rho \rho^{2n+1} [N_f(\rho)]^2 \mu(\rho) d\rho = n! ([f(n)]!)^2; \quad \forall n \in \mathbb{Z}^+. \tag{124}$$

These are actually an infinity of moment equations for the measure μ . For the usual coherent states, as well as for the eigenstates of the q -deformed annihilation operators, such measures exist [25] and in both cases the integral is over the entire complex plane.

As far as the harmonious states of Section 7 are concerned, applying the general resolution of identity (62) one obtains

$$1 = -\frac{1}{4\pi^2} \oint \oint \frac{d\alpha d\beta^*}{\alpha\beta^* - 1} \frac{1}{\sqrt{(1-|\alpha|^2)(1-|\beta|^2)}} |\alpha, f_h\rangle \langle \beta, f_h| \tag{125}$$

after having inserted the normalization constants (98) for the harmonious states $|\alpha, f_h\rangle$ and $|\beta, f_h\rangle$. We can make this integral along two contours as an integral over the phases of α and β , if the contours are chosen according to

$$\alpha = ae^{i\phi}; \quad \beta = ae^{i\psi}, \tag{126}$$

and $0 < a < 1$. Then the double contour integral transforms into

$$1 = -\frac{a^2}{4\pi^2(1-a^2)^2} \int_0^{2\pi} \int_0^{2\pi} d\phi d\psi \times \frac{e^{i(\phi-\psi)}}{a^2 e^{i(\phi-\psi)} - 1} |ae^{i\phi}, f_h\rangle \langle ae^{i\psi}, f_h|. \tag{127}$$

We conclude remarking that for each $a \in (0, 1)$ there is a completeness relation in terms of projectors on harmonious states.

It is possible, however, to have a resolution of the identity with the integral depending on one parameter only, once states with norm not necessarily strictly positive are allowed. In [23] such states have been considered after having introduced the following scalar product

$$\langle \alpha | \beta \rangle = (1 - \alpha^* \beta).$$

Then, since

$$|\alpha, f_h\rangle = (1 - |\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \alpha^n |n\rangle; \quad |\alpha| < 1,$$

we have

$$\int ||\alpha|e^{i\theta}\rangle\langle\alpha|e^{i\theta}| \frac{d\theta}{2\pi} = \sum_0^\infty |\alpha|^{2n}(1-|\alpha|^2)^{-1/2}|n\rangle\langle n|$$

and

$$\int ||\alpha|^{-1}e^{i\theta}\rangle\langle\alpha|e^{i\theta}| \frac{d\theta}{2\pi} = (1-|\alpha|^2)^{-1/2}(1-|\alpha|^{-2})^{-1/2} \sum_0^\infty |n\rangle\langle n|.$$

Hence,

$$(2-|\alpha|^2-|\alpha|^{-2})^{1/2} \int ||\alpha|^{-1}e^{i\theta}\rangle\langle\alpha|e^{i\theta}| \frac{d\theta}{2\pi} = 1.$$

12. Two-mode f-Coherent States

For usual multimode harmonic oscillators, there exist generalized correlated states [29, 30] in which the mode quadratures are statistically dependent. The quasidistributions for these states have Gaussian form and the photon distribution function is described by multivariable Hermite polynomials [31]. It is possible to extend nontrivially the construction of one-mode f-oscillator to many modes. In particular, for two-mode state we consider the two operators

$$A_i = a_i f(\hat{n}); \quad i = 1, 2, \tag{128}$$

where

$$\hat{n} = \hat{n}_1 + \hat{n}_2; \quad \hat{n}_i = a_i^\dagger a_i \tag{129}$$

(the operators a_i satisfy boson commutation relation). These operators commute and for this reason we can construct algebraically the two-mode f-coherent state $|\alpha_1, \alpha_2, f\rangle$ defined by the following equations

$$A_i |\alpha_1, \alpha_2, f\rangle = \alpha_i |\alpha_1, \alpha_2, f\rangle, \quad i = 1, 2. \tag{130}$$

Considering the series expansion

$$|\alpha_1, \alpha_2, f\rangle = \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty c_{n_1, n_2} |n_1, n_2\rangle, \tag{131}$$

where the Fock states $|n_1, n_2\rangle$ satisfy

$$a_1^\dagger a_1 |n_1, n_2\rangle = n_1 |n_1, n_2\rangle; \quad n_1 \in \mathbf{Z}^+ \tag{132}$$

and

$$a_2^\dagger a_2 |n_1, n_2\rangle = n_2 |n_1, n_2\rangle; \quad n_2 \in \mathbf{Z}^+. \tag{133}$$

The solution of the recurrence relation which is obtained for the $c_{n_1 n_2}$'s is

$$c_{n_1 n_2} = c_{00} \frac{\alpha_1^{n_1} \alpha_2^{n_2}}{\sqrt{n_1! n_2!} [f(n)]!} \tag{134}$$

with c_{00} , fixed as before by normalization, being

$$c_{00} = \left(\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty |\alpha_1|^{2n_1} |\alpha_2|^{2n_2} ([f(n)]!)^{-2} (n_1! n_2!)^{-1} \right)^{-1/2}. \tag{135}$$

The two-mode f-coherent state can be now defined as

$$|\alpha_1, \alpha_2, f\rangle = c_{00} \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \frac{\alpha_1^{n_1} \alpha_2^{n_2}}{\sqrt{n_1! n_2!} [f(n)]!} \times |n_1, n_2\rangle; \quad \alpha_i \in \mathbf{C}; \quad i = 1, 2. \tag{136}$$

It should be remarked that in this form there is a coupling between the two modes, as there is a dependence of each of them on the total energy, this interaction between the two modes in general is nonlinear.

Another generalization for two-mode coherent states is of course obtained by means of the product of two one-mode f-coherent states, so that there is no interaction between the two modes. After defining

$$A_i = a_i f_i(n_i); \quad i = 1, 2 \tag{137}$$

and finding their eigenstates, we can make the tensor product obtaining

$$|\alpha_1, \alpha_2, f_1, f_2\rangle = \left(\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty |\alpha_1|^{2n_1} |\alpha_2|^{2n_2} (n_1! n_2!)^{-1} \times ([f_1(n_1)]! [f_2(n_2)]!)^{-2} \right)^{-1/2} \otimes \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \frac{\alpha_1^{n_1} \alpha_2^{n_2}}{\sqrt{n_1! n_2!} [f_1(n_1)]! [f_2(n_2)]!} \times |n_1, n_2\rangle. \tag{138}$$

In the case $f_1 = f_2 = 1$, we have the usual two-mode coherent states, namely,

$$|\alpha_1, \alpha_2, 1, 1\rangle = e^{-(|\alpha_1|^2 + |\alpha_2|^2)/2} \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \frac{\alpha_1^{n_1} \alpha_2^{n_2}}{\sqrt{n_1! n_2!}} |n_1, n_2\rangle. \tag{139}$$

13. Physical Application of f-coherent States

We give the examples of how the notions so far discussed might be of some interest in dealing with physical problems. With the choice made and repeatedly illustrated, we can continue interpreting $|n\rangle$ as the state containing n quanta, also when it appears in a series representing an f-coherent state. Recalling the wide use made in quantum optics of the harmonic oscillator formalism, denoting by $|n\rangle$ a state containing n photons, our examples will all be related to this interpretation.

In the usual coherent state $|\alpha, 1\rangle$, the particle distribution is given by the Poissonian function

$$P_{1, \alpha}(n) = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \tag{140}$$

with mean photon number

$$\langle n \rangle_1 = |\alpha|^2 \tag{141}$$

and dispersion

$$\sigma_{1, n} = \langle n^2 \rangle_1 - \langle n \rangle_1^2 = |\alpha|^2. \tag{142}$$

Then the ratio of the above quantities $\sigma_{1, n} / \langle n \rangle_1 = 1$.

In the case of f-coherent state the above equations become

$$P_{f, \alpha}(n) = |c_n|^2 = \left(\sum_{j=0}^\infty \frac{|\alpha|^{2j}}{[f(j)]!} \right)^{-1} \frac{|\alpha|^{2n}}{[f(n)]!}. \tag{143}$$

The mean photon number and dispersion are

$$\langle n \rangle_f = \sum_{n=0}^{\infty} n P_{f, \alpha}(n) \quad (144)$$

and

$$\sigma_{f, n} = \sum_{n=0}^{\infty} n^2 P_{f, \alpha}(n) - \left(\sum_{n=0}^{\infty} n P_{f, \alpha}(n) \right)^2, \quad (145)$$

while their ratio $\sigma_{f, n}/\langle n \rangle_f$ can be less or greater than 1, producing for a given *f* either sub-Poissonian or super-Poissonian statistics.

For the case considered in Section 10, where *f* has zeroes, the photon distribution function *P*(*n*) vanishes for $n \leq n_0$ and

$$P(n) = \frac{\tilde{N}^2 |\alpha|^{2n}}{\tilde{n}! ([f(n)]!)^2}; \quad n > n_0. \quad (146)$$

The photon distribution in the first of the two generalization for two modes (128) is

$$P_{f, \alpha_1, \alpha_2}(n_1, n_2) = c_{00}^2 \frac{|\alpha_1|^{2n_1} |\alpha_2|^{2n_2}}{n_1! n_2! ([f(n)]!)^2} \quad (147)$$

and for the second one (137) reads

$$P_{f_1, f_2, \alpha_1, \alpha_2} = P_{f_1, \alpha_1} P_{f_2, \alpha_2}. \quad (148)$$

We emphasize the fact that in the latter case the two modes are independent and there is no correlation.

On the contrary, in the previous two-mode generalization a correlation exists between the modes: in facts if one gives the interpretation to the *f*-coherent states as the states related to deformation of the annihilation operator due to a specific *f*-nonlinearity, in the first case, this happens through the total energy and then we can now conclude that this particular nonlinearity of the field produces a correlation among the modes. In the case of *q*-oscillators, we obtain for the photon distribution in the one-mode *q*-coherent state

$$P_{f_q, \alpha}(n) = \left(\sum_{j=0}^{\infty} \frac{|\alpha|^{2j}}{[\sinh \lambda j / \sinh \lambda]!} \right)^{-1} \times \frac{|\alpha|^{2n}}{[\sinh \lambda n / \sinh \lambda]!}. \quad (149)$$

The property of this distribution is that, for large $n (n \gg \lambda^{-1})$, the probability to have *n* photons differs essentially from Poissonian distribution due the exponentially decreasing of the denominator. The mean photon number $\langle n \rangle_q$ is given for *q*-nonlinear field by

$$\langle n \rangle_q = \left(\sum_{j=0}^{\infty} \frac{|\alpha|^{2j}}{[\sinh \lambda j / \sinh \lambda]!} \right)^{-1} \times \sum_{n=0}^{\infty} \frac{n |\alpha|^{2n}}{[\sinh \lambda n / \sinh \lambda]!}, \quad (150)$$

the second moment by

$$\langle n^2 \rangle_q = \left(\sum_{j=0}^{\infty} \frac{|\alpha|^{2j}}{[\sinh \lambda j / \sinh \lambda]!} \right)^{-1} \times \sum_{n=0}^{\infty} \frac{n^2 |\alpha|^{2n}}{[\sinh \lambda n / \sinh \lambda]!}, \quad (151)$$

and the dispersion depends of course on the nonlinearity parameter λ .

The photon distribution demonstrates the fast decreasing of the distribution function in comparison with the Poisson distribution function; the *q*-nonlinearity makes photon statistics sub-Poissonian. The *q*-nonlinearity and some other types of *f*-nonlinearities may influence the statistical properties that can be checked by Hanbury Brown–Twiss-like experiments and the presumed detection of sub-Poissonian counting distribution in quantum optics.

14. Squeezing and Correlation

Now we will calculate the squeezing and correlation of the quadrature components in the introduced *f*-coherent states. We face, in fact, the problem since the discussed nonlinearity (for example, *q*-nonlinearity) of the field produces the state which is *f*-coherent state. Then this nonlinearity yields the phenomenon of squeezing and correlation of the field (photon) quadrature components. It is possible to calculate the dispersion and correlation of the quadratures explicitly. To do this we will take advantage of eq. (66), which gives the expression

$$a = \frac{1}{f(\hat{n} + 1)} A; \quad a^\dagger = A^\dagger \frac{1}{f(\hat{n} + 1)}. \quad (152)$$

Then the quadrature mean values are expressed through

$$\langle \alpha, f | a | \alpha, f \rangle = \alpha N_{\alpha, f}^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n! \{ [f(n)]! \}^2}, \quad (153)$$

yielding

$$\langle \alpha, f | x | \alpha, f \rangle = \frac{\alpha + \alpha^*}{\sqrt{2}} N_{\alpha, f}^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n! \{ [f(n)]! \}^2}, \quad (154)$$

$$\langle \alpha, f | p | \alpha, f \rangle = \frac{\alpha - \alpha^*}{i\sqrt{2}} N_{\alpha, f}^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n! \{ [f(n)]! \}^2}. \quad (155)$$

The dispersions may be calculated from the relations

$$\langle \alpha, f | a^2 | \alpha, f \rangle = \alpha^2 N_{\alpha, f}^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n! \{ [f(n)]! \}^2}, \quad (156)$$

$$\langle \alpha, f | a^\dagger a | \alpha, f \rangle = |\alpha|^2 N_{\alpha, f}^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f^2(n+1)n! \{ [f(n)]! \}^2}, \quad (157)$$

as

$$\langle \alpha, f | x^2 | \alpha, f \rangle = \frac{1}{2} \left\{ 1 + N_{\alpha, f}^2 \left[2|\alpha|^2 + \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f^2(n+1)n! \{ [f(n)]! \}^2} + (\alpha^2 + \alpha^{*2}) \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n! \{ [f(n)]! \}^2} \right] \right\}, \quad (158)$$

$$\begin{aligned} \langle \alpha, f | p^2 | \alpha, f \rangle &= \frac{1}{2} \left\{ 1 + N_{\alpha, f}^2 \left[2|\alpha|^2 \right. \right. \\ &\times \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f^2(n+1)n! \{ [f(n)]! \}^2} - (\alpha^2 + \alpha^{*2}) \\ &\left. \left. \times \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n! \{ [f(n)]! \}^2} \right] \right\}. \end{aligned} \quad (159)$$

Thus, for quadrature dispersion $\sigma_x = \langle \alpha, f | x^2 | \alpha, f \rangle - \langle \alpha, f | x | \alpha, f \rangle^2$, we have

$$\sigma_x = \frac{1}{2} + \mu_x \alpha^2 + \mu_x^* \alpha^{*2} + \nu_x \alpha \alpha^*, \quad (160)$$

where

$$\begin{aligned} \mu_x &= \frac{1}{2} N_{\alpha, f}^2 \left\{ \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n! \{ [f(n)]! \}^2} \right. \\ &\left. - N_{\alpha, f}^2 \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n! \{ [f(n)]! \}^2} \right)^2 \right\} \end{aligned} \quad (161)$$

and

$$\begin{aligned} \nu_x &= N_{\alpha, f}^2 \left\{ \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f^2(n+1)n! \{ [f(n)]! \}^2} \right. \\ &\left. - N_{\alpha, f}^2 \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n! \{ [f(n)]! \}^2} \right)^2 \right\}. \end{aligned} \quad (162)$$

For the other quadrature dispersion $\sigma_p = \langle \alpha, f | p^2 | \alpha, f \rangle - \langle \alpha, f | p | \alpha, f \rangle^2$, we have

$$\sigma_p = \frac{1}{2} + \mu_p \alpha^2 + \mu_p^* \alpha^{*2} + \nu_p \alpha \alpha^*, \quad (163)$$

where

$$\begin{aligned} \mu_p &= \frac{1}{2} N_{\alpha, f}^2 \left\{ \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n! \{ [f(n)]! \}^2} \right. \\ &\left. - N_{\alpha, f}^2 \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n! \{ [f(n)]! \}^2} \right)^2 \right\} \end{aligned} \quad (164)$$

and

$$\begin{aligned} \nu_p &= N_{\alpha, f}^2 \left\{ \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f^2(n+1)n! \{ [f(n)]! \}^2} \right. \\ &\left. - N_{\alpha, f}^2 \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n! \{ [f(n)]! \}^2} \right)^2 \right\}. \end{aligned} \quad (165)$$

Depending on the function $f(n)$ the dispersion $\sigma_x(\sigma_p)$ may become less than 1/2. It means squeezing. One can calculate correlation of the quadrature components in f-coherent states as

$$\begin{aligned} \sigma_{xp} &= \frac{\alpha^2 - \alpha^{*2}}{2i} N_{\alpha, f}^2 \left[\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n! \{ [f(n)]! \}^2} \right. \\ &\left. - N_{\alpha, f}^2 \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n! \{ [f(n)]! \}^2} \right)^2 \right]. \end{aligned} \quad (166)$$

Then the quadrature correlation coefficient $r = \sigma_{xp} / \sqrt{\sigma_x \sigma_p}$ is not equal to zero. Thus, the f-coherent state has the property of being a correlated state [29]. The invariant $\sigma_x \sigma_p$

– σ_{xp}^2 is larger than 1/4. Thus, the f-coherent states do not minimize the Schrödinger uncertainty relation [32, 33].

15. Deformation of Planck Formula

We will discuss what physical consequences may be found if the considered f-nonlinearity influences the vibrations of the real field mode oscillators like, for example, electromagnetic field oscillators or the oscillations of the nuclei in polyatomic molecules.

First, it will be seen that this nonlinearity changes the specific heat behaviour. To show this, we have to find the partition function for a single f-oscillator corresponding to the Hamiltonian $H = (AA^\dagger + A^\dagger A)/2$

$$Z(T) = \sum_{n=0}^{n=\infty} \exp(-\beta E_n), \quad (167)$$

where the variable β is the inverse temperature T^{-1} and E_n was given in eq. (42). For the evaluation of the quantum partition function for an ensemble of q-oscillators [9] we first note that in this case

$$E_n = \frac{1}{\sinh \lambda} [\sinh \lambda(n+1) + \sinh \lambda n]; \quad \lambda = \log q,$$

and obtain that the specific heat decreases for $T \rightarrow \infty$ as

$$C \propto \frac{1}{\ln T}. \quad (168)$$

Thus, the behaviour of the specific heat of the q-oscillator is different from the behaviour of the usual oscillator in the high temperature limit. This property may serve as an experimental check of the existence of vibrational nonlinearity of the q-oscillator fields.

The q-deformed Bose distribution can be obtained by the same method and one has [9]

$$\begin{aligned} \langle n \rangle &= \bar{n}_0 - \beta \frac{\lambda^2}{6} \left[\frac{1}{2} (\bar{n}^2)_0 - (\bar{n})_0^2 \right. \\ &\left. + \frac{3}{2} (\bar{n}^3)_0 - 3 \bar{n}_0 (\bar{n}^2)_0 + (\bar{n}^4)_0 - \bar{n}_0 (\bar{n}^3)_0 \right], \end{aligned} \quad (169)$$

in which \bar{n}_0 is the usual Bose distribution function and

$$(\bar{n}^k)_0 = 2 \sinh \frac{\beta}{2} \sum_{n=0}^{\infty} n^k e^{-\beta(n+1/2)} \quad (170)$$

Calculating the partition function for small q-nonlinearity parameter we have also the following q-deformed Planck distribution formula

$$\langle n \rangle = \frac{1}{e^{\hbar\omega/kT} - 1} - \lambda^2 \frac{\hbar\omega}{kT} \frac{e^{3\hbar\omega/kT} + 4e^{2\hbar\omega/kT} + e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT} - 1)^4} \quad (171)$$

It means that q-nonlinearity deforms the formula for the mean photon numbers in black body radiation [19].

One can write down the high and low temperature approximations for the deformed Planck distribution formula [34]. For small temperature, the behaviour of the deformed Planck distribution differs from the usual one

$$\langle n \rangle - \bar{n}_0 = -\lambda^2 \frac{\hbar\omega}{kT} e^{-\hbar\omega/kT}. \quad (172)$$

For the high temperature, the nonlinear correction to the usual mean photon number also depends on temperature

$$\langle n \rangle - n_0 = -\lambda^2 \left(\frac{\hbar\omega}{kT} \right)^{-3}. \quad (173)$$

As it was seen, the discussed *q*-nonlinearity produces a correction to the Planck distribution formula (mean oscillator energy) and this may also be subjected to an experimental test.

As it was suggested in [10] the *q*-nonlinearity of the field vibrations produces blue shift effect which is the effect of the frequency increase with the field intensity. For small nonlinearity parameter λ and for large number of photons n in a given mode, the relative shift of the light frequency is

$$\frac{\delta\omega}{\omega} = \frac{\lambda^2 n^2}{2}.$$

This consequence of the possible existence of a *q*-nonlinearity may be relevant for models of the early stage of the Universe.

Another possible phenomenon related to the *q*-nonlinearity was considered in [35], where it was shown, that if one deforms the electrostatics equation using the method of deformed creation and annihilation operators, a point charge acquires a form factor due to *q*-nonlinearity.

16. Conclusion

Starting with the example of the harmonic oscillator, we have exhibited a family of associated nonlinear systems which are completely integrable, both in classical and quantum physics.

We have shown that *q*-nonlinearity, associated with quantum groups, is a subclass of a more general class of possible nonlinearity. These aspects are related to the existence of alternative Hamiltonian descriptions for the harmonic oscillator.

A class of states has been considered in the Fock space through the deformation process applied to the harmonic oscillator operators. Such states have been described as *f*-coherent states (or nonlinear coherent states), harmonious states and *q*-coherent states being particular examples of them. Their different representation have been constructed, like the Wigner–Moyal and Husimi–Kano distributions. It is shown how nonlinear couplings between different modes are easy to obtain.

Keeping unaltered the current physical identification of the Fock states, a possible use is presented in the field of quantum optics, obtaining deformed photon distributions and related physical quantities. Physical consequences of the deformed vibrations, like the Planck distribution deformation, are then reviewed for the *q*-oscillators, where a blue shift effect exists. The related phenomena were studied recently [36–38]. In [36] the estimation of upper limit of *q*-nonlinearity of the electromagnetic field vibration was done.

The studied nonlinearities, if they exist, for the electromagnetic field or for the gluons, may influence the particle decays, correlations in particle multiplicities, and a change in the Hanbury Brown–Twiss experiment results. It would naturally affect the stimulated emission rates and hence the radiative equilibrium in the presence of matter.

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