The Spin-Statistics Theorem in Arbitrary Dimensions

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Abstract

We investigate the spin-statistics connection in arbitrary dimensions for hermitian spinor or tensor quantum fields with a rotationally invariant bilinear Lagrangian density. We use essentially the same simple method as for space dimension $D = 3$. We find the usual connection (tensors as bosons and spinors as fermions) for $D = 8n+3,\ 8n+4,\ 8n+5$, but only bosons for spinors and tensors in dimensions $8n \pm 1$ and $8n$. In dimensions $4n+2$ the spinors may be chosen as bosons or fermions.

The argument hinges on finding the identity representation of the rotation group either on the symmetric or the antisymmetric part of the square of the field representation.

1 Introduction

The spin-statistics connection is an essential ingredient in our description of the world with quantized fields, which assures on one hand the existence of macroscopic fields (like the radiation field), and on the other hand gives rise through anticommuting fields (Pauli principle) to the valence electrons, the chemical bonds etc., and therefore to the existence of forms and structures:

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the Pauli principle is really the *differentiating principle* in Nature.

The connection asserts that the wavefunction of several identical particles in $D = 3$ with integer spin remains unchanged under an arbitrary permutation of the arguments, in which case bosons (Bose-Einstein statistics, BE) obtains, whereas for half-integer spin the permuted wavefunction picks up a minus sign whenever one performs a transposition of the order (or, more generally, an odd sign permutation) (Fermi-Dirac statistics, FD). This translates in the usual way in the commutation (BE) or anticommutation (FD) of fields at different space points.

With hindsight, we can say that historically the first case of such correlation appeared in the statistical mechanics of *Lichtquanten* or photons (from Planck 1900 to Einstein 1905, to Bose 1924). But the main application came with the interpretation of the Pauli exclusion principle (1925) by Heisenberg and by Dirac (1926) in terms of the (anti-)symmetry of the many particle wavefunctions under transposition, and subsequent application to many electron atoms. This relation between spin and statistics is fundamental; it led to the periodic system of chemical elements, peculiar intensity rules in band spectra of homonuclear diatomic molecules, non-classical scattering of alpha particles, or the Ehrenfest-Oppenheimer theorem [1] (1931) on the statistics of compound systems, among the oldest applications, and more recently to the selection rules for the decay of unstable particles e.g. positronium, coherent boson states and the existence of the laser, superfluidity of helium four (Kapitza 1938) and later superconductivity as coherent states of Cooper pairs (BCS theory, 1957) and even superfluidity in helium three through condensation of pairs of atoms.[2]

There is more to the quantum states that mere rays. The wavefunction is really a section on a vector bundle with base the space of rays (projective space), and a sign change after permutation on the arguments of the wavefunction is allowed as long as the associated density matrix is invari-

[2] The symmetry or antisymmetry of the wave functions obtains only in a quantum field theory. We find remarkable that a healthy positivistic attitude, namely that the permutation of identical particles should produce no observable effects, would have different ways to be implemented in the quantum theory! This is possible at all, of course, because of the projective or rather the bundle nature of quantum states, which we have already emphasized [2].

ant: in particular, a pure state lifted to a vector repesentative could acquire a plus or minus sign, corresponding to the two unique one-dimensional irreducible representations (*irreps*) of the symmetric group; the sign cancels going down to the base space: in this context, that is the precise form of the common statement that quantum states are state vectors up to a phase, here up to a sign. In mathematical terms, statistics sign is just a Schur multiplier.

The same argument shows also [2] why half-integer angular momentum could exist in quantum theory in the first place; namely, the pertinent projective representations of the rotation group (say, $SO(n)$) come from the linear representations of the double covering (universal covering for $n > 2$), the $Spin(n)$ group. That is, spin 1/2 is an admissible (projective) representation of the $3D$ rotation group, although it does not come from a linear one, and picks up a minus sign under a full rotation.

We find it logical that (in space dimension 3, and as we shall see also in $D = 8n + 3, +4$ or $+5$) these two nonclassical properties "compensate each other"; namely, the case of Fermi statistics goes along with half-integer spin whereas the Bose statistics occurs with integer spin. There are two compensating minus signs for spinors (permutations of fermions *and* 2π rotation), but none in the Bose (tensor) case, which therefore looks more, but it is altogether not, wholly classical. This connection is essentially the spin-statistics relation.

In this paper we are going to see whether the spin-statistics connection holds in arbitrary dimensions. The motivation to study this question is fairly clear today: unification of forces by the Kaluza-Klein mechanism, supersymmetry and superstrings, extended objects and $M-$theory, etc., all point to the necessity of higher dimensions, whether invisible or macroscopic; in $F-$theory we even face the case of (2, 10) spacetime dimensions, that is, two times. As these theories are quantum theories, one needs to see how the usual argument, i.e. the symmetry of the bilinear scalar product under $3D$ rotations, extends now to other $D > 3$ dimensions. Although at the moment the question is rather academic, if one of these higher dimensional theories stands in the future, the question will be an important one, and so we believe that the present investigation is justified.

2 Review of the usual proof

There are many proof of the Spin-Statistics relation in relativistic quantum field theory, starting with the original one by Pauli in 1940 [3]; for a thorough review of the situation up to the year 2000 see the book [4]. For our purposes we shall recall here the proof of the theorem as given by Sudarshan many years ago [5], which starts from a $3D$ rotationally invariant field Lagrangian density and contains the essential features. The manifold applications of the theorem in nonrelativitic contexts claims for a demonstration not requiring relativistic invariance. Axiomatic formulations of quantum field theory, which do not use Lagrangians, do need special relativity to prove commutativity properties of the fields at distant points [6]. However, the requirement of relativistic invariance is somewhat inappropiate, since most of the manifestations of this relationship are in the nonrelativistic domain: atoms, nuclei, condensed matter situations, quantum liquids, phonons in solids, etc. Also the key topological feature, namely the symmetry group not being simply connected, appears already in the pure space part.

The fundamental principle of field dynamics is the Action Principle, as established by Weiss (1938) and in its quantized form by Schwinger (1951) [7]. This presentation of the quantum theory demands that the variation of any object Φ in the theory be given by its commutator with the variation of S, the action of the system. That is

$$\delta\Phi = [\Phi, -i\delta S], \tag{1}$$

which is simply the generalization of the quantum rule $[q, p] = i\hbar$. It characterizes the action as universal generator of variations. The action is the time integral of the Lagrangian; we shall describe now a *classical* mechanical theory in the first order formalism in which the Lagrangian is a function on the TT^*Q manifold, where Q (dim $Q = n$) is the configuration space, T^*Q the phase space (or cotangent bundle) and TM means the tangent bundle to any manifold M. In the first order formalism we have the Lagrangian function $L_0 \in F(TT^*Q)$, with

$$L_0 = p_a\dot{q}_a - H(q_b, p_b) \tag{2}$$

with summation on $a, 1 \leq a \leq n$, and where $\dot{A} = (\partial/\partial t)A$.

The equations of motion are not altered by the (anti)symmetrization

$$L = 1/2(p_a\dot{q}_a - q_a\dot{p}_a) - H(q_b, p_b) \tag{3}$$

So, defining $\xi := (q_a, p_a)$ as a column vector, we can take the Lagrangian as (${}^tB$ is the transpose of B)

$$L = 1/2({}^t\xi\, C\, (\partial/\partial t)\xi) - H(\xi) \tag{4}$$

where $C = -{}^tC$ is a purely numerical (invertible) real antisymmetric matrix. Notice the "symplectic" character of this first order Lagrangian associated to the use of first order time derivative. The variable ξ (with $2n$ number of components) depends in time, and the dynamical term $H(\xi)$ does not contain time derivatives. Notice also in this formalism the kinetic term is bilinear in the fields.

Inspired by this, we know write our Action and the Lagrangian density operators for arbitrary *quantum* fields χ as

$$S[\chi] = \int_{t_0}^{t} dt \int d^3x \mathcal{L}[\chi],\ \mathcal{L} = (1/4)({}^t\chi K(\partial/\partial t\chi) - (\partial/\partial t^t\chi)K\chi) - \mathcal{H}[\chi] \tag{5}$$

following also Schwinger [7] [8]. Here $\chi = \chi(\mathbf{x}, t)$ is a finite-dimensional hermitian quantum field, $\mathcal{H}$ is the Hamiltonian density, and K is an *antihermitian* numerical matrix: The dynamical variable becomes hermitian, and K should be taken antihermitian, $K = -K^\dagger$.

But now there are naturally two possibilities: the matrix K can be real antisymmetric or purely imaginary amd symmetric, as already Schwinger said half a century ago; and these two possibilities would fix the commutation/anticommutation properties of the field variation $\delta\chi$ with the fields contained in χ, leading finally to the sought-for connection between spin and statistics.

Here we shall use for Φ just the fields χ. The complete Lagrangian would have many pieces, viz.:

$$\mathcal{L} = \mathcal{L}_1(kinetic,\ time) + \mathcal{L}_2(kinetic,\ space) + \mathcal{L}_3(mass\ ) + \mathcal{L}_4(\ interactions) \tag{6}$$

where we know already that the first term, from rotational invariance alone, should be a scalar and so we shall impose $SO(3)$ invariance in the quantum mechanical sense of above, that is, linear $SU(2)$ invariance. The form of the temporal kinetic term, with the imposed rotational invariance, *is the only ingredient we need for our proof of the theorem*. Whether the remaining terms in the Lagrangian, specially the dynamics encoded in the Hamiltonian, would spoil the arguments, we leave open at this point and will comment later on.

The general variation δS contains three terms: variation in the content of the integral inside the fixed boundary, which gives the equations of motion, variations of the limits of integration, and thirdly variations of the field quantities at the fixed boundary. For our case only the *third* variation is pertinent, namely the variations of the fields at the boundaries, which can be taken as two spacelike surfaces at times t_0 and t, respectively: we consider the variation only on the "future", at time t, and omit the (repeated) t label.

The equation becomes

$$4i\delta\chi_a(x) = [\chi_a(x), \int d^3y \, \{\delta\, {}^t\chi_b(y) K_{bc}\chi_c(y) -^t \chi_b(y) K_{bc}\delta\chi_c(y)\}] \tag{7}$$

where $x = (\mathbf{x}, t), y = (\mathbf{y}, t)$,etc.

This is completely general. Now we require that

" The field variation $\delta\chi_a(x)$ either conmmutes or anticommutes with the field itself": this is equivalent to restricting ourselves to fermi or bose statistics (we specifically exclude parastatistics; the only kind of parastatistics that is valid is the reducible parastatistics as introduced by H. S. Green [9]; see also [10]).

A) The field variation COMMUTES with the field itself. Then we obtain in the usual way (see [5])

$$2i\delta^3(\mathbf{x} - \mathbf{y}) = [\chi_a(\mathbf{x}), \chi_b(\mathbf{y})] K_{ab} \tag{8}$$

where K has to be real *antisymmetric*. This matrix K might be degenerate: call K_0 the restriction of K to the minimal components of a particular spin in χ: K_0 is then regular (invertible). We can then write symbolically

$$2iK_0^{-1} = [\chi, \chi] \tag{9}$$

where

$$K_0 = -{}^tK_0 \quad \text{with} \quad \det \mathrm{K}_0 \neq 0 \tag{10}$$

which is the most general way of expressing the fundamental commutation relations characteristic of Bose fields: different field components commute, but field and momentum components have an "i" as their commutator.

B) The field variation ANTICOMMUTES with the field itself. Then from the previous eq. we obtain, (with $\{a, b\} := ab + ba$):

$$2\delta^3(\mathbf{x} - \mathbf{y}) = \{\chi_a(\mathbf{x}), \chi_b(\mathbf{y})\}K_{ab} \tag{11}$$

with K now a *real* symmetric matrix. Again, by restricting to minimum fields, we can write the anticommutation rules for Fermi fields in a form similar as before:

$$2K_0^{-1} = \{\chi, \chi\}. \tag{12}$$

But the character of K can be obtained also from the kinetic term of the Lagrangian by appealing to rotational invariance: namely ${}^t\chi K\partial/\partial t\chi$ has to be a $SO(3)$ scalar (invariant), as K connects only pieces of χ with the same spin. Recalling that the kinetic term involves the *anti*symmetric time derivative, for *integer spin* the matrix K has to be *antisymmetric*, whereas for *halfinteger* spin K is *symmetric*:

In three space dimensions the squares of the irreducible representations of $SU(2)$ are well known; for example, for l integer

$$D_l \otimes D_l = D_{0+} + D_{1-} + D_{2+} + ... + D_{2l+} \tag{13}$$

whereas for s half-integer

$$D_s \otimes D_s = D_{0-} + D_{1+} + ... + D_{2s+} \tag{14}$$

where $(+)$ indicates the symmetric, and $(-)$ the antisymmetric, parts of the product. This says that for tensors, the Identity *irrep* (scalar product) is in the symmetric part, whereas for spinors is in the antisymmetric part (e.g. $D_{1/2} \otimes D_{1/2} = D_1 + D_0 = 3(sym) + 1(asym)$). This crucial result comes really from the *symplectic character* of the fundamental, spin 1/2 *irrep* of $SO(3)$, as $Spin(3) = SU(2) = SpU(1)$.

This encompasses the spin-statistics theorem in 3 space dimensions: the specific form of $K = -K^\dagger$ in the lagrangian implies that integer spin would have K as real antisymmetric, hence the commutation relations and Bose statistics. With half-integer spin fields is the other way around: symmetric K would imply anticommutators, hence Fermi statistics and the Pauli exclusion principle.

The argument can be reversed, namely starting from this result, we would conclude the symmetry/antisymmetry of K, and from this the rules Bose or Fermi respectively for BE or FD, recalling that the time derivative is an antisymmetric operator.

Our argument in $3D$ is really in consonance with special relativity: namely, the use of a spacelike surface to state initial operator conditions is a wholly Lorentz invariant statement. It is still valid for, say, Galilean invariant theories, as long as one deals with quantum field theories, in which particles can be created and destroyed.

We address now the question whether the validity of the proof could not be spoiled by the neglected terms in the Lagrangian. The space part of the kinetic density should cause no problems, and indeed our argument should be a proof of the spin-statistics theorem for nonrelativistic *field* theories, in which particle creation/destruction is allowed. Wightman has emphasized that simple, quantum-mechanical many-body systems with fixed number of particles need not obey the standard spin-statistics relation. The reason for the sufficiency of the time derivative comes from the variation principle refering, in our case, to two spacelike surfaces.

What about limitations coming from peculiar Hamiltonians? We do not have a full answer to this, but would like to make the following remarks: in some cases, in which the Hamiltonian is not bounded from below, as in the naive case of the Dirac equation, the right statistics comes to the rescue, and makes sense of such a Hamiltonian, as the "anticommutator" statistics incorporates the exclusion principle, and the equivalent of hole theory and redefinition of the vacuum makes the rest. A general Hamiltonian with no lower bound would be of course unacceptable already at the classical level.

Another question is the applicability of the method to composite systems; it is a bit striking that e.g. protons and neutrons, being fermions, make up compound systems like the deuteron or the alpha particle with tested boson character. Here we only remind the reader of the old quoted Ehrenfest-Oppenheimer paper [1], making very plausible that composite even/odd number of fermions should enjoy corresponding bose/fermi statistics. But one should admit frankly that the whole issue of statistics of composite systems deserves a closer look.

3 Particle statistics in arbitrary dimensions

Before going into technical details we would like to show why the symmetry type of K in $3D$ is *not* to be expected for 8 space dimensions.

The reason is this: in $3D$, the Id *irrep* of $Spin(3)$ appears in the antisymmetric part of the square, viz.:

$$D_{1/2} \wedge D_{1/2} = D_0 : \; 2 \otimes 2 = 3(sym) + 1(asym) \tag{15}$$

whereas in $8D$ the two chiral irreps of $Spin(8)$, $\underline{8}_{s,c}$ behave like the vector irrep $\underline{8}_v$, also of dim 8, because triality (see e.g. [11]) permutes the three *irreps*, so the Id *irrep* appears necessarily in the symmetric square of *any* of the three *irreps* :

$$\underline{8}_v \vee \underline{8}_v = \underline{1} + \underline{35}, \; \underline{8}_v \wedge \underline{8}_v = \underline{28} \tag{16}$$

$$\underline{8}_s \vee \underline{8}_s = \underline{1} + \underline{35'}, \; \underline{8}_s \wedge \underline{8}_s = \underline{28} \text{ same for } \underline{8}_c; \tag{17}$$

here $\underline{28}$ is the adjoint, $\underline{35}$ the 2−symmetric traceless, $\underline{35'}$ the (anti-)self-dual 4-form, etc. Thus for $D = 8$ the identity (Id) *irrep* appears in the symmetric part of the square of either chiral *irrep*, contrary to the situation in $3D$; so they can only describe bose fields, according the arguments above. In the *Appendix* we delve more deeply in the dimension 8 case.

Indeed, from the properties of Clifford algebra we can see that the $8D$ case is a case of *real* type for the spin *irreps*, whereas in $3D$ the type is *quaternionic* (pseudoreal). The general result is now easily obtained from the Clifford periodicity-8 theorem for spin groups, which itself can be easily obtained from the *finite* Clifford groups [12]. The result for the Type T of $Spin(n)$ irrep is

$$Dim \, 8n+3, 8n+4, 8n+5 \; : T = -1 \, (peudoreal) \tag{18}$$

$$Dim\, 4n+2, : T = 0\, (complex) \tag{19}$$

$$Dim\, 8n+1, 8n, 8n-1\ : T = +1\, (real) \tag{20}$$

In the first case, because $Spin(n)$ group lies inside the symplectic part, the normal situation obtains. The Id *irrep* is in the antisymmetric part. In the third case, is the opposite: the Spin group lies in the orthogonal part, and the Id *irrep* is in the symmetric part. This generalizes the cases $D = 3$ and $D = 8$ respectively.

In the complex case, 2 mod 4, the Id *irrep*, being real, has to be in the (mixed) product of the two complex conjugate *irreps*; by putting them together we get real fields (Majorana). There are two Id *irreps*, and we can always arrange to have one in the antisymmetric part, if we wish, but it is not forced upon us. In other words, spinors in $4n + 2$ dimensions can be either bosons or fermions.

Since the governing crtiterion is the Type, whether R (real, +1), C (complex, 0) or H (quasireal, -1), the general result, as far as the argument depends on the group of the space only, is:

For $8n + 3$, +4, +5 , the usual spin-statistics connection obtains, and spinors are fermions.

For $8n \pm 1$, 0, a wrong connection extants (i.e., tensors and spinors have to be bosons).

For $4n + 2$ (complex case), spinors can be fermions or bosons, it is up to us. Tensors are bosons in all cases (correspondence principle).

4 Concluding remarks

We see that the proof of the commutation rules in arbitrary dimension is very simple; it uses only the temporal part if the kinetic term in the Lagrangian. This is in the spirit of Neuenschwander query [13] regarding a simple proof of the spin-statistics connection, extended now to arbitrary dimensions.

We find surprisingly few references to the $n-$dimensional spin-statistics question in the published literature; the reason might be that statistics deals with two or more particles, but in higher dimensions we still have to find one! Weinberg [14] is one of the few references to higher dimension spin-statistics connection; see also [15].

The derived results are for wavefunctions of theories with quantized fields, allowing variable number of particles. For a *fixed* number of identical particles one could use either symmetrized or antisymmetrized wavefunctions [16].

We are not considering dimensions 1 and 2. There is no little group in $D = 1$, as $SO(1) = \{1\}$, hence no spin, and indeed there is some freedom to chose the quantization rules: recall quantized solitons in $1 + 1$ dimension should behave as fermions (Coleman, Mandelstam 1975). Also, the covering group of $SO(2)$ is R: then there is a vast margin for statistics. The large literature for space dimension 2, where anyons live, han been reviewed e.g. by Forte [17].

5 Appendix

We include here for completeness some mathematical results regarding spin groups and spin representations; see [11] and [12]

The first eitght spin groups already reflect the *Irrep* Type as stated above, because the isomorphisms

$$\begin{array}{ccccccc} & Spin(1) & Spin(2) & Spin(3) & Spin(4) & Spin(5) & Spin(6) \\ & || & || & || & || & || & || \\ & O(1) & U(1) & SpU(1) & SpU(1)^2 & SpU(2) & SU(4) \\ Type & +1 & 0 & -1 & -1 & -1 & 0 \end{array} \tag{21}$$

and again $Spin(7)$ and $Spin(8)$ are real, Type (+1).

In three dimensions, the spin group $Spin(3)$ has a faithful *irrep* of complex dimension 2, isomorphic indeed to $SU(2) = SpU(1)$. The relation with the rotation group

$$1 \to Z_2 \to SU(2) \to SO(3) \to 1 \tag{22}$$

implies spinors rotate 1/2 turn when vectors make a full turn. In 8 space dimensions the situation is different. The group $Spin(8)$ has as center $Z_2 \times Z_2$, so it has *three* order two subgroups, whose quotients coincide with the two real chiral spin representations $\underline{8}_s = \Delta_L$, $\underline{8}_c = \Delta_R$ and the vector one, $\underline{8}_v$. A further quotient by the remaining Z_2 produces the projective group $PO(8)$ from either:

$$Spin(8) \left\{ \begin{array}{lll} Spinor_L & = \Delta_L, \dim 8 & real \\ Vector & = SO(8) & \\ Spinor_R & = \Delta_R, \dim 8 & real \end{array} \right\} \quad PO(8) \tag{23}$$

Now in 8 dimensions the three real dim-8 representations are permuted by the symmetric group in three symbols S_3 (Cartan's triality, [11]). Recall $O(8)$, with symbol D_4 is the *unique* simple Lie algebra with a large than Z_2 automorphism group. Therefore the square of either *irrep* of the three should be similar, so it is impossible that the chiral *irrep* and the vector *irreps* differ in the symmetry type of the product. Indeed, the products of these 8$-$dim irreps are (with + sym, $-$ asym parts)

$$\begin{array}{ll} Vector^2 & = graviton(35+) + dilaton(1+) + 2 - \text{form}(28-) \\ Spinor_L^2 & = selfdual + 4{-}form(35+) + scalar(1+) + 2{-}form(28-) \\ Spinor_R^2 & = antiselfdual\ 4{-}form(35+) + scalar(1+) + 2{-}form(28-) \\ Vector \times Spinor & = Gravitino(56) + vector(8) \\ Spinor_L \times Spinor_R & = 3{-}form(56) + vector(8) \end{array}$$

We use the particle content of SuperGravity $\mathcal{N} = 2$ in ten dimensions, whose massless little group is $O(8)$.

Indeed, the exceptional Lie algebra F_4 contains the four fundamental *irreps* of D_4: dim $F_4 = 52 = 8 \times 7/2 + 3 \times 8$. The Weyl group of F_4 (order 1152) is the symmetry group of the 24$-$cell, the most symmetric of the regular polytopes, living in 4 dimensions [18], and tessellating S^3.

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