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What price the spin–statistics theorem?

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Abstract. We examine a number of recent proofs of the spin–statistics theorem. All, of course, get the target result of Bose–Einstein statistics for identical integral spin particles and Fermi–Dirac statistics for identical half-integral spin particles. It is pointed out that these proofs, distinguished by their purported simple and intuitive kinematic character, require assumptions that are outside the realm of standard quantum mechanics. We construct a counterexample to these non-dynamical kinematic 'proofs' to emphasize the necessity of a dynamical proof as distinct from a kinematic proof. Sudarshan's simple non-relativistic dynamical proof is briefly described. Finally, we make clear the price paid for any kinematic 'proof'.

Keywords. Spin–statistics; rotational invariance; non-relativistic fields.

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1. Introduction

A number of proofs of the connection between spin and statistics which have recently appeared in the literature are reviewed and shown to contain an arbitrary phase factor. In these proofs, the correct connection is characteristically the result of a choice which is not dictated by quantum mechanics in its usual form, as already acknowledged by the proponents of perhaps the most sophisticated of these proofs [1,2]. We review a simpler proof which has long been available [3–5]; (for a review and extensive references, see [6,7]), based on the action principle and the symmetry (antisymmetry) of the scalar product of two tensors (spinors), which retains all the precepts of canonical quantum theory.

It is generally accepted that identical particles in quantum theory have a unique connection between spin and statistics: identical integral spin particles obey Bose–Einstein statistics and have a wave function symmetric under the exchange of any two particles. Identical half-integral spin particles obey Fermi–Dirac statistics and have a wave function antisymmetric under the exchange of any two particles. The 'standard' proof of this result [8] is based on relativistic quantum field theory which might seem inappropriate and possibly even irrelevant for such important and extremely non-relativistic systems as atoms, molecules, conduction electrons, lasers, liquid helium, micro-degree Bose–Einstein condensates, phonons, etc. As a consequence, there is strong motivation to derive the spin–statistics connection without making use of relativistic invariance of field theory [9,10].

The most explicit discussion has been given recently by Berry and Robbins who consider two isolated identical particles with spin S, spin projections m_1 and m_2, located at $\vec{r}$ and $-\vec{r}$, corresponding to the state

$$\Psi = |\psi(\vec{r}); m_1, m_2\rangle.$$

Under exchange of the particles, this state goes into

$$\Psi =\sim |\psi(-\vec{r}); m_2, m_1\rangle.$$

Berry and Robbins seek an intuitive exchange operator which is related to – but cannot be exactly equal to – a simple rotation through an angle π around the center of mass. As pointed out by Bacry [11], when the configuration is particularly simple an ordinary rotation through π can accomplish this transformation. To circumvent complications which arise when the spins are not antiparallel and perpendicular to their separation vector $\vec{r}$ [12], Berry and Robbins construct an exchange operator which is an element of an Abelian unitary group in a space of dimension $(4S+1)(4S+2)(4S+3)/6$, enlarged from the $(2S+1)^2$-dimensional physical space. With this construction, the 'rotation' through $\theta = \pi$ transforms Ψ into Ψ_{ex} and is to be identified as the exchange operator. As θ increases from 0 to π, it is a parallel transport of the spins which, however, is accomplished at the price of making excursions into unphysical states. But for $\theta = 0$ and $\theta = \pi$, physical states *are* mapped onto physical states, and the particle exchanged state is identified as the transported state. In what sense a transformation which must be invisible except at its end-points can be regarded as intuitive and physical or continuous and single valued is not obvious. Projected from the higher dimensional space of unphysical states to the space of physical states, it need be neither differentiable nor single valued nor unitary. It is at the least trans-quantum mechanics.

2. Simple derivations of the spin–statistics connection

Bacry considers the simplest case of two particles of spin $S = \frac{1}{2}$, z-projections $m_{1,2} = \pm\frac{1}{2}$ at $\vec{r}_{1,2} = \pm a\hat{y}$ on the y-axis. The system is rotated through an angle π around the x-axis by the operator

$$\mathscr{R}_x(\pi) = \mathrm{e}^{i\pi(L_{x,1}+(1/2)\sigma_{x,1}+L_{x,2}+(1/2)\sigma_{x,2})} \tag{1}$$

which changes $\vec{r}_{1,2}$ to $\vec{r}_{2,1}$, $m_{1,2}$ to $m_{2,1}$ and changes the overall sign. In eq. (1),

$$\begin{aligned} L_{x,1} + L_{x,2} &= \vec{q}_1 \times \vec{p}_1 + \vec{q}_2 \times \vec{p}_2 \\ &= L_{\text{CM}} + L_{\text{rel}} = \vec{Q} \times \vec{P} + \vec{q} \times \vec{p}, \end{aligned}$$

where

$$\vec{Q} = \frac{\vec{q}_1 + \vec{q}}{2} \qquad \text{and} \qquad \vec{P} = \vec{p}_1 + \vec{p}_2$$

are the center of mass variables;

$$\vec{q} = \vec{q}_1 - \vec{q}_2 \qquad \text{and} \qquad \vec{p} = \frac{\vec{p}_1 - \vec{p}_2}{2}$$

are the relative variables. Since rotation is about the CM, $\vec{P} = 0$ and $L_{\mathrm{CM}} = 0$, so the rotation operator becomes

$$\mathscr{R}_x(\pi) = \mathrm{e}^{i\pi((1/2)\sigma_{x,1}+(1/2)\sigma_{x,2}+L_{\mathrm{rel},x})}. \tag{2}$$

(The CM wave function is not taken into account.) That is,

$$\mathscr{R}_x(\pi)\Psi(1,2) = -\Psi(2,1) \to (-)^{2S}\Psi(2,1) \equiv \Psi_{\mathrm{ex}}(1,2), \tag{3}$$

as required, antisymmetric under the exchange of two spin-$\frac{1}{2}$ particles, and exhibiting the characteristic 'Pauli' phase $(-)^{2S}$. However, all that can *really* be said is that the rotated state and the exchanged state are the same within a phase factor. So

$$\mathscr{R}_x(\pi)\Psi(1,2) = -\Psi(2,1) = \mathrm{e}^{i\phi}\Psi_{\mathrm{ex}}(1,2). \tag{4}$$

If we demand that a 2π rotation should leave everything unchanged, then $\mathrm{e}^{2i\phi} = 1$ and the phase factor $\mathrm{e}^{i\phi} = \pm 1$. Bacry and also Berry and Robbins make the assumption that the phase factor must be $+1$, but there is no other reason to do so than to obtain the spin–statistics connection. (See the counterexample in §5.) A spin-$\frac{1}{2}$ particle is a ready example of a case where a 2π rotation returns to the same physical state, but changes the sign of the wave function, so we cannot simply assume that identical configurations have identical wave functions. Their wave functions are related by a phase which depends on their histories. We [3–7] have derived the spin–statistics connection using the kinetic Lagrangian density of second-quantized hermitian fields. The two simplest cases are the scalar Klein–Gordon field in the linearized form

$$\mathscr{L}_{\mathrm{KG}} \sim \frac{1}{2}\{-\dot{\pi}\phi + \phi\dot{\phi}\} + \sum_{jk}\phi_j m_{jk}^2\phi_k + \sum_j \nabla\phi_j\nabla\phi_j$$

and the electromagnetic field with the electric term

$$\mathscr{L}_{\mathrm{EM}} \sim \frac{1}{2}\{\dot{E}_j A_k - E_j\dot{A}_k\}g_{jk}$$

and the magnetic term

$$(\nabla\times A)_j(\nabla\times A)_k g_{jk}$$

and the constraint $\nabla\cdot E = 0$. Both of these kinetic Lagrangian densities exhibit characteristic antisymmetric time derivative terms and symmetric 'mass' terms. Hermiticity and exchange invariance of the Lagrangian establish the symmetry type of the numerical matrices m^2 and the metric g_{jk}, and rotational invariance of the Lagrangian density (or equally well, the Hamiltonian density) combine to require these integral spin fields to commute. Bose–Einstein statistics and symmetric wave functions are a necessary consequence.

An analogous result holds for the (hermitian-)Dirac field where the mass term in the Dirac Lagrangian is

$$\mathscr{L}_{\mathrm{D}} \sim \sum_{r,s} M_{r,s}\psi_r\psi_s.$$

Again hermiticity and exchange invariance are required of the Lagrangian density. The difference for half-integral spin particles is that the rotationally invariant combination of two spinors is antisymmetric. Witness the spin-singlet combination $\alpha\beta - \beta\alpha$ of the usual Pauli spinors. This is in contrast with the symmetric rotationally invariant product of, for example, vector components $x^2 + y^2 + z^2$ (symmetry of g_{jk} above) for integral spin particles. The net result for half-integral spins is that hermiticity, exchange invariance and rotational invariance of the kinetic Lagrangian combine to require anticommutation for the half-integral spin fields. Fermi–Dirac statistics and antisymmetric wave functions follow. The kinetic Lagrangian for the Schrödinger field is

$$\psi_j^\dagger i \overleftrightarrow{\partial_t} A_{jk}\psi_k$$

leading to

$$A_{jk} i\partial_t \psi_k = -\left(\frac{\delta\mathscr{L}_{\text{int}}}{\delta\psi_j^\dagger}\right) = H_{jk}\psi_k. \tag{5}$$

For our proof of the spin–statistics theorem ψ and $\psi^\dagger$ should be written in terms of hermitian fields ϕ, χ of the same spinor or tensor character, and their transpose ϕ^{T}, χ^{T},

$$\psi = \phi + i\chi \quad \text{and} \quad \psi^\dagger = \phi^{\mathrm{T}} - i\chi^{\mathrm{T}}.$$

Finally the hermitian fields must be 'stacked' to give

$$\Phi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

For spin-$\frac{1}{2}$ this is a hermitian spinor (majorana) field of dimension $2(2 \times \frac{1}{2} + 1) = 4$. Rewriting the Schrödinger kinetic Lagrangian, we obtain

$$\mathscr{L} = \Phi^{\mathrm{T}} \mathscr{A}\, i \overleftrightarrow{\partial_t}\, \Phi$$

in an obvious notation. Hermiticity and rotational invariance dictate the symmetry type of the numerical matrix $\mathscr{A}$ and exchange invariance of the kinetic Lagrangian then establishes the spin–statistics connection as in the above examples.

There are loose ends for which we refer to our earlier paper [7]. These include:

1. The use of somewhat unfamiliar hermitian fields, which demands the introduction of $2 \times 2(2S+1)$ components obeying linear equations, to represent complex charged fields of higher spin.
2. The above arguments follow from either the mass terms or the kinetic energy terms in the Lagrangian.
3. The possibility of antisymmetrising on 'charge' degrees of freedom, thereby defeating the proof. We show that if the charge degrees of freedom are not the unit matrix, they can be diagonalized with equal and opposite energy spectra, implying an energy spectrum unbounded below, which we reject. The requirement of a lower bound to the spectrum (and positivity of the norm) is as close as we come to special relativity but it clearly can be made independently (from the second law of thermodynamics) leaving a fully non-relativistic proof.
4. The Lagrangian density will usually possess a Lorentz invariance or a Galilean invariance, but only the Euclidean (rotations and translations in R^3) invariance plus time translation (Newtonian group) invariance is needed in the proof; but no boosts (neither Lorentz nor Galilei).

5. The neglect of interactions and the use only of the quadratic kinetic Lagrangian. Here we do require that 'the tail wags the dog', and that all terms in the Lagrangian must respect the exchange properties of the kinetic energy and mass terms.
6. Unusual cases such as fractional statistics (in two dimensions) are excluded from consideration of the Schwinger action principle presented here. Green's parastatistics can be consistent only with multicomponent Hilbert spaces ('vector' quantum mechanics), and so they are excluded here. Nor do we address topological extended structures in this paper.
7. It is important to emphasize that this is an optimal proof in that counterexamples appear if any postulate is relaxed or omitted.

3. Homotopy and multivalued wave functions

For closed loop excursions in parameter space, quantum amplitudes undergo phase changes of a subtle kind [13]. If the space is not simply connected, there is a non-trivial homotopy group [14] which can associate a 'winding'-number and a net phase change with non-contractible 2π excursions. The state vectors belong to a one-dimensional representation of the (Abelianized) first homotopy group. For two identical particles in R^3-space with coincident points Δ removed and exchanged points identified and counted once, the configuration space is

$$\mathscr{M} = (R^3 \times R^3 - \Delta)/S_2 \quad \text{with} \quad \pi_i(\mathscr{M}) = S_2. \tag{6}$$

There are two kinds of closed loops: contractible and non-contractible. There are only two irreducible representations of S_2: the first assigns a $(+1)$ to both elements; the other (which is faithful) assigns $(+1)$ to contractible loops but (-1) to non-contractible loops. For spatial wave functions this odd parity is for odd orbital momentum. The 'trivial' representation of the homotopy group – assumed in all the above proofs based on particle transport – is not demanded for any other reason, since the other choice is consistent with all other requirements of quantum mechanics. In many-body physics (in dimension three) the N-particle manifold is

$$\mathscr{M}_N = (R^3 \times R^3 \times \cdots \times R^3 - \Delta)/S_N,$$

and the homotopy group is $\pi_1(\mathscr{M}_N) = S_N$. Because we use only the 'scalar' Hilbert spaces only one-dimensional representations are allowed. These are the Bose and Fermi systems which are the two representations of the Abelianized π_1, i.e., Abelianized S_N which gives S_2 (just Z_2, the cyclic group of order two). The generic Green ansatz and parastatistics are in conflict with ordinary ('scalar') quantum mechanics. By taking reducible representations of S_N we can include an auxiliary (color) label in the constituent fields.

4. Physical transformation for spin exchange

The 'parallel transport' operator constructed by Berry and Robbins to accomplish exchange has the blemish that it involves extra degrees of freedom which result in excursions outside the space of physical states, although the completed rotations by π, 2π ... associated with exchange do not. In this section we construct an alternative operator for the exchange of arbitrary spins which uses only physical operators and remains within the space of physical states. The net result, however, is the same phase ambiguity already met by Bacry, Berry and Robbins, and Balachandran *et al*, which – in our view – precludes any conclusion about the spin–statistics connection. A spin exchange operator for two spin - $\frac{1}{2}$ particles A and B is

$$F = \frac{1}{2}\left(1^A \cdot 1^B + \vec{\sigma}^A \cdot \vec{\sigma}^B\right) \tag{7}$$

with

$$F^2 = 1^A \cdot 1^B. \tag{8}$$

On symmetric spin states ($S = 1$, triplet), $F = +1$; and on antisymmetric spin states ($S = 0$, singlet), $F = -1$. This operator can be embedded in the one-parameter group

$$G_{\pm}(\theta) = \left\{\frac{1 \mp F}{2} + \mathrm{e}^{i\theta}\frac{1 \pm F}{2}\right\}\mathrm{e}^{iL\theta}. \tag{9}$$

For $\theta = 0$, $G(O) = 1$; for $\theta = \pi$,

$$G(\pi) = \left\{\frac{1 \mp F}{2} - \frac{1 \pm F}{2}\right\}(-1)^L. \tag{10}$$

The factor $(-1)^L$ provides for the inversion of the Bacry-type space wave function to complete the exchange. If we chose the upper sign we would get a factor $+1$ for the antisymmetric state and -1 for the symmetric state. To get the usual connection we must choose the lower sign

$$G(\theta) = \left\{\frac{1 + F}{2} + \mathrm{e}^{i\theta}\frac{1 - F}{2}\right\}\mathrm{e}^{iL\theta}. \tag{11}$$

Note that $G(\theta)$ does not move out of the space of physical states as was the case with the parallel transport operator of Berry and Robbins. It is defined to act only on the two spins in their symmetric or antisymmetric states, where it produces the correct sign. Observe that we could equally well, reverse the sign of $G(\pi)$ with a factor $\mathrm{e}^{-i\theta}$ and get the wrong connection. Nor are we obligated to consider a continuous transformation, but the spin-exchange operator $G(\pi)$ is sufficient. One can generalize this construction for arbitrary spin S. The spin matrices would now be $(2S+1)\times(2S+1)$ dimension. The unit matrix and the spin matrices no longer constitute a complete set, but one could add the five quadrupole matrices, the seven sextupole matrices, etc., up to the $(2S)$th-pole matrices. These are

$$1 + 3 + 5 + 7 + \cdots = (2S+1)^2$$

independent matrices which form a basis for the $(2S+1)^2$-dimensional Hilbert–Schmidt operators. From the fact that the product of any two of these matrices has vanishing trace, there is a unique decomposition for any $(2S+1)\times(2S+1)$ matrix. It follows that

$$\mathbf{F} = \frac{1}{(2S+1)}\left\{\mathbf{1}^A \cdot \mathbf{1}^B + \mathbf{S}^A \cdot \mathbf{S}^B + \mathbf{Q}^A \cdot \mathbf{Q}^B + \cdots\right\}. \tag{12}$$

F has eigenvectors in the space of the two spin wave functions which are symmetric or antisymmetric. The projection operators to these states are

$$\mathbf{\Pi}_{\pm} = \frac{\mathbf{1} \pm \mathbf{F}}{2}. \tag{13}$$

We embed these into

$$\mathbf{G}_{\mp} = \left\{ \mathbf{\Pi}_{\pm} + \mathrm{e}^{i\theta} \mathbf{\Pi}_{\mp} \right\} \mathrm{e}^{iL\theta}. \tag{14}$$

The upper sign $\mathbf{G}_{-}$ gives the normal connection for integral spins, the lower sign $\mathbf{G}_{+}$ for half-integral spins. This follows from the fact that for integral spin, **1**, **Q**,... are symmetric and **S**, ... are antisymmetric; and the opposite for half-integral spins. This construction generalizes Bacry's picture in the way sought by Broyles. In this construction we do not go outside the $(2S+1) \times (2S+1)$-dimensional physical spin-space. The construction is natural and complete without further embedding, in contrast to the 'physically incoherent' examples discussed in Appendix D of [1]. In these derivations [1,2,9,11], there is still a choice to be made. In our construction, the choice of phase for $G(\theta)$ gave the normal connection. Should we include the non-trivial representations of the homotopy (loop) group? Without the assumptions of Bacry and of Berry and Robbins, we can make no statement about the spin–statistics connection. The discussion of the properties of extended systems under interchange by Balachandran *et al* [10] also suffers from the need to choose the trivial representation of the homotopy group [15].

A recent attempt to obtain the spin–statistics connection from arguments of continuity and differentiability of the wave function has been presented by Murray Peshkin [16]. Peshkin has required that the wave function on the (non-contractible) loop from $(x, y, 0)$ to $(-x, -y, 0)$ should be continuous. It is precisely this assumption that all the loops including the non-contractible ones should have the wave function return to its original value that was made by Berry and Robbins; they therefore beg the question.

5. Counterexample to non-dynamical derivations of the spin–statistics connection

To remove the ambiguity between the normal and the abnormal spin–statistics connection, we must go beyond the kinematical transport of the spins, or paths which interchange the two particles. As a demonstration of this necessity, consider two spin-0 creation and annihilation operators which anticommute:

$$\{a, a^{\dagger}\}_{+} = \{b, b^{\dagger}\}_{+} = 1 \quad \text{and} \quad \{a, b\}_{+} = \{a^{\dagger}, b^{\dagger}\}_{+} = 0. \tag{15}$$

Now construct a state of two spinless fermions

$$f(\vec{r}) a^{\dagger} b^{\dagger} |0\rangle.$$

The continuous transformation which interchanges these particles can be accomplished according to Bacry or Berry and Robbins by the trivial rotation $\mathrm{e}^{iL\theta} = (-1)^{L}$ for $\theta = \pi$, and presents no barrier to the abnormal spin–statistics connection. Similarly we can construct spinor bosons.

We need more than just the choice of the spins to deduce commutation or anticommutation. This is illustrated by the Schwinger action principle [17] which defines a fundamental bracket for the ψs. (Note that we are assuming hermitian fields, so the usual $\psi^\dagger$ is a linear combination of the ψs appearing here.) The Schwinger action principle requires

$$(\psi_r, \psi_s) = \left(A^{-1}\right)_{rs}$$

where A is the numerical matrix in the kinetic Lagrangian (for restrictions originating in second class constraints, see [18]). It follows from the necessary symmetry or antisymmetry of A that the commutator bracket is required for tensor fields, and the anticommutator for spinor fields. The dynamics defined by the (kinetic part of the) Lagrangian $\psi A i \partial_t \psi$ specifies the commutation properties and thereby the statistics of the field by the symmetry-type of A. This structure of the kinetic Lagrangian establishes, not by fiat but by derivation, that only the trivial representation of the homotopy group is realized.

6. Conclusions

In our simple non-relativistic proof, we trace the spin–statistics connection to the symmetry (antisymmetry) of the scalar product of two tensors (spinors). The quadratic kinetic Lagrangian density must define the exchange type for the complete Lagrangian.

No purely kinematic derivation of the spin–statistics connection can be carried through without assuming the trivial representation of the homotopy group as our counterexample of scalar fermions shows. No such assumption is required if the kinematics is imposed by the action principle and the symmetry of the scalar product for tensor fields and its antisymmetry for spinor fields.

Finally, to the invocation to ‘consider two isolated identical particles’, our response ultimately must be ‘No, we will not!’. To do so is to abandon the formal structure of quantum mechanics based on the action principle [19–21]; and the powerful machinery of second-quantization [22] for treating the many-body problem in quantum mechanics. Without this formal foundation of quantum theory, what is left is a world only superficially and at first glance simpler, but in fact arbitrarily complicated by the absence of any constraining – we prefer to think in terms of ‘guiding’ – principle.

For the specific problem of identical particle exchange, we maintain that the attempts to represent the exchange operation as the end-point of a continuous physical transformation fails for reasons even more fundamental than the phase ambiguity already discussed at length. These attempts fail because (in the world of many-body quantum mechanics) the requisite operators do not exist. There is no physical operator which affects electrons 1 and 2 but not 3. So it is impossible to transport, and thereby permute, 1 and 2 and leave 3 untouched. The rotation operator $J_{1,2} = J_1 + J_2$ does not exist in isolation of J_3. These attempts fail because any physical transportation must be generated by the total conjugate operator including all identical particles symmetrically, and not just any subset.

So what then about exchange as the end result of an actual physical transportation? Exchange is similar to but essentially different from other discrete transformations like space, time, and charge-reversal: similar in being discrete, different in being a proper transformation which leads to the temptation to embed it in a continuous ‘transport’. But different from all other transformations in one essential way: it deals with identical particles a pair at a time, in a piecemeal transformation of the identical particle state vectors which is intrinsically inaccessible to any transformation generated by the dynamical variables of the canonical quantum mechanics of identical particles.

The paradox would not arise if we were to exercise perfect semantic precision. The words 'exchange' and even 'permute' give rise to the vision of classically and manually moving pieces on a chessboard, a physical actionable transformation. A better word would be 'relabel', which is an arithmetic but not a physical act, inevitably an alias-transformation, but (in this one case) not an *alibi*-transformation.

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