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Frictional trajectories near a barrier: A dissipationless Newtonian approach

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Abstract. We address the problem of classical frictional motion under a potential V possessing a barrier, apart from other possible confining and nonstationary terms. It is pointed out that the Green's solution of the exact equation of motion can be reduced (under suitable conditions) either to an improved Rayleigh form or a non-Rayleigh form, the latter being outside the scope of the standard large-friction treatment of the Fokker–Planck equation. The resulting dissipationless dynamics involves an appropriately scaled potential which may have promising applications to quantum stochastic phenomena. Genuine dissipative corrections in regions far away from the barrier can be accounted for by the higher-order terms in our asymptotic expansions.

Keywords. Friction; barrier; Rayleigh; Green's solution; dissipationless.

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1. Introduction

The properties of *single-particle*, nonconservative, frictional motion in classical [1], quantum [2], stochastic [3] and statistical [4] mechanics are very important conceptually and quite interesting application-wise. Damping effects on classical trajectories are theoretically introduced via a linear-velocity term [1a] in an extension of Newton's law or a quadratic dissipation function [1b] in a generalization of Lagrange's equations or an explicitly time-dependent factor [1c] in the Bateman–Caldirola–kanai (BCK) Hamiltonian. Numerous applications of frictional trajectories include the calculation of deterministic collision [1d] of nuclear heavy ions and probabilistic description of phenomena-like Brownian movement [3a] of the harmonic oscillator, signal-to-noise ratio for stochastic resonance [3b] in a double well etc. It may be noted that the problems tackled in refs, [1d] and [3b] involve a potential energy which has a pronounced barrier (see figure 1) apart from other possible nonlinear/nonstationary terms.

In the above context, previous workers have employed two different types of theoretical modelling, viz., *the standard frictional equation* and its *overdamped Rayleigh version*. We review their salient features in § 2 and point out that generally these equations have been solved [1d, 3b] by numerical computation/analog simulation and their associated Lagrangians are inconvenient to use in practice. In § 3 we carefully set up the formal Green's solution of the exact equation of motion and show analytically that the

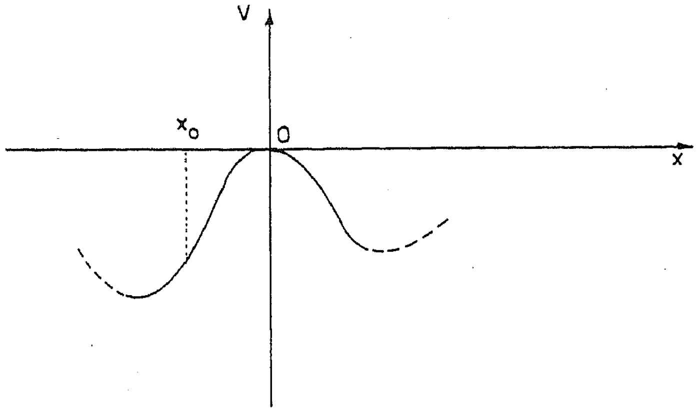

Figure 1. Schematic plot of the potential energy as a function of position (at a specific time). The barrier is explicitly shown. The dashed portions represent other confining and nonstationary parts of the potential.

non-Newtonian dissipational trajectory near the barrier approximately coincides with a Newtonian dissipationless trajectory provided the applied potential is scaled by a suitable factor. Our concluding remarks appear in § 4 where it is emphasized that a Lagrangian employing such a scaled potential can be useful both for path-integral based quantization and for the treatment of quantum stochastic resonance. The algebraic details of our formulation are relegated to the Appendix for convenience.

2. Review of existing models

Consider a test particle moving in an environment and let the symbols

$$m, t, x, v = \mathrm{d}x/\mathrm{d}t, V, F = -\partial V/\partial x, \gamma \tag{1a}$$

respectively denote the mass, time, position, velocity, external potential, applied force and the coefficient of friction. This V may contain possible nonlinear terms in position and/or nonstationary terms in time. The symbols

$$V_I = V_I(x); \quad F_I = -\partial V_I(x)/\partial x \tag{1b}$$

will refer to the particular situation when the potential and force do not contain the time explicitly.

2.1 *Standard frictional equation*

The basic equation of motion for the unknown trajectory $x(t)$ valid for arbitrary damping, reads

$$\mathrm{d}v/\mathrm{d}t + \gamma V = F/m \tag{2a}$$

subject to the initial conditions

$$x(t=0) = x_0; \quad v(t=0) = v_0. \tag{2b}$$

Table 1. Salient features of the standard frictional model (cf. eq. 2). The symbols V_I and F_I refer to a potential and force not depending on t explicitly.

Item	Standard frictional model
Motivation	Dissipative force proportional to velocity is motivated experimentally
Parameters	The barrier frequency Ω and the coefficient of friction γ are arbitrary
Estimated velocity	For general potentials with a barrier v is obtained by numerical computation/analog simulation
Effective acceleration	$\frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma v + \frac{F}{m}$
Branch of motion	Both types of branches are present depending on whether the particle is sliding down (Rayleigh-like branch) or climbing up (non-Rayleigh-like branch) the barrier
Lagrangian	$L = \mathrm{e}^{\gamma t}(mv^2 - V)$ due to Bateman–Caldirola–Kanai which is explicitly time-dependent
Mechanical energy	$E_I = mv^2/2 + V_I$ $\mathrm{d}E_I/\mathrm{d}t = -m\gamma v^2 < 0$

As far as x as a function of t is concerned, eq. (2a) is a second-order differential equation of the deterministic (stochastic) type according as the temporal of the force F is definite (random). Equation (2a) is equivalent to the *formal* expression

$$v = v_0\,\mathrm{e}^{-\gamma t} + \mathrm{e}^{-\gamma t}\int_0^t \mathrm{d}t'\,\mathrm{e}^{\gamma t'} F'/m, \tag{2c}$$

where $F' = F(x(t'), t')$ is the force at the integration time t' when the position becomes $x(t')$. Other known features of the standard frictional equation are summarized in table 1. It is seen that irreversible dissipation of the mechanical energy $E_I = mv^2/2 + V_I$ is an inherent property of the model so that a dissipationless approximation to the motion is never attempted. Furthermore, since the underlying BCK [1c, 2a] Lagrangian $L = (mv^2/2 - V)\,\mathrm{e}^{\gamma t}$ is explicitly time-dependent, its path integration in general, would require complicated numerical matrix multiplication [5a].

2.2 *Overdamped Rayleigh equation*

If the coefficient of friction is large enough, the right-hand-side of eq. (2c) can be evaluated by repeated partial integration to yield

$$v \approx F/m\gamma - \dot{F}/m\gamma^2 + \cdots; \quad \gamma t \gg 1 \tag{3a}$$

where $\dot{F} = \mathrm{d}F/\mathrm{d}t$. On the so-called Rayleigh branch of motion (labelled by the superscript R), the leading term of eq. (3a) is picked up, giving a velocity parallel to the applied force,

$$v^{(R)} = F/m\gamma. \tag{3b}$$

As regards the trajectory $x^{(R)}$ as a function of t, this is a first-order differential equation of the deterministic (stochastic) variety depending on whether F has a definite (random) temporal profile. Table 2 summarizes the main features of the overdamped Rayleigh

Table 2. Salient features of the overdamped Rayleigh model (cf. eq. 3). Nonleading terms of order $\dot{F}/m\gamma$ have been suppressed.

Item	Conventional Rayleigh model (label R)
Motivation	The frictional force balances the applied force
Parameter	γ large, $\gamma t \gg 1$
Estimated velocity	$v^{(R)} = \dfrac{F}{m\gamma}$
Effective acceleration	$\dfrac{dv^{(R)}}{dt} \approx -\gamma v^{(R)} + \dfrac{F}{m} \approx 0$
Branch of motion	Emphasis is on those branches where v and F are almost parallel as happens when the particle is sliding down the barrier in figure 1
Lagrangian	$L^{(R)}$ does not exist because the acceleration has become trivial
Potential energy	$V_I^{(R)}(x)$ such that $dV_I^{(R)}/dt = -m\gamma v^{(R)^2} < 0$

equation. It is observed that there is no recipe available to handle the so-called non-Rayleigh branch on which the velocity would be antiparallel to the external force. Furthermore, eq. (3b) being of the first order can not follow from a usual Lagrangian in which $x^{(R)}$ is the only degree of freedom.

3. The dissipationless Newtonian model

3.1 *Preliminaries*

In view of what has been said above it is worth examining (for finite γ) questions such as the possibility of a dissipationless picture, existence of a non-Rayleigh branch, construction of a convenient Lagrangian, etc. This task will be accomplished analytically for those problems [1d, 3b] in which the input potential includes a parabolic barrier. Then the full external force on the particle can be decomposed as

$$F = F_B + F_p = m\Omega^2 x + F_p(x,t) \tag{4}$$

where F_B is the linear force due to the barrier, Ω the corresponding angular frequency and all the remaining contributions are lumped together in the perturbation force F_p which is supposed to be confining at large distances. In the sequel we shall also need the useful symbols

$$\bar{\Omega} = (\Omega^2 + \gamma^2/4)^{1/2}; \quad \Omega_\pm = \bar{\Omega} \pm \gamma/2. \tag{5}$$

3.2 *Formulation*

The standard frictional equation (2a) is rewritten as

$$\left[\frac{d^2}{dt^2} + (\Omega_+ - \Omega_-)\frac{d}{dt} - \Omega^2\right] x = \frac{F_p}{m}. \tag{6}$$

In the Appendix we construct its formal solution via Green's functions and impose the condition that the absolute magnitude of the trajectory x must not grow exponentially as $t \to \infty$. This leads to a pair of exact, formal relations involving integrals (see Appendix, eqs (A8), (A9a, b) between v, x and F_p, which can be suitably approximated depending upon the mechanical branch of interest.

3.3 *The 'improved' Rayleigh branch*

Suppose the particle is sliding down the potential hill of figure 1 so that its velocity is essentially parallel to the applied force. The particle then takes relatively less time to cover a given distance down the hill. After the system has relaxed with respect to the time scale Ω_+^{-1} one employs the asymptotic expansion (All) and finds for the velocity

$$v^{(R)^*} = F/m\Omega_+ - \cdots; \quad \Omega_+ t \gg 1. \tag{7a}$$

Here the dots ... stand for nonleading terms of order $\dot{F}_\mathrm{p}/m\Omega_+^2$ and the superscript $(R)^*$ labels the Rayleigh branch as improved by our analysis. Of course, in the limit $\gamma/\Omega \gg 1$ the above result coincides with the usual Rayleigh estimate $F/m\gamma$. The improved expression for the acceleration is given by

$$\frac{\mathrm{d}v^{(R)^*}}{\mathrm{d}t} = -\gamma v^{(R)^*} + \frac{F}{m} = q^{(R)^*}\frac{F}{m}; \quad q^{(R)^*} = \frac{\Omega_-}{\Omega+} \tag{7b}$$

which indeed has the appearance of a dissipationless Newtonian equation under a force that has been scaled down by a factor $q^{(R)^*} < 1$. Finally, the improved equation of motion (7b) does follow conveniently from a Lagrangian

$$L^{(R)^*} = mv^2/2 - V^{(R)^*} \quad \text{where } V^{(R)^*} = q^{(R)^*}V, \tag{7c}$$

in sharp contrast to the customary Rayleigh model. Clearly, if V does not contain t explicitly the improved energy $E_I^{(R)^*} = mv^2/2 + V_I^{(R)^*}$ is approximately conserved, unless one goes far from the barrier.

3.4 *The non-Rayleigh branch*

Next, we turn to the very interesting case of the particle climbing up the potential hill in figure 1, so that the velocity is essentially antiparallel to the applied force. The particle then takes a relatively long time to cover a given distance up the hill. For times large compared to $1/\Omega_-$ employing the asymtotic expansion (A12) we obtain for the velocity

$$v^{(N)^*} = -F/m\Omega_- - \cdots; \quad \Omega_- t \gg 1. \tag{8a}$$

Here the dots ... represent nonleading terms of order $\dot{F}_\mathrm{p}/m\Omega_-^2$ and the superscript $(N)^*$ labels the non-Rayleigh branch under consideration. The corresponding acceleration is deduced from

$$\frac{\mathrm{d}v^{(N)^*}}{\mathrm{d}t} = -\gamma v^{(N)^*} + \frac{F}{m} = q^{(N)^*}\frac{F}{m}; \quad q^{(R)^*} = \frac{\Omega_+}{\Omega-} \tag{8b}$$

which again has the look of a dissipationless Newtonian equation but under a force that has been scaled up by a factor $q^{(N)^*} > 1$. Of course, the above equation of motion follows from a Lagrangian

$$L^{(N)^*} = mv^2/2 - v^{(N)^*}; \quad V^{(N)^*} = q^{(N)^*}V, \tag{8c}$$

which will not depend on t explicitly if V does not, in sharp contrast to the BCK [1c, 2a] model. The corresponding energy $E_I^{(N)^*} = mv^2/2 + V_I^{(N)^*}$ becomes an approximate constant of motion unless we go far from the barrier.

3.5 *Physical interpretation and numerical estimates*

From the conceptual viewpoint we can say that the effect of the $-\gamma v$ term in the exact frictional equation for dv/dt is roughly simulated by altering the force in the dissipationless formulation. The scale factor $q^{(N)^*}$ in eq. (8b) is greater than unity and is an increasing function of γ. This physically implies that as the coefficient of friction increases, the effective potential hill $V^{(N)^*}$ to be surmounted by the particle becomes higher and the deceleration grows. Let us estimate a typical numerical range for the ratio γ/Ω for which the asymptotic expansion (8a) on the non-Rayleigh branch holds. Suppose $\Omega + t \approx 100$, implying that the particle has completely relaxed with respect to the time scale Ω_+^{-1}. Relaxation with respect to the other time scale, viz. Ω_+^{-1}, would also have been achieved during this time provided $\Omega_- t > 1$, i.e.,

$$\Omega_-/\Omega_+ > 1/100, \quad \text{i.e. } 0 \leq \gamma/\Omega < 10. \tag{9}$$

In other words the analysis based on eqs (8a–c) remains valid over a fairly wide range of underdamped as well as overdamped motions, because $\gamma/\Omega = 0$ means no-damping and $\gamma/\Omega = 10$ implies very heavy damping.

3.6 *Comparison with statistical large-friction expansion*

When the input force includes a random component we must examine any possible link between our theory and the 'large γ expansions' in statistical mechanics, especially in the context of the Fokker–Planck [4a, b] equation. It is well known [4a] that the kramers distribution function $P(x, v, t)$ in phase space can be systematically represented by a series in inverse powers of γ (for very large values of γ) as

$$P = P^{(0)} + r^{-1}P^{(1)} + O(r^{-2}). \tag{10}$$

The first two terms of this expansion, after integration over the velocity, lead directly to the Smoluchowski distribution function $W(x, t)$ in configuration space. Reverting to the trajectory language one may say that the asymptotic reduction from Kramers to Smoluchowski distribution is equivalent to the asymptotic reduction from the underlying Langevin [4b] (cf. eqs ((2a), (b)) to the overdamped Rayleigh [4b] (cf. eqs (3b), 7(a)) stochastic differential equation (provided the noise is white Gaussian). In this context, the main results of our theory (cf. eqs (8), (9)) are entirely new because they pertain to the non-Rayleigh branch of motion, the large parameters involved are $\Omega_\pm$, and the domain of validity extends even to the extreme underdamped limit.

4. Conclusions

All our important results are collected in table 3 which should be compared against the entries of tables 1 and 2. Some additional points which deserve mention are as follows:

(a) *Limited equivalence:* We do not claim perfect equivalence between the actual frictional eq. (2a) and its proposed dissipationless versions viz. eqs (7b, 8b). As a matter of fact, for *time-independent* static potentials the exact x trajectories would eventually come to rest at a potential minimum whereas our $x^{(N)^*}$ trajectories would go on oscillating between static confining walls. However, our approximation (8b) is most effective when the particle is climbing up the hill in figure 1, and hence it can be profitably employed to calculate the first transit time from the bottom to the top of the barrier. Far away from the barrier, i.e., in the region of the confining wells, the one-term approximations (7b, 8b) may become poor. In these regions nontrivial corrections due to dissipation may be systematically made by including the $\dot{F}_{\mathrm{P}}$ terms of the asymptotic expansions (AII, 12).

(b) *Time-dependent situations*: Of greater physical interest is the case when the perturbation force F_{p} contains *explicit time-dependence* due to applied modulation, white noise, etc. It is known that at times large compared to Ω^{-1} the frictional energy loss suffered by the actual x trajectories tends to be compensated by the energy gain from the time-dependent perturbation so that the particle tends to follow the profile of F_{p} (see Appendix, eq. (A8)). Obviously, in the absence of net loss our dissipationless formulation based on the $x^{(N)^*}$ trajectories should also become equally valid.

Table 3. Salient features of the dissipationless Newtonian model (cf. eqs (7), (8)). Useful abbreviations are $\bar{\Omega} = (\Omega^2 + \gamma^2/4)^{1/2}$, $\Omega_+ = \bar{\Omega} + \gamma/2$, $\Omega_- = \bar{\Omega} - \gamma/2$, $q^{(R)^*} = \Omega_-/\Omega_+$ and $q^{(N)^*} = \Omega_+/\Omega_-$.

Item	Dissipationless Newtonian model
Motivation	The effect of deleting the $-\gamma v$ term in the exact equation of motion is sought to be partially restored by suitably scaling the external force
Parameters	$(R)^*$ branch: Ω_+large, i.e. $\Omega_+ t \gg 1$ $(N)^*$ branch: Ω_-large, i.e $\Omega_- t \gg 1$. Even the limit of zero friction is included in these inequalities
Estimated velocity	$v^{(R)^*} = F/m\Omega_+$, $v^{(N)^*} = -F/m\Omega_-$
Effective acceleration	$\mathrm{d}v^{(R)^*}/\mathrm{d}t = q^{(R)^*}F$, $\mathrm{d}v^{(N)^*}/\mathrm{d}t = q^{(N)^*}F$
Branch of motion	On the improved Rayleigh branch v and f are essentially parallel as happens while sliding down the barrier. On the non-Rayleigh branch reverse is the case
Lagrangian	$(L)^{(R)^*} = mv^2/2 - q^{(R)^*}V$, $(L)^{(N)^*} = mv^2/2 - q^{(N)^*}V$
Mechanical energy	$(E)_I^{(R)^*} = mv^2/2 + q^{(R)^*}V_I$, $(E)_I^{(N)^*} = mv^2/2 + q^{(N)^*}V_I$ which are approximately conserved in regions not too far from the barrier (i.e., not inside the confining wells)

The superscript $(R)^*$ labels the improved Rayleigh branch and $(N)^*$ labels the non-Rayleigh branch.

(c) *Quantization and stochastic applications:* As regards the frictional motion of a quantum wave packet near the barrier, our dissipationless Lagrangian $L^{(N)^*}$ (see eq. (8c)) may have bright applicational prospects. Indeed, if the input V represents time-independent mean field, the path integral corresponding to the scaled-potential $V^{(N)^*}$ can be readily computed via the short-time propagator method [5b] of Sethia *et al.* Next, if V includes time-dependent random components, the ensuing dynamics based on $V^{(N)^*}$ can be treated using the recently developed concept [6] of quantum dissipationless random motion. Finally, a detailed application of these ideas to study the quantum stochastic resonance phenomenon [7] in a double well potential under the simultaneous influence of thermal temperature, linear friction, external noise and sinusoidal modulation is in progress and the results will be reported elsewhere.

Appendix:

Solution of eq. (6)

(i) *Green's function.* Given the initial condition x_0 and v_0 we wish to solve eq. (6) written in the form

$$\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}+\Omega_+\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\Omega_-\right)\right]x=\frac{F_\mathrm{p}}{m}, \tag{A1}$$

where $\bar{\Omega}$, Ω_+ and Ω_- are given by eqs (5). The differential operator within square brackets has the causal Green's function

$$G(t,t')=\frac{-\theta(t-t')}{2\bar{\Omega}}[\mathrm{e}^{-\Omega_+(t-t')}-\mathrm{e}^{\Omega_-(t-t')}], \tag{A2}$$

where $\theta(t-t')$ is the unit step function.

(ii) *Formal solution.* If the perturbative force F_p were absent, the trajectory x_B in presence of the barrier alone would have been

$$x_B=\frac{1}{2\bar{\Omega}}[\mathrm{e}^{-\Omega_+t}(\Omega_-x_0-v_0)+\mathrm{e}^{\Omega_-t}(\Omega_+x_0+v_0)]. \tag{A3}$$

When F_p is also present the use of eqs (A2, 3) permits us to convert eq. (A1) into the integral equation

$$\begin{aligned} x &= x_B+\int_0^\infty \mathrm{d}t'G(t,t')F'_\mathrm{p}/m \\ &= x_B-\frac{1}{2m\bar{\Omega}}\left\{\mathrm{e}^{-\Omega_+t}\int_0^t \mathrm{d}t'\mathrm{e}^{\Omega_+t'}F'_\mathrm{p}-\mathrm{e}^{\Omega_-t}\left[\int_0^\infty+\int_t^\infty\right]\mathrm{d}t'\mathrm{e}^{-\Omega_-t'}F'_\mathrm{p}\right\}, \end{aligned} \tag{A4}$$

where $F'_\mathrm{p}=F_\mathrm{p}(x(t'),\ t')$ is the perturbative force at the integration time t' when the position becomes $x(t')$.

The algebraic structure of eq. (A4) shows that there are five types of terms appearing viz., $\mathrm{e}^{-\Omega_+t}$, e^{Ω_-t} and three integrals. Therefore, it will be convenient to introduce the following notation:

$$K = \Omega_- x_0 - v_0; \quad \rho_\mathrm{p} = \int_0^\infty \mathrm{d}t' \mathrm{e}^{-\Omega_- t'} F'_\mathrm{p}$$

$$N = \Omega_+ x_0 + v_0 + \rho_\mathrm{p}/m; \quad \xi_\mathrm{p} = \mathrm{e}^{-\Omega_+ t} \cdot \int_0^t \mathrm{d}t' \mathrm{e}^{\Omega_+ t'} F'_\mathrm{p}$$

$$\lambda_\mathrm{p} = \mathrm{e}^{\Omega_- t} \cdot \int_t^\infty \mathrm{d}t' \mathrm{e}^{-\Omega_- t'} F'_\mathrm{p}. \tag{A5}$$

Then the formal solution (A4) may be recast into the compact form

$$x = \frac{1}{2\bar{\Omega}} \left\{ \left[K\mathrm{e}^{-\Omega_+ t} - \frac{\xi_\mathrm{p}}{m} \right] + \left[N\mathrm{e}^{\Omega_- t} - \frac{\lambda_\mathrm{p}}{m} \right] \right\} \tag{A6}$$

(iii) *Asymptotic condition.* We shall suppose that the static part of the applied force is confining at large distances as happens, for instance, in the case of a double-well potential. Then, as $t \to \infty$ the position x can not blow up exponentially, implying that the coefficient of $\mathrm{e}^{\Omega_- t}$ in eq. (A6) should vanish, i.e.,

$$N = 0; \quad v_0 = \Omega_+ x_0 + v_0 - \rho_\mathrm{p}/m. \tag{A7}$$

(iv) *Final expressions.* Inserting the conditions (A7) into eq. (A6) we get the following exact, general expression for the position:

$$x = x_0 \mathrm{e}^{-\Omega_+ t} - [\xi_\mathrm{p} - \rho_\mathrm{p} \mathrm{e}^{-\Omega_+ t} + \lambda_\mathrm{p}]/2m\bar{\Omega}. \tag{A8}$$

Differentiation with respect to t yields two mutually equivalent representations for the velocity, viz.,

$$v = \Omega_- x - 2\bar{\Omega} x_0 \mathrm{e}^{-\Omega_+ t} + [\xi_\mathrm{p} - \rho_\mathrm{p} \mathrm{e}^{-\Omega_+ t}]/m \tag{A9a}$$

$$= -\Omega_+ x - \lambda_\mathrm{p}/m. \tag{A9b}$$

It is not difficult to verify explicitly that this v satisfies the standard frictional equation (2a) of the text along with the specified initial conditions (2b).

(v) *Existence.* At this stage some comments on the mathematical validity of the above analysis are in order. Although the adjective 'perturbative' was used to describe F'_p, our eqs (A8), (9) are exact because no assumption apart from confinement has been made as regards the shape and strength of F_P. Also, the functions ξ_p, ρ_p and λ_p in eq. (A5) exist throughout the interval $0 \le t \le \infty$ provided the integrand $\mathrm{e}^{-\Omega_- t'} F'_\mathrm{p}$ remains bounded at all finite t' and drops faster than $1/t'$ as $t' \to \infty$. Finally, although the top of the barrier in figure 1 corresponds to a point of unstable equilibrium, our solutions (A8, 9) are quite stable with respect to choice of the initial location x_0 because of the important identity (A7) satisfied by v_0.

(vi) *Lowest order estimates.* We shall now evaluate the integrals appearing in eq. (A5) approximately without employing any specific model for F_p, but making a few reasonable assumptions. Suppose that $F_\mathrm{p}(x, t)$ is a slowly varying function of space and time so that its derivatives may be neglected – a situation which is somewhat reminiscent of the potential smoothness assumption in the quantum WKB method. Also, suppose that the particle has relaxed with respect to the time-scale Ω_-^{-1} implying that $\Omega_\pm t \gg 1$. Then, partial integration in eq. (A5) yields

$$\xi_p \approx \dot{F}_p/\Omega_+; \quad \lambda_p \approx F_p + \dot{F}_p/\Omega_-, \tag{A10}$$

where terms of order $\dot{F}_p/\Omega_\pm^2$ have been dropped. With the help of these estimates we can describe motion on two different types of branches as explained below.

(vii) *The improved Rayleigh branch.* If the particle starts sliding down the barrier in figure 1 its speed may be relatively substantial. Relaxation with respect to the time scale Ω_+^{-1} happens soon so that one can take $\Omega_+ t \gg 1$. Neglecting the $e^{-\Omega_+ t}$ terms in the first representation (cf. eq. (A9a)) for the velocity and using the estimate (A10) one obtains

$$v^{(R)^*} \approx \Omega_- x + \frac{\xi_p}{m} \approx \frac{F}{m\omega_+} - \frac{\dot{F}_p}{m\Omega_+^2}, \tag{A11}$$

where the superscript $(R)^*$ labels the 'improved' Rayleigh branch and $F = m\Omega^2 x + F_p$ is the full external force. Note that the customary Rayleigh velocity is just $F/m\gamma$ instead (cf. eq. (3b)). Other dynamical functions of interest, viz., the effective acceleration, Lagrangian and conserved energy associated with $v^{(R)^*}$ are reported explicitly in table 3.

(viii) *The non-Rayleigh branch.* If the particle starts climbing up the hill in figure 1 its speed may get reduced substantially. Since the particle will take a relatively long time to cover a given distance, relaxation with respect to the time scale Ω_-^{-1} may occur so that we can take $\Omega_t \gg 1$. From the second representation (cf. eq. (A9b)) of the velocity we obtain the asymptotic expansion

$$v^{(N)^*} = -\Omega_+ x - \frac{\lambda_p}{m} \approx \frac{-F}{m\Omega_-} - \frac{\dot{F}_p}{m\Omega_-^2}, \tag{A12}$$

where use has been made of the estimate (A10), and the superscript $(N)^*$ labels the non-Rayleigh branch. Note that $v^{(N)^*}$ is outside the scope of the conventional Rayleigh velocity. The relevant acceleration, Lagrangian and conserved energy are again displayed explicitly in table 3.

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