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Nonequilibrium phase transition in a radiation-driven Josephson junction

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We predict that a nonequilibrium phase transition, analogous to optical bistability, occurs when coherent radiation is applied to an unbiased Josephson junction with an external resistance across it. The order parameter is the self-consistently developed dc voltage, and the drive parameter is the applied radiation intensity. The order parameter exhibits jump and hysteresis behavior characteristic of a first-order phase transition. The size of the hysteresis region can be tuned by varying the resistance. An approach based on the Fokker-Planck equation is adopted. The extremum of the stationary probability yields the self-consistency equation for the mean-field order parameter. Relaxation and decay times are calculated, the decay times being identified with the first passage time. Estimates of parameters show that the bistable regime could be experimentally accessible.

I. INTRODUCTION

Nonequilibrium superconductivity is the focus of increasing interest[1,2] as part of a general rise in activity in driven systems and in nonequilibrium phase transitions[3-5] in general. It has been shown[1,2] theoretically and proved experimentally that irradiating thin superconducting strips with photons of less than pair-breaking energies can enhance the order parameter by depopulating the quasiparticles near the gap edge. A nonequilibrium phase transition to a nonuniform gap state has been predicted[6] and tested[7] in thin superconducting films with quasiparticles injected through a tunnel junction. The drive in such a nonequilibrium system directly acts on, and modifies, the quasiparticle distribution. The pairs and the gap are modified indirectly through the BCS relation. In this paper we consider a situation where the pairs tunneling through a Josephson junction are directly acted upon by external photons, and a first-order nonequilibrium phase transition results.[8] The radiation intensity is the drive, and the self-consistently developed voltage across the junction (with a resistance R across it) is the order parameter. The system is predicted to exhibit bistability and hysteresis.

That radiation can affect the current in Josephson junctions is of course well-known; the Shapiro effect[9,10] involves jumps in the current when the Josephson frequency (i.e., applied voltage) is an integral multiple of the irradiation frequency. Similar current steps are also seen in the Fiske effect[9,10] where the oscillating current interferes with its own oscillating radiation field, trapped in the oxide layer of the junction. These effects are essentially complicated beat phenomena due to matching of frequencies, and dissipation terms are not essential for the effects to be seen, although they may influence their magnitude. Furthermore, an externally applied voltage is necessary for both effects.

Our predicted effect, by contrast, is a true nonequilibrium phase transition between steady states determined by a balance between drive and dissipation terms, and involves hysteresis between the self-consistently developed (not applied) voltage and the applied radiation intensity. It is distinct from either effect. Although "free-running" irradiated junctions with Shapiro-type steps have also been studied,[11] we find that our bistability effect is also distinct from this effect, being applicable in an opposite parameter regime, as seen in Appendix A.

The situation here is quite analogous, though there are major differences due to the different type of physics involved, to optical bistability which has been extensively studied[3] both experimentally and theoretically. In optical bistability one finds that the transmission of light through a system of two-level atoms contained in a cavity shows bistable behavior under certain conditions. It may also be noted that since the superconductor can be formally regarded as a collection of two-level quasispins,[12] many analogs of well-known optical effects have been suggested[13] for superconducting systems; in this paper, we show[8] the analog of optical bistability. However, such an analog has peculiarities, as shown in the text, that are unique to the superconducting system.

The plan of the paper is as follows. In Sec. II we briefly outline the derivation of the Hamiltonian and the damping terms, deriving the coupled Langevin equations. In Sec. III we obtain the Fokker-Planck equation for the self-consistent voltage and its stationary solution. In Sec. IV we show that the equation for the mean-field order parameter (most probable dc voltage) has two-valued stable solutions, so the steady states exhibit bistability and hysteresis.

The Fokker-Planck equation of Sec. III is used in Sec. V to calculate the relaxation $(1/T_1)$ and intrinsic decay $(1/T_2)$ rates of the states in the competing minima of the generalized potential. The estimates of Secs. IV and V show that the hysteresis characteristics are indeed accessible to experiment. The effect of a separate temperature bath for the resistance is examined in Appendix B.

II. THE HAMILTONIAN AND THE DAMPING TERMS—EQUATIONS OF MOTION

The system considered is a Josephson junction irradiated coherently and with an external resistance R across it, schematically depicted in Fig. 1. There is no externally applied voltage. The tunneling Hamiltonian describing the transfer of superconducting pairs through the oxide layer[14] is obtained by a canonical transformation[15] from the single electron tunneling Hamiltonian. It may be written (g is a coupling constant) as[16]

$$H_T = g\sum_{kq}(\sigma_k^\dagger \sigma_q + \sigma_q^\dagger \sigma_k)\ , \tag{2.1}$$

where Anderson's pseudospin notation[12] has been used,

$$\sigma_k^\dagger = C_{k\uparrow}^\dagger C_{-k\downarrow}^\dagger\ , \quad \sigma_k^z = \tfrac{1}{2}(1 - C_{k\uparrow}^\dagger C_{k\uparrow} - C_{k\downarrow}^\dagger C_{k\downarrow}), \tag{2.2}$$

and the levels $k(q)$ refer to electrons in the left (right) electrodes. Throughout the paper we will assume that temperatures are so low that quasiparticle tunneling may be neglected, and $\Omega < 2\Delta/\hbar$ so that the radiation does not itself produce quasiparticles. Δ represents the gap and Ω the frequency of the external radiation.

We note that (2.1) may be written in terms of the following operations[16]:

$$S^- = S_L^- S_R^+, \quad S^+ = S_R^- S_L^+, \quad S^z = \tfrac{1}{2}(S_L^z - S_R^z)\ , \tag{2.3}$$

where

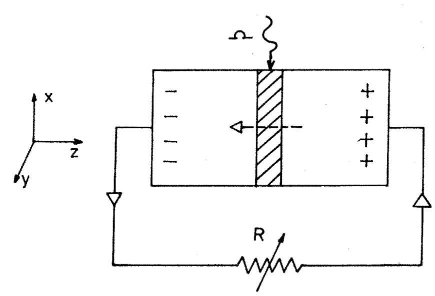

FIG. 1. Schematic diagram of the system with the arrows denoting the direction of pair flow.

$$S_L^- = \frac{1}{\sqrt{2}\, m}\sum_k \sigma_k\ , \quad S_L^z = -\sum_k \sigma_k^z \tag{2.4}$$

and similar equations hold for the right electrode. Clearly S^- (S^+) represents a pair transfer from left (right) to right (left) and S^z counts the number of excess pairs in the left superconductor. The constant m is the number of electron states within $\pm\omega_D$ of the Fermi surface and is very large. The normalization is such that in the BCS ground state

$$\langle S_R^+ S_R^- \rangle = \langle S_L^+ S_L^- \rangle = 1\ . \tag{2.5}$$

The commutation relations of $S^\pm$ and S^z follow from those of the fermion operators out of which they are constructed and are

$$[S^z, S^\pm] = \pm S^\pm\ ,$$
$$[S^+, S^-] = \frac{1}{m^2}(S_R^+ S_R^- S_L^z - S_L^+ S_L^- S_R^z) \approx \frac{2}{m^2} S^z\ . \tag{2.6}$$

Then the pair tunneling Hamiltonian may be written[16] in the following simple form with the coupling constant related to the experimentally measured maximal Josephson tunneling current I_J:

$$H_T = -\frac{\hbar I_J}{4e}(S^+ + S^-)\ . \tag{2.7}$$

The Josephson current is, using (2.6) and taking expectation values in the BCS ground state ($N_L = \sum_{k\sigma} C_{k\sigma}^\dagger C_{k\sigma}$),

$$\langle I \rangle = -2e\langle \dot{N}_L \rangle = -\frac{2ei}{\hbar}\langle [H_T, N_L] \rangle = I_J \frac{\langle S^+ - S^- \rangle}{2i}\ . \tag{2.8}$$

Defining

$$\langle S^- \rangle = e^{-i\theta}\ , \tag{2.9}$$

we obtain the familiar dc Josephson effect, θ being the phase difference between left and right electrodes.

The normalization of the operators could, of course, be chosen so that (2.6) becomes the usual angular momentum algebra, but this would cause m to reappear in the coupling constant, which at present is a directly measurable quantity I_J. We therefore use this normalization which explicitly displays the very real physical difference between an actual spin system and this system of interacting fermions.

Since we have allowed for a situation where there may be a pair imbalance between the left and right (identical) electrodes, a term in the total Hamiltonian must be included representing the electrostatic energy of the charged electrode-oxide–electrode capacitor

$$H_C = \frac{1}{2}\frac{(2eS^z)^2}{C} . \qquad (2.10)$$

Here C is the capacitance of the junction, $C = l_x l_y \epsilon / 4\pi l_z$ of cross section $l_x l_y$, oxide thickness l_z, and oxide dielectric constant ϵ.

Finally, since the tunneling current can emit and absorb photons, we must have the interaction term $(1/c)\int d^3r j(\vec{r}) A(\vec{r})$. The oxide region forms a cavity of resonant frequency ω_c with boundary conditions $H=0$ applied at the surface, giving standing waves. The quantized electromagnetic field for a particular polarization and a particular mode $k=n\pi/l_x$ is[16,17]

$$A_z = \left(\frac{2\pi\hbar c^2}{\epsilon\omega_c}\right)^{1/2}(a+a^\dagger)\phi(x,z) , \qquad (2.11)$$

where

$$\phi(x,z) = \begin{cases} \dfrac{\sqrt{2}}{(l_x l_y l_z)^{1/2}} \cos kx, & |z| \le l_z/2 \\ 0 & \text{otherwise.} \end{cases} \qquad (2.12)$$

The speed of electromagnetic waves in the oxide layer is reduced by a factor of $\bar{c}/c$:

$$\omega_c = \bar{c}\, k, \quad \bar{c}/c = (l_z/\epsilon d)^{1/2} , \qquad (2.13)$$

where $d = 2\lambda + l_z$, λ being the penetration depth. Clearly the Josephson tunneling current density must have a spatial variation in the x direction in order to couple to this mode. This is produced by an applied magnetic field H_0 in the y direction, so that

$$\langle j_z(\vec{r})\rangle = \frac{I_J}{l_x l_y}\sin(\theta + qx), \quad q = \frac{2eH_0 d}{\hbar c} . \qquad (2.14)$$

The junction size l_x, l_y is assumed to be smaller than the Josephson length[18] λ_J, so that the current in the absence of the field is uniform across the junction. Doing the integrals and reexpressing θ's in terms of $S^\pm$, the Hamiltonian for cavity photons and their interaction with the Josephson current that produces them is

$$H_{\text{cav}} = \hbar[T_r(k,\omega_c)(S^- + S^+) + iT_i(k,\omega_c)(S^- - S^+)](a+a^\dagger) + \hbar\omega_c a^\dagger a , \qquad (2.15)$$

where

$$T_i(k,\omega_c) = T(k,\omega_c)\left(\frac{2q}{(q^2-k^2)l_x}\sin(q-k)l_x + \frac{2\cos ql_x \sin kl_x}{(q+k)l_x}\right) , \qquad (2.16a)$$

$$T_r(k,\omega_c) = T(k,\omega_c)\left(\frac{2q}{(q^2-k^2)l_x}\sin^2(q-k)\frac{l_x}{2} + \frac{2\sin ql_x \sin kl_x}{(q+k)l_x}\right) , \qquad (2.16b)$$

and where

$$T(k,\omega_c) = \frac{1}{2}\left(\frac{\pi l_z}{\epsilon\hbar\omega_c l_x l_y}\right)^{1/2} I_J . \qquad (2.16c)$$

Note that for coupling to cavity photons for which $k=\pi/l_x$ the last term in both T_i and T_r is zero. For $H_0 = 0$, there is no coupling. For $q=k$, $T_i(k,\omega_c) = T(k,\omega_c)$, while T_r vanishes.

A similar interaction may be written between the coherent radiation in the cavity of frequency Ω and wave number K, $(\Omega = \bar{c}K)$ coming from the external source that drives the junction away from equilibrium. We assume that this coherent radiation has some finite but narrow bandwidth, with $\Omega/\omega_c \sim 1$, for the range of frequencies in the incident radiation. The interaction Hamiltonian between the current and photons in the cavity that enter from outside is then $(K' = \Omega'/\bar{c})$

$$H_{\text{ext}} = \int d\Omega' W(\Omega',\Omega)[a_{\text{ext}}(t) + \text{H.c.}] \times \hbar[T_r(K',\Omega')(S^- + S^+) + iT_i(K',\Omega')(S^- - S^+)] ,$$

$$a_{\text{ext}} = N_{\text{ext}}^{1/2}\cos(\Omega' t) , \qquad (2.17)$$

where W is the spectral function of the external radiation, centered at Ω and relatively flat in a narrow range around it. The finite bandwidth covering the hysteresis frequency range is necessary so that even when the Josephson frequency ω jumps, there is a component of radiation at ω available to continue to drive it. The total Hamiltonian for the system is now

$$H = H_C + H_J + H_{\text{cav}} + H_{\text{ext}} . \qquad (2.18)$$

The Hamiltonian (2.18) describes the coherent interactions. The incoherent processes can be described either by writing down the quantum Langevin equations for a, $a^\dagger$, $S^\pm$, S^z or equivalently in terms of the density matrix of the combined system consisting of the field and the pair operators $S^\pm$ and S^z. The important incoherent processes are

(i) The leakage of the photons from the cavity at a rate $\kappa = \omega_c/2Q$ where Q represents the quality factor of the cavity. This leads to a decay term $-\kappa\langle a\rangle$ in the equation for $\langle a\rangle$.

(ii) The discharge rate of the capacitor as the current is passing through the external resistance R. Thus $\langle S^z\rangle$ will have a decay term like $-\langle S^z\rangle/RC$.

Each of the incoherent decay processes is, in general, known to lead to a random force operator in the equation of motion. For example, the quantum Langevin equation for the photon operator will have the structure[19] (in the absence of any other interactions)

$$\dot{a} = -i\omega_c a - \kappa a + f_a \ , \tag{2.19a}$$

where the operator f_a has the properties

$$\begin{aligned} \langle f_a(t) f_a(t')\rangle &= \langle f_a^\dagger(t) f_a^\dagger(t')\rangle = \langle f_a^\dagger(t) f_a(t')\rangle \\ &= \langle f_a(t)\rangle = \langle f_a^\dagger(t)\rangle = 0, \\ \langle f_a(t) f_a^\dagger(t')\rangle &= 2\kappa\delta(t-t') \ . \end{aligned} \tag{2.19b}$$

Any temperature-dependent contribution to f_a has been ignored since we are working at a temperature close to zero. We will in our discussion in Sec. III ignore all the terms corresponding to thermal noise.

The equations of motion (quantum Langevin equations) are now found to be

$$\dot{a} = -i\omega_c a - \kappa a - T(S^+ - S^-) + f_a \ , \tag{2.20a}$$

$$\dot{S}^\pm = \pm i \frac{2eV}{\hbar} S^\pm + f^\pm, \quad V = \frac{2eS^z}{C} \tag{2.20b}$$

$$\begin{aligned} \dot{S}^z = &-\frac{S^z}{RC} - \frac{I_J}{4ei}(S^+ - S^-) - T(k,\omega_c)(a+a^\dagger)(S^- + S^+) \\ &- 2N^{1/2}T(k,\omega_c)\int d\Omega'\, W(\Omega',\Omega)\cos(\Omega' t)(S^+ + S^-) \\ &+ f^z \ . \end{aligned} \tag{2.20c}$$

In Eqs. (2.20b) and (2.20c), $f^\pm$ and f^z are again delta-correlated random forces which take care of the operator nature of the variables $S^\pm$ and S^z and any thermal-noise factors. The properties of f^z are discussed in Appendix B. In the above equations, we have assumed that the bandwidth of the incident radiation is such that $T_{r,i}(K,\Omega) \approx T_{r,i}(k,\omega_c)$ and that H_0 is such that $q \sim k = \pi/l_x$, so that $T_i(k,\omega_c) = T(k,\omega_c)$, $T_r = 0$.

In deriving Eqs. (2.20), we have ignored all the terms that arise from the commutator $[S^+, S^-]$ as such terms are of the order S^z/m^2. The damping terms in $S^\pm$ equations are also quite small and hence ignored.[17]

Since $C \sim l_x l_y$ we have $C \sim N_L^{2/3}$ (we assume $N_L \sim N_R$), so for intensive voltages independent of the length of the electrodes, we have $\langle S^z\rangle \sim N_L^{2/3}$. Since $m \sim N_L$, the neglected terms are indeed small. For the neglected terms to be important, we would have to consider a regime where $\langle S^z\rangle \sim m^2 \sim N_L^2$ and the voltage $\sim N_L^{4/3}$ is then size dependent. This solution however seems less physical, and we shall not pursue it further. Note that in order that the mean-field approximation, which is used extensively in Sec. III, be effective, we must have the relations

$$|\langle S^+\rangle|\,|\langle S^z\rangle| \gg |\langle S^+\rangle|, \ |\langle S^+\rangle|\,|\langle S^-\rangle| \gg \frac{|\langle S^z\rangle|}{m^2},$$

i.e.,

$$|\langle S^z\rangle| \gg 1 \quad |\langle S^z\rangle| \ll m^2 \ .$$

In our regime $\langle S^z\rangle \sim N_L^{2/3}$, the above conditions are easily met. However, if $\langle S^z\rangle \sim m^2$ then the mean-field approximation is likely to break down.

III. THE FOKKER-PLANCK EQUATION FOR THE DISTRIBUTION OF SELF-CONSISTENTLY DEVELOPED VOLTAGE

The quantum Langevin equations (2.20), as presented in the last section, do not appear to be soluble in general. However, the physics of Josephson junctions permits considerable simplifications, so that an analytical expression for the self-consistently developed voltage may be obtained. In what follows we make the following assumptions: (i) Ignore all the operator characteristics of $S^\pm$ and S^z. (ii) Ignore all the thermal-noise terms since the system is assumed to be at a temperature close to zero. (iii) The photon leakage rate κ is assumed to be much greater than $1/RC$; this implies that the photon mode can be adiabatically eliminated from the problem. Under assumptions (i) and (ii), the variables $S^\pm$ and S^z can be treated as numbers. Note that in view of (2.20b), we can express $S^\pm$ as

$$S^\pm = e^{\pm i\theta}, \tag{3.1}$$

with

$$\dot{\theta} = \frac{2eV}{\hbar} = \frac{2e}{\hbar}\frac{2e}{C} S^z \ . \tag{3.2}$$

In the above we assume that the current flow is in a particular direction. It can be shown later that this can be maintained by external radiation. We look for a solution of the form

$$\theta(t)=\int_0^t d\tau\,\omega(\tau)+\theta_0, \qquad (3.3)$$

whence

$$\omega=(4e^2/\hbar C)\,S^z\,. \qquad (3.4)$$

Here θ_0 is a phase difference between the electrodes at $t=0$, in which we also absorb the phase difference between the applied radiation and the current it produces. θ_0 is treated later as a variational parameter. ω can be identified with the steady-state value of S^z and would essentially give the dc voltage. On introducing the variable $\tilde{a}$ by

$$a=\tilde{a}e^{-i\theta}, \qquad (3.5)$$

we find the Langevin equation for $\tilde{a}$:

$$\dot{\tilde{a}}=-[i(\omega_c-\omega)+\kappa]\tilde{a}-2iT\sin\theta e^{i\theta}+e^{i\theta}f_a\,. \qquad (3.6)$$

On being time-averaged over the interval of order ω^{-1}, Eq. (3.6) reduces to

$$\dot{\tilde{a}}=-[i(\omega_c-\omega)+\kappa]\tilde{a}+T+e^{i\theta}f_a\,. \qquad (3.7)$$

In view of our assumption (iii), i.e., that the "a" is a fast mode, we can set $\dot{\tilde{a}}=0$, so that (3.7) leads to

$$\tilde{a}=[\kappa+i(\omega_c-\omega)]^{-1}T+\frac{e^{i\theta}f_a}{[\kappa+i(\omega_c-\omega)]}\,. \qquad (3.8)$$

We now examine the S^z equation. When time-averaging is carried out again, terms like $I_J\sin\theta$ drop out and terms like

$$2T(a+a^\dagger)\cos\theta=T(\tilde{a}e^{-i\theta}+\tilde{a}e^{i\theta})(e^{i\theta}+e^{-i\theta})$$

lead to terms like $T(\tilde{a}+\tilde{a}^\dagger)$. Hence the time-averaged S^z equation looks like

$$\dot{S}^z=-\frac{S^z}{RC}-T(\tilde{a}+\tilde{a}^\dagger)-2N^{1/2}T_{\text{ext}}W(\omega,\Omega)\cos\theta_0+f^z\,. \qquad (3.9)$$

On substituting (3.8) in (3.9) we finally obtain the equation

$$\dot{S}^z=-\frac{S^z}{RC}-\frac{2T^2\kappa}{\kappa^2+(\omega_c-\omega)^2}-2T_{\text{ext}}N^{1/2}W(\omega,\Omega)\cos\theta_0-T[(\kappa+i(\omega_c-\omega)^{-1}e^{i\theta}f_a+\text{c.c.}]+f^z\,. \qquad (3.10)$$

Hence the final equation for ω becomes

$$\dot{\omega}=\frac{4e^2}{\hbar C}\dot{S}^z=-\frac{\omega}{RC}-\frac{2T^2\kappa}{\kappa^2+(\omega_c-\omega)^2}\left(\frac{4e^2}{\hbar C}\right)-2T_{\text{ext}}N^{1/2}W(\omega,\Omega)\cos\theta_0\left(\frac{4e^2}{\hbar C}\right)+f_\omega\,. \qquad (3.11)$$

The properties of the random force f_ω can be calculated from the properties of f_a, f^z:

$$\langle f_\omega(t)\rangle=0,$$
$$\langle f_\omega(t)f_\omega(t')\rangle=\left(\frac{8e^2T}{\hbar C\kappa}\right)^2\frac{\kappa}{2}\langle[1+(\omega_c-\omega)^2/\kappa^2]^{-1}\rangle+\frac{4e^2}{\hbar^2C^2R}\hbar\omega_c\coth\hbar\frac{\hbar\omega_c}{2k_BT}\delta(t-t')\,. \qquad (3.12)$$

It is evident from (3.11) that the mean value of ω satisfies the equation

$$\langle\dot{\omega}\rangle=-\frac{\langle\omega\rangle}{RC}-\frac{4e^2}{\hbar C}\frac{2T^2\kappa}{\kappa^2+(\omega_c-\omega)^2}-\frac{8e^2}{\hbar C}T_{\text{ext}}N^{1/2}W(\omega,\Omega)\cos\theta_0\,. \qquad (3.13)$$

Equation (3.13) has a simple physical interpretation if we rewrite it as an equation for the pair charge imbalance

$$\begin{aligned}2e\langle\dot{S}^z\rangle&=-\frac{1}{R}\left(\frac{2e}{C}\langle S^z\rangle\right)-4eT_{\text{ext}}N^{1/2}W\cos\theta_0-\frac{4eT^2\kappa}{\kappa^2+(\omega_c-\omega)^2}\\&=-\frac{V}{R}-4e\,T_{\text{ext}}N^{1/2}W(\omega,\Omega)\cos\theta_0-4e\kappa\langle\tilde{a}\rangle\langle\tilde{a}^\dagger\rangle,\end{aligned} \qquad (3.14)$$

where the average $\langle\tilde{a}\rangle$ is given by the first term in (3.8). Each term in (3.14) has a physical interpretation. The second term is the externally driven change of the pair charge imbalance $\langle S^z\rangle$; the first term is a decay of S^z through which current flows in the external circuit that removes (adds) pairs from (to) the left (right) electrode. The third term can be understood as follows: In the absence of losses a pair can tunnel to the lower energy electrode emitting a cavity photon, which is then reabsorbed lifting the pair back, the repetition of the cycle contributing to the ac Josephson current. If the photons leak out at a rate $2\kappa\langle a^\dagger a\rangle$, such pairs will not be lifted back, resulting in a pair charge change at a rate $4e\kappa\langle a^\dagger a\rangle\sim 4e\kappa\langle a^\dagger\rangle\langle\tilde{a}\rangle$.

It will be noted that the time-averaged Josephson term $I_J\sin\theta(t)$ has been neglected in Eq. (3.10). In fact, if we had allowed for the possibility of sinusoidal contributions to $\theta(t)$, then dc contributions would survive the time-averaging precisely as in the Shapiro effect[9,10] (as shown in Appendix A). The size of these contributions is found to be quite negligible in the regime in which we work.

On introducing scaled variables (f is always restricted to $f<[2eI_JR/(\hbar\omega_c)]$, $2\Delta/\hbar\omega_c$)

$$f=\frac{\omega}{\omega_c}\ ,\quad \alpha=\frac{8e^2}{\hbar}\,R\left(\frac{T(k,\omega_c)}{\omega_c}\right)^2\ ,$$
$$N_c^{-1/2}=\frac{8e^2}{\hbar}\,R\left(\frac{T(K,\Omega)}{\omega_c}\right)W(\omega,\Omega)\ . \tag{3.15}$$

In (3.11), we obtain the following Langevin equation for f:

$$\dot{f}=-\frac{1}{RC}\left[f+\left(\frac{N}{N_c}\right)^{1/2}\cos\theta_0+\frac{2Q\alpha}{1+4Q^2(f-1)^2}\right]+F_f\ , \tag{3.16}$$

where the relation $\kappa=\omega_c/2Q$ has been used. The random force now has the property

$$\langle F_f(t)F_f(t')\rangle=\tau_f^{-1}\delta(t-t'),$$
$$\tau_f^{-1}\cong\frac{8e^2Q\alpha}{\hbar\omega_c RC^2}\left(1+\frac{1}{2Q\alpha}\right)\ . \tag{3.17}$$

The stochastically equivalent Fokker-Planck equation for the distribution P of f will be

$$\frac{\partial P}{\partial t}=\frac{1}{RC}\frac{\partial}{\partial f}\left[f+\left(\frac{N}{N_c}\right)^{1/2}\cos\theta_0+\frac{2Q\alpha}{1+4Q^2(f-1)^2}\right]P$$
$$+\frac{1}{2\tau_f}\frac{\partial^2 P}{\partial f^2}\ . \tag{3.18}$$

The stationary solution of (3.18) is given by (N_{st} = normalizing const)

$$P_{st}(f)=N_{st}\exp\left(\frac{-2\tau_f}{RC}\,\Phi(f)\right)\ , \tag{3.19}$$

where Φ is the generalized potential given by

$$\Phi(f)=\int^f\left[f-\left(\frac{N}{N_c}\right)^{1/2}+\frac{2Q\alpha}{1+4Q^2(f-1)^2}\right]df$$
$$=\Phi(1)+\tfrac{1}{2}(f^2-1)-\left(\frac{N}{N_c}\right)^{1/2}(f-1)$$
$$+\alpha\tan^{-1}2Q(f-1)\ . \tag{3.20}$$

Here we have chosen the value $\cos\theta_0=-1$ that makes Φ the smallest. Physically this means that the relative phase of the electrodes and the phase difference between the current and the radiation it produces adjust instantaneously to minimize the generalized potential Φ.

IV. BISTABILITY IN THE CHARACTERISTICS OF THE IRRADIATED JOSEPHSON JUNCTION

The probability distribution (3.19) is sharply peaked at the minima of $\Phi(f)$ provided $2\tau_f/RC\gg1$, a regime that is realized in practice, as seen later. Then the most probable values of f are the (stable) solutions of

$$\mu-f=\frac{2Q\alpha}{1+4Q^2(f-1)^2}\ , \tag{4.1}$$

where

$$\mu\equiv(N/N_c)^{1/2}\ . \tag{4.2}$$

This can be solved graphically as shown in Fig. 2; for a range of parameters there are three solutions for a given $(N/N_c)^{1/2}$. This is given by the S-shaped curve in Fig. 3, where, for a given $\alpha=0.02$, $Q=5$, the solution(s) f of (4.1) is (are) plotted versus the drive (μ). Taking a typical value $\mu=1.21$ in the multiple-root region, we see from the generalized potential (Fig. 4) that two solutions are locally stable minima, while the third root that separates them (on the dashed line of Fig. 3) is unstable corresponding to a local probability minimum. The critical points $f_{c1,2}$ are points where one or other of the two minima approaches the intervening maxima and both disappear. Once μ exceeds, for example, μ_{c1}, the system jumps to the branch f_2, where in the region $\mu>\mu_{c1}$, only one stable root exists corresponding to one minimum of Φ (Fig. 4). The location of the critical points and hence the size of the whole bistability region depends on the values of the parameters. For a given junction, the external field H_0 (entering T) resistance R are the most easily varied thereby changing α. The size and shape of the hysteresis region may thus be tuned by varying these parameters. The locus of the critical points as α is varied is the spinodal curve in this problem, and at the tip of the curve the bistable

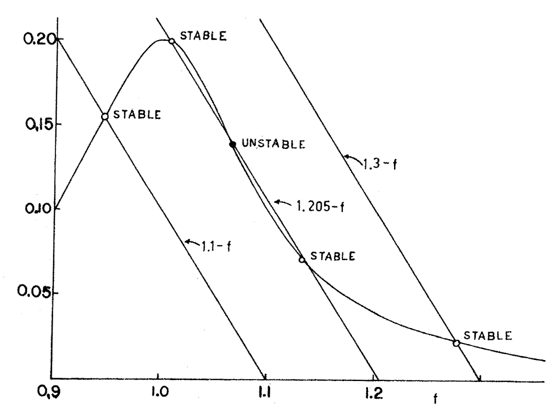

FIG. 2. Graphical solution of order parameter equation (4.1) for various values of the drive parameter $\mu\equiv(N/N_c)^{1/2}$ for $\alpha=0.02$, $Q=5$. Open circles denote stable solutions; the solid circle denote the unstable solution.

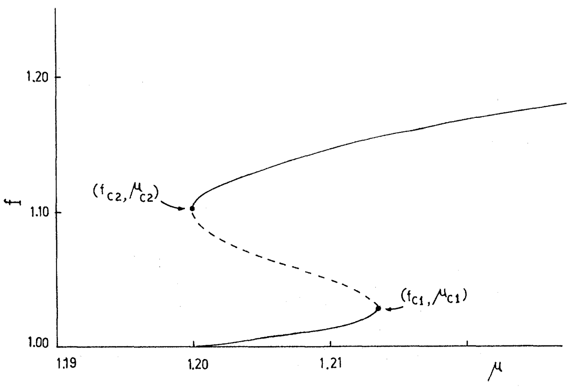

FIG. 3. Plot of the scaled dc voltage $f = 2\ \mathrm{eV}/\hbar\omega_c$ versus the scaled (square root of the) externally supplied photon number in the cavity $\mu = (N/N_c)^{1/2}$. The hysteresis region is bounded by the critical points $(f_{c1,c2}, \mu_{c1,c2})$. Here $\alpha = 0.02$, $Q = 5$.

region shrinks to zero at the nonequilibrium second-order transition point. The locus of equal Φ well-depth points lies outside the spinodal curve and meets it at the second-order point (Fig. 5).

The analogy between Fig. 3 and the optical bistability curve is striking, and the spinodal and equal well-depth curves have their counterpart in the spinodal and coexistence curves of the liquid-gas transition and other systems.[20]

The condition for the critical points (f_{c1}, μ_{c1}) is

$$\left(\frac{\partial f}{\partial \mu}\right)^{-1} = 0 \tag{4.3}$$

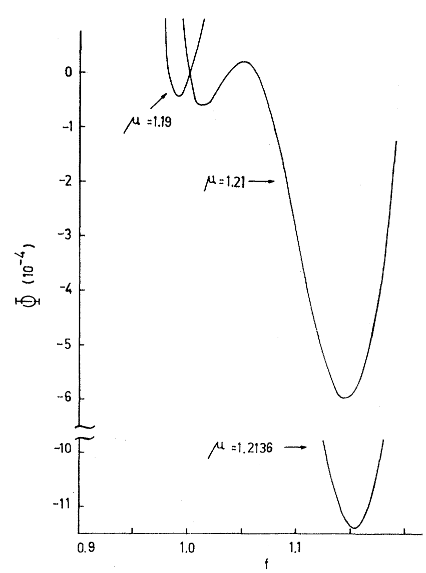

FIG. 4. Plots of generalized potential Φ versus scaled self-consistent voltage f for three values of the drive μ. The double-well behavior ($\mu = 1.21$) occurs in the bistable region of Fig. 3.

and from (4.1) the bistable region cannot exist unless

$$Q^2\alpha \geq 2/3\sqrt{3} \tag{4.4}$$

$$\mu_{c1} > \mu > \mu_{c2}\ . \tag{4.5}$$

Equation (4.4) is equivalent to requiring that $\langle a^\dagger a\rangle$ the number of current-produced cavity photons, must exceed a certain value. Equation (4.5) requires that the externally supplied photon number must fall within a certain window.

For a given junction there is a critical value R_0 of the external resistance (and hence a critical $\alpha = \alpha_0$) below which bistable behavior is not seen, and is defined by

$$Q^2\alpha_0 = 2/3\sqrt{3}\ . \tag{4.6}$$

For resistance close to the value R_0, $R = R_0(1+\delta)$ $0 < \delta \ll 1$, we can obtain explicit series solutions for the critical points. We find from (4.1) and (4.3) that $f_{c1,2} = f_0 \mp \frac{1}{3}\delta^{1/2}/Q$,

$$\mu_{c1,2} = \mu_0 \pm \tfrac{1}{9}\delta^{3/2}/Q,$$

where

$$f_0 = 1 + (2\sqrt{3}\,Q)^{-1}, \quad \mu_0 = 1 + \sqrt{3}(1 + \tfrac{2}{3}\delta)/2Q\ .$$

We now examine the feasibility of the bistability. Lossy cavities make the dissipation rate due to photon leakage $(n\hbar\omega_c\kappa)$ non-negligible, compared to the Joule losses $\sim V^2/R$. Since $V \sim (\hbar\omega_c/2e)$ for the bistable region, large resistances make the power requirements low. Thus we take $R = 136$

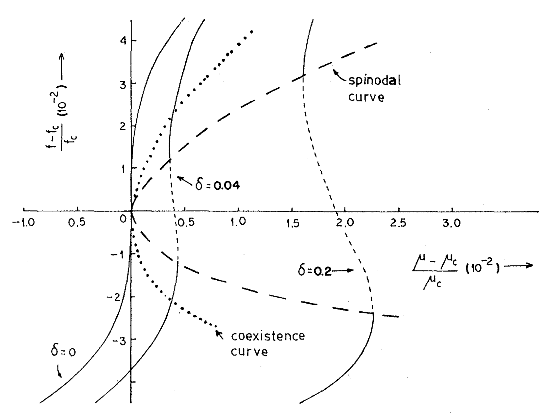

FIG. 5. Plots of the voltage versus drive parameter curves for various values of δ. (The critical value is $\delta = 0$ corresponding to $\alpha_0 = 2/3\sqrt{3}Q^2$.) The dashed line is the spinodal curve passing through the points $(f_{c1,2}, \mu_{c1,2})$. The "second-order" transition point is where $f_{c1} = f_{c2} = f_c = 1 + (2\sqrt{3}Q)^{-1}$; $\mu_{c1} = \mu_{c2} = \mu_c = 1 + \sqrt{3}/2Q$. The dotted line, analogous to a "coexistence curve" passes through points where the double-well depths of Φ are equal.

ohms, $Q=5.$ and junction parameters $l_x=l_y=0.1$, cm, $l_z=20$ Å, $\epsilon=5$, $I_J=30\ \mu$A, $\lambda=3.7\times10^{-6}$ cm. This gives $\bar{c}/c=0.07$ and $\omega_c=68$ GHz. We assume the magnetic field is such that $q=k$, which corresponds to $H_0\sim 0.14$ G. We also assume that, with Ω falling in the middle of the frequency hysteresis region ($\Omega\sim\omega_c f_0$ for $\delta\ll1$), the spectral weight falls off so that $W(\omega,\Omega)\sim\frac{1}{15}$ a constant[21] along the hysteresis branches. The scaling parameters are then $\alpha=0.02$ ($\delta=0.3$) and $N_c=4.3\times10^4$ photons, corresponding to an external photon density in the cavity at least $N_c/(2\lambda+l_z)\,l_x l_y\sim6\times10^{11}$ cm^{-3}. The available power in the cavity due to external photons is $N_c\hbar\Omega\bar{c}/l_x=7\times10^{-9}$ W. The power requirements drawn by the system, however, are less and typically are $V^2/R=\hbar^2\omega_c^2 f^2/4e^2R\sim10^{-12}$ W (Joule losses) and $n\hbar\omega_c\kappa=\hbar\omega_c^2\alpha/[1+4Q^2(f-1)^2]\sim10^{-14}$ W (photon leakage). Assuming a 10% transmission factor, the external photon power per area required is $10N_c\hbar\omega_c\bar{c}/(2\lambda+l_z)l_x\sim0.1$ W cm^2. The capacitance is 0.022 μF so that the lifetimes $RC\sim3\times10^{-6}$ sec and $\kappa^{-1}\sim10^{-10}$ sec satisfy the original fast-mode assumption for photons. Voltages and currents in the bistable region are typically $\sim$0.02 mV, 0.1 μA, and the jumps are around 10%. $S^z\sim10^6$ pairs, which as a fraction of the total number of pairs within a distance $\xi=10^{-4}$ cm of the oxide layer attempting tunneling, is roughly one pair in 10^9. The magnitude of the neglected S^z/m^2 terms will be even smaller.

The power requirements and bistability conditions[22] (4.4) and (4.5) are clearly sensitive to the junction dimensions and to Q, R, and I_J. This could explain why the predicted phenomena have not previously been seen.

The actual behavior of the systems within the multiple-root region depends on how the rate of change of the drive $\dot{\mu}$ compares with the intrinsic decay and relaxation rates in the system. As seen in the next section for μ variation within a window, hysteresis behavior will be seen, analogous to superheating and supercooling. The point at which the actual jump occurs is determined by when the inequalities that define the window are violated.

V. RELAXATION AND DECAY TIMES AND CONDITIONS FOR HYSTERESIS

There are two characteristic times associated with the system: the relaxation time T_1 of a state towards a given minimum of Φ, and, within the bistable region, the time T_2 for a given minimum of Φ to decay into the other, competing minimum by being kicked over the intervening barrier by intrinsic fluctuations. We have elsewhere[23] considered these times and given hysteresis conditions in terms of them for a general system. We will apply those results to the Josephson junction.

The relaxation time T_1 [of $f(t)$] to a steady-state f is given by

$$\dot{f}(t)=-\frac{1}{T_1}\,[f(t)-f], \tag{5.1}$$

and is explicitly

$$T_1=RC/\Phi''(f), \tag{5.2}$$

where f is a solution of (4.1). The relaxation rate $1/T_1$ vanishes at critical points $(f_{c1,2},\mu_{c1,2})$ that define the spinodal curve.[20] The usual concept of critical slowing down at a second-order transition can thus be extended to these metastable states, the "second-order" critical point defined by $f_{c1}=f_{c2}$, $\mu_{c1}=\mu_{c2}$ then being just a particular case. The behavior of the time T_1 as a function of μ is shown in Fig. 6.

The decay time T_2 may be identified with[23,24] the mean first-passage time for transitions across the Φ barrier and, in terms of the probability of f, is

$$T_2=2\tau_f\int_{f_1}^{f_3}\frac{df}{P(f)}\int^{f}P(f')df', \tag{5.3}$$

where f_1 (f_3) is one of the minima (maxima) of Φ. For steady states away from the critical points, the time T_2 for decay from a state f_1 can be written as[25]

$$T_2=\frac{\pi RC\exp\left(\dfrac{2\tau_f}{RC}\Delta\Phi\right)}{(\Phi''(f_1)|\Phi''(f_3)|)^{1/2}},\quad \Delta\Phi=\Phi(f_3)-\Phi(f_1). \tag{5.4}$$

The decay time T_2 is expected to vanish at the

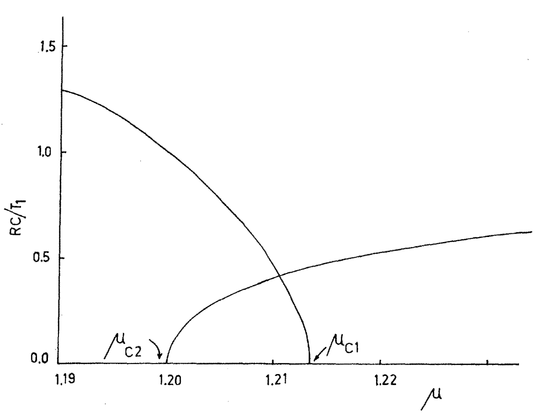

FIG. 6. Plot of the (scaled) relaxation rate RC/T_1 versus the drive parameter μ. Relaxation on both stable branches shows the effects of "critical slowing down" at the appropriate critical points (μ_{c1} or μ_{c2}) on the spinodal curve.

critical points where the maxima and minima of Φ approach each other, as can be seen from (5.3). Within a critical region defined by $\Phi(f_3)-\Phi(f_1) \ll \Phi(f_1)$, T_2 can be written as

$$\frac{T_2}{RC} = (\sqrt{2})\,(0.83)\,6^{1/3}\left(\frac{2\tau_f}{RC}\right)^{2/3}\left(\frac{\partial^2\mu}{\partial f_{c1}^2}\right)^{1/2} \times |\Phi'''(f_{c1})|^{-1/3}|\mu-\mu_{c1}|^{1/2}\,. \tag{5.5}$$

The corresponding expressions for $1/T_1$ in the critical region near f_{c1} is

$$RC\,T_1^{-1} = \sqrt{2}\left(\frac{\partial^2\mu}{\partial f_{c1}^2}\right)^{-1/2}|\Phi'''(f_{c1})||\mu-\mu_{c1}|^{1/2}\,. \tag{5.6}$$

The critical region is estimated to be

$$\Delta\mu \ll \left(\frac{3}{8\sqrt{2}}\,\frac{RC/2\tau_f}{|\Phi'''(f_{c1})|}\right)^{2/3}\left(\frac{\partial^2\mu}{\partial f_{c1}^2}\right). \tag{5.7}$$

Putting in the parameters of Sec. IV we get $T_1 \sim 10^{-6}$ sec and[26] $T_2 \sim 10^{86}$ sec (see, however, Appendix B). Within $\Delta\mu \sim 10^{-4}$, $T_2 \sim$ msec.

Note that we have treated μ as a steady, continuous variable; fluctuations in the photon number, if appreciable, would tend to reduce the size of the hysteresis region. In practice, such effects might have to be considered for probing the critical region.[27]

The conditions for the occurrence of hysteresis[23] depend on the rate of change $\dot{\mu}$ relative to $1/T_1$ and $1/T_2$. Physically the following conditions must be obeyed:

(a) The minima f must adiabatically follow the values fixed from the order parameter equation with a varying $\mu(t)$.

(b) In the bistable region, a given minimum in which the system sits must move along faster than the hopping rate $1/T_2$.

(c) The hopping rate itself must not be drastically increased by the time variation of μ. For this system, the three inequalities are

$$\frac{1}{RC}\left|\frac{\partial^2\Phi}{\partial\mu^2}\,\delta\mu\right| \ll \frac{1}{T_1}\,, \tag{5.8}$$

$$\dot{\mu} \gg \left(\frac{2\tau_f}{RC}\,T_2\,\frac{\partial\Phi}{\partial\mu}\right)^{-1}, \tag{5.9}$$

$$\frac{\delta T_2}{T_2} = \frac{2\tau_f}{RC}\left|\frac{\partial\Delta\Phi}{\partial\mu}\,\delta\mu\right| \ll 1\,. \tag{5.10}$$

Taking $\delta\mu \sim \dot{\mu}\Delta t$ and arbitrarily choosing a rise time $\Delta t \sim 10\,T_1$ we find a window for μ that for the parameters of Sec. IV gives

$$4\times 10^{11} \gg \dot{\mu} \gg 3\times 10^{-80}\,, \tag{5.11}$$

in units of $\sec^{-1}$ with the third condition then automatically satisfied, $\dot{\mu} \ll 3\times 10^{12}\ \sec^{-1}$. For a range of reasonable μ within this window, therefore, almost the entire hysteresis can be explored, jumps occurring within the critical region, where the $T_2 \to 0$ and $T_1 \to \infty$ behavior causes the inequalities to be violated.

In this paper we predict that an irradiated[27] Josephson junction with suitably chosen parameters exhibits bistable behavior with jumps and hysteresis in the voltage across it as a function of the radiation intensity. Through the ac Josephson effect, the frequency of the ac current will also exhibit such jumps. We have shown that the system possesses very interesting switching characteristics which could be tuned, for example, by changing the resistance across the circuit or the applied magnetic field. Estimates indicate that such bistable behavior may be within the reach of present experimental capabilities. The driven Josephson junction is thus worth investigating not only as an interesting new bistable system in its own right, but also as a means to deepen our understanding of nonequilibrium phase transitions in general.

APPENDIX A: SMALL CORRECTIONS TO THE ORDER PARAMETER EQUATION DUE TO $I_J\sin\theta$ TERM

In this appendix, we show that in the parameter regime of interest the neglect of the time-averaged $I_J\sin\theta$ term in Eq. (3.10) is fully justified as it leads to very small correction terms in the order parameter equation. Let us assume a somewhat more general steady state solution of (2.20) than in the text, namely,

$$\theta(t) = \omega t + \theta_0 + \sum_{n=1} \frac{\omega_n}{n\omega_0}\sin n\omega_0 t\,, \tag{A1}$$

where ω_0, ω_n, and ω are to be determined self-consistently. Then we have from $\dot{\theta} = 2eV(t)/\hbar$, that

$$V(t) = V + \sum_n V_n \cos n\omega_0 t\,, \tag{A2}$$

where

$$V = \hbar\omega/2e, \quad V_n = \hbar\omega_n/2e.$$

Substituting (A1) and (A2) into (2.20c) for the average of $\langle S^{\bm{*}}\rangle$ [noting that $\langle f^{\bm{*}}\rangle = 0$]

$$\langle \dot{S}^{\bm{*}}\rangle = -\frac{\langle S^{\bm{*}}\rangle}{RC} - \frac{\langle I_J\sin\theta\rangle}{2e} - 2T\langle a+a^\dagger\rangle\cos\theta - 4T_{\rm ext}\,N^{1/2}\int W(\Omega',\Omega)\cos\Omega' t\,\cos\theta\, d\Omega' \tag{A3}$$

and time-averaging (so $\omega_0 = \omega$) we obtain a Fourier-Bessel series on the right. Keeping only the leading contribution from ω_1, (assuming $\omega \gg \omega_1 \gg \omega_2 \cdots$) we get

$$\frac{\omega_1}{\omega} = \frac{2eI_J}{\hbar C\omega^2} J_0\left(\frac{\omega_1}{\omega}\right)\cos\theta_0 \tag{A4}$$

and

$$-\frac{V}{R} = I_J\, J_1\left(\frac{\omega_1}{\omega}\right)\sin\theta_0 + 4eN^{1/2}T_{\rm ext}\, WJ_0\left(\frac{\omega_1}{\omega}\right)\cos\theta_0 + 4eJ_0\left(\frac{\omega_1}{\omega}\right)\langle a^\dagger\rangle\langle a\rangle\,\kappa\,. \tag{A5}$$

In terms of scaled variables, this means that the order parameter equation

$$f - \mu + \frac{2Q\alpha}{1+4Q^2(f-1)^2} = 0$$

will acquire a correction term of the form

$$\frac{eR}{\hbar\,\omega_c}\left(\frac{eI_J^2\sin 2\theta_0}{\hbar C\omega^2}\right),$$

as well as $J_0(\omega_1/\omega)$ coefficients, which are nearly unity if $\omega_1/\omega \ll 1$. For the range of parameters used in the text, we find the correction term $\sim 10^{-2}$, i.e., since $f \sim 1$, the correction term is of the order of 1%. Note that $\omega_1/\omega \sim O(10^{-4}-10^{-5})$ which justifies our perturbation expansion. Thus the effect of the $I_J \sin\theta$ term in (3.10) is negligibly small in our parameter regime.

There are other effects in free-running junctions, besides the ones considered here, where the $I_J \sin\theta$ term may be non-negligible. Chen, Todd, and Kim[11] (CTK) have found Shapiro-type step behavior in the I -V characteristics of irradiated junctions without a dc bias. They have analyzed this behavior by considering only the $I_J \sin\theta$ term, without the photon drive and leakage terms on which our analysis depends. Thus the CTK effect is distinct from the effect considered here and occurs in the opposite regime $\omega_1/\omega \sim O(1)$, as can be shown by putting in their parameters. The difference between the two effects can also be seen on physical grounds: The CTK effect does not depend on the leakage, while bistability is suppressed if κ is set equal to zero. Bistability is a nonequilibrium phase transition involving a competition and jump between two dissipative states and is not merely a "step" phenomenon from the beating of two time-dependent voltages.

APPENDIX B: REDUCTION OF T_2 BY A "HOT" RESISTANCE

We have so far considered the entire Josephson-junction–resistor system as maintained at a common temperature $T \ll T_c$. In this appendix we consider the effect of placing the resistance in a separate temperature bath $T_R > T$, while the junction is maintained at $T \ll T_c$. Thermal voltage fluctuations across the resistor may then become important. Equation (3.10) for S^z has a Johnson noise term f^z, with[17]

$$\langle f^z\rangle = 0;\ \ \langle f^z(t) f^z(0)\rangle = \frac{\hbar\omega_c}{4e^2R}\coth\left(\frac{\hbar\omega_c}{2R_BT_R}\right)\delta(t)\,. \tag{B1}$$

(An electrical noise source of this strength could also be applied.) We continue to ignore quasiparticle effects, since $eV < \Delta$, $T \ll T_c$. The scaled Langevin equation of (3.16)

$$\dot f = \frac{-1}{RC}\left(f + \mu\cos\theta_0 + \frac{2Q\alpha}{1+4Q^2(f-1)^2}\right) + F_f \tag{B2}$$

then has a random force F_f obeying

$$\langle f(t) f(0)\rangle = \tau_f^{-1}\delta(t) \tag{B3}$$

and with the modified strength

$$\tau_f^{-1} = \frac{8e^2Q\alpha}{\hbar\omega_c RC^2}\left(1 + \frac{B_3T_R}{\alpha Q\hbar\omega_c}\right). \tag{B4}$$

Note that, as in (3.12), we have replaced the f-dependent diffusion constant τ_f by its value at $f = 1$. The full form would produce correction factors of order unity, making T_2 quite different on different branches.

The decay time T_2 is exponentially dependent on τ_f [see (5.4)] and is thus sensitive to the resistor temperature T_R. For our parameters, the second term enclosed in parentheses is $\approx (1 + 20T_R)$. A considerable reduction in T_2 could thus be achieved by this "hot" resistor method, keeping the junction parameters fixed. High temperatures would tend to reduce the hysteresis region. The critical region would be relatively expanded by a factor $(1+20T_R)^{2/3}$. A measurement of lifetimes would enable a direct estimate of T_2 to be made.

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[21]Note that a slow ω dependence would lead to a slight bending of the straight lines in Fig. 2; however, the bistability remains as long as a triple-root region exists.
[22]In a recent paper M. Milani, R. Bonifacio, and M. Scully [Lett. Nuovo Cimento 26, 353 (1979)] have also predicted bistability in a different model which involves a biased Josephson junction rather than an unbiased one as in the present case.
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[26]It is interesting to note that the time T_2 for a model of optical bistability has been estimated to be of the order of 10^{27} sec [K. Kondo, M. Mabuchi, and H. Hasegawa, Opt. Commun. 32, 136 (1980)].
[27]Other drive parameters could be used, separately or together. For example, a voltage source V_0 could be included in the circuit. The resistive relaxation term in (2.20c) then becomes $-(S_z - CV_0)/R$. In scaled form, the voltage enters in the same way as $(N/N_c)^{1/2}$. μ is then $2eV_0/\hbar\omega_c$ and the bistability conditions carry over with $V_0 \sim 0.1$ mV in the hysteresis region. A voltage drive might be simpler to control close to a second-order critical point, e.g., when looking for deviations from mean-field critical exponents $[\Delta f \sim (\Delta\mu)^{1/3}]$.