Applications of the Collective Field Theory for the Calogero-Sutherland Model

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Abstract

We use the collective field theory known for the Calogero-Sutherland model to study a variety of low-energy properties. These include the ground state energy in a confining potential upto the two leading orders in the particle number, the dispersion relation of sound modes with a comparison to the two leading terms in the low temperature specific heat, large amplitude waves, and single soliton solutions. The two-point correlation function derived from the dispersion relation of the sound mode only gives its nonoscillatory asymptotic behavior correctly, demonstrating that the theory is applicable only for the low-energy and long wavelength excitations of the system.

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1. INTRODUCTION

The Calogero-Sutherland-Moser model (CSM) [1-3] has attracted much attention in recent years due to its relation to a wide variety of interesting problems. Some examples are random matrix theory [4], quantum spin chains with long-range interactions [5], generalized exclusion statistics [6-12], Gaussian conformal field theories [13], edge states in a quantum Hall system [14], and nonlinear internal waves in a stratified fluid [15].

The CSM has been known to be exactly solvable and integrable, both classically and quantum mechanically, for quite some time [1-3, 16]. However detailed investigations into its collective properties have begun only recently [17-19]. A collective field theory to study the excitations of a superfluid, as well as the ground state of a condensed Bose-Einstein gas was developed long back [20]. In such a theory, the fundamental coordinate is the density field [21-24]. For the CSM, the results obtained so far include the ground state energy of the model placed in a harmonic oscillator potential, waves of arbitrary amplitude for strong coupling, and isolated solitons on an uniform background density.

In our paper, we will study essentially the same aspects but in more detail and for arbitrary coupling, thereby generalizing the earlier results in several ways. Wherever appropriate, we will compare our results with those obtained earlier by other methods [11, 25]. This will illustrate that certain properties of the model can be derived more easily and generally from collective field theory. These properties include the dispersion relation for small amplitude and long wavelength sound modes, the low temperature specific heat, and the two-point correlation function. The collective field theory yields a dispersion relation for the sound mode that terminates exactly in the second order of the wave number, which is adequate only for small wave numbers. This gives the correct *nonoscillatory* behavior of the two-point correlation function for asymptotically large distances, and it fails at shorter distances. The collective field theory formulation is thus seen to be a useful description for the low-energy, or small wave number excitations of the CSM system.

2. CALOGERO-SUTHERLAND MODEL AND COLLECTIVE FIELD THEORY

The simplest form of the CSM consists of particles on a line which interact pairwise through an inverse-square potential. The model can also be defined on a circle with periodic boundary condition [2]; the two versions of the model have identical physical properties in the thermodynamic limit in which the the number of particles N and the length L of the line (or circle) are simultaneously taken to infinity keeping the particle density $\rho_0 = N/L$ fixed. The Hamiltonian for particles on a line is given by

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{\hbar^2 \lambda(\lambda-1)}{m} \sum_{i<j} \frac{1}{(x_i - x_j)^2} , \tag{1}$$

where the dimensionless coupling $\lambda \geq 0$. To make the problem well-defined quantum mechanically, we have to add the condition that the wave functions Ψ goes to zero as $|x_i - x_j|^\lambda$ whenever two particles i and j approach each other. For $\lambda = 0$ and 1, the model describes free bosons and free fermions respectively. Since the two-body potential is singular enough to prevent particles from crossing each other, we can choose the wave functions to be either symmetric (bosonic) or antisymmetric (fermionic). The energy spectrum is the same in the two descriptions.

Let us briefly summarize some of the exactly known results for this model [2, 17]. If E_0 denotes the ground state energy, then the chemical potential at zero temperature is given by $\mu = \partial E_0 / \partial N$ in the thermodynamic limit. This takes the form

$$\mu \ = \ \frac{\pi^2 \hbar^2 \lambda^2 \rho_0^2}{2m} \ . \tag{2}$$

In a fermionic description of the model, it is natural to define a Fermi momentum

$$p_0 \ = \ \pi \hbar \rho_0 \ . \tag{3}$$

(We should point out that some papers in this field find it more convenient to define the Fermi momentum to be $\pi \hbar \lambda \rho_0$). The low-energy excitations of (1) are known in detail. They can be thought of as being made up of particle and hole excitations. Let us first define the sound velocity by the relation

$$v_s^2 \ = \ \frac{\rho_0}{m} \left(\frac{\partial \mu}{\partial \rho_0} \right) \ . \tag{4}$$

Then

$$v_s \ = \ \frac{\pi \hbar \lambda \rho_0}{m} \ . \tag{5}$$

It is known that the particle excitations necessarily have $|p| \geq p_0$, with the dispersion

$$\epsilon_p (p) \ = \ \frac{1}{2m} \ (|p| - p_0) \ [\ |p| + (2\lambda - 1) p_0 \ ] \ + \ \mu \ . \tag{6}$$

The hole excitations have $|p| \leq p_0$, with the dispersion

$$\epsilon_h (p) \ = \ \frac{\lambda}{2m} \ (p_0^2 - p^2) \ - \ \mu \ . \tag{7}$$

If we define group velocities $v = \hbar\partial\omega/\partial p$, we find that particles have $|v| \geq v_s$, while holes have $|v| \leq v_s$. The sound velocity in (5) can be obtained by considering a sound mode to be made up of a particle with energy-momentum (ϵ_1, p_1) and a hole with energy-momentum (ϵ_2, p_2). Then $v_s = (\epsilon_1 + \epsilon_2)/(p_1 - p_2)$ in the limit $p_1, p_2 \to p_0$.

The collective field theory for the CSM is obtained by changing variables from the particle coordinates x_i to the density field $\rho(x)$ defined as

$$\rho(x) \; = \; \sum_i \; \delta(x - x_i) \; . \tag{8}$$

As emphasized in reference [21], such a change of variables is meaningful only if the particle number $N \to \infty$. We therefore have to check at various stages whether the results obtained from collective field theory for *finite* values of N are indeed correct for the model defined in (1). For this reason, we will compare the collective field theory results with those obtained by other methods whenever possible.

After changing variables, the quantum Hamiltonian takes the form [22, 23]

$$\begin{aligned} H \; = \; \frac{\hbar^2}{2m} \int dx \quad [\; \rho \; (\partial\theta)^2 \; + \; \frac{\pi^2\lambda^2}{3} \; \rho^3 \\ + \; \lambda(\lambda - 1) \; \rho_H \partial\rho \; + \; \frac{(\lambda - 1)^2}{4} \; \frac{(\partial\rho)^2}{\rho} \;] \; , \end{aligned} \tag{9}$$

where $\partial = \partial/\partial x$, and ρ_H is proportional to the Hilbert transform of ρ defined as the principal part integral [27],

$$\begin{aligned} \rho_H(x) \quad &= \quad \int dy \; \rho(y) \; \frac{P}{x - y} \; , \\ \text{where} \quad \frac{P}{x - y} \quad &\equiv \quad \frac{1}{2} \lim_{\epsilon \to 0} \; (\; \frac{1}{x - y + i\epsilon} \; + \; \frac{1}{x - y - i\epsilon} \;) \; . \end{aligned} \tag{10}$$

The field $\hbar\theta$ is canonically conjugate to ρ, and they satisfy the equal-time commutation relation

$$[\ \rho(x)\ ,\ \hbar\theta(y)\ ]\ =\ i\hbar\ \delta(x-y)\ . \tag{11}$$

We may therefore set $\theta(x) = -i\delta/\delta\rho(x)$, and try to find eigenstates of the Hamiltonian (9) as functionals of $\rho(x)$. Although this is sometimes possible [23], it is generally very difficult to find exact eigenstates.

We will therefore take the simpler route of studying (9) *classically.* For this purpose, it is useful to rewrite the collective theory in terms of a fluid. Let us introduce a complex field

$$\psi(x,t)\ =\ \sqrt{\rho(x,t)}\ e^{i\theta(x,t)}\ . \tag{12}$$

We define a Lagrangian $L = \int dx\mathcal{L}$, where the Lagrangian density

$$\begin{aligned}
\mathcal{L}\ &=\ \frac{i\hbar}{2}\ (\psi^\star\dot{\psi}\ -\ \dot{\psi}^\star\psi)\ -\ \frac{\hbar^2}{2m}\ \partial\psi^\star\partial\psi\ -\ U[\rho(x)] \\
&=\ -\ \hbar\rho\dot{\theta}\ -\ \frac{\hbar^2}{2m}\ \Big[\ \frac{(\partial\rho)^2}{4\rho}\ +\ \rho(\partial\theta)^2\ \Big]\ -\ U[\rho]\ , \\
\text{where}\quad U[\rho]\ &=\ \frac{\hbar^2}{2m}\ \Big[\ \frac{\pi^2\lambda^2}{3}\ \rho^3\ +\ \lambda(\lambda-1)\ \rho_H\partial\rho\ +\ \frac{\lambda(\lambda-2)}{4}\ \frac{(\partial\rho)^2}{\rho}\Big]\ ,
\end{aligned} \tag{13}$$

and a dot denotes $\partial/\partial t$. From (13) we see that ρ and $\hbar\theta$ are canonically conjugate to each other, and we can recover the Hamiltonian (9) by the usual methods.

It is interesting to note that the Lagrangian (13) is quadratic in ψ and $\psi^\star$, and is therefore noninteracting for $\lambda = 0$ (free bosons). This is understandable because the collective field theory is a bosonic theory, as is clear from (11). On the other hand, the collective theory is interacting for $\lambda = 1$ (which is free in terms of a *fermionic* theory).

We now proceed to study the theory (13) classically. At a classical level, Eq. (9) (for the static case) may be regarded as the energy density functional of the single-particle density $\rho(x)$. This is analogous to the density functional theory of a correlated many-particle system, a highly sucessful formalism in many branches of physics [26]. Since we wish to study the system in an external potential $V(x)$ in Section 3, let us add

$$\int dx \, [\, \mu - V(x) \,] \, \rho(x) \; - \; \mu N \tag{14}$$

to the Lagrangian, where μ is the chemical potential. The Euler-Lagrange equations of motion obtained by extremizing the action $S = \int dt L$ are given by

$$\begin{aligned} \frac{\pi^2 \hbar^2 \lambda^2}{2m} \, \rho^2 \quad &- \quad \frac{\hbar^2 \lambda (\lambda - 1)}{m} \, \partial \rho_H \; + \; \frac{\hbar^2 (\lambda - 1)^2}{8m} \Big[\, \Big(\frac{\partial \rho}{\rho} \Big)^2 \; - \; 2 \frac{\partial^2 \rho}{\rho} \, \Big] \\ &+ \quad \hbar \dot{\theta} \; + \; \frac{\hbar^2}{2m} \, (\partial \theta)^2 \; - \; \mu \; + \; V(x) \; = \; 0 \; , \end{aligned} \tag{15}$$

and

$$\dot{\rho} \; + \; \frac{\hbar}{m} \, \partial (\rho \partial \theta) \;\; = \;\; 0 \; . \tag{16}$$

In addition, the density must satisfy the constraint

$$\int dx \; \rho \;\; = \;\; N \; . \tag{17}$$

Eq. (16) will be recognized as the equation of continuity since $(\hbar/m)\partial\theta$ is the velocity field; this can be seen from the expression for momentum given below.

Our system has three conserved quantities, namely, the particle number N, the momentum (if there is no external potential)

$$P \;=\; -\;\frac{i\hbar}{2}\int dx\;(\psi^\star\partial\psi - \partial\psi^\star\psi) \;=\; \hbar\int dx\;\rho\;\partial\theta\;, \tag{18}$$

and the energy

$$\begin{aligned} E = \quad & \frac{\hbar^2}{2m}\int dx\;\Big[\;\rho(\partial\theta)^2 + \frac{\pi^2\lambda^2}{3}\rho^3 + \lambda(\lambda-1)\rho_H\partial\rho + \frac{(\lambda-1)^2}{4}\;\frac{(\partial\rho)^2}{\rho}\;\Big] \\ & + \int dx\;V(x)\;\rho\;. \end{aligned} \tag{19}$$

There are probably an infinite number of conserved quantities in addition to the three above since our original system is integrable; however explicit field theoretic expressions for these other quantities are not known.

In the absence of an external potential $V(x)$, Eqs. (15) and (16) are invariant under scaling and Galilean transformations. Under scaling by a factor α, we have

$$\begin{aligned} \rho(x,t) &\quad\rightarrow\quad \alpha\;\rho(\alpha x, \alpha^2 t)\;, \\ \theta(x,t) &\quad\rightarrow\quad \theta(\alpha x, \alpha^2 t)\;, \\ \mu &\quad\rightarrow\quad \alpha^2\mu\;. \end{aligned} \tag{20}$$

Under a Galilean transformation by velocity v,

$$\begin{aligned} \rho(x,t) &\quad\rightarrow\quad \rho(x - vt, t)\;, \\ \theta(x,t) &\quad\rightarrow\quad \theta(x - vt, t)\;+\;\frac{mv}{\hbar}\;(x - \frac{1}{2}vt)\;, \\ \mu &\quad\rightarrow\quad \mu\;. \end{aligned} \tag{21}$$

Thus

$$P \quad\rightarrow\quad P\;+\;mNv\;,$$

$$E \quad \rightarrow \quad E \ + \ Pv \ + \ \frac{1}{2} m N v^2 \ . \tag{22}$$

It is quite remarkable that if $V = 0$, all values of $\lambda > 1$ are equivalent to each other according to Eqs. (15) and (16). Namely, if we redefine

$$\begin{aligned}
\tilde{x} \quad &= \quad x \ , \quad \tilde{t} \ = \ t(\lambda - 1) \ , \\
\tilde{\rho} \quad &= \quad \frac{\lambda \rho}{\lambda - 1} \ , \quad \tilde{\theta} \ = \ \frac{\theta}{\lambda - 1} \ , \\
\tilde{\mu} \quad &= \quad \frac{\mu}{(\lambda - 1)^2} \ ,
\end{aligned} \tag{23}$$

then $\lambda - 1$ can be completely scaled out of (15) and (16). Similarly, all values of $\lambda < 1$ (but not equal to 0) are equivalent to each other; we can carry out the same redefinitions as in (23), followed by $\tilde{\rho} \rightarrow -\tilde{\rho}$ in order to keep $\tilde{\rho}$ positive. If we redefine the energy functional as $\tilde{E} = \lambda E / |\lambda - 1|^3$, we see from (19) that

$$\tilde{E} \ = \ \frac{\hbar^2}{2m} \int dx \ \Big[\ \tilde{\rho} (\partial \tilde{\theta})^2 \ + \ \frac{\pi^2}{3} \tilde{\rho}^3 \ \pm \ \tilde{\rho}_H \partial \tilde{\rho} \ + \ \frac{(\partial \tilde{\rho})^2}{4 \tilde{\rho}} \ \Big] \ , \tag{24}$$

where the $\pm$ signs are for $\lambda > 1$ and $\lambda < 1$ respectively. Thus it is sufficient to study the collective field theory for just two values of λ, one less than 1 and the other greater than 1. This property of the collective field theory clearly shows that it is a rather coarse description of the CSM. This is to be contrasted with the exact solution of the model (1) some of whose features (for instance, the dynamical correlation functions [28]) are sensitively dependent on number theoretic properties of λ.

In passing, it may be noted that formally the scaled energy density given by Eq. (24) is of the same form as the so-called Madelung fluid [29], which is a hydrodynamical description of the one-particle Schrödinger equation. In this picture, the first term is the classical kinetic energy of the fluid, the next two represent the potential energy, and the last term arises from the quantum kinetic energy. The latter gives rise to the Bohm potential [30] in the equations of motion. This interpretation also holds if there are N particles in the same quantum state. The normalization of $\tilde{\rho}$ in Eq. (24), however, is *not* N. To pursue this line of thought more carefully, it is necessary to modify transformations (23), and scale the x-coordinate to demand $\int d\tilde{x}\tilde{\rho} = N$. The Bohm term in (24) remains unaffected, but the interaction terms become λ-dependent. We will not elaborate further along these lines.

We will now study various solutions of the equations of motion (15) and (16), with an external potential in Section 3 and without an external potential in Sections 4-6. We are interested in two kinds of solutions, (a) static solutions in which ρ depends only on x and $\theta = 0$ (in particular, the ground state is always of this form), and (b) time-dependent solutions in which ρ and θ depend on both x and t.

3. GROUND STATE IN AN EXTERNAL POTENTIAL

For any external potential, Eq. (15) gives the exact quantum ground state energy and density if $\lambda = 0$. In that case, let $\Psi_0(x)$ and e_0 denote the exact one-particle ground state wave function (normalized to unity) and energy obtained by solving the Schrödinger equation with a potential $V(x)$. Then the solution of (15) and (16) is given by

$$\begin{aligned} \rho(x) &= N\,|\Psi_0(x)|^2 \ , \\ E_0 &= N e_0 \ . \end{aligned} \tag{25}$$

The question therefore is how well collective field theory does for *nonzero* values of λ.

To begin with, let us consider the case of a simple harmonic potential, with

$$V(x) \; = \; \frac{1}{2} m\omega^2 x^2 \; . \tag{26}$$

This turns out to be a rather special case because the ground state energy of the collective field theory can be found exactly. Since this is a static solution with $\theta = 0$, we can use the principal part identity

$$\frac{P}{x-y}\,\frac{P}{x-z} \; + \; \frac{P}{y-z}\,\frac{P}{y-x} \; + \; \frac{P}{z-x}\,\frac{P}{z-y} \; = \; \pi^2 \delta(x-y)\delta(x-z) \; , \tag{27}$$

to write (19) as a perfect square [23]

$$E = \frac{\hbar^2}{2m} \int dx \; \rho \Big(\lambda \rho_H + \frac{\lambda - 1}{2} \frac{\partial \rho}{\rho} - \frac{m\omega}{\hbar} x \Big)^2 + \frac{\hbar \omega}{2} \left[\lambda N^2 + (1{-}\lambda) N \right] . \tag{28}$$

Thus if ρ satisfies

$$\lambda \rho_H \; + \; \frac{\lambda - 1}{2} \; \frac{\partial \rho}{\rho} \; = \; \frac{m\omega}{\hbar} \; x \; , \tag{29}$$

then it minimizes (19) and is therefore a solution of the equation of motion (15). Further, the ground state energy follows from (28),

$$E_0 \; = \; \frac{\hbar \omega}{2} \; [\lambda N^2 \; + \; (1-\lambda) N] \; . \tag{30}$$

This is in fact the exact answer for the Hamiltonian (1).

Eq. (29) for the density can be solved analytically only if $\lambda = 0$ or 1. We get

$$\begin{aligned} \rho \quad &= \quad N \left(\frac{m\omega}{4\pi\hbar}\right)^{1/2} \; \exp \; \left(\; - m\omega x^2/\hbar \; \right) \quad \text{if} \quad \lambda = 0 \\ &= \quad \frac{mw}{\pi\hbar} \; \Big(\; \frac{2N\hbar}{m\omega} \; - \; x^2 \; \Big)^{1/2} \quad \text{if} \quad \lambda = 1 \; . \end{aligned} \tag{31}$$

We can show analytically that the collective field theory density has a Gaussian tail of the form

$$\rho \;\sim\; x^{2\lambda N/(1-\lambda)} \;\exp\;\Big[\;-\;\frac{m\omega x^2}{\hbar(1-\lambda)}\;\Big] \tag{32}$$

if $\lambda < 1$, and has a sharp cutoff $x = \pm x_0$ beyond which ρ vanishes if $\lambda \geq 1$. For large values of N, we can also show that the second term on the left hand side of (29) is generally much smaller than the first term; the scaling argument is indicated below. If we ignore the second term altogether, we get the leading behavior of ρ to be a semicircle for all nonzero values of λ,

$$\begin{aligned} \rho \;&=\; \frac{mw}{\pi\hbar\lambda}\,\Big(x_0^2 \;-\; x^2\,\Big)^{1/2} \qquad \text{for} \quad |x| \leq x_0, \\ &=\; 0 \quad \text{for} \quad |x| \;> x_0 \;. \end{aligned} \tag{33}$$

Here x_0 is defined by

$$x_0 \;=\; \Big(\frac{2N\hbar\lambda}{m\omega}\Big)^{1/2} \;. \tag{34}$$

The relations (33) and (34) are identical to the Thomas-Fermi result obtained in [12], and x_0 is just the classical turning point. Note that the form of ρ in (33) is essentially a statement of exlusion statistics for the CSM; the occupation number in each state in phase space $dxdp = 2\pi\hbar$ is given by $1/\lambda$. One should, however, be wary of using the expression (33) for the density $\rho(x)$. For example, if we indiscriminantly substitute this $\rho(x)$ in the static energy density functional

$$E = \int dx\; V(x)\;\rho\; +\; \frac{\hbar^2}{2m}\;\int dx\;\Big[\;\frac{\pi^2\lambda^2}{3}\rho^3 +\; \lambda(\lambda-1)\rho_H\partial\rho + \frac{(\lambda-1)^2}{4}\;\frac{(\partial\rho)^2}{\rho}\;\Big]\;, \tag{35}$$

the integrals with $V(x)$ and ρ^3 on the right-hand side together yield the correct N^2-dependent term in E_0, and the third integral gives the right N-dependent term (see Eq. (30), but the last integral involving $(\partial\rho)^2/\rho$ diverges. It may be easily checked, however, that this divergent term goes like N^0, i.e., of order 1 in the large-N expansion. Such terms in the expansion will be dropped.

We should point out that the $1/N$ expansion of ρ within the collective field theory cannot be taken too seriously; we recall the cautionary remarks following Eq. (8). For instance, the absence of a Gaussian tail if $\lambda \geq 1$ is an artifact of collective field theory. If N is finite, the ground state of (1) has a Gaussian tail for all values of λ; this can be seen from the exact expression

$$\Psi_0 \, [x_i] \;\sim\; \prod_{i<j} \, |x_i - x_j|^\lambda \; \exp \; [\; - \frac{m\omega}{2\hbar} \; \sum_i \; x_i^2 \;] \; . \tag{36}$$

On the other hand, we can generally trust the next to leading term in the *energy* given by collective field theory. We have already seen this for the harmonic oscillator potential, where the N^2 and N-dependent terms both came out correctly. We will now show this for a somewhat more general class of potentials.

We formally define the first two terms in a $1/N$ expansion as follows. We assume that the ground state energy and chemical potential have expansions of the form

$$\begin{aligned} \mu &= \mu^{(0)} + \mu^{(1)} \; , \\ E_0 &= E_0^{(0)} + E_0^{(1)} \; . \end{aligned} \tag{37}$$

where $E_0^{(1)}/E_0^{(0)}$ and $\mu^{(1)}/\mu^{(0)}$ are of order $1/N$. For the density, we have an expansion of the form

$$\rho(x) \;=\; \rho^{(0)}(x) \;+\; \rho^{(1)}(x) \;. \tag{38}$$

We will state the procedure for obtaining the expansion (38) shortly. But we can point out immediately that it is not a $1/N$ expansion for *all* positions x; although $\rho^{(1)}(x)/\rho^{(0)}(x)$ will *generally* be order $1/N$, that will not be true near the turning point x_0. The first term $\rho^{(0)}$ in (38) is defined by considering only the ρ^3 and $V\rho$ terms in the energy functional (35) and hence in Eq. (15). The leading term in the chemical potential $\mu^{(0)}$ and the turning point x_0 are then fixed by the particle number constraint (17). Next, we define the term $\rho^{(1)}$ in (38) by considering the ρ^3, $V\rho$ and $\rho_H \partial \rho$ terms in (35) and the corresponding terms in (15). Once again, $\mu^{(1)}$ is fixed by the constraint (17). We will not go beyond the two leading terms in Eqs. (37) and (38), and will therefore not need to consider the $(\partial\rho)^2/\rho$ term in (35); this term actually diverges even more severely for $\rho^{(1)}$ than for $\rho^{(0)}$.

As a specific example of the $1/N$ expansion, let us now consider a confining potential of the power-law form

$$V(x) \;=\; \frac{\hbar^2}{2ma^2} \left(\frac{|x|}{a} \right)^p \;, \tag{39}$$

where $p > 0$, and a is a measure of the width of the potential. (A harmonic potential corresponds to the case $p = 2$). For such a potential, we can prove that the $V\rho + \rho^3$, $\rho_H \partial\rho$, and $(\partial\rho)^2/\rho$ terms in the energy functional (35) are successively of higher order in $1/N$. To show this, let us define the dimensionless variables

$$\tilde{x} \quad = \quad \frac{1}{N^{2/(p+2)}} \, \frac{x}{a} \;,$$

$$\tilde{\rho}(\tilde{x}) = \frac{1}{N^{p/(p+2)}} \, a \, \rho(x) \, , \tag{40}$$

so that $\int d\tilde{x}\tilde{\rho} = 1$. Then Eq. (35) takes the form

$$\begin{aligned} E \; = \; \frac{\hbar^2}{2ma^2} \, N^{(3p+2)/(p+2)} \int d\tilde{x} \, \Big[\, |\tilde{x}|^p \, \tilde{\rho} \; &+ \; \frac{\pi^2\lambda^2}{3}\tilde{\rho}^3 \; + \; \frac{\lambda(\lambda-1)}{N}\tilde{\rho}_H\tilde{\partial}\tilde{\rho} \\ &+ \; \frac{(\lambda-1)^2}{4N^2}\frac{(\tilde{\partial}\tilde{\rho})^2}{\tilde{\rho}} \, \Big] \, . \end{aligned} \tag{41}$$

This justifies the form of the $1/N$ expansion given in the previous paragraph. Following that procedure, the leading order terms in μ and ρ are found to be

$$\begin{aligned} \mu^{(0)} &= \frac{\hbar^2}{2ma^2} \, \Big(\frac{x_0}{a}\Big)^p \, , \\ \rho^{(0)} &= \frac{2}{\pi\hbar\lambda} \, [\, 2m(\mu^{(0)} - V(x) \,) \,]^{1/2} \quad \text{if} \quad |x| \le x_0 \, . \end{aligned} \tag{42}$$

From the constraint (17) we find

$$\begin{aligned} \Big(\frac{x_0}{a}\Big)^{1+p/2} &= \frac{\pi\lambda N}{2I_1} \, , \\ I_1 &= \int_0^1 dy \, \sqrt{1-y^p} \; = \; \frac{\Gamma(1+\frac{1}{p}) \, \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+\frac{1}{p})} \, . \end{aligned} \tag{43}$$

The leading terms ρ^3 and $V\rho$ in the energy (35) then give

$$\begin{aligned} E_0^{(0)} &= \frac{2}{3\pi\hbar\lambda} \int_0^{x_0} dx \, [\, 2m(\mu^{(0)} - V(x) \,]^{1/2} \, (\mu^{(0)} + 2V(x)) \\ &= \frac{\hbar^2}{2ma^2} \, \frac{p+2}{3p+2} \, \Big(\frac{\pi\lambda}{2I_1}\Big)^{2p/(p+2)} \, N^{(3p+2)/(p+2)} \, . \end{aligned} \tag{44}$$

We now go to next order in $1/N$ by including the terms in (15) and (35) which contain the Hilbert transform ρ_H; we get

$$\frac{\pi^2\hbar^2\lambda^2}{m} \, \rho^{(0)}\rho^{(1)} \; - \; \frac{\hbar^2\lambda(\lambda-1)}{m} \, \partial\rho_H^{(0)} \; = \; \mu^{(1)} \, . \tag{45}$$

The structure of (45) shows that $\rho^{(1)}$ must be taken to be zero outside the turning point x_0, just like $\rho^{(0)}$. Since $\rho^{(0)}$ is normalized to N, we fix $\mu^{(1)}$ using the constraint

$$\int_{-x_0}^{x_0} dx \; \rho^{(1)} \;=\; 0 \; , \tag{46}$$

and we then determine $E_0^{(1)}$ by including the $\rho_H \partial \rho$ term in (35). Interestingly, we find that $\rho^{(1)} = 0$ for all λ for the simple harmonic case ($p = 2$); that is why we get the energy correct to order N by just substituting $\rho^{(0)}$ in (35). However $\rho^{(1)}$ is not zero for a general value of p. We will omit the final expression for $E_0^{(1)}$ for general λ, and will now specialize to $\lambda = 1$ where we can compare with the results of a WKB approximation. (Although this is a free fermion theory, it is an interacting bosonic theory. Hence agreement at $\lambda = 1$ is a nontrivial check of collective field theory). We find that both $\mu^{(1)}$ and $\rho^{(1)}(x)$ are zero for $\lambda = 1$. Hence there is *no* correction to E_0 at the next order after (44), i.e., at order $N^{2p/(p+2)}$. We will now show that this result agrees with WKB.

If e_n denote the single-particle energy levels obtained by solving the Schrödinger equation in the potential $V(x)$, then the exact ground state energy at $\lambda = 1$ is given by

$$E_0 \;=\; \sum_{n=0}^{N-1} \; e_n \; . \tag{47}$$

For large n, we can obtain the two leading order terms in e_n using the WKB formula

$$\int_{-x_n}^{x_n} dx \; [\; 2m \; (\; e_n - V(x) \;) \;]^{1/2} \;=\; \Big(\; n + \frac{1}{2} \; \Big) \; \pi \hbar \; , \tag{48}$$

where x_n denotes the classical turning point for energy e_n. We thus obtain the expansion

$$e_n \;=\; \frac{\hbar^2}{2ma^2} \Big(\frac{\pi n}{2I_1}\Big)^{2p/(p+2)} \; [\; 1 \;-\; \frac{p}{(p+2)n} \;+\; \cdots \;] \; . \tag{49}$$

On substituting this in (47), we find that E_0 is indeed given by (44) and that there is no correction to the next order in $1/N$.

4. SMALL AMPLITUDE WAVES, CORRELATION FUNCTIONS, AND SPECIFIC HEAT

In this Section, we will study the small amplitude density fluctuations about an uniform background density ρ_0. We will show that these exhaust the low-energy excitations upto some order, both in a sum rule and in the low-temperature specific heat.

For an uniform density ρ_0, the chemical potential is given by (2) or (15) to be

$$\mu \;=\; \frac{\pi^2 \hbar^2 \lambda^2 \rho_0^2}{2m} \; . \tag{50}$$

Let us now study (15) and (16) to linear order in an amplitude $a << 1$. We assume

$$\begin{aligned} \rho \;&=\; \rho_0 \;+\; a \rho_0 \; \cos(kx - \omega t) \; , \\ \theta \;&=\; a \, \frac{m\omega}{\hbar k^2} \; \sin(kx - \omega t) \; , \end{aligned} \tag{51}$$

where k denotes the wave number; the second equation in (51) follows from the first due to the equation of continuity (16). Eq. (15) then yields the dispersion relation

$$\omega_k \;=\; | \; \frac{\pi \hbar \lambda \rho_0 |k|}{m} \;-\; \frac{(\lambda - 1) \hbar k^2}{2m} \; | \; . \tag{52}$$

The sound velocity is given by the group velocity $\partial\omega/\partial k$ at $k = 0$; the result agrees with the exact value given in (5). Note that (52) gives the correct single-particle dispersion for $\lambda = 0$, as expected for a free boson theory.

We see that the dispersion (52) vanishes not only at $k = 0$, but also at

$$|k| \;=\; \frac{2\pi\lambda\rho_0}{\lambda - 1} \tag{53}$$

if $\lambda > 1$. However the latter point where ω vanishes seems to be an artifact of collective field theory; it does not agree with known results. For instance, reference [25] defines a dispersion relation called the "Feynman spectrum" as follows. Consider the dynamical correlation function and its Fourier transform

$$\begin{aligned} G(x,t) \;&=\; \langle\ \rho(x,t)\rho(0,0)\ \rangle \;-\; \rho_0^2 \ , \\ S(k,\omega) \;&=\; \frac{1}{2\pi\rho_0}\int dx \int dt\ G(x,t)\ e^{-i(kx-\omega t)} \ . \end{aligned} \tag{54}$$

$S(k,\omega)$ can be represented in terms of all the states of the system $|n\rangle$ with energies E_n as

$$\begin{aligned} S(k,\omega) \;&=\; \frac{1}{N}\sum_n\ |\langle n|\rho_k|0\rangle|^2\ \delta(\hbar\omega - E_n + E_0)\ , \\ \rho_k \;&=\; \sum_{n=1}^{N}\ e^{-ikx_n}\ . \end{aligned} \tag{55}$$

One can then define various moments of $S(k,\omega)$ as

$$I_n(k) \;=\; \int d\omega\ \omega^n\ S(k,\omega)\ , \tag{56}$$

where $I_1(k) = k^2/2m$. The Feynman dispersion is defined as

$$\omega_F(k) \;=\; \frac{I_1(k)}{I_0(k)}\ . \tag{57}$$

Now this dispersion is known from random matrix theory [2, 4] for three special values of $\lambda = 1/2, 1$ and 2. For these three values, it is found [25] that $\omega_F(k)$ agrees with (52) upto second order for k close to 0. We therefore believe that $\omega_F(k)$ is given by (52) to order k^2 for *all* values of λ. However, the agreement between $\omega_F(k)$ and (52) does not persist to higher orders in k even for the three special values of λ; in particular, $\omega_F(k)$ does not vanish at any nonzero values of k, although it does have a roton-like minimum at $|k| = 2\pi\rho_0$ for $\lambda > 1$ [25]. This discrepancy between $\omega_F(k)$ and (52) seems to indicate that collective field theory cannot be trusted for large values of the wavenumber; it seems to work only upto order k^2.

Our statement that the low-energy dispersion is correctly given by (52) upto k^2 near $k = 0$ also agrees with the known low-temperature specific heat of the CSM to second order in the temperature T [11]. We can compute the free energy per unit length f from (52) taking the sound modes to have zero chemical potential. Thus

$$\beta f \; = \; \int_{-\infty}^{\infty} \frac{dk}{2\pi} \; \ln \; (\; 1 \; - \; e^{-\beta \hbar \omega_k} \;) \; , \tag{58}$$

where $\beta = 1/k_B T$. After evaluating this, we can obtain the specific heat per unit length $C_V = -T \partial^2 f / \partial T^2$ to second order in T. We find

$$\begin{aligned} C_V \quad &= \quad \frac{\pi k_B^2 T}{3 \hbar v_s} \; + \; \frac{6 \zeta(3) (\lambda - 1)}{\pi} \; \frac{k_B^3 T^2}{m \hbar v_s^3} \; , \\ \zeta(3) \quad &= \quad \sum_{n=1}^{\infty} \frac{1}{n^3} \; . \end{aligned} \tag{59}$$

This agrees with the result in reference [11].

We note that the linear terms in (52) and (59) are typical of a system whose low-energy and long-wavelength excitations are governed by a conformal field theory [13]. The quadratic terms in those two equations indicate deviations from conformal field theory which start appearing when the wavelength is no longer much longer than the typical particle spacing $1/\rho_0$.

Finally, it may be useful to see what we get if we compute the correlation function $G(x,t)$ defined in (54) by *quantizing* the collective field theory. Using the commutation relation (11) and the equations of motion (15) and (16) to linear order, we find that ρ and θ have the following second quantized expressions,

$$\begin{aligned} \rho &= \rho_0 + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, f_k \left[a_k \, e^{i(kx-\omega_k t)} + a_k^\dagger \, e^{-i(kx-\omega_k t)} \right] , \\ \theta &= \frac{i}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \frac{1}{f_k} \left[- a_k \, e^{i(kx-\omega_k t)} + a_k^\dagger \, e^{-i(kx-\omega_k t)} \right] , \\ f_k &= \left(\frac{\hbar \rho_0 k^2}{2m\omega_k} \right)^{1/2} , \end{aligned} \tag{60}$$

where

$$[\, a_k \, , \, a_{k'}^\dagger \,] = 2\pi \, \delta(k-k') \, . \tag{61}$$

From this we find that

$$G(x,t) = \frac{\hbar \rho_0}{2m} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \frac{k^2}{\omega_k} \, e^{i(kx-\omega_k t)} \, . \tag{62}$$

If we now use the collective field theory dispersion (52), the integral will diverge at the nonzero values of $|k|$ where ω vanishes. We will therefore assume, as stated above, that (52) can only be trusted in the region near $k=0$. The asymptotic form of $G(x,t)$ at large values of $x \pm v_s t$ only gets a contribution from that region in the integral (62); further, only the linear term in the dispersion (52) is required to derive the asymptotic expression.

We then find that

$$G(x,t) \sim - \frac{1}{4\pi^2\lambda} \left[\frac{1}{(x-v_s t)^2} + \frac{1}{(x+v_s t)^2} \right] . \tag{63}$$

This agrees with the leading nonoscillating term in the exact expression given in reference [28]. However the exact expression also has oscillating terms; in fact, such a term dominates over (63) if $\lambda > 1$. The fact that collective field theory is unable to reproduce these oscillating terms clearly shows its limitation.

5. LARGE AMPLITUDE WAVES

Following a method given in reference [18], we can find exact solutions which describe waves with arbitrary amplitude. We will consider the cases $\lambda > 1$ and $0 < \lambda < 1$ separately.

For $\lambda > 1$, the solutions are given by

$$\begin{aligned} \rho(x,t) &= \frac{(\lambda-1)k}{2\pi\lambda} \left[c + \frac{\sinh\alpha}{\cosh\alpha - \cos k(x-vt)} \right] , \\ \theta(x,t) &= \frac{mv}{\hbar} \Big[- (c+1) \int_{-\infty}^{x} dy \, \frac{\cosh\alpha - \cos k(y-vt)}{c\cosh\alpha + \sinh\alpha - c\cos k(y-vt)} \\ &\quad + x - \frac{1}{2} vt \Big] , \end{aligned} \tag{64}$$

We choose the wavenumber k and the parameter α to be positive. The phase velocity v in (64) satisfies

$$v^2 = \frac{(\lambda-1)^2\hbar^2 k^2 c^2}{4m^2(c+1)^2} \, (c^2 + 2c\coth\alpha + 1) , \tag{65}$$

where

$$c \geq \frac{1-\cosh\alpha}{\sinh\alpha} . \tag{66}$$

The average density for this solution is found to be

$$\rho_0 \ = \ \frac{(\lambda - 1)k}{2\pi\lambda} \ (c+1) \ . \tag{67}$$

We can define the dimensionless amplitude a of the wave to be the fractional difference between the maximum and minimum densities. Thus

$$a \ = \ \frac{\rho_{max} - \rho_{min}}{\rho_{max} + \rho_{min}} \ = \ \frac{1}{c \sinh\alpha + \cosh\alpha} \ . \tag{68}$$

For $\lambda < 1$, the solutions are again given by Eqs. (64) and (65), but

$$c \ \leq \ \frac{-1 - \cosh\alpha}{\sinh\alpha} \ . \tag{69}$$

The average density for this solution is the same as in Eq. (67), while the amplitude a is

$$a \ = \ - \ \frac{1}{c \sinh\alpha + \cosh\alpha} \ . \tag{70}$$

The solutions above are characterized by three independent parameters which may be considered to be the average density ρ_0, the wavenumber k, and the amplitude a. If we hold ρ_0 and k fixed and let $\alpha \to \infty$, we recover the small amplitude waves discussed in the previous Section. Let us now look at the conditions under which the frequency $\omega = |vk|$ can vanish; this corresponds to stationary waves. We can see from Eq. (65) that ω vanishes if $k = 0$ or if $c = 0$; the latter is allowed if $\lambda > 1$ in which case k satisfies (53). These two conditions for $\omega = 0$ are therefore the same as those found for the sound modes in Section 4. However we now see that ω also vanishes if

$$c \ = \ \frac{1 - \cosh\alpha}{\sinh\alpha} \tag{71}$$

for $\lambda > 1$, or if

$$c = \frac{-1 - \cosh\alpha}{\sinh\alpha} \tag{72}$$

for $\lambda < 1$. In these two cases, we get staionary waves with the largest possible amplitude $a = 1$ since $\rho_{min} = 0$.

The interpretation of all these large amplitude waves, including the new kinds of stationary waves in (71) and (72), in terms of the exact solutions of the CSM model remains an open question. It is possible that some of the solutions obtained here are peculiar to the collective field theory and do not correspond to anything in the CSM. For instance, there are no exact solutions of the CSM which have arbitrary nonzero values of k with $\omega = 0$.

In concluding, we would like to mention that the density waves studied in this Section and in the previous Section were known earlier [18] for *large* λ. In addition, the stationary waves (72) for $\lambda < 1$ were found in reference [19]. Our own results describe both stationary and moving waves, and are valid for all values of λ.

6. SINGLE SOLITONS

We will now describe the single soliton solutions of the collective field theory [18, 19]. Starting from the large amplitude waves in Section 5, we can find these solutions for any λ different from 0 and 1 as follows. In Eq. (64), we take the limit $k, \alpha \to 0$ keeping $\alpha/k = b$ fixed. Simultaneously, we let $c \to \infty$ for $\lambda > 1$ or $-\infty$ for $\lambda < 1$, keeping ρ_0 fixed according to (67). Since the wavelength $2\pi/k \to \infty$, we obtain a solution describing an isolated lump. We find the following expressions for ρ and θ in terms of the width b

and velocity v of the soliton.

$$\begin{aligned}
\rho(x,t) &= \rho_0 + \frac{\lambda-1}{\pi\lambda} \frac{b}{(x-vt)^2+b^2} , \\
\theta(x,t) &= \pm (\lambda-1) \tan^{-1} \Big(\frac{x-vt}{b\,\eta} \Big) - \frac{1}{2} \frac{mv^2 t}{\hbar} , \\
\text{where} \quad v &= \pm \frac{\pi\hbar\lambda\rho_0}{m} \eta , \\
\eta &= \Big[1 + \frac{\lambda-1}{\pi\lambda\rho_0 b} \Big]^{1/2} .
\end{aligned} \tag{73}$$

Thus the velocity and width are related to each other. From Eq. (5) we see that $|v| \geq v_s$ and any value of $b\rho_0$ is allowed for $\lambda > 1$. For $\lambda < 1$, we must have $b\rho_0 \geq (1-\lambda)/\pi\lambda$; then $|v| \leq v_s$. These ranges of velocity agree with the exact results known for the particle and hole respectively, as discussed after Eq. (7). The identification with particle for $\lambda > 1$ and hole for $\lambda < 1$ may be justified by considering the *sign* of the integrated density for the soliton,

$$\int dx \, [\, \rho(x,t) - \rho_0 \,] = \frac{\lambda-1}{\lambda} . \tag{74}$$

The magnitude of this number is generally not an integer; the physical meaning of this is not clear to us. It is interesting to note at this point that if we perform the scaling (23) which eliminates λ, then the redefined soliton number is 1 for all λ. Hence (73) describes a one particle solution in the redefined theory.

The momentum and energy (obtained after subtracting the background value) of the soliton (73) are given by Eqs. (18) and (19).

$$\begin{aligned}
P &= \frac{\lambda-1}{\lambda} mv , \\
E &= \frac{\lambda-1}{\lambda} (\frac{1}{2} mv^2 - \mu) .
\end{aligned} \tag{75}$$

This dispersion relation does *not* agree with the exact dispersion relations given in (6) and (7). It therefore seems that the interpretation of solitons as particles or holes has some difficulties which need to be resolved.

It is interesting to observe that we can also go in the opposite direction and recover the large amplitude waves by superposing a number of single soliton solutions in a periodic way [18]. For this purpose, it is useful to recall that

$$2\alpha \sum_{n=-\infty}^{\infty} \frac{1}{(2\pi n + kx)^2 + \alpha^2} ~=~ \frac{\sinh \alpha}{\cosh \alpha - \cos ~(kx)} ~. \tag{76}$$

It is worth noting that for $\lambda > 1$, there is no lower bound on ρ_0; in particular, we can set $\rho_0 = 0$. We then get a new solution corresponding to a stationary and isolated soliton with no background density. This may also be seen for the static case from Eq. (29) with no external harmonic potential,

$$\lambda \rho_H ~+~ \frac{\lambda - 1}{2} \frac{\partial \rho}{\rho} ~=~ 0 ~. \tag{77}$$

The solution of this is found to be

$$\rho(x) ~=~ \frac{\lambda - 1}{\pi \lambda} \frac{b}{x^2 + b^2} ~, \tag{78}$$

with eigenvalue $E_0 = 0$. We may now boost this solution using (21). The general solution is therefore

$$\begin{aligned} \rho(x,t) ~&=~ \frac{\lambda - 1}{\pi \lambda} \frac{b}{(x - vt)^2 + b^2} ~, \\ \theta(x,t) ~&=~ \frac{mv}{\hbar} \left[~x ~-~ \frac{1}{2} vt ~\right] ~, \end{aligned} \tag{79}$$

where the width and velocity are now independent parameters. The particle number, momentum and energy of this soliton are given by

$$N ~=~ \frac{\lambda - 1}{\lambda} ~,$$

$$P = \frac{\lambda - 1}{\lambda} mv \ ,$$

$$E = \frac{\lambda - 1}{\lambda} \frac{mv^2}{2} \ . \tag{80}$$

Note that if we had used the scaled variables given by Eq. (23), the normalized $dx \int \tilde{\rho}(x) = 1$, with the momentum and energy like a classical particle. Even though it is an exact solution of the collective field theory, it not a genuine solution of the CSM. This is because the collective field theory is meaningful only for large N.

Finally, we note that we may rewrite Eq. (77) for the scaled density $\tilde{\rho}$ as

$$\frac{1}{2} \frac{\partial^2 \tilde{\rho}}{\partial x^2} + \frac{\partial}{\partial x} (\tilde{\rho} \tilde{\rho}_H) = 0 \ . \tag{81}$$

Formally, this equation has the same form as the steady-state Coulomb gas model of Dyson [31]. In the diffusion problem, it is known as the Smoluchowski equation with a singular kernel, and describes the Brownian motion of a particle immersed in a fluid, with friction-limited velocity. A description of this equation is given by Andersen and Oppenheim [32]. The analogous single-soliton solution (78) of the equation for the diffusion problem was obtained by Satsuma and Mimura [33]. These authors also found the soliton with the hyperbolic kernel, and the periodic solution appropriate for the Sutherland Hamiltonian on a circle [2].

7. DISCUSSION

We have seen that collective field theory is a powerful technique from which many properties of the CSM can be derived without having to solve the N-particle Schrödinger equation (1). We can consider other applications of collective field theory. For instance, it should be possible to solve for the low-energy excitations of the CSM in a slowly time-varying harmonic potential. Analogous calculations have been performed for a trapped Bose-Einstein condensate in the three-dimensional problem [34].

It is evident that there are several issues which are either not clear or beyond the reach of collective field theory. We list some of these problems below; they are of course related to each other.

(a) The dispersion relation of small amplitude waves whose wavelengths are comparable to the average particle spacing remains unknown. The difficulties mentioned in Section 4 seem to suggest that the collective field theory discussed in this paper is not complete; perhaps one needs higher derivative terms in the energy functional (19) to obtain better results at short wavelengths.

(b) The interpretation of the large amplitude waves and soliton solutions is not clear. The ideal way to resolve these difficulties would be to set up a precise correspondence between the collective excitations and the known solutions of the Schrödinger equation. Refs. [17, 18] make some suggestions in this direction, but a quantitative mapping is still missing.

(c) It is not clear why we only get exact soliton solutions which correspond to particles for $\lambda > 1$ and holes for $\lambda < 1$. It would be desirable to complete the story by finding, perhaps numerically, solutions corresponding to particles for $\lambda < 1$ and holes for $\lambda > 1$.

(d) Finally, it would be very interesting to quantize the collective field theory and study it more carefully than we have done in Section 4. This may lead to an alternative way of deriving the oscillating terms in the dynamical correlation functions.

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