UCSBTH-95-27, TIFR/TH/95-46, hep-th/9509108

ROTATING BLACK HOLES WHICH SATURATE A BOGOMOL'NYI BOUND

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September 1995

Abstract

We construct and study the electrically charged, rotating black hole solution in heterotic string theory compactified on a $(10-D)$ dimensional torus. This black hole is characterized by its mass, angular momentum, and a $(36-2D)$ dimensional electric charge vector. One of the novel features of this solution is that for $D>5$, its extremal limit saturates the Bogomol'nyi bound. This is in contrast with the $D=4$ case where the rotating black hole solution develops a naked singularity before the Bogomol'nyi bound is reached. The extremal black holes can be superposed, and by taking a periodic array in $D>5$, one obtains effectively four dimensional solutions without naked singularities.

I. INTRODUCTION

Among the infinite tower of states in string theory, those which saturate a Bogomol'nyi bound are of particular interest, since they do not receive quantum corrections to their mass or charges [1–3]. During the past few years, there has been considerable discussion over whether these states can be identified with black holes [4–8]. A necessary condition for this to be the case is clearly that there exist black hole solutions with the same values of the mass and charges. For compactifications of the heterotic string down to four dimensions on a torus, it has been shown that for all spherically symmetric BPS saturated states with $N_L \neq 0$, there are extremal black hole configurations satisfying this condition [9–12].[1] This is encouraging, but there are also nonspherically symmetric states saturating the bound which one would like to identify with rotating black holes. Unfortunately, in four dimensions, the rotating black hole solutions become extremal before the Bogomol'nyi bound is saturated. The solutions which do saturate the bound contain naked singularities, making their physical significance highly questionable.

We will show that in higher dimensions, this problem disappears (at least for black holes with a single component of angular momentum). This is directly related to an unusual property of the higher dimensional Kerr solution. As shown by Myers and Perry [16], in dimensions greater than five, there are rotating (uncharged) black hole solutions with any value of the ratio a/m where a is the angular momentum parameter and m is the mass. In other words, there is no extremal limit in this case. (In four dimensions, solutions with $a > m$ contain naked singularities.) We will find that when one adds charges to this solution appropriate to heterotic string theory, there is an extremal limit precisely when the Bogomol'nyi bound is saturated.

The massless fields in heterotic string theory compactified on a $(10 - D)$ dimensional torus consist of the string metric $G_{\mu\nu}$, the anti-symmetric tensor field $B_{\mu\nu}$, $(36 - 2D)$ U(1) gauge fields $A^{(j)}_\mu$ $(1 \leq j \leq 36 - 2D)$, the scalar dilaton field Φ, and a $(36-2D)\times(36-2D)$ matrix valued scalar field M satisfying,

$$MLM^T = L, \qquad M^T = M. \tag{1.1}$$

Here L is a $(36 - 2D) \times (36 - 2D)$ symmetric matrix with $(26 - D)$ eigenvalues -1 and $(10 - D)$ eigenvalues $+1$. For definiteness we shall take L to be,

$$L = \begin{pmatrix} -I_{26-D} & \\ & I_{10-D} \end{pmatrix}, \tag{1.2}$$

where I_n denotes an $n \times n$ identity matrix. The action describing the effective field theory of these massless bosonic fields is given by [17],

$$S = C \int d^D x \sqrt{-\det G}\, e^{-\Phi} \Big[R_G + G^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu ML \partial_\nu ML) \\ -\frac{1}{12} G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} H_{\mu\nu\rho} H_{\mu'\nu'\rho'} - G^{\mu\mu'} G^{\nu\nu'} F^{(j)}_{\mu\nu} (LML)_{jk} F^{(k)}_{\mu'\nu'} \Big], \tag{1.3}$$

[1]The case $N_L = 0$ has recently been discussed [13–15].

where,

$$F^{(j)}_{\mu\nu} = \partial_\mu A^{(j)}_\nu - \partial_\nu A^{(j)}_\mu \,, \tag{1.4}$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + 2A^{(j)}_\mu L_{jk} F^{(k)}_{\nu\rho} + \text{cyclic permutations of } \mu,\, \nu,\, \rho \,, \tag{1.5}$$

and R_G denotes the scalar curvature associated with the metric $G_{\mu\nu}$. C is an arbitrary constant which does not affect the equations of motion and can be absorbed into the dilaton field Φ.

The general rotating black hole in D dimensions is characterized by $[(D-1)/2]$ different angular momentum parameters (where $[x]$ denotes the integer part of x). This just corresponds to the components of angular momentum in different orthogonal two-planes. We will consider here the simplest case of a single nonzero angular momentum parameter a. The uncharged rotating black hole in pure Einstein gravity in D dimensions is given by [16]:

$$\begin{aligned} ds^2 = & -\frac{\rho^2 + a^2\cos^2\theta - 2m\rho^{5-D}}{\rho^2 + a^2\cos^2\theta} dt^2 + \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho^{5-D}} d\rho^2 + (\rho^2 + a^2\cos^2\theta) d\theta^2 \\ & + \frac{\sin^2\theta}{\rho^2 + a^2\cos^2\theta}[(\rho^2 + a^2)(\rho^2 + a^2\cos^2\theta) + 2m\rho^{5-D} a^2 \sin^2\theta] d\phi^2 \\ & - \frac{4m\rho^{5-D} a \sin^2\theta}{\rho^2 + a^2\cos^2\theta} dt d\phi + \rho^2\cos^2\theta d\Omega^{D-4} \,. \end{aligned} \tag{1.6}$$

Here t, ρ, θ, ϕ denote four of the space-time coordinates and $d\Omega^{D-4}$ is the square of the line element on a $D-4$ dimensional unit sphere. When $D = 4$, this metric reduces to the familiar Kerr solution. Notice that, aside from the $d\Omega^{D-4}$ factor, the D dependence always appears multiplied by the mass in the combination $m\rho^{5-D}$. The event horizon is located where the $\rho =$ constant surfaces become null. This implies $G^{\rho\rho} = 0$ or

$$\rho^2 + a^2 - 2m\rho^{5-D} = 0 \,. \tag{1.7}$$

One can immediately see the qualitative difference between $D > 5$ and $D \leq 5$. For $D > 5$, and $m > 0$, this equation always has a solution where ρ is positive. Thus an event horizon exists for all values of m, a. Since (1.7) has only one (positive) solution in this case, there is no inner horizon. In contrast, for $D \leq 5$, there is a maximal value of a beyond which the event horizon disappears. The curvature singularity also changes its character in higher dimensions. For $D > 5$, since there is no inner horizon, the singularity is spacelike. Furthermore, it is easy to see that the norm of the Killing vector $\partial/\partial t$ diverges at $\rho = 0$, showing that this surface is singular. This is in contrast to the situation in four dimensions where the singularity is timelike and is concentrated on a ring at $\rho = 0$ and $\theta = \pi/2$. One feature of this metric which does not change in higher dimensions is that the vector $\partial/\partial t$ becomes null on a surface outside the horizon, showing that an ergosphere is present.

Since the metric (1.6) is Ricci flat, it is automatically a solution of the equations of motion derived from the action (1.3) if we also set

$$\Phi = 0, \qquad B_{\mu\nu} = 0, \qquad A^{(j)}_\mu = 0, \qquad M = I_{36-2D} \,. \tag{1.8}$$

In the next section, we will add a general electric charge to this solution and discuss its properties. In section 3, we investigate the extremal limit of this charged, rotating black hole, and show that it saturates a Bogomol'nyi bound. We also discuss how one can superpose these higher dimensional extremal black holes to obtain effectively four dimensional solutions. Section 4 contains some concluding remarks.

II. ROTATING, CHARGED BLACK HOLES IN $D \geq 4$

One can add a general charge to the rotating black hole solution of the previous section by applying the solution generating transformations $(O(26-D,1)/O(26-D)) \times (O(10-D,1)/O(10-D))$.[2] This generates a nontrivial $\Phi, B_{\mu\nu}$ and M, as well as $A^{(j)}_\mu$. Since the analysis is identical to the one given in ref. [12] we shall not give the details here, but only quote the final result. The solution is given by,

$$\begin{aligned} ds^2 &\equiv G_{\mu\nu} dx^\mu dx^\nu \\ &= (\rho^2 + a^2\cos^2\theta)\Big\{ -\Delta^{-1}(\rho^2 + a^2\cos^2\theta - 2m\rho^{5-D})dt^2 + (\rho^2 + a^2 - 2m\rho^{5-D})^{-1} d\rho^2 + d\theta^2 \\ &\quad + \Delta^{-1}\sin^2\theta[\Delta + a^2\sin^2\theta(\rho^2 + a^2\cos^2\theta + 2m\rho^{5-D}\cosh\alpha\cosh\beta)]\,d\phi^2 \\ &\quad -2\Delta^{-1} m\rho^{5-D} a\sin^2\theta(\cosh\alpha + \cosh\beta)dtd\phi + \rho^2\cos^2\theta(\rho^2 + a^2\cos^2\theta)^{-1} d\Omega^{D-4}\Big\}\,, \end{aligned} \tag{2.1}$$

where,

$$\Delta = (\rho^2 + a^2\cos^2\theta)^2 + 2m\rho^{5-D}(\rho^2 + a^2\cos^2\theta)(\cosh\alpha\cosh\beta - 1) + m^2\rho^{10-2D}(\cosh\alpha - \cosh\beta)^2\,, \tag{2.2}$$

$$\Phi = \frac{1}{2}\ln\frac{(\rho^2 + a^2\cos^2\theta)^2}{\Delta}\,, \tag{2.3}$$

$$\begin{aligned} A^{(j)}_t &= -\frac{n^{(j)}}{\sqrt{2}}\Delta^{-1} m\rho^{5-D}\sinh\alpha\{(\rho^2 + a^2\cos^2\theta)\cosh\beta + m\rho^{5-D}(\cosh\alpha - \cosh\beta)\} \\ &\qquad \text{for } 1 \leq j \leq 26 - D\,, \\ &= -\frac{p^{(j-26+D)}}{\sqrt{2}}\Delta^{-1} m\rho^{5-D}\sinh\beta\{(\rho^2 + a^2\cos^2\theta)\cosh\alpha + m\rho^{5-D}(\cosh\beta - \cosh\alpha)\} \\ &\qquad \text{for } j \geq 27 - D\,, \end{aligned} \tag{2.4}$$

$$A^{(j)}_\phi = \frac{n^{(j)}}{\sqrt{2}}\Delta^{-1} m\rho^{5-D} a\sinh\alpha\sin^2\theta\{\rho^2 + a^2\cos^2\theta + m\rho^{5-D}\cosh\beta(\cosh\alpha - \cosh\beta)\}$$

[2]This solution was partially constructed by Peet [7].

for $1 \leq j \leq 26-D$,

$$= \frac{p^{(j-26+D)}}{\sqrt{2}} \Delta^{-1} m\rho^{5-D} a \sinh\beta \sin^2\theta \{\rho^2 + a^2\cos^2\theta + m\rho^{5-D}\cosh\alpha(\cosh\beta - \cosh\alpha)\}$$

for $j \geq 27-D$,

(2.5)

$$B_{t\phi} = \Delta^{-1} m\rho^{5-D} a \sin^2\theta(\cosh\alpha - \cosh\beta)\{\rho^2 + a^2\cos^2\theta + m\rho^{5-D}(\cosh\alpha\cosh\beta - 1)\} \quad (2.6)$$

$$M = I_{36-2D} + \begin{pmatrix} Pnn^T & Qnp^T \\ Qpn^T & Ppp^T \end{pmatrix} , \quad (2.7)$$

where,

$$P = 2\Delta^{-1} m^2 \rho^{10-2D} \sinh^2\alpha \sinh^2\beta ,$$
$$Q = -2\Delta^{-1} m\rho^{5-D} \sinh\alpha \sinh\beta \{\rho^2 + a^2\cos^2\theta + m\rho^{5-D}(\cosh\alpha\cosh\beta - 1)\} . \quad (2.8)$$

Here α and β are two boost angles, $\vec{n}$ is a $(26-D)$ dimensional unit vector, and $\vec{p}$ is a $(10-D)$ dimensional unit vector.

There are several consistency checks on the solution (2.1) - (2.8). First, the solution generating transformation applied to a metric like (1.6) only changes the t and ϕ components of the string metric. Comparing (2.1) with (1.6) we see that indeed the $\rho\rho, \theta\theta$ and additional $D-4$ components of the metric are identical. Second, since the $tt, t\phi$ and $\phi\phi$ components of the higher dimensional Kerr metric depend on D and m only through the combination $m\rho^{5-D}$, one should be able to obtain the general solution (2.1) - (2.8) by starting with the general four dimensional rotating black hole solution [12] and replacing $m\rho$ with $m\rho^{5-D}$. This is indeed the case. Finally, setting the angular momentum parameter a to zero, one obtains the general electrically charged nonrotating black hole in D dimensions [7].

From eqs.(2.1) and (2.3) we can also find an expression for the canonical Einstein metric $g_{\mu\nu} \equiv e^{-2\Phi/(D-2)} G_{\mu\nu}$:

$$\begin{aligned} ds_E^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu \\ &= \Delta^{\frac{1}{D-2}} (\rho^2 + a^2\cos^2\theta)^{\frac{D-4}{D-2}} \Big\{ -\Delta^{-1}(\rho^2 + a^2\cos^2\theta - 2m\rho^{5-D}) dt^2 \\ &\quad + (\rho^2 + a^2 - 2m\rho^{5-D})^{-1} d\rho^2 + d\theta^2 \\ &\quad + \Delta^{-1}\sin^2\theta[\Delta + a^2\sin^2\theta(\rho^2 + a^2\cos^2\theta + 2m\rho^{5-D}\cosh\alpha\cosh\beta)]\, d\phi^2 \\ &\quad - 2\Delta^{-1} m\rho^{5-D} a\sin^2\theta(\cosh\alpha + \cosh\beta) dt d\phi + \rho^2\cos^2\theta(\rho^2 + a^2\cos^2\theta)^{-1} d\Omega^{D-4} \Big\} . \end{aligned} \quad (2.9)$$

The total mass M, angular momentum J, and electric charge $Q^{(j)}$ of these black holes can be obtained from the asymptotic form of the solution and are given by

$$M = \frac{1}{2} m[1 + (D-3)\cosh\alpha\cosh\beta] , \quad (2.10)$$

$$J = \frac{1}{2} ma(\cosh\alpha + \cosh\beta)\,, \tag{2.11}$$

$$\begin{aligned} Q^{(j)} &= \frac{m}{\sqrt{2}}(D-3)\sinh\alpha\cosh\beta\, n^{(j)} \qquad \text{for} \quad 1 \le j \le 26-D \\ &= \frac{m}{\sqrt{2}}(D-3)\sinh\beta\cosh\alpha\, p^{(j-26+D)} \qquad \text{for} \quad j \ge 27-D\,. \end{aligned} \tag{2.12}$$

Let us define,

$$Q^{(j)}_{\substack{L\\R}} = \frac{1}{2}(I_{36-2D} \mp L)_{jk} Q^{(k)}\,. \tag{2.13}$$

$\vec{Q}_L$ and $\vec{Q}_R$ may be regarded as $(26-D)$ and $(10-D)$ dimensional vectors respectively. Eq.(2.12) gives,

$$(\vec{Q}_R)^2 = \frac{m^2}{2}(D-3)^2\sinh^2\beta\cosh^2\alpha, \qquad (\vec{Q}_L)^2 = \frac{m^2}{2}(D-3)^2\cosh^2\beta\sinh^2\alpha. \tag{2.14}$$

We now consider some properties of these black holes. Since the $\rho\rho$ component of the string metric is not changed by the addition of charge, the event horizon of the solution is again given by (1.7): $\rho_H^2 + a^2 = 2m\rho_H^{5-D}$. (Since the conformal factor is regular on the horizon, this is also the location of the event horizon in the Einstein metric.) The area of the event horizon, i.e. $D-2$ volume, can be computed from (2.9). This calculation is simplified by noticing that, on the horizon, the expression in brackets in $g_{\phi\phi}$ reduces to $(\rho_H^2+a^2)^2(\cosh\alpha+\cosh\beta)^2/4$. The area turns out to be

$$A_H = m\rho_H\Omega_{D-2}(\cosh\alpha + \cosh\beta)\ , \tag{2.15}$$

where Ω_{D-2} is the volume of a $(D-2)$ sphere of unit radius. The angular velocity Ω_H of the black hole is defined by the condition that $\xi = \partial/\partial t + \Omega_H \partial/\partial\phi$ be null on the horizon. One finds that

$$\Omega_H = \frac{a\rho_H^{D-5}}{m(\cosh\alpha + \cosh\beta)}\ . \tag{2.16}$$

We next consider the surface gravity κ of the black hole. This can be obtained from $\xi^\mu\xi_\mu \equiv -\lambda^2$ via $\kappa^2 = \lim_{\rho\to\rho_H}\nabla_\mu\lambda\nabla^\mu\lambda$. Since κ is a constant on the horizon, one can evaluate it at any point. It is convenient to choose a point where $\theta = 0$. One obtains

$$\kappa = \frac{(D-3)\rho_H^2 + (D-5)a^2}{2m\rho_H^{6-D}(\cosh\alpha + \cosh\beta)}\ . \tag{2.17}$$

The Hawking temperature is related to this surface gravity by $T = \kappa/2\pi$.

Finally, we consider the singularity structure of these black holes for $D > 5$. We first discuss the general case $\alpha \ne \beta$. Unlike the vacuum solution, the norm of the Killing vector $\partial/\partial t$ does not diverge at $\rho = 0$ but now vanishes there in both the string and Einstein metrics. However, one can see that this surface still contains a curvature singularity as follows. Eq. (2.3) implies that the dilaton diverges at $\rho = 0$ like $\Phi \approx (D-5)\ln\rho$. In the

Einstein metric, this corresponds to a diverging energy density, which implies a divergence in the Ricci tensor as $\rho \to 0$. This singularity remains in the string metric as well. For $\alpha = \beta$, the Killing vector $\partial/\partial t$ diverges at $\rho = 0$ in the Einstein metric, showing that this surface is again singular. In contrast, all components of the string metric now have regular limits at $\rho = 0$, $\theta \neq \pi/2$, except for the $D-4$ sphere which shrinks to zero volume (and $G_{\rho\rho}$ which can be made regular by introducing a new radial coordinate). However the shrinking spheres are sufficient to cause the curvature to diverge at $\rho = 0$. Notice that the string coupling $g = e^{\Phi/2} \to 0$ near the singularity as expected for an electrically charged black hole.

III. THE EXTREMAL LIMIT AND BOGOMOL'NYI BOUND

Supersymmetry of the toroidally compactified heterotic string theory implies an upper limit on the ratio of the charge to the mass known as the Bogomol'nyi bound. It is saturated when

$$M^2 = \frac{1}{2}\vec{Q}_R^2 \,. \tag{3.1}$$

Eqs.(2.10) and (2.14) show that the only way to satisfy this equation is to take the limit $m \to 0$, $\beta \to \infty$ keeping $m_0 \equiv m \cosh\beta$, a and α fixed. For $D \leq 5$, there is a minimum value of m required in order for a solution to (1.7) to exist. Thus the horizon disappears before the Bogomol'nyi bound is reached. However, for $D > 5$, a horizon exists for all $m > 0$. As we take the limit $m \to 0$, the location of the horizon, ρ_H, approaches the singularity at $\rho = 0$, showing that one cannot increase the charge beyond this limit. *So for $D > 5$, the solution saturating the Bogomol'nyi bound is an extreme black hole.* From the above formulas, it is clear that in this limit, the horizon area and angular velocity go to zero. The behavior of the surface gravity (or Hawking temperature) is rather surprising. If we first set $a = 0$ in (2.17) and consider the nonrotating black hole, we see that the surface gravity approaches a nonzero constant in $D = 4$ and vanishes for $D > 4$. However, for the rotating black hole, we see that the "critical dimension" is increased by two: the surface gravity approaches a nonzero constant for $D = 6$ and vanishes for $D > 6$. This is another illustration of the fact that an arbitrarily small amount of angular momentum can qualitatively change the properties of extreme dilatonic black holes [18,19].

The solution for $M^2 = \vec{Q}_R^2/2$ is given by

$$\begin{aligned} ds^2 = (\rho^2 + a^2\cos^2\theta)\Big\{ &-\Delta^{-1}(\rho^2 + a^2\cos^2\theta)dt^2 + (\rho^2 + a^2)^{-1}d\rho^2 + d\theta^2 \\ &+\Delta^{-1}\sin^2\theta[\Delta + a^2\sin^2\theta(\rho^2 + a^2\cos^2\theta + 2m_0\rho^{5-D}\cosh\alpha)]\, d\phi^2 \\ &-2\Delta^{-1}m_0\rho^{5-D}a\sin^2\theta dt d\phi + \rho^2\cos^2\theta(\rho^2 + a^2\cos^2\theta)^{-1}d\Omega^{D-4}\Big\} \,. \end{aligned} \tag{3.2}$$

$$\Delta = (\rho^2 + a^2\cos^2\theta)^2 + 2m_0\rho^{5-D}\cosh\alpha\,(\rho^2 + a^2\cos^2\theta) + m_0^2\rho^{10-2D} \,, \tag{3.3}$$

$$\Phi = \frac{1}{2}\ln\frac{(\rho^2 + a^2\cos^2\theta)^2}{\Delta} \,, \tag{3.4}$$

$$A_t^{(j)} = -\frac{n^{(j)}}{\sqrt{2}}\Delta^{-1} m_0 \rho^{5-D} \sinh\alpha(\rho^2 + a^2\cos^2\theta) \quad \text{for } 1 \le j \le 26 - D\,,$$
$$= -\frac{p^{(j-26+D)}}{\sqrt{2}}\Delta^{-1} m_0 \rho^{5-D}\{(\rho^2 + a^2\cos^2\theta)\cosh\alpha + m_0\rho^{5-D}\} \quad \text{for } j \ge 27 - D\,, \tag{3.5}$$

$$A_\phi^{(j)} = -\frac{n^{(j)}}{\sqrt{2}}\Delta^{-1}(m_0)^2\rho^{10-2D} a \sinh\alpha \sin^2\theta \quad \text{for } 1 \le j \le 26 - D\,,$$
$$= \frac{p^{(j-26+D)}}{\sqrt{2}}\Delta^{-1} m_0 \rho^{5-D} a \sin^2\theta\{\rho^2 + a^2\cos^2\theta + m_0\rho^{5-D}\cosh\alpha\} \quad \text{for } j \ge 27 - D\,, \tag{3.6}$$

$$B_{t\phi} = -\Delta^{-1} m_0 \rho^{5-D} a \sin^2\theta\{\rho^2 + a^2\cos^2\theta + m_0\rho^{5-D}\cosh\alpha\} \tag{3.7}$$

$$M = I_{36-2D} + \begin{pmatrix} Pnn^T & Qnp^T \\ Qpn^T & Ppp^T \end{pmatrix}\,, \tag{3.8}$$

$$P = 2\Delta^{-1} m_0^2 \rho^{10-2D} \sinh^2\alpha\,,$$
$$Q = -2\Delta^{-1} m_0 \rho^{5-D} \sinh\alpha\{\rho^2 + a^2\cos^2\theta + m_0\rho^{5-D}\cosh\alpha\}\,. \tag{3.9}$$

Notice that the ergosphere has disappeared; $\partial/\partial t$ is now timelike everywhere. What is the nature of the singularity at $\rho = 0$? A key property is whether this singularity is timelike or null. Since there is no event horizon, a timelike singularity would be naked and classical evolution would break down (although see [20]). A null singularity is much more mild. The criterion for a singularity to be timelike is the existence of null geodesics which reach it staying in the past of a $t =$ constant surface. Consider first the nonrotating case, obtained by setting $a = 0$ in the string metric (3.2). Radial null geodesics satisfy $dt = \pm\Delta^{1/2}\rho^{-2}d\rho \approx \rho^{3-D}d\rho$ near $\rho = 0$. Thus, for all $D \ge 4$, t diverges along the null geodesic as $\rho \to 0$ showing that the singularity is null. For the rotating solution with $a \ne 0$, one can consider radial null geodesics along the rotation axis $\theta = 0$. These satisfy $dt = \pm\Delta^{1/2}(\rho^2 + a^2)^{-1}d\rho \approx \rho^{5-D}d\rho$ near $\rho = 0$. Thus the singularity is null (at least in this direction) only for $D \ge 6$. These are precisely the dimensions for which (3.2) describes the extremal limit of a black hole. The $D = 4, 5$ solutions contain naked singularities.

Since objects which saturate a Bogomol'nyi bound have no force between them, we can also construct stationary multiple black hole solutions. To do this, it is convenient to bring the above solution into the IWP form [21,22]. For simplicity, we shall consider only the case $\alpha = 0$; the $\alpha \ne 0$ solution can be found by starting from the $\alpha = 0$ solution and performing a boost along one of the internal directions of the $(10 - D)$ dimensional torus. We introduce new Cartesian coordinates $x^1, \ldots x^{D-1}$ through the relations:

$$x^1 = \sqrt{\rho^2 + a^2}\sin\theta\cos\phi\,,$$
$$x^2 = \sqrt{\rho^2 + a^2}\sin\theta\sin\phi\,,$$

$$
\begin{aligned}
x^3 &= \rho\cos\theta\cos\psi^1\,, \\
x^4 &= \rho\cos\theta\sin\psi^1\cos\psi^2\,, \\
&\;\;. \\
&\;\;. \\
x^{D-2} &= \rho\cos\theta\sin\psi^1\cdots\sin\psi^{D-5}\cos\psi^{D-4}\,, \\
x^{D-1} &= \rho\cos\theta\sin\psi^1\cdots\sin\psi^{D-5}\sin\psi^{D-4}\,,
\end{aligned}
\tag{3.10}
$$

where $\psi^1, \ldots \psi^{D-4}$ are the angles labeling points on the $(D-4)$ sphere. In this coordinate system the $\alpha = 0$ solution may be written as,

$$
ds^2 = -F^2(\vec{x}))[dt + \omega_i(\vec{x})dx^i]^2 + d\vec{x}^2\,, \tag{3.11}
$$

$$
\Phi = \ln F(\vec{x})\,, \tag{3.12}
$$

$$
\begin{aligned}
A_t^{(j)} &= 0 \qquad \text{for} \qquad 1 \le j \le 26 - D\,, \\
&= \frac{p^{(j-26+D)}}{\sqrt{2}}[F-1] \qquad \text{for} \qquad j \ge 27 - D\,,
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
A_i^{(j)} &= 0 \qquad \text{for} \qquad 1 \le j \le 26 - D\,, \\
&= \frac{p^{(j-26+D)}}{\sqrt{2}}F\omega_i \qquad \text{for} \qquad j \ge 27 - D\,,
\end{aligned}
\tag{3.14}
$$

$$
B_{ti} = -F\omega_i\,, \qquad M = I_{36-2D}\,, \tag{3.15}
$$

where

$$
F^{-1} = 1 + \frac{m_0\rho^{5-D}}{\rho^2 + a^2\cos^2\theta}\,, \qquad \omega = \frac{m_0\rho^{5-D}a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}d\phi\,. \tag{3.16}
$$

The explicit form of F and ω in terms of the Cartesian coordinates x^i can be obtained by inverting (3.10). Let R be the radial distance from the origin in the $D-1$ dimensional Euclidean space, and let r be the radial distance in the $D-3$ dimensional subspace orthogonal to x^1 and x^2. Then

$$
R^2 \equiv \sum_{i=1}^{D-1}(x^i)^2 = \rho^2 + a^2\sin^2\theta\,, \qquad r^2 = \sum_{i=3}^{D-1}(x^i)^2 = \rho^2\cos^2\theta\,. \tag{3.17}
$$

These can be inverted to yield

$$
\begin{aligned}
\rho^2 &= \frac{(R^2 - a^2) + \sqrt{(R^2-a^2)^2 + 4a^2r^2}}{2} \\
a^2\cos^2\theta &= \frac{(a^2 - R^2) + \sqrt{(R^2-a^2)^2 + 4a^2r^2}}{2}\,.
\end{aligned}
\tag{3.18}
$$

Also, from (3.10), $\phi = \tan^{-1}(x^2/x^1)$. Note that $\rho^2 = 0$ everywhere on the two dimensional disk $r = 0,\ R \le a$, while $\rho^2 + a^2\cos^2\theta = 0$ only on the ring $r = 0,\ R = a$. Thus in the $\vec{x}$ coordinate system, the singular surface ($\rho = 0$) corresponds to

$$x^i = 0 \quad \text{for} \quad i \ge 3, \qquad (x^1)^2 + (x^2)^2 \le a^2 \,. \tag{3.19}$$

In the string metric, this corresponds to a disk of radius a.

F^{-1} is a harmonic function

$$\sum_{i=1}^{D-1} \partial_i \partial_i F^{-1} = 0 \tag{3.20}$$

while ω satisfies the equation

$$\sum_{i=1}^{D-1} \partial_i \partial_{[i}\omega_{j]} = 0 \ . \tag{3.21}$$

Conversely, any solution of the form given above, where F^{-1} is an arbitrary harmonic function and ω is an arbitrary 1-form satisfying eq.(3.21) will be a solution of the equations of motion. Since both F^{-1} and ω satisfy linear equations, it is now easy to construct multi-black hole solutions by superposing single black hole solutions. If we write the above single black hole solution as $F^{-1} = 1 + f(\vec{x}, m_0, a)$, and $\omega = g(\vec{x}, m_0, a)$, a general linear superposition is given by

$$F^{-1}(\vec{x}) = 1 + \sum_{s=1}^{n} f(\vec{x} - \vec{x}_s, m_s, a_s)\,, \tag{3.22}$$

and

$$\omega(\vec{x}) = \sum_{s=1}^{n} g(\vec{x} - \vec{x}_s, m_s, a_s)\,. \tag{3.23}$$

m_s, a_s and $\vec{x}_s$ are arbitrary parameters labeling the mass, angular momentum and the position of individual black holes. In this superposition, all of the black holes are spinning in the same x^1, x^2 plane. Also, the electric charge vectors associated with all the black holes are parallel to each other.

The ten dimensional form of these D dimensional solutions falls into a class of configurations called 'chiral null models' which were introduced in [23] and further studied in [13,24]. They are characterized by a null translational symmetry and chiral coupling of the world-sheet to the background. It was shown in [23] that these configurations are exact string solutions and do not receive α' corrections (in a particular renormalization scheme).

A particularly interesting class of multi-black hole solutions is an infinite periodic array of black holes. This can be interpreted as a solution in a theory where one of the $D-1$ spatial directions has been compactified [25–27][3]. In order to see how this works, let us consider a periodic array of these solutions along the direction x^{D-1}. Let us define,

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[3]We wish to thank J. Schwarz for this suggestion.

$$s^2 = (x^1)^2 + (x^2)^2 \,, \tag{3.24}$$

and

$$\widetilde{R}^2 = \sum_{i=1}^{D-2} (x^i)^2 \,, \qquad \widetilde{r}^2 = \sum_{i=3}^{D-2} (x^i)^2 \,, \qquad \widetilde{s}^2 = s^2 = (x^1)^2 + (x^2)^2 \,. \tag{3.25}$$

Now, for a single extremal black hole solution, the asymptotic values of F^{-1} and ω_ϕ are given by,

$$F^{-1} \simeq 1 + \frac{m_0}{R^{D-3}} \,, \qquad \omega_\phi \simeq \frac{m_0 a s^2}{R^{D-1}} \,. \tag{3.26}$$

Thus, for a periodic array of black holes in the x^{D-1} direction with periodicity one, the asymptotic values of F^{-1} and ω_ϕ are given by,

$$\widetilde{F}^{-1} \simeq 1 + \sum_{n=-\infty}^{\infty} \frac{m_0}{\{\widetilde{R}^2 + (x^{D-1} - n)^2\}^{\frac{D-3}{2}}} \,, \tag{3.27}$$

$$\widetilde{\omega}_\phi \simeq \sum_{n=-\infty}^{\infty} \frac{m_0 a \widetilde{s}^2}{\{\widetilde{R}^2 + (x^{D-1} - n)^2\}^{\frac{D-1}{2}}} \,. \tag{3.28}$$

For large $\widetilde{R}$ the summand is a slowly varying function of n and hence we can replace the sum over n by an integration. Thus we may write

$$\widetilde{F}^{-1} \simeq 1 + \int_{-\infty}^{\infty} dt \frac{m_0}{\{\widetilde{R}^2 + (t - x^{D-1})^2\}^{\frac{D-3}{2}}} \,, \tag{3.29}$$

$$\widetilde{\omega}_\phi \simeq \int_{-\infty}^{\infty} dt \frac{m_0 a \widetilde{s}^2}{\{\widetilde{R}^2 + (t - x^{D-1})^2\}^{\frac{D-1}{2}}} \,. \tag{3.30}$$

Using a change of variable $t \equiv x^{D-1} + \widetilde{R} u$ we can rewrite the above equations as,

$$\widetilde{F}^{-1} \simeq 1 + \frac{C_1 m_0}{\widetilde{R}^{D-4}} \,, \qquad \widetilde{\omega}_\phi \simeq 1 + \frac{C_2 m_0 a \widetilde{s}^2}{\widetilde{R}^{D-2}} \,, \tag{3.31}$$

where,

$$C_1 = \int_{-\infty}^{\infty} \frac{du}{(u^2+1)^{\frac{D-3}{2}}} \,, \qquad C_2 = \int_{-\infty}^{\infty} \frac{du}{(u^2+1)^{\frac{D-1}{2}}} \,, \tag{3.32}$$

are two numerical constants. These asymptotic forms are identical to those given in (3.26) with D replaced by $(D-1)$ and with appropriate redefinition of m_0 and a. Thus we see that the periodic array of the D dimensional solution has the same asymptotic field configuration as a $D-1$ dimensional rotating black hole.

This procedure can be used to construct solutions in five and four dimensional theories by taking periodic and doubly periodic arrays of extremal black holes in six dimensions. The asymptotic form of these solutions is the same as that of rotating black hole solutions in five and four dimensions saturating the Bogomol'nyi bound which have naked singularities. But now the solution near the black hole becomes six dimensional, and the singularity is that of the extreme rotating black holes that we have described here. This implies that the gyromagnetic ratios of the four dimensional solution, which are computed from the asymptotic form of the gauge field configuration, are identical to those of the elementary string states with the same quantum numbers, since this equality is known to hold for the singular solution in four dimension [12]. Thus these solutions are good candidates for describing the field configuration around elementary string states.

IV. CONCLUDING REMARKS

In this paper, we have considered only rotating black holes with one component of angular momentum. More general solutions can be constructed by starting with the higher dimensional Kerr solution with all components of the angular momentum nonzero [16], and applying the solution generating transformation. It is likely that only some of these black holes will saturate the Bogomol'nyi bound in their extremal limit. This needs to be investigated further. It remains to be seen whether these black holes can be fruitfully identified with nonspherically symmetric BPS saturated string states. One unusual feature is that there appears to be no upper bound on the magnitude of the angular momentum. Since a is an arbitrary parameter, the solutions we have constructed can saturate the Bogomol'nyi bound with any value of J.

Although we have considered solutions to heterotic string theory compactified on a torus, our $D = 6$ rotating black hole can easily be transformed into a Type IIA string solution. This is because the low energy effective action for the Type IIA string theory compactified on K3 is related to that of the heterotic string compactified on a torus by a simple field redefinition. This may be useful in testing the string-string duality conjecture in six dimensions.

Acknowledgment: We wish to thank J. Gauntlett and J. Schwarz for useful discussions. We also acknowledge the hospitality of the Aspen Center for Physics where this work was initiated. GTH was supported in part by NSF Grant PHY-9507065.

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