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MARGINAL DEFORMATIONS OF WZNW AND COSET MODELS FROM $O(d,d)$ TRANSFORMATION

S.F. Hassan and Ashoke Sen[⋆]

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

ABSTRACT

We show that $O(2,2)$ transformation of $SU(2)$ WZNW model gives rise to marginal deformation of this model by the operator $\int d^2z J(z)\bar{J}(\bar{z})$ where J, $\bar{J}$ are $U(1)$ currents in the Cartan subalgebra. Generalization of this result to other WZNW theories is discussed. We also consider $O(3,3)$ transformation of the product of an $SU(2)$ WZNW model and a gauged $SU(2)$ WZNW model. The three parameter set of models obtained after the transformation is shown to be the result of first deforming the product of two $SU(2)$ WZNW theories by marginal operators of the form $\sum_{i,j=1}^{2} C_{ij} J_i \bar{J}_j$, and then gauging an appropriate $U(1)$ subgroup of the theory. Our analysis leads to a general conjecture that $O(d,d)$ transformations of any WZNW model correspond to marginal deformation of the WZNW theory by an appropriate combination ofleft and right moving currents belonging to the Cartan subalgebra; and $O(d,d)$ transformations of a gauged WZNW model can be identified to the gauged version of such marginally deformed WZNW models.

⋆ e-mail addresses: FAWAD@TIFRVAX.BITNET, SEN@TIFRVAX.BITNET

1. Introduction

$O(d,d)$ transformations [1] [2] have been used in many recent papers to generate new classical solutions of string theory equations of motion from known ones $[3-12]$. In this paper we shall discuss a new application of these transformations, namely, generating marginal deformations of Wess-Zumino-Novikov-Witten (WZNW) and coset models.

The motivation for studying marginally deformed WZNW models and their gauging is as follows. The propagation of a string in a background is described by a conformally invariant non-linear sigma model in two dimensions. The background is constructed as a solution to the vanishing beta function equations which enforce the condition of conformal invariance perturbatively . The low energy equations of motion for the background fields have been solved in various situations and a wide range of solutions has been obtained. Particular examples are solvable conformal field theories like WZNW models, or the "blackhole" type solutions obtained by gauging WZNW models [13]. In general, these models form a very small subset in the space of all conformal field theories, and it is of particular interest to know the interrelation between various exactly solvable conformal field theories of this kind, e.g. the question of whether they can be obtained from each other by marginal deformations. In order to address such questions, we need to know the form of the σ-model action of the theory after a finite marginal deformation, and this is precisely the question we address in this paper.

It has been argued [14] that WZNW theories have exact marginal deformations, generated by operators of the form $\sum_{i,j} C_{ij} \int d^2z J_i(z) \bar{J}_j(\bar{z})$ where J_i and $\bar{J}_i$ are holomorphic and anti-holomorphic currents belonging to the Cartan subalgebra. The σ-model action of the corresponding perturbed theory can be written down easily to first order in the perturbing parameter, but there is no general method for writing it down for a finite value of the perturbation parameter. We shall show that $O(d,d)$ transformations provide a way out of this problem. More specifically, we shall show that if we start from an unperturbed SU(2) WZNW theory, and perform a (finite) $O(2,2)$ transformation on it, the result is a σ-model describing an SU(2) WZNW model deformed by the marginal operator $\int d^2z J(z)\bar{J}(\bar{z})$. The form of the σ-model is exact to all orders in the perturbing parameter, but only to lowest order in $1/k$, where k is the central charge of the current algebra. (This shortcoming is only due to the fact that the explicit $O(d,d)$ transformation rules are known only to lowest order in the derivatives.) We also consider the generalization of this result for more general WZNW theories; in particular we discuss the case of $SU(2) \otimes SU(2)$ theory, and show that the four parameter family of marginal deformations in this theory is again given by $O(4,4)$ transformation of the unperturbed theory.

Besides the WZNW models, another type of conformal field theories have been the subject of intense investigation in recent years; these are the coset models and are obtained by gauging one or more subgroups of some WZNW model. In this context, a natural question to ask would be, what kind of models can we get if we gauge a marginally deformed WZNW model. We show that the answer is reasonably simple; these models can be identified with the $O(d,d)$ deformations of the gauged unperturbed WZNW models. In other words, we show, through various examples, that gauging an $O(d,d)$ transformed WZNW model generates an $O(d,d)$ transformed coset model.

The paper is organized as follows. In section 2 we consider an $O(2,2)$ transformation of an $SU(2)$ WZNW model, and show that the resulting σ-model can be identified to the marginal deformation of the original model by the $\int d^2z J(z)\bar{J}(\bar{z})$ perturbation. We also consider $O(4,4)$ transformation of $SU(2) \otimes SU(2)$ WZNW model and show that the result can be identified to marginal deformation of the original theory by perturbations of the form $\int d^2z J_i(z)\bar{J}_j(\bar{z})$ $(1 \leq i,j \leq 2)$. We then discuss $O(d,d)$ transformations of more general WZNW models, and their relation to marginal deformations of these models.

In sect.3 we consider $O(3,3)$ transformations of the product of an $SU(2)$ WZNW model, and an $SU(2)/U(1)$ coset model. The result is a three parameter family of conformal field theories. The physical interpretation of this conformal field theory is provided by the analysis of sect.4, where we start with an $O(4,4)$ transformed $SU(2)\otimes SU(2)$ WZNW model (which, by the result of sect.2 is a marginal deformation of the $SU(2)\otimes SU(2)$ WZNW model), and then gauge a particular $U(1)$ subgroup of the model. The result is shown to be the model of sect.3, thereby showing that an $O(d,d)$ transformation of the coset model is equivalent to gauging a marginally deformed WZNW model. We also comment on the observations of refs.[9] and [12] in the context of $SU(2)$ WZNW model.

We conclude in sect.5 with a summary of the results and some comments.

2. $O(d,d)$ Trasformations and Marginal Perturbations in the WZNW Models

To explore the connection between $O(d,d)$ transformations and marginal perturbations in WZNW models, we first consider an $SU(2)$-WZNW model defined by the action

$$S[g]=\frac{k}{8\pi}\int\limits_{\partial B} d^2x\, Tr(\partial_\mu g^{-1}\partial^\mu g)+\frac{k}{12\pi}\int\limits_{B} d^3x\,\epsilon^{ijk}\, Tr(\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g g^{-1}) \quad (2.1)$$

where g is an $SU(2)$ group element and the constant k specifies the level of the associated Kac-Moody (KM) algebra. B is a solid ball in three dimensions with boundary ∂B. This model describes a conformal field theory (CFT) of central charge $c=\frac{3k}{k+2}$ and gives the well known *2-d* blackhole on gauging. If we parametrize g as

$$g=e^{i\theta_L\sigma_2/2}e^{i\phi\sigma_1/2}e^{i\theta_R\sigma_2/2} \quad (2.2)$$

then the action takes the form

$$S[\phi,\theta_L,\theta_R]=\frac{k}{2\pi}\int d^2z(\bar\partial\phi\partial\phi+\bar\partial\theta_L\partial\theta_L+\bar\partial\theta_R\partial\theta_R+2\cos\phi\bar\partial\theta_L\partial\theta_R) \quad (2.3)$$

which is related to the level k $SL(2,R)$ model by the replacements $\phi\to ir$, $k\to -k$.

The action (2.3) has chiral invariances

$$\begin{aligned} \delta\theta_L &= v_L(z); \qquad & \delta\theta_R &= 0 \\ \bar{\delta}\theta_L &= 0; \qquad & \bar{\delta}\theta_R &= \bar{v}_R(\bar{z}) \end{aligned} \tag{2.4}$$

which give rise to the conserved chiral currents

$$\begin{aligned} J &= \frac{1}{2}k(\partial\theta_L + \cos\phi\,\partial\theta_R) \\ \bar{J} &= \frac{1}{2}k(\bar{\partial}\theta_R + \cos\phi\,\bar{\partial}\theta_L) \end{aligned} \tag{2.5}$$

respectively.

A small deformation of the $SU(2)$-WZNW model can be obtained by adding a marginal perturbation of the form

$$O = \frac{\delta\lambda}{2\pi}\int d^2z\, J\bar{J} \tag{2.6}$$

to the action (2.3). The integrability of this perturbation⋆ indicates the existence of a continuous family of CFT's of the same central charge, parametrized by λ, such that two theories corresponding to two adjacent values of λ are related by a generalization of the operator (2.6). Let us denote the action of the deformed theory by $S_{(\lambda)}$. Except for some relatively simple cases, like the theory of a free boson compactified on a circle, the form of $S_{(\lambda)}$ cannot be obtained in a straightforward way. In the following we show that for WZNW models a solution to this problem is provided by $O(d,d)$ transformations.

The WZNW models are special cases of non-linear σ-models which are conformally invariant. As a consequence, the σ-model coupling constants in these theories automatically satisfy the zero β-function conditions and correspond to specific classical configurations of background fields described by the string theory

⋆ In a CFT with a KM symmetry, all marginal operators of the form $\sum_{i,j} C_{ij} J_i \bar{J}_j$, where J_i, $\bar{J}_i$ belong to the Cartan subalgebra, are integrable [14].

effective action. It has been shown [3]- [11] that the string theory low-energy effective action, when restricted to background fields that are independent of d of the space dimensions, is invariant under an $O(d,d)$ group of transformations. Since this group of transformations is not necessarily a symmetry of the unrestricted action, it can relate classical field configurations which are not equivalent, giving rise to different conformally invariant world-sheet theories. In the following we briefly describe the action of this group on the background fields $G_{\mu\nu}, B_{\mu\nu}$ and Φ of the closed bosonic string theory. For all our purposes in the present paper it is sufficient to consider backgrounds of the form†

$$G = \begin{pmatrix} \widetilde{G}_{\alpha\beta} & 0 \\ 0 & \widehat{G}_{mn} \end{pmatrix} \quad , \quad B = \begin{pmatrix} \widetilde{B}_{\alpha\beta} & 0 \\ 0 & \widehat{B}_{mn} \end{pmatrix} \tag{2.7}$$

where the indices m and n span the d-dimensions on which the fields do not depend. Let us now construct a $2d \times 2d$ matrix M as

$$M = \begin{pmatrix} \widehat{G}^{-1} & -\widehat{G}^{-1}\widehat{B} \\ \widehat{B}\widehat{G}^{-1} & \widehat{G} - \widehat{B}\widehat{G}^{-1}\widehat{B} \end{pmatrix} \tag{2.8}$$

In terms of this matrix and the dilaton field Φ, the action of the $O(d,d)$ group is given by

$$\begin{aligned} M &\longrightarrow M' = \Omega M \Omega^T \\ \Phi &\longrightarrow \Phi' = \Phi + \frac{1}{2}\ln\left[\frac{\det \widehat{G}'}{\det \widehat{G}}\right] \end{aligned} \tag{2.9}$$

where Ω is an $O(d,d)$ group element defined by

$$\Omega \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \Omega^T = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \tag{2.10}$$

The components of the fields G and B which do not appear in M remain unchanged under the transformation.

† An extention to general backgrounds is given in ref.[5] in the context of heterotic string theory

An Ω lying in the $O(d) \otimes O(d)$ subgroup of $O(d,d)$ can be parametrized as

$$\Omega_1 = \frac{1}{2} \begin{pmatrix} R+S & R-S \\ R-S & R+S \end{pmatrix} \tag{2.11}$$

with $R, S \in O(d)$. The diagonal part $R = S$ corresponds to rotations in d - dimensions. Thus the non-trivial transformations are generated by matrices of the form Ω_1 modulo the diagonal subgroup; they form a coset $O(d) \otimes O(d)/O(d)$. A general $O(d,d)$ transformation may be expressed as the product of an element of the coset $O(d) \otimes O(d)/O(d)$ (which, in turn, may be labelled by an element of the group $O(d)$ by making the specific choice $S = R^T$), and the group generated by matrices of the form [4]:

$$\Omega_2 = \begin{pmatrix} (A^T)^{-1} & 0 \\ 0 & A \end{pmatrix}, \qquad \Omega_3 = \begin{pmatrix} 1_d & 0 \\ C & 1_d \end{pmatrix} \tag{2.12}$$

where A and C are constant matrices in d-dimensions and C is antisymmetric. They generate transformations of the form $\widehat{G} \to A\widehat{G}A^T, \widehat{B} \to A\widehat{B}A^T$ and $\widehat{B} \to \widehat{B} + C$. These are implemented by general coordinate transformations of the form $X'^m = A^m{}_n X^n$ and gauge transformations of B_{mn} with gauge parameter $\Lambda_m = C_{mn}X^n$. The elements of the form (2.11) with $R = S^T$, on the other hand, act non-linearly on the background fields and, for non-compact coordinates, are the only elements which give rise to inequivalent backgrounds. For WZNW models the coordinates are compact, and a new background obtained by a transformation of coordinates is not equivalent to the old background if we assume the same periodicity for the new and the old coordinates. In fact, since these coordinates are angular variables, the new background generically will correspond to a singular metric even if the original metric was non-singular.* As we shall see, the field configuration obtained by transforming the WZNW theory by the elements of the

$\star$ A trivial example of this is the metric $dr^2 + r^2 d\theta^2$ in polar coordinates. If we define $\theta' = \theta/C$, and consider the case where θ' is an angular variable with period 2π, then the new metric $dr^2 + C^2 d(\theta')^2$ has conical singularity at $r = 0$.

form given in eq.(2.11) with $R = S^T$ is quite often singular, and needs to be followed by a general coordinate transformation in order to give a non-singular metric.

Although $O(d,d)$ symmetry survives to all orders in the σ-model perturbation theory [4] [5], the explicit form given above is correct only to the lowest order in the σ-model loop expansion parameter. In applications to WZNW models, therefore, the explicit results obtained are valid in the large-k limit. We will always assume this to be the case.

Returning to the action (2.3) we note that the target space metric and anti-symmetric tensor fields of the σ-model are independent of the two coordinates θ_L and θ_R. This allows us to perform an $O(2,2)$ transformation on the model. We first define new coordinates

$$\theta = \frac{1}{2}(\theta_L - \theta_R)\,, \qquad \widetilde{\theta} = \frac{1}{2}(\theta_L + \theta_R) \tag{2.13}$$

in terms of which the metric $G_{\mu\nu}$ is diagonal. After shifting the B-field by a constant matrix, the action (2.3) becomes

$$S(\phi,\theta,\widetilde{\theta}) = \frac{k}{2\pi}\int d^2z[\,\frac{1}{4}\bar{\partial}\phi\partial\phi + \sin^2\frac{\phi}{2}\bar{\partial}\theta\partial\theta + \cos^2\frac{\phi}{2}\bar{\partial}\widetilde{\theta}\partial\widetilde{\theta} + \cos^2\frac{\phi}{2}(\bar{\partial}\theta\partial\widetilde{\theta} - \bar{\partial}\widetilde{\theta}\partial\theta)\,] \tag{2.14}$$

from which, we can read off the σ-model coupling constants as

$$G = \begin{pmatrix} k/4 & 0 & 0 \\ 0 & k\sin^2\frac{\phi}{2} & 0 \\ 0 & 0 & k\cos^2\frac{\phi}{2} \end{pmatrix}\,, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k\cos^2\frac{\phi}{2} \\ 0 & -k\cos^2\frac{\phi}{2} & 0 \end{pmatrix} \tag{2.15}$$

and $\Phi = 0$. Since θ and $\widetilde{\theta}$ have periodicities of 2π, the metric does not have coordinate singularities at $\phi = 0$ and $\phi = \pi$.

Now we consider a transformation of the above backgrounds by the elements

Ω_1 of the $O(2,2)$ group as given in (2.11), with the parametrization

$$R = S^{-1} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{2.16}$$

The new metric has conical singularities at $\phi = 0$ and π which are removed by scalings $\theta \to \theta/(\cos\alpha - k\sin\alpha)$ and $\widetilde{\theta} \to \widetilde{\theta}/\cos\alpha$ (This restores the periodicity of θ and $\widetilde{\theta}$ to 2π). We also make a further transformation $B_{\tilde{\theta}\theta} \to B_{\tilde{\theta}\theta} + L$, where L is a constant given by

$$L = \cos\alpha(k\cos\alpha + \sin\alpha) \tag{2.17}$$

This corresponds to locally adding a total derivative term to the action and does not change the β-function of the theory. We will elaborate more on this choice of L while discussing the gauging of the $O(2,2)$ transformed model. The new σ-model action obtained after these transformations is

$$\begin{aligned} S_{(\alpha)}(\phi,\theta,\widetilde{\theta}) = \frac{k}{2\pi}\int d^2z[\,&\frac{1}{4}\bar{\partial}\phi\partial\phi + \frac{1}{\Delta}(\cos\alpha - k\sin\alpha)^2\sin^2\frac{\phi}{2}\bar{\partial}\theta\partial\theta \\ &+ \frac{1}{\Delta}\cos^2\alpha\cos^2\frac{\phi}{2}\bar{\partial}\widetilde{\theta}\partial\widetilde{\theta} - \frac{1}{\Delta}\cos^2\alpha\sin^2\frac{\phi}{2}(\bar{\partial}\theta\partial\widetilde{\theta} - \bar{\partial}\widetilde{\theta}\partial\theta)\,] \end{aligned} \tag{2.18}$$

with a dilaton field Φ given by*

$$\Phi = -\ln\Delta \tag{2.19}$$

where,

$$\Delta = \cos^2\alpha + k\,(k\sin^2\alpha - 2\sin\alpha\cos\alpha)\cos^2\frac{\phi}{2} \tag{2.20}$$

Although the explicit form of the transformed action is valid only in the large k limit, the existence of the $O(2,2)$ transformation, and hence of the transformed action can be shown to all orders in k. The action (2.18) along with the dilaton

$\star$ There is a freedom of adding an overall constant to the expression for Φ, which we have not displayed explicitly. While comparing different solutions we should keep this in mind.

(2.19) describes a one-parameter (α) family of CFT's which include the $SU(2)$ WZNW model ($\alpha = 0$). For small α the change in the action (2.14) under the $O(2,2)$ transformation is

$$\delta S = S_{(\alpha)} - S_{(0)} = \frac{\alpha}{\pi} \int d^2z\, J\bar{J} \tag{2.21}$$

where, J and $\bar{J}$ are the currents (2.5) now written in terms of θ and $\widetilde{\theta}$. A comparison with (2.6), for $\lambda = 2\alpha$, shows that the $O(2,2)$ transformed action for small α is the same as the action obtained after a marginal perturbation. To generalize this result to finite α, we notice that

$$S_{(\alpha+\delta\alpha)} = S_{(\alpha)} + \frac{\delta\alpha}{\pi} \int d^2z\, [\frac{1}{\Delta^2} \cos\alpha(\cos\alpha - k\sin\alpha) J\bar{J}] \tag{2.22}$$

with J, $\bar{J}$ as defined in eq.(2.5). The equations of motion for θ and $\widetilde{\theta}$, obtained from $S_{(\alpha)}$, are $\bar{\partial}(J/\Delta) = \partial(\bar{J}/\Delta) = 0$. The significance of these equations may be understood in the following way. The action $S_{(\alpha)}(\phi, \theta, \widetilde{\theta})$ has chiral invariances

$$\begin{aligned} \delta\theta &= \frac{1}{2}\frac{1}{(\cos\alpha - k\sin\alpha)^2} v(z), \qquad & \delta\widetilde{\theta} &= \frac{1}{2}\frac{1}{\cos^2\alpha} v(z) \\ \bar{\delta}\theta &= -\frac{1}{2}\frac{1}{(\cos\alpha - k\sin\alpha)^2} \bar{v}(\bar{z}), \qquad & \bar{\delta}\widetilde{\theta} &= \frac{1}{2}\frac{1}{\cos^2\alpha} \bar{v}(\bar{z}) \end{aligned} \tag{2.23}$$

The corresponding conserved chiral currents have the form

$$\begin{aligned} J_{(\alpha)} &= \frac{k}{\Delta}(\sin^2\frac{\phi}{2}\partial\theta + \cos^2\frac{\phi}{2}\partial\widetilde{\theta}) = J/(\Delta) \\ \bar{J}_{(\alpha)} &= \frac{k}{\Delta}(-\sin^2\frac{\phi}{2}\bar{\partial}\theta + \cos^2\frac{\phi}{2}\bar{\partial}\widetilde{\theta}) = \bar{J}/(\Delta) \end{aligned} \tag{2.24}$$

In terms of $J_{(\alpha)}$ and $\bar{J}_{(\alpha)}$ (2.22) becomes

$$S_{(\alpha+\delta\alpha)} = S_{(\alpha)} + \frac{\delta\alpha}{\pi} \cos\alpha(\cos\alpha - k\sin\alpha) \int d^2z\, J_{(\alpha)}\bar{J}_{(\alpha)} \tag{2.25}$$

The above equation shows that appropriate $O(2,2)$ transformations of a $SU(2)$-WZNW model generate a continuous line of conformal field theories which are related by marginal perturbations. The precise relationship is as follows. The $O(2,2)$ transformation involves two of the target space coordinates, θ and $\widetilde{\theta}$, on which the background fields do not depend. Global shifts in these coordinates are therefore commuting isometries of the backgrounds, giving rise to conserved isometry currents. If this symmetry is extendable to local shifts with only holomorphic or anti-holomorphic dependences on the world-sheet coordinates, as in (2.23), then the theory contains a pair of chiral currents, and hence, a marginal operator. Equation (2.25) shows that perturbations by this operator are reproduced by appropriate $O(2,2)$ transformations involving θ and $\widetilde{\theta}$. The integrability of the marginal perturbation (which was proved [14] to all orders in α by working at the $SU(2)$ point where the conformal structure of the theory is known) insures that the new theory $S_{(\alpha+\delta\alpha)}$ also contains chiral currents and can be perturbed further. We have proved that an infinite series of such perturbations can effectively be added up by a finite $O(2,2)$ transformation. This transformation also gives the expression for the new dilaton field without any extra calculations.

Next, we consider the $SU(2)\otimes SU(2)$ model defined by the action

$$S = S_1(\,\phi,\theta,\widetilde{\theta}\,) + S_2(\,r,t,\widetilde{t}\,) \tag{2.26}$$

Here S_1 is the action given in equation (2.14) with $k = k_1$ and S_2 is obtained from S_1 by the replacements $(k_1,\phi,\theta,\widetilde{\theta}\,) \to (k_2,r,t,\widetilde{t}\,)$. The backgrounds in S are independent of the four coordinates $\theta,\widetilde{\theta},t$ and $\widetilde{t}$ and can be transformed by an $O(4,4)$ group of transformations. As stated before, a general $O(4,4)$ transformation may be represented by an $O(4)$ transformation followed by a gauge and general coordinate transformation. To study the connection between these transformations and marginal deformations in the $SU(2)\otimes SU(2)$ model, it is sufficient to consider the action of various $O(2)$ subgroups of $O(4)$ individually. Note that $O(4)$ contains a subgroup $O(2)\otimes O(2)$ which transforms each one of the $SU(2)$ theories separately. The action of this subgroup (followed by an appropriate scaling of the coordinates) is a trivial extension of the $O(2,2)$ transformation of the $SU(2)$ model considered above and the σ-model action of the transformed theory is given by,

$$S_{(\alpha,\beta)} = S_{1\,(\alpha)}(\,\phi,\theta,\widetilde{\theta}\,) + S_{2\,(\beta)}(\,r,t,\widetilde{t}\,) \tag{2.27}$$

As an example of the remaining four $O(2,2)$ subgroups, we consider the one which twists the θ and t coordinates and parametrize the elements Ω_1, given in (2.11) , by choosing

$$R = S^{-1} = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\gamma & 0 & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.28}$$

The transformed backgrounds are obtained from equations (2.8) and (2.9) . The new metric has conical singularities at $\phi = 0$ and $r = 0$ which are removed by scalings $\theta \to \theta/\cos\gamma$ and $t \to t/\cos\gamma$, restoring the periodicities of $\theta, \widetilde{\theta}, t$ and $\widetilde{t}$ to 2π. After shifting the B-field by a constant, $B_{\theta t} \to B_{\theta t} - \sin\gamma\cos\gamma$, the transformed σ-model action and the dilaton field are given by

$$\begin{aligned}
S_{(\gamma)} = \frac{1}{2\pi}\int d^2z \Big[& \frac{k_1}{4}\bar\partial\phi\partial\phi + \frac{k_2}{4}\bar\partial r\partial r \\
& + \frac{1}{\Delta}\Big(k_1\cos^2\gamma\sin^2\frac{\phi}{2}\bar\partial\theta\partial\theta + k_2\cos^2\gamma\sin^2\frac{r}{2}\bar\partial t\partial t \\
& + k_1(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{r}{2})\cos^2\frac{\phi}{2}\bar\partial\widetilde{\theta}\partial\widetilde{\theta} \\
& + k_2(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{\phi}{2})\cos^2\frac{r}{2}\bar\partial\widetilde{t}\partial\widetilde{t} \\
& + k_1\cos^2\gamma\cos^2\frac{\phi}{2}(\bar\partial\theta\partial\widetilde{\theta} - \bar\partial\widetilde{\theta}\partial\theta) + k_2\cos^2\gamma\cos^2\frac{r}{2}(\bar\partial t\partial\widetilde{t} - \bar\partial\widetilde{t}\partial t) \\
& - k_1k_2\sin\gamma\cos\gamma\sin^2\frac{\phi}{2}\sin^2\frac{r}{2}(\bar\partial\theta\partial t - \bar\partial t\partial\theta) \\
& - k_1k_2\sin\gamma\cos\gamma\sin^2\frac{\phi}{2}\cos^2\frac{r}{2}(\bar\partial\theta\partial\widetilde{t} + \bar\partial\widetilde{t}\partial\theta) \\
& + k_1k_2\sin\gamma\cos\gamma\cos^2\frac{\phi}{2}\sin^2\frac{r}{2}(\bar\partial t\partial\widetilde{\theta} + \bar\partial\widetilde{\theta}\partial t) \\
& + k_1k_2\sin\gamma\cos\gamma\cos^2\frac{\phi}{2}\cos^2\frac{r}{2}(\bar\partial\widetilde{\theta}\partial\widetilde{t} - \bar\partial\widetilde{t}\partial\widetilde{\theta})\Big)\Big] \\
\Phi = -\ln\Delta &
\end{aligned} \tag{2.29}$$

where,

$$\Delta = \cos^2\gamma + k_1 k_2 \sin^2\gamma \sin^2\frac{\phi}{2}\sin^2\frac{r}{2} \tag{2.30}$$

For an infinitesimal $O(2,2)$ transformation, the change in the action (2.26) is

$$\delta S = \frac{-\gamma}{2\pi}\int d^2z\,\left(J_1\bar{J}_2 - J_2\bar{J}_1 \right) \tag{2.31}$$

where, $J_1, \bar{J}_1$ and $J_2, \bar{J}_2$ are the chiral $U(1)$ currents of the two $SU(2)$ theories in (2.26). This, clearly, is an integrable marginal perturbation of the untransformed action S. To identify the theory given by the action $S_{(\gamma)}$ for finite γ, we note that the action (2.29) has the following chiral invariances:

$$\begin{aligned}
&\delta\theta = \frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{2\cos^2\gamma}v(z)\ ,\delta\widetilde{\theta} = \frac{1}{2}v(z)\ ,\delta t = 0\ ,\delta\widetilde{t} = \frac{1}{2}k_1\tan\gamma\, v(z)\\
&\bar{\delta}\theta = -\frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{2\cos^2\gamma}\bar{v}(\bar{z})\ ,\bar{\delta}\widetilde{\theta} = \frac{1}{2}\bar{v}(\bar{z})\ ,\bar{\delta}t = 0\ ,\bar{\delta}\widetilde{t} = -\frac{1}{2}k_1\tan\gamma\,\bar{v}(\bar{z})\\
&\delta\theta = 0\ ,\delta\widetilde{\theta} = -\frac{1}{2}k_2\tan\gamma\, u(z)\ ,\delta t = \frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{2\cos^2\gamma}u(z)\ ,\delta\widetilde{t} = \frac{1}{2}u(z)\\
&\bar{\delta}\theta = 0\ ,\bar{\delta}\widetilde{\theta} = \frac{1}{2}k_2\tan\gamma\,\bar{u}(\bar{z})\ ,\bar{\delta}t = -\frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{2\cos^2\gamma}\bar{u}(\bar{z})\ ,\bar{\delta}\widetilde{t} = \frac{1}{2}\bar{u}(\bar{z})
\end{aligned} \tag{2.32}$$

These invariances give rise to the following conserved chiral currents

$$\begin{aligned}
J_{1\,(\gamma)} &= \frac{1}{\Delta}\big[(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{r}{2})\,J_1 + \sin\gamma\cos\gamma k_1\cos^2\frac{\phi}{2}J_2\,\big]\\
\bar{J}_{1\,(\gamma)} &= \frac{1}{\Delta}\big[(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{r}{2})\,\bar{J}_1 - \sin\gamma\cos\gamma k_1\cos^2\frac{\phi}{2}\bar{J}_2\,\big]\\
J_{2\,(\gamma)} &= \frac{1}{\Delta}\big[(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{\phi}{2})\,J_2 - \sin\gamma\cos\gamma k_2\cos^2\frac{r}{2}J_1\,\big]\\
\bar{J}_{2\,(\gamma)} &= \frac{1}{\Delta}\big[(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{\phi}{2})\,\bar{J}_2 + \sin\gamma\cos\gamma k_2\cos^2\frac{r}{2}\bar{J}_1\,\big]
\end{aligned} \tag{2.33}$$

In terms of these currents, the variation of $S_{(\gamma)}$ under a small $O(2,2)$ transformation is given by

$$S_{(\gamma+\delta\gamma)} = S_{(\gamma)} + \frac{\delta\gamma}{2\pi(\cos^2\gamma + k_1k_2\sin^2\gamma)^2}\int d^2z\big[2\sin\gamma\cos\gamma(k_2J_{1\,(\gamma)}\bar{J}_{1\,(\gamma)} + k_1J_{2\,(\gamma)}\bar{J}_{2\,(\gamma)})$$
$$- (\cos^2\gamma - k_1k_2\sin^2\gamma)(J_{1\,(\gamma)}\bar{J}_{2\,(\gamma)} - J_{2\,(\gamma)}\bar{J}_{1\,(\gamma)})\big] \tag{2.34}$$

This shows that in the continuous one-parameter family of conformal field theories described by the action $S_{(\gamma)}$, two adjacent theories are related by a marginal perturbation constructed from the product of a holomorphic and an anti-holomorphic current.

The action (2.29) was obtained by an $O(2,2)$ transformation which twists the coordinates θ and t. The full transformation group also contains three other subgroups which mix the two $SU(2)$ theories in (2.26) . The corresponding transformed actions can be directly obtained from (2.29) by the following replacements:

$$\begin{aligned} \theta - \widetilde{t}\ \text{ twisting } &: \ r \rightarrow r + \pi,\ t \leftrightarrow \widetilde{t} \\ \widetilde{\theta} - t\ \text{ twisting } &: \ \phi \rightarrow \phi + \pi,\ \theta \leftrightarrow \widetilde{\theta} \end{aligned}$$

By combining the above two replacements, one obtains the transformed action for $\widetilde{\theta} - \widetilde{t}$ twisting.

The result can be generalised for WZNW models based on other groups G. If r is the rank of the group, and N denotes the total number of generators, then we can choose a parametrisation of the group element (at least locally) of the form:

$$g = exp(i\sum_{i=1}^{r}\theta_{iL}H_i)exp(i\sum_{a=1}^{N-2r}\alpha_aT_a)exp(i\sum_{j=1}^{r}\theta_{jR}H_j) \tag{2.35}$$

where H_j are the generators of the Cartan subalgebra, and $\{T_a\}$ denote a specific set of $N-2r$ generators outside the Cartan subalgebra. In this case, when the action of the WZNW theory is written in the form of the conventional σ-model action in two dimensions, the background metric and the antisymmetric tensor field components will be independent of the $2r$ coordinates θ_{iL}, θ_{iR} $(1 \le i \le r)$. Thus there is an $O(2r,2r)$ transformation which can generate new conformally invariant background, and we would expect that all the r^2 marginal deformations generated by taking products of holomorphic and anti-holomorphic currents in the Cartan subalgebra will be generated by the $O(2r,2r)$ transformation.

3. $O(3,3)$ Transformation of $SU(2)\otimes(SU(2)/U(1))$ Model

In this section we shall start with a conformal field theory that is a product of an $SU(2)$ WZNW model and an $SU(2)/U(1)$ coset model; and then construct the most general model obtained by $O(3,3)$ transformation of this model. The starting model has the metric [13]:

$$ds^2 = k_2(\frac{1}{4}dr^2 + \sin^2\frac{r}{2}dt^2 + \cos^2\frac{r}{2}d\tilde{t}^2) + k_1(\frac{1}{4}d\phi^2 + \tan^2\frac{\phi}{2}d\theta^2) \tag{3.1}$$

and the dilaton and the antisymmetric tensor fields:

$$\Phi = -\ln\cos^2\frac{\phi}{2}, \qquad B_{t\tilde{t}} = -k_2\sin^2\frac{r}{2} \tag{3.2}$$

with all other components of the anti-symmetric tensor field being zero. The fields are independent of the three coordinates ϕ, t and $\tilde{t}$, and hence we can generate other conformally invariant background via an $O(3,3)$ transformation in general. Let us define,

$$x^1 = \theta, \qquad x^2 = t, \qquad x^3 = \tilde{t} \tag{3.3}$$

As discussed in sect.2, new conformally invariant background can be generated from the one given in eqs.(3.1), (3.2) via $O(3,3)$ transformation. A general $O(3,3)$ transformation can be written as a product of an $O(3)$ transformation and a three dimensional general coordinate transformation and gauge transformation involving the antisymmetric tensor field. We shall first perform the $O(3)$ transformation of the background given in eqs.(3.1), (3.2). This is generated by matrices of the form

(2.11) with $R = S^T$. In the present case ($d = 3$), a general R may be taken to be of the form:

$$R = R_3 R_2 R_1 \tag{3.4}$$

where,

$$R_1 = \begin{pmatrix} \cos\tilde{\alpha} & -\sin\tilde{\alpha} & 0 \\ \sin\tilde{\alpha} & \cos\tilde{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_2 = \begin{pmatrix} \cos\tilde{\beta} & 0 & \sin\tilde{\beta} \\ 0 & 1 & 0 \\ -\sin\tilde{\beta} & 0 & \cos\tilde{\beta} \end{pmatrix} \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tilde{\gamma} & -\sin\tilde{\gamma} \\ 0 & \sin\tilde{\gamma} & \cos\tilde{\gamma} \end{pmatrix} \tag{3.5}$$

Calculation of the transformed fields is relatively straightforward. It turns out that the final metric obtained this way is singular at $r = 0$, $r = \pi$ and $\phi = 0$ if we assume conventional periodicities in the variables θ, t and $\tilde{t}$. In order to remove these singularities, we need to perform a linear set of coordinate transformations of the form:

$$\begin{pmatrix} \theta \\ t \\ \tilde{t} \end{pmatrix} = A \begin{pmatrix} \theta' \\ t' \\ \tilde{t}' \end{pmatrix} \tag{3.6}$$

where A is a 3×3 matrix, such that the transformed metric satisfies the following conditions near $r = 0$, $r = \pi$ and $\phi = 0$:

$$\begin{aligned} \text{For } r \simeq 0 \quad & G'_{tt} \simeq k_2 \frac{r^2}{4} \\ & G'_{t\tilde{t}}, G'_{t\theta} \propto r^2 \\ \text{For } r \simeq \pi \quad & G'_{\tilde{t}\tilde{t}} \simeq k_2 \frac{(r-\pi)^2}{4} \\ & G'_{\tilde{t}t}, G'_{\tilde{t}\theta} \propto (r-\pi)^2 \\ \text{For } \phi \simeq 0 \quad & G'_{\phi\phi} \simeq k_1 \frac{\phi^2}{4} \\ & G'_{\phi t}, G'_{\phi\tilde{t}} \propto \phi^2 \end{aligned} \tag{3.7}$$

Such a metric is non-singular if we assume θ', t' and $\tilde{t}'$ to be angular coordinates with period 2π each.

It turns out that these requirements fix the matrix A completely. The dilaton, metric and the antisymmetric tensor field components after the transformation are given by (for convenience of writing we have dropped the primes),

$$
\begin{aligned}
\Phi &= -\ln\Delta \\
G_{\theta\theta} &= \Delta^{-1}k_1\sin^2\frac{\phi}{2}\{A^2\cos^2\frac{r}{2}+\sin^2\frac{r}{2}\} \\
G_{tt} &= \Delta^{-1}k_2\sin^2\frac{r}{2}\{B^2\sin^2\frac{\phi}{2}+A^2\cos^2\frac{\phi}{2}\} \\
G_{\tilde{t}\tilde{t}} &= \Delta^{-1}k_2\cos^2\frac{r}{2}\{C^2\sin^2\frac{\phi}{2}+\cos^2\frac{\phi}{2}\} \\
G_{\theta t} &= \Delta^{-1}\sqrt{k_2k_1}B\sin^2\frac{\phi}{2}\sin^2\frac{r}{2} \\
G_{\theta\tilde{t}} &= \Delta^{-1}\sqrt{k_2k_1}AC\sin^2\frac{\phi}{2}\cos^2\frac{r}{2} \\
G_{t\tilde{t}} &= 0 \\
B_{\theta t} &= \Delta^{-1}\sqrt{k_2k_1}AC\sin^2\frac{\phi}{2}\sin^2\frac{r}{2} \\
B_{\theta\tilde{t}} &= \Delta^{-1}\sqrt{k_2k_1}B\sin^2\frac{\phi}{2}\cos^2\frac{r}{2} \\
B_{t\tilde{t}} &= -\Delta^{-1}k_2\sin^2\frac{r}{2}\{\cos^2\frac{\phi}{2}+C^2\sin^2\frac{\phi}{2}\}
\end{aligned}
\tag{3.8}
$$

where,

$$
\begin{aligned}
A =& \frac{\cos\tilde{\alpha}\cos\tilde{\beta}\cos\tilde{\gamma}}{k_2\sin\tilde{\gamma}-\cos\tilde{\alpha}\cos\tilde{\beta}\cos\tilde{\gamma}} \\
B =& \sqrt{k_2k_1}\frac{\sin\tilde{\beta}\cos\tilde{\gamma}}{k_2\sin\tilde{\gamma}-\cos\tilde{\alpha}\cos\tilde{\beta}\cos\tilde{\gamma}} \\
C =& \sqrt{k_2k_1}\frac{\sin\tilde{\alpha}\cos\tilde{\beta}\cos\tilde{\gamma}}{k_2\sin\tilde{\gamma}-\cos\tilde{\alpha}\cos\tilde{\beta}\cos\tilde{\gamma}} \\
\Delta =& \cos^2\frac{\phi}{2}\{\sin^2\frac{r}{2}+A^2\cos^2\frac{r}{2}\}+\sin^2\frac{\phi}{2}\{B^2\cos^2\frac{r}{2}+C^2\sin^2\frac{r}{2}\}
\end{aligned}
\tag{3.9}
$$

Various special cases of this solution have been discussed in refs.[15][12]. In the next section we shall see that these models represent conformal field theories obtained after gauging a marginally deformed $SU(2)\otimes SU(2)$ WZNW theory.

4. Gauging the $O(d,d)$ Transformed WZNW-Models:

In this section we describe the gauging of commuting isometries of a non-linear σ-model with chiral symmetries and in which the background metric, antisymmetric tensor and dilaton fields do not depend on d of the coordinates. For isometries which involve these d coordinates, we show that, upto a total derivative term, the gauged action can be obtained by a covariant derivative replacement. We then apply the procedure to gauge the marginally deformed $SU(2)$ and $SU(2)\otimes SU(2)$ models of section 2 and comment on the results. This gives us a method to construct classes of solutions of the string theory low-energy equations of motion as exact conformal field theories.

Consider a non-linear σ-model with a Wess-Zumino term given by the action

$$S=\frac{1}{2\pi}\int d^2z(\,g_{ij}+b_{ij}\,)\bar{\partial}X^i\partial X^j \tag{4.1}$$

The model may also contain a background dilaton field, Φ, which does not explicitly appear in the action. The above action is invariant under a global transformation

$$\delta X^i=v^\alpha\xi^i_\alpha(X) \tag{4.2}$$

if $\nabla_i\xi_{j,\alpha}+\nabla_j\xi_{i,\alpha}=0\,,\xi^i_\alpha\partial_i\Phi=0$ and provided there exists a 1-form $K_\alpha=K_{i,\alpha}dX^i$ given by

$$b_{ik}\,\partial_j\xi^k_\alpha+b_{kj}\,\partial_i\xi^k_\alpha+\partial_k b_{ij}\,\xi^k_\alpha=\partial_iK_{j,\alpha}-\partial_jK_{i,\alpha} \tag{4.3}$$

The isometry currents associated with the above invariance are

$$\begin{aligned}I_\alpha&=[\,(\,g_{ij}+b_{ij}\,)\xi^i_\alpha-K_{j,\alpha}\,]\partial X^j\\ \bar{I}_\alpha&=[\,(\,g_{ij}-b_{ij}\,)\xi^i_\alpha+K_{j,\alpha}\,]\partial X^j\end{aligned} \tag{4.4}$$

with $\bar{\partial}I_\alpha+\partial\bar{I}_\alpha=0$. In refs.[16][17] it was shown that to gauge the isometry (4.2), the following conditions must be satisfied

$$
\begin{aligned}
b_{jk}\,\xi^k_\beta\partial_i\xi^j_\alpha + \partial_j(b_{ik}\,\xi^k_\beta)\xi^j_\alpha + \partial_j K_{i,\beta}\xi^j_\alpha + K_{j,\beta}\partial_i\xi^j_\alpha &\\
= f^\gamma_{\alpha\beta}(\,b_{ij}\xi^j_\gamma + K_{i,\gamma}) &\\
K_{i,\alpha}\xi^i_\beta = -\,K_{i,\beta}\xi^i_\alpha &
\end{aligned}
\tag{4.5}
$$

We restrict ourselves to commuting isometries and set $f^\gamma_{\alpha\beta} = 0$. If the gauge field A^α_μ transforms as $A^\alpha_\mu \to A^\alpha_\mu + \partial_\mu v^\alpha$, then the gauge invariant action is given by

$$
\begin{aligned}
S^{\text{gauged}} = \frac{1}{2\pi}\int d^2z [\,(\,g_{ij} + b_{ij}\,)\bar\partial X^i\partial X^j - A^\alpha \bar I_\alpha - \bar A^\alpha I_\alpha &\\
+ ((\,g_{ij} + b_{ij}\,)\xi^i_\alpha\xi^j_\beta - \frac{1}{2}(K_{i,\alpha}\xi^i_\beta - K_{i,\beta}\xi^i_\alpha))\bar A^\alpha A^\beta\,] &
\end{aligned}
\tag{4.6}
$$

This action has an arbitrariness stemming from the fact that equations (4.3) and (4.5) do not determine $K_{i,\alpha}$ completely. Moreover, it cannot in general be obtained from (4.1) by a covariant derivative replacement.

To fix the arbitrariness of the gauged action, we note that for general $K_{i,\alpha}$, the action (4.6) does not necessarily describe a conformal field theory even if the field theory corresponding to the ungauged action (4.1) is conformally invariant. However, if the action (4.1) has chiral symmetries, and if the gauge fields in (4.6) couple to the corresponding chiral currents, then the resulting theory may be described as a coset of the original CFT by $U(1)$ current algebra theories, and hence describes a new conformal field theory. The $O(d,d)$ transformed WZNW models considered in sect.2 still have residual chiral invariance given in eqs.(2.23), (2.32). This enables us to break up the transformation (4.2) into

$$
\delta_L X^i = v^a_L\xi^i_{La}\,, \qquad \delta_R X^i = v^a_R\xi^i_{Ra} \tag{4.7}
$$

such that its chiral extension with $v^a_L(z)$ and $v^a_R(\bar z)$ is also a symmetry of the ungauged action. The isometry currents I^μ_{La} and I^μ_{Ra} associated with (4.7) are obtained from (4.4) on replacing the pair $(\xi^i_\alpha, K_{i,\alpha})$ by $(\xi^i_{La}, L_{i,a})$ and $(\xi^i_{Ra}, R_{i,a})$

respectively. If we choose to gauge a diagonal subgroup of (4.7) with $v_L^a = v_R^a = v^a(z,\bar{z})$, and therefore $A_\mu^{La} = A_\mu^{Ra} \equiv A_\mu^a$, then the gauged action is given by (4.6) with the index α replaced by the index a, where, now

$$\xi_a^i = \xi_{La}^i + \xi_{Ra}^i \,, \qquad K_{i,a} = L_{i,a} + R_{i,a} \tag{4.8}$$

and, therefore, $I_a^\mu = I_{La}^\mu + I_{Ra}^\mu$. It can be easily seen that the chiral invariance of the ungauged action allows us to choose

$$L_{j,a} = -(\, g_{ij} - b_{ij}\,)\xi_{La}^i \qquad R_{j,a} = (\, g_{ij} + b_{ij}\,)\xi_{Ra}^i \tag{4.9}$$

satisfying eq.(4.3). This gives $\bar{I}_{La} = I_{Ra} = 0$ and $\partial\bar{I}_{Ra} = \bar{\partial}I_{La} = 0$. (We shall denote these chiral currents by J_a, $\bar{J}_a$.) If this choice is made in the gauged action, the gauge fields will couple to chiral currents of the ungauged theory, leading to a new CFT as argued above. This can be checked explicitly in the specific examples we have, by verifying that the backgrounds obtained after gauge fixing and integrating out the gauge fields satisfy the β-function vanishing equations. This observation has been made earlier in ref.[18].

Note that after obtaining $L_{j,a}$ and $R_{j,a}$ using eq.(4.9), we need to verify that they satisfy eqs.(4.5). This further restricts the choice of ξ_{aL}^i and ξ_{aR}^i to anomaly free subgroups. In all the cases we shall discuss, we shall make appropriate choices of ξ_{aL}^i and ξ_{aR}^i so that these conditions are satisfied.

Next, we explore the possibility of writing the gauged action (4.6) in terms of covariant derivatives. This can be done, provided, by adding appropriate total derivative terms to the Lagrangian density, we can ensure that the Noether currents associated with the chiral symmetries are the same as the chiral currents J_a, $\bar{J}_a$. If in (4.1) b_{ij} is replaced by $b_{ij} + 2\partial_{[i}w_{j]}$ and $\partial_\mu X^i$ by $D_\mu X^i = \partial_\mu X^i - \xi_a^i A_\mu^a$, then we get

$$S^{\text{gauged}} = \frac{1}{2\pi}\int d^2z(\, g_{ij} + b_{ij} + 2\partial_{[i}w_{j]}\,)\bar{D}X^i DX^j \tag{4.10}$$

This is the correct gauged action (4.6) provided one can find a w_j such that

$$2\partial_{[i}w_{j]}\,\xi^j_a = L_{i,a} + R_{i,a} \tag{4.11}$$

This is possible if we restrict ourselves to backgrounds of the form (2.7) and to shift isometries in the coordinates on which the backgrounds do not depend ($\xi^\alpha_a = 0$, $\partial_i \xi^m_a = 0$). Equations (4.3) and (4.5) then imply that $L_{m,a}$ and $R_{m,a}$ are constant while $L_{\alpha,a} = R_{\alpha,a} = 0$. $2\partial_{[i}w_{j]}$ is, therefore, a constant matrix with non-zero elements only in the subspace spaned by the coordinates the isometries in which are gauged. Combining (4.11) with eq. (4.9) gives

$$2\partial_{[i}w_{j]}\,\xi^j_a = g_{ij}(\xi^j_{Ra} - \xi^j_{La}) - b_{ij}(\xi^j_{La} + \xi^j_{Ra}) \tag{4.12}$$

In the following we use equations (4.10) and (4.12) to gauge the shift isometries of $O(d,d)$ transformed WZNW models. First, we consider the axial gauging of the deformed $SU(2)$ model (2.18). Reading out ξ^i_L and ξ^i_R from (2.23) and substituting in (4.12), we get $\partial_{[\theta}w_{\tilde\theta]} = 0$,* leading to the gauged action

$$S^{\rm gauged}_{(\alpha)}(\phi,\theta,\widetilde\theta,A) = S_{(\alpha)}(\phi,\theta,\widetilde\theta) - \frac{1}{2\pi}\int d^2z\big(\,A\bar J_{(\alpha)} + \bar A J_{(\alpha)} - \frac{k\cos^2\frac{\phi}{2}}{\Delta\cos^2\alpha}A\bar A\,\big) \tag{4.13}$$

where $J_{(\alpha)}$, $\bar J_{(\alpha)}$ have been defined in eq.(2.24). Surprisingly, after gauge fixing ($\widetilde\theta = 0$) and integrating out the gauge field, one obtains the $SU(2)/U(1)$ blackhole of the untransformed theory

$$\begin{aligned} ds^2 &= (k/4)d\phi^2 + k\tan^2(\phi/2)\,d\theta^2 \\ e^{-\Phi} &= \cos^2\frac{\phi}{2} \end{aligned} \tag{4.14}$$

The reason for the disappearence of α lies in the freedom to redefine the gauge

* This is a consequence of the appropriate choice of L in (2.17). For vector gauging the proper choice is $L = \cos\alpha(k\cos\alpha + \sin\alpha) - k$. Since the added term is independent of α, it does not affect equation (2.22).

fields. In fact, substituting

$$A = \cos\alpha\sqrt{\Delta}A' + \cos\alpha(\cos\alpha - \sqrt{\Delta})\frac{J}{k\cos^2\frac{\phi}{2}} \tag{4.15}$$

in (4.13), with a similar expression for $\bar{A}$, we find that

$$S^{\text{gauged}}_{(\alpha)}(\phi,\theta,\widetilde{\theta},A) = S^{\text{gauged}}_{(\alpha=0)}(\phi,\theta,\widetilde{\theta},A') \tag{4.16}$$

The Jacobian of the transformation modifies the path integral measure and readjusts the dilaton field to zero.

This result may also be understood in a qualitative manner by noting that gauging the deformed WZNW model corresponds to taking the coset of the conformal field theory before gauging by the $U(1)$ current algebra theory. The action of the operators $J_{(\alpha)}$, $\bar{J}_{(\alpha)}$ becomes trivial in the coset theory, and hence one would expect that the effect of perturbing the original theory by the $J_{(\alpha)}\bar{J}_{(\alpha)}$ operator before gauging the theory will be washed out after gauging. This, in turn, implies that the final theory will be independent of the parameter α, which is indeed the case here.

Before proceeding further, we use equation (4.16) to clarify the meaning of the observations made in refs.[9] and [12]. In [9] it was observed that the gauged action of ref.[19] can be obtained from the corresponding ungauged action by a constant $O(d,d)$ transformation. In the present context, equation (2.18) , before performing the scalings following (2.16), shows that an $O(2,2)$ transformation of the $SU(2)$ model with $\alpha = \pi/2$ gives the $SU(2)/U(1)$ coset model and a free field $\widetilde{\theta}$. To understand why the coset model is obtained by such an $O(2,2)$ transformation, it is sufficient to look for some $\alpha = \alpha_0$ for which $\widetilde{\theta}$ decouples as a free field

$$S_{(\alpha=\alpha_0)}(\phi,\theta,\widetilde{\theta}) = S'_{(\alpha=\alpha_0)}(\phi,\theta) + \frac{1}{2\pi}\int d^2z\bar{\partial}\widetilde{\theta}\partial\widetilde{\theta} \tag{4.17}$$

Now, quotienting by the axial $U(1)$ subgroup simply eliminates $\widetilde{\theta}$ on the right hand side, leaving $S'_{(\alpha=\alpha_0)}(\phi,\theta)$. On the other hand, this, by (4.16), must be the same as $S^{\rm gauged}_{(\alpha=0)}$. This shows that once we know that $S_{\alpha=\alpha_0}$ can be written in the form of eq.(4.17), then $S'_{\alpha=\alpha_0}(\phi,\theta)$ must be the action of the gauged WZNW model ($SU(2)/U(1)$ coset model). Though the situation considered in ref.[9] is more general, the results can be understood in the same way. The above argument is valid provided the $O(d,d)$ transformed theory is written in a form that can be gauged by covariant derivative replacement. In ref.[12] it was observed that by adding a free field to the singular $SL(2,R)/U(1)$ background and an $O(2,2)$ transformation, one can "boost away" the singularity. This can be understood by inverting the argument based on eqn (4.17) : the singularities which are boosted away are the ones that appear as a consequence of a $U(1)$ gauging of the boosted theory, for which (4.16) holds. The argument can also be generalized to the case of the 4-dimensional solution of ref.[13] considered in [12].

Next, we consider the gauging of the $O(4,4)$ transformed $SU(2)\otimes SU(2)$ model in which the two $SU(2)$ sectors have not been mixed by $O(4,4)$. As shown above, a naive gauging of a $U(1)$ subgroup belonging to either of the two $SU(2)$'s will simply give the direct product of a marginally deformed $SU(2)$ WZNW model and an $SU(2)/U(1)$ coset model. To obtain a non-trivial result, we gauge a $U(1)$ subgroup which acts simultaneously on both sectors of the theory. The action of this subgroup is given by

$$\delta\theta = 0\,, \quad \delta\widetilde{\theta} = \frac{v(z,\bar z)}{\cos^2\alpha}\,, \quad \delta t = 0\,, \quad \delta\widetilde{t} = \lambda\frac{v(z,\bar z)}{\cos^2\beta} \tag{4.18}$$

where, λ is a new parameter. The gauged action is given by

$$\begin{aligned} S^{\rm gauged}_{(\alpha,\beta)} = S_{(\alpha)}(\,\phi,\theta,\widetilde{\theta}\,) + S_{(\beta)}(\,r,t,\widetilde{t}\,) - \frac{1}{2\pi}\int d^2z \Big[\, A\,(\bar J_{1\,(\alpha)} + \lambda\,\bar J_{2\,(\beta)}) \\ + \bar A\,(J_{1\,(\alpha)} + \lambda\,J_{2\,(\beta)}) - A\bar A\,(\,\frac{\cos^2\frac{\phi}{2}}{\cos^2\alpha\Delta_1} + \lambda^2\frac{\cos^2\frac{r}{2}}{\cos^2\beta\Delta_2}\,)\,\Big] \end{aligned} \tag{4.19}$$

where, the subscripts 1 and 2 refer to the theories $(\,\phi,\theta,\widetilde{\theta}\,)$ and $(\,r,t,\widetilde{t}\,)$, respectively. Choosing a gauge $\widetilde{\theta}=0$ and integrating out the gauge field, we obtain the following background fields:

$$
\begin{aligned}
G_{\theta\theta} &= D^{-1} k_1 \sin^2 \frac{\phi}{2} (\sin^2 \frac{r}{2} + Q \cos^2 \frac{r}{2}) \\
G_{tt} &= D^{-1} k_2 \sin^2 \frac{r}{2} (k_1 k_2 P^2 \sin^2 \frac{\phi}{2} + Q \cos^2 \frac{\phi}{2}) \\
G_{\tilde{t}\tilde{t}} &= D^{-1} k_2 \cos^2 \frac{\phi}{2} \cos^2 \frac{r}{2} \\
G_{\theta t} &= D^{-1} k_1 k_2 P \sin^2 \frac{\phi}{2} \sin^2 \frac{r}{2} \\
G_{\theta\tilde{t}} &= G_{t\tilde{t}} = B_{\theta t} = 0 \\
B_{\theta\tilde{t}} &= D^{-1} k_1 k_2 P \sin^2 \frac{\phi}{2} \cos^2 \frac{r}{2} \\
B_{t\tilde{t}} &= -D^{-1} k_2 \sin^2 \frac{r}{2} \cos^2 \frac{\phi}{2} \\
\Phi &= -\ln D
\end{aligned} \tag{4.20}
$$

where,

$$
\begin{aligned}
D &= \cos^2 \frac{\phi}{2} + k_1 k_2 P^2 \cos^2 \frac{r}{2} - (1 - Q + k_1 k_2 P^2) \cos^2 \frac{\phi}{2} \cos^2 \frac{r}{2} \\
P &= \frac{\lambda}{k_1} \left(\frac{\cos\alpha}{\cos\beta} \right)^2 \\
Q &= (1 - k_2 \tan\beta)^2 + \lambda^2 \frac{k_2}{k_1} \left(\frac{\cos\alpha}{\cos\beta} \right)^4 (1 - k_1 \tan\alpha)^2
\end{aligned} \tag{4.21}
$$

These fields depend on only two independent parameters P and Q. The reason for the disappearance of the third parameter is again to be sought in the freedom to redefine the gauge fields. In fact, if A and $\bar{A}$ are redefined such that the terms linear in the gauge fields in (4.19) are eliminated, the resulting action will depend only on P and Q. Now we can compare the backgrounds in (3.8) and (4.20). In fact, if we put $C = 0$ in (3.8), and identify P and Q of (4.20) as

$$
P = \frac{B}{\sqrt{k_2 k_1}}, \quad Q = A^2 \tag{4.22}
$$

then the two sets of background fields in (3.8) and (4.20) turn out to be the same.

To generate a class of solutions (3.8) with non-zero C, we cosider the gauging of the marginally deformed $SU(2) \otimes SU(2)$ theory obtained by an $O(2,2)$ which twists θ and $\widetilde{t}$.⋆ The action $S'_{(\gamma)}$, chiral symmetries, and chiral currents, $J'_{i\,(\gamma)}$ and $\bar{J}'_{i\,(\gamma)}$, of this theory are obtained from equations (2.29),(2.32) and (2.33) by the replacements $r \rightarrow r+\pi$ and $t \leftrightarrow \widetilde{t}$. The transformation to be gauged is $\delta\theta = 0$, $\delta\widetilde{\theta} = v(z,\bar{z})$, $\delta t = \lambda v(z,\bar{z})$ and $\delta\widetilde{t} = 0$. From (4.12) we get $\partial_{[\theta} w_{\tilde{\theta}]} = -k_1$, $\partial_{[t} w_{\tilde{t}]} = k_2$ which, using (4.10), leads to the following gauged action

$$\begin{aligned} S'^{\text{gauged}}_{(\gamma)} = S'_{(\gamma)} - \frac{1}{2\pi} \int d^2 z \Big[& A\,(\bar{J}'_{1\,(\gamma)} + \lambda\,\bar{J}'_{2\,(\gamma)}) + \bar{A}\,(J'_{1\,(\gamma)} + \lambda\,J'_{2\,(\gamma)}) \\ & - \frac{1}{\Delta} A\bar{A} \Big(k_1 \cos^2\frac{\phi}{2}\,(\cos^2\gamma + k_1 k_2 \sin^2\gamma\,\cos^2\frac{r}{2}) \\ & + \lambda^2\,k_2 \sin^2\frac{r}{2}\,(\cos^2\gamma + k_1 k_2 \sin^2\gamma\,\sin^2\frac{\phi}{2}) \Big) \Big] \end{aligned} \tag{4.23}$$

where,

$$\Delta = \cos^2\gamma + k_1 k_2 \sin^2\gamma \sin^2\frac{\phi}{2} \cos^2\frac{r}{2} \tag{4.24}$$

On fixing the gauge $\widetilde{\theta} = 0$ and integrating out the gauge fields, we obtain the following expressions for the metric, antisymmetric tensor field and the dilaton field†:

⋆ The background fields obtained on gauging the $\theta - t$ twisted model are still of the form (4.20).

† These fields can also be obtained from (4.20) after replacing r by $r+\pi$ and interchanging t and $\widetilde{t}$. This corresponds to gauging a combination of axial and vector $U(1)$'s.

$$
\begin{aligned}
G_{\theta\theta} &= D^{-1}k_1 \sin^2\frac{\phi}{2}[(1+k_1k_2MN^2)\sin^2\frac{r}{2} + M\cos^2\frac{r}{2})\\
G_{tt} &= D^{-1}k_2M\sin^2\frac{r}{2}\cos^2\frac{\phi}{2}\\
G_{\tilde{t}\tilde{t}} &= D^{-1}k_2\cos^2\frac{r}{2}[k_1k_2MN^2\sin^2\frac{\phi}{2} + (1+k_1k_2MN^2)\cos^2\frac{\phi}{2}]\\
G_{\theta\tilde{t}} &= D^{-1}k_1k_2MN\sin^2\frac{\phi}{2}\cos^2\frac{r}{2}\\
G_{\theta t} &= G_{t\tilde{t}} = B_{\theta\tilde{t}} = 0\\
B_{\theta t} &= D^{-1}k_1k_2MN\sin^2\frac{\phi}{2}\sin^2\frac{r}{2}\\
B_{t\tilde{t}} &= D^{-1}k_2M\cos^2\frac{\phi}{2}\cos^2\frac{r}{2}\\
\Phi &= -\ln D
\end{aligned}
\tag{4.25}
$$

where,

$$
\begin{aligned}
&D = k_1k_2MN^2 + \cos^2\frac{\phi}{2} - k_1k_2MN^2\cos^2\frac{r}{2} + (M-1)\cos^2\frac{\phi}{2}\cos^2\frac{r}{2}\\
&M = \frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{\cos^2\gamma - \lambda^2k_2\sin^2\gamma}\,, \qquad N = \lambda/k_1
\end{aligned}
\tag{4.26}
$$

To compare the two sets of background fields in (3.8) and (4.25) (up to an overall shift of the dilaton field), we have to set $B = 0$ and $B_{t\tilde{t}} \to B_{t\tilde{t}} + k_2$ in (3.8) and identify its remaining parameters in terms of the parameters M and N of (4.25) as

$$
M = \frac{A^2}{1-C^2}\,,\ \ N = \frac{C}{A\sqrt{k_2k_1}}
\tag{4.27}
$$

This shows that various classes of solutions of the low energy equations of motion which are obtained by the action of the $O(3,3)$ group on the $\big(SU(2)/U(1)\big)\otimes SU(2)$ coset model, can be constructed as $U(1)$ cosets of the marginally deformed $SU(2)\otimes SU(2)$ WZNW theory.

5. Summary and Conclusion

In this paper we have studied σ-models obtained by $O(d,d)$ transformations of WZNW models and have shown that they correspond to finite marginal deformations of the original WZNW models. These marginal deformations are generated by a product of holomorphic and anti-holomorphic currents belonging to the Cartan subalgebra of the underlying current algebra. We have also studied the gauging of $U(1)$ subgroups of the marginally deformed theory and have shown that the results can be obtained by $O(d,d)$ transformations of the gauged unperturbed WZNW model.

Our analysis provides a way to give a σ-model description of marginally deformed WZNW models and their cosets. The existance of such deformed models was proved to all orders in perturbation theory in [14],where, it was shown that marginal perturbations by products of holomorphic and anti-holomorphic currents from the Cartan subalgebra are integrable. Though, it is easy to write down the form of these models to first order in the perturbation parameter, obtaining the σ-model for a finite deformation is in no way a straightforward task without the help of the $O(d,d)$ transformations.

Although the form of the σ-model actions that we have derived for the deformed WZNW models and their gauging is valid to all orders in the deformation parameter, the expression is valid only to the lowest order in the derivatives (in the target space). In other words, the σ-model β-functions are zero only to one loop order, or in the large k limit. This limitation stems from the fact that the explicit $O(d,d)$ transformation laws of various fields are known only to this order. The existence of an $O(d,d)$ transformation that converts a conformally invariant background to another conformally invariant background has, however, been established to all orders [4] [5]. This, in turn, shows the existence of σ-models which represents deformed WZNW models (and their gauging) to all orders in the σ-model loop expansion.

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