

hep-th/9609176
MRI-PHY/96-28

Unification of String Dualities

Ashoke Sen^{1 2}

*Mehta Research Institute of Mathematics
and Mathematical Physics
10 Kasturba Gandhi Marg, Allahabad 211002, INDIA*

Abstract

We argue that all conjectured dualities involving various string, M - and F - theory compactifications can be ‘derived’ from the conjectured duality between type I and $SO(32)$ heterotic string theory, T-dualities, and the definition of M - and F - theories.

(Based on a talk given at the conference on ‘Advanced Quantum Field Theory’, Le Londe-les-Maures, France, Aug.31-Sept.5, 1996, in memory of Claude Itzykson)

¹On leave of absence from Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

²E-mail: sen@mri.ernet.in, sen@theory.tifr.res.in

1 Introduction

During the last two years, there has been an explosion of conjectures describing new relations between apparently unrelated string theories[1, 2, 3, 4]. The purpose of this article will be not to add new conjectures to this list, but to provide a link between these various duality conjectures. In particular I shall try to identify a minimal set of duality conjectures from which all other duality conjectures can be ‘derived’.

There are five known string theories in ten dimensions which have been named type IIA, type IIB, type I, SO(32) heterotic and $E_8 \times E_8$ heterotic. In order to relate these five theories, we need at least four relations. One of them comes from the observation[5, 6, 7, 8] that the type I theory can be regarded as the quotient of the type IIB theory by the world sheet parity transformation Ω . Two more duality relations follow from the T-duality relations that can be verified order by order in perturbation theory. These can be stated as follows:

- Type IIA theory compactified on a circle of radius R is dual to type IIB theory compactified on a circle of radius $1/R$ [9, 7].
- SO(32) heterotic theory compactified on a circle of radius R , with gauge group broken down to $SO(16) \times SO(16)$ by Wilson line is dual to $E_8 \times E_8$ heterotic theory compactified on a circle of radius $1/R$ with gauge group broken down to $SO(16) \times SO(16)$ by Wilson line[10, 11].

Using these three relations we can divide the five string theories into two families, the type II family which includes the two type II and the type I theories, and the heterotic family that includes the two heterotic string theories. The members within each family are related to each other by T-duality transformations or quotienting. Thus in order to relate all five string theories, we need at least one more relation between these two families. This is provided by the type I - SO(32) duality conjecture which states that[12]

- In ten dimensions type I theory with coupling constant g is dual to SO(32) heterotic string theory with coupling constant $1/g$.

We shall see that all other conjectured dualities among various string theories and their compactifications can be ‘derived’ from these four relations between the five string theories.

Before I go on, I would like to make two cautionary remarks. First of all, we should recall that since duality in general relates a weakly coupled string theory to a strongly coupled string theory, any ‘derivation’ of a duality requires manipulations involving strongly coupled string theory. In the absence of a non-perturbative formulation of string theory, the rules that we must follow while carrying out these manipulations are not known *a priori* and need to be discovered on the way. We shall see that there are a reasonable set of rules that one can follow during these manipulations, using which we can ‘derive’ all known string dualities from a few simple ones.

The second remark that I would like to make is that one should not interpret the result of this paper by regarding these ‘parent’ dualities as more fundamental than any other duality. One of the main lessons that we have learned during the last two years is that no string theory is more fundamental than any other, although some theory might be more suitable than others for describing a specific region of the moduli space. I believe the same is true for string dualities, since there are complicated interconnections between various dualities. Nevertheless it might be useful to identify a minimal set of dualities from which others can be ‘derived’.

The paper will be organized as follows. In section 2 I shall outline the set of rules that we shall be using for ‘deriving’ one duality relation from another. In section 3 we shall see how these rules can be used to derive all conjectured dualities involving conventional string compactification by starting from the $SO(32)$ heterotic - type I duality in ten dimensions. In particular, I shall discuss in some detail the ‘derivation’ of the six dimensional string-string duality between heterotic string theory on T^4 and type IIA string theory on K3, and also of the $SL(2,Z)$ self-duality of type IIB string theory in ten dimensions. Sections 4 and 5 will be devoted to dualities involving compactification of M - and F -theories respectively. During our analysis in sections 3-5 we shall be making use of T-duality symmetries indiscriminately, but in the appendix we shall identify the minimal set of T-dualities from which all other T-duality symmetries can be derived.

2 Rules for Derivation

In this section I shall review the various rules that we shall be following in order to ‘derive’ new duality relations among string theories from a given duality relation. I should mention here that most of the manipulations discussed in this section are valid

in the form stated only for theories for which the local structure of the moduli space is not corrected by quantum effects. This is automatic for theories with sixteen or more supercharges since for these theories supersymmetry determines the local structure of the moduli space completely. Since the detailed application of these methods, discussed in the next three sections, involve theories of this kind only, we are on a firm ground as far as these examples are concerned. For theories with less number of supersymmetries the moduli space gets quantum corrected, and hence even the T-duality symmetries are modified by these corrections[13, 14]. We do expect however that the methods of this section can still be used to determine dualities between such theories, although we shall not be able to reproduce the precise map between the quantum corrected moduli spaces of the two theories by these methods. A notable example of this is the ‘derivation’ of the duality between type IIA theory on a Calabi-Yau manifold and heterotic string theory on $K3 \times T^2$. We shall discuss this briefly in the next section.

2.1 T-Dualities

T-dualities are the best understood duality symmetries in string theory. They hold order by order in string perturbation theory to all orders, and even though they have not been established for full non-perturbative string theory,³ we shall use them indiscriminately during our analysis by assuming that they are symmetries of the full non-perturbative string theory.

It is nevertheless useful to identify a minimal set of T-dualities from which others can be derived. Part of the T-duality symmetries in various string theories can be identified as the global diffeomorphism group of the manifold on which the string theory has been compactified. This is part of the general coordinate transformation in the corresponding string theory before compactification. Similarly part of the T-duality symmetries might be associated with Yang-Mills gauge symmetries of string theory. Validity of these T-duality symmetries in the full non-perturbative string theory is therefore on a firm footing. In the appendix we shall identify the minimal set of T-duality conjectures which need to be added to these gauge symmetries in order to derive the full set of T-duality relations in string theory.

³This point has been emphasized in ref.[15].

2.2 Unifying S - and T -Dualities

Suppose a theory A compactified on a manifold K_A (this might represent a string, M - or F -theory compactification) is known to have a self-duality group G . Let us now further compactify this theory on another manifold \mathcal{M} , and suppose that this theory has a T -duality group H . It is reasonable to assume – and we shall assume – that this theory is also invariant under the duality symmetry group G of the theory before compactification on \mathcal{M} . Quite often one finds that the elements of G and H don't commute with each other and together generate a much larger group \mathcal{G} . From this we can conclude that the full duality symmetry group of theory A compactified on $K_A \times \mathcal{M}$ is at least \mathcal{G} . This method has been successfully used to derive the duality group in many cases. For example, it was used to show[16] that the full duality symmetry group of heterotic string theory compactified on T^7 is $O(8,24;\mathbb{Z})$ which includes the T -duality group $O(7,23;\mathbb{Z})$ of this compactification, as well as the $SL(2,\mathbb{Z})$ S -duality group of heterotic string theory compactified on T^6 . This method has also been used to derive the duality groups of type II theories compactified on T^n from the knowledge of T -duality group of the corresponding compactification, and the $SL(2,\mathbb{Z})$ S -duality group of type IIB theory in ten dimensions[17]. For example type IIB theory compactified on T^4 has a T -duality group $SO(4,4;\mathbb{Z})$. These $SO(4,4;\mathbb{Z})$ transformations do not in general commute with the $SL(2,\mathbb{Z})$ S -duality transformation of the type IIB theory, and together they generate a duality symmetry group $SO(5,5;\mathbb{Z})$ which has been conjectured to be the full duality symmetry group in this theory.

2.3 Duality of Dualities

This is a generalization of the procedure described in the previous subsection to the case of dualities relating two different theories. Suppose theory A on K_A is dual to theory B on K_B . Let us now compactify both theories further on a smooth manifold \mathcal{M} . In the spirit of the previous subsection we shall assume that these two theories are still dual to each other. Let H_A be the T -duality group of the first theory and H_B be the T -duality group of the second theory. Typically H_A and H_B are not isomorphic; in fact quite often the image of H_B in the first theory involves transformations which act non-trivially on the coupling constant. The same is true for the image of H_A in the second theory. Thus H_B (H_A) in general contains information about non-perturbative duality symmetries in the first (second) theory. In particular, the full duality group of the first theory must

include H_A , as well as the image of H_B . By studying the full group generated by H_A and the image of H_B we can quite often determine the full duality group of the theory.

The most well known example of this kind is the ‘derivation’ of S -duality of heterotic string theory on T^6 from string-string duality in six dimensions[18, 12]. We start from the duality between type IIA on K3 and heterotic on T^4 [17], and compactify both theories on T^2 . The first theory has an $SL(2,Z)$ T-duality symmetry associated with the torus compactification of the type IIA theory. Under string-string duality this transformation gets mapped to the S-duality of the heterotic string theory on T^6 . Thus the six dimensional string-string duality conjecture together with the T-duality of type IIA on T^2 implies S-duality of heterotic string theory on T^6 .

2.4 Taking Large Size Limit of Compact Manifolds

Suppose a theory A on K_A is known to have a self duality group \mathcal{G}_A . Let us now try to recover the theory A in flat space-time by taking the large size limit of the manifold K_A . In general, the duality group \mathcal{G}_A will not commute with this limit, since a typical element of the duality group will map a large size K_A to a small or finite size K_A . Thus not all of \mathcal{G}_A will appear as duality symmetry of the theory A in flat space-time. However suppose G_A is the subgroup of \mathcal{G}_A that commutes with this limit; *i.e.* elements of G_A , acting on a large size K_A , produces a large size K_A . In that case we can conclude that G_A is the duality symmetry group of the theory A in flat space-time[19, 20, 21]. (Sometime, if the duality group of the compactified theory contains global diffeomorphism symmetries of the compact manifold K_A , then in the large volume limit it becomes part of the general coordinate transformation of the non-compact manifold, and does not give any new information).

I shall illustrate this idea through an example. Let us consider type IIA and type IIB theories compactified on a circle. These two theories are related by T-duality, and both have duality group $SL(2,Z)$ which act on the moduli fields as

$$\lambda_{IIB} \rightarrow \frac{p\lambda_{IIB} + q}{r\lambda_{IIB} + s}, \quad \Psi \rightarrow \Psi, \quad (2.1)$$

where p, q, r, s are integers satisfying $ps - qr = 1$, λ_{IIB} is a complex scalar and Ψ is a real scalar field. In terms of variables of the type IIB theory, λ_{IIB} and Ψ are given by

$$\lambda_{IIB} = a_{IIB} + ie^{-\Phi_{IIB}/2}, \quad \Psi = \Phi_{IIB} - 8 \ln R_{IIB}, \quad (2.2)$$

where a_{IIB} denotes the scalar arising in the Ramond-Ramond (RR) sector, Φ_{IIB} denotes the ten dimensional dilaton, and R_{IIB} is the radius of the circle measured in the type IIB metric. On the other hand, in terms of the type IIA variables, we have

$$\lambda_{IIB} = (A_{IIA})_9 + ie^{-\Phi_{IIA}/2 + \ln R_{IIA}}, \quad \Psi = \Phi_{IIA} + 6 \ln R_{IIA}, \quad (2.3)$$

where $(A_{IIA})_\mu$ denotes the μ -th component of the RR vector field, x^9 denotes the direction of the circle, Φ_{IIA} is the ten dimensional dilaton of the type IIA theory and R_{IIA} is the radius of the circle measured in the type IIA metric.

Let us now try to recover the ten dimensional type IIB theory by taking the limit $R_{IIB} \rightarrow \infty$ keeping Φ_{IIB} and a_{IIB} fixed. As is clear from (2.2), in this limit λ_{IIB} is fixed, and $\Psi \rightarrow -\infty$. The $SL(2, Z)$ duality transformations described in eq.(2.1) does not affect this limit. Thus we can conclude from this that the $SL(2, Z)$ duality of type IIB theory on S^1 implies $SL(2, Z)$ duality of the ten dimensional type IIB theory.

Let us now try to do the same thing for the type IIA theory by taking the limit $R_{IIA} \rightarrow \infty$ keeping Φ_{IIA} fixed. We see from eq.(2.3) that in this limit $Im(\lambda_{IIB}) \rightarrow \infty$. But an $SL(2, Z)$ transformation of the form given in eq.(2.1) does not preserve this condition; in fact acting on a configuration with large $Im(\lambda_{IIB})$ an $SL(2, Z)$ transformation with $r \neq 0$ will produce a configuration with small $Im(\lambda_{IIB})$. Thus we see that in this case $SL(2, Z)$ duality of type IIA theory compactified on a circle cannot be used to conclude that the type IIA theory in ten dimensions has an $SL(2, Z)$ duality symmetry.

A variant of this idea can also be used for deriving dual pairs of theories. Suppose theory A on K_A is dual to theory B on K_B . If we now take the large size limit of K_A , then typically it will correspond to small or finite size of K_B . However if it so happens that in the limit when the size of K_A approaches infinity, the size of K_B approaches infinity as well, with the coupling constant remaining finite, then we can conclude that the two theories A and B are dual to each other in flat space-time.

2.5 Fiberwise Application of Duality Transformation

Suppose we are given that a theory A on a compact manifold K_A is dual to another theory B on a compact manifold K_B . (More precisely the theory A on $K_A \times R^{n,1}$ is dual to theory B on $K_B \times R^{n,1}$, where $R^{n,1}$ is a Minkowski space of signature $(n, 1)$.) This duality relation automatically comes with a map between the moduli spaces of the two theories, with the moduli specifying the geometry of K_A (K_B) as well as various

background field configurations in the two theories. Let us now consider compactification of the theories A and B on two new manifolds E_A and E_B respectively, obtained by fibering K_A and K_B on another manifold \mathcal{M} . This means that E_A (E_B) is obtained by erecting at every point in \mathcal{M} a copy of K_A (K_B), with the provision that the moduli of K_A (K_B) (including possible background fields that arise upon compactifying A (B) on K_A (K_B)) could vary as we move on \mathcal{M} . Thus if \vec{y} denote the coordinates on \mathcal{M} , and \vec{m}_A and \vec{m}_B denote the coordinates on the moduli spaces of theory A on K_A and theory B on K_B respectively, then specific compactification of A and B on E_A and E_B are specified by the functions $\vec{m}_A(\vec{y})$ and $\vec{m}_B(\vec{y})$ respectively. The original duality between A on K_A and B on K_B then gives rise to a map between the moduli spaces of A on E_A and B on E_B as follows: given a function $\vec{m}_A(\vec{y})$, we construct $\vec{m}_B(\vec{y})$ by applying the original duality map at every point \vec{y} on \mathcal{M} .

It has been argued by Vafa and Witten[22] that the theory A on E_A is dual to the theory B on E_B under this map. The basis of this argument is as follows. As long as the moduli of K_A and K_B vary slowly on \mathcal{M} , near a local neighbourhood of any point P on \mathcal{M} the two manifolds effectively look like $K_A \times R^{n,1}$ and $K_B \times R^{n,1}$ respectively. Thus we can relate the two theories by applying the original duality between A on $K_A \times R^{n,1}$ and B on $K_B \times R^{n,1}$. This argument breaks down when the moduli of K_A and K_B vary rapidly on \mathcal{M} , in particular near singular points on \mathcal{M} where the fiber degenerates. However, in many examples that have been studied, the duality between A on E_A and B on E_B continues to hold even in the presence of such singular points. Perhaps the lesson to be learned from here is that the presence of singular points of ‘measure zero’ does not affect the duality between the two theories. Put another way, the fact that in the bulk of the manifolds E_A and E_B the two theories are equivalent by the original duality relation, forces them to be equivalent even on these singular subspaces of codimension ≥ 1 , even though the original argument breaks down on these subspaces. In any case, we shall henceforth assume that for E_A and E_B constructed this way, A on E_A is dual to B on E_B .

This method was used to ‘derive’ the conjectured dualities[23, 24, 25, 26] between type IIA string theory on a Calabi-Yau manifold and the heterotic string theory on $K3 \times T^2$ from the string-string duality conjecture in six dimensions that relates type IIA on $K3$ and heterotic on T^4 . This is done by representing the Calabi-Yau manifold as $K3$ fibered over CP^1 and the corresponding $K3 \times T^2$ as T^4 fibered over CP^1 .

A special case of this is a pair of Z_2 orbifolds constructed as follows. Suppose we have a dual pair of theories (A on K_A) and (B on K_B). Let us compactify both theories further on a smooth manifold \mathcal{M} , and let us assume that both these theories have a Z_2 symmetry group that are related to each other under the duality map. We can divide the action of the Z_2 transformation into two parts: let s represent the geometric action on the manifold \mathcal{M} which is identical in the two theories, and h_A (h_B) denote the geometric action on the manifold K_A (K_B) as well as internal symmetry transformation in theory A (B). We now compare the two quotient theories

$$(A \text{ on } K_A \times \mathcal{M}/h_A \cdot s) \text{ and } (B \text{ on } K_B \times \mathcal{M}/h_B \cdot s)$$

where by an abuse of notation we have denoted the Z_2 group by its generator $h_A \cdot s$ ($h_B \cdot s$). A little bit of mental exercise shows us that the first theory has the structure of A compactified on a fibered space E_A with base \mathcal{M}/s , and fiber K_A , with twist h_A on the fiber as we go from any point P on \mathcal{M} to $s(P)$. (Note that this is a closed cycle on \mathcal{M}/s .) Similarly the second theory has the structure of B compactified on E_B , where E_B is a fibered space with base \mathcal{M}/s and fiber K_B , h_B being the twist on the fiber as we go from the point P to $s(P)$ on \mathcal{M} . Thus our previous argument will tell us that these two theories are dual to each other. Note that if P_0 denotes a fixed point of s , then at P_0 the fiber degenerates to K_A/h_A (K_B/h_B), and the original duality between A on K_A and B on K_B fails to relate these two quotient theories. But as has been stated before, as long as these are isolated fixed points (hyperplanes) of codimension ≥ 1 we expect the duality between the two resulting theories to hold. This method was used in ref.[27] to construct a dual pair of string theories in four dimensions with N=2 supersymmetry.

The two extreme cases of this construction are:

1. s acts on \mathcal{M} without fixed points. In this case \mathcal{M}/s has no fixed points and the fiber never degenerates. The case for duality between the two resulting theories is on a firmer footing in this case.
2. s acts trivially on \mathcal{M} . In other words s is just the identity transformation. In this case the fiber is everywhere K_A/h_A (K_B/h_B) and there is no reason to expect the resulting theories to be dual to each other. In fact there are many known examples of this kind where the resulting theories are *not* dual to each other.

In principle the above construction should extend to more general orbifolds, but there are interesting subtleties involved in such extensions[28, 29, 30, 31].

3 Minimal Set of String Dualities

I begin by briefly reviewing various known duality conjectures involving conventional string compactifications. In ten dimensions there are two non-trivial duality conjectures: the $SL(2,Z)$ self-duality of type IIB string theory and the duality between type I and $SO(32)$ heterotic string theories. Upon compactifying type IIB or type IIA theory on an n dimensional torus, we get a large duality symmetry group, but it can be derived by combining the T-duality symmetries of type IIA/IIB string theories and the $SL(2,Z)$ self-duality of the ten dimensional type IIB string theory[17]. As we go down in dimensions, we encounter the next non-trivial duality conjecture in six dimensions: the duality between type IIA string theory compactified on $K3$ and heterotic string theory compactified on T^6 , also known as the string-string duality conjecture[17]. As has already been pointed out, starting from this duality conjecture, we can ‘derive’ the S -duality of heterotic string theory on T^6 by mapping it to a T -duality symmetry of type IIA on $K3 \times T^2$. Furthermore, all the conjectured dualities involving type IIA on Calabi-Yau manifolds and heterotic string theory on $K3 \times T^2$ can be recovered from the string-string duality conjecture by representing the Calabi-Yau manifold as $K3$ fibered over CP^1 and $K3 \times T^2$ as T^4 fibered over CP^1 [23, 24, 25, 26]. Finally there are strong-weak coupling duality conjectures involving compactification of $E_8 \times E_8$ heterotic string theory on $K3$ [32]. By using a T-duality transformation one can relate this theory to $SO(32)$ heterotic string theory on $K3$, and the type I - $SO(32)$ heterotic duality in ten dimension relates this further to an orientifold compactification[33]. Under this duality, the strong-weak coupling duality in the original theory can be mapped to a T-duality transformation in the orientifold theory[33]. Thus the strong-weak coupling duality of $E_8 \times E_8$ heterotic string theory on $K3$ can also be ‘derived’ in terms of other duality conjectures.

Thus at this stage it would appear that there are three independent strong-weak coupling duality conjectures involving string compactification:

1. Duality between type I and $SO(32)$ heterotic string theories in ten dimensions.
2. $SL(2,Z)$ self-duality of type IIB string theory in ten dimensions.
3. Duality between type IIA string theory on $K3$ and heterotic string theory on T^4 .

There are many other duality conjectures involving string compactifications, but all of them can be ‘derived’ from these using the procedures outlined in section 2.

We shall now see that

- (3) can be ‘derived’ from (1) and (2), and
- (2) can be ‘derived’ from (1),

using the rules given in section 2. Thus the only independent non-perturbative duality conjecture is that between type I and SO(32) heterotic string theory in ten dimensions.

3.1 Derivation of String-String Duality

The analysis in this subsection will follow closely that in ref.[34]. I begin with a review of the symmetries of type IIB string theory in ten dimensions. This theory has two global Z_2 symmetries which hold order by order in string perturbation theory. The first one involves changing the sign of all Ramond sector states on the left, and is denoted by $(-1)^{F_L}$. Acting on the bosonic fields of the theory it changes the sign of all the fields coming from the RR sector, leaving all the fields in the Neveu-Schwarz-Neveu-Schwarz (NS) sector invariant. The second Z_2 symmetry involves a world-sheet parity transformation, and is denoted by Ω . It changes the sign of the anti-symmetric rank two tensor field $B_{\mu\nu}$ coming from the NS sector, and the scalar a_{IIB} and the rank four anti-symmetric tensor field $D_{\mu\nu\rho\sigma}$ coming from the RR sector, leaving all the other massless bosonic fields invariant.

Besides these perturbative duality symmetries, the ten dimensional type IIB theory also has a conjectured $SL(2,Z)$ symmetry. This transformation leaves the canonical metric invariant and acts on the other massless bosonic fields in the theory as:

$$\lambda_{IIB} \rightarrow \frac{p\lambda_{IIB} + q}{r\lambda_{IIB} + s}, \quad \begin{pmatrix} B'_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix} \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} B'_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix}, \quad (3.1)$$

where p, q, r, s are integers satisfying

$$ps - qr = 1. \quad (3.2)$$

$B'_{\mu\nu}$ is the rank two antisymmetric tensor field arising in the RR sector of the theory, and λ_{IIB} has been defined in eq.(2.2). The special $SL(2,Z)$ transformation generated by

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.3)$$

is denoted by S .

Studying the action of the transformations $(-1)^{F_L}$, Ω and S on the various massless fields in this theory one can show that

$$S(-1)^{F_L}S^{-1} = \Omega. \quad (3.4)$$

We shall now construct a new dual pair of theories following the orbifolding procedure described in section 2. According to this procedure, if theory A on K_A is dual to theory B on K_B , then A on $K_A \times \mathcal{M}/h_A \cdot s$ is dual to B on $K_B \times \mathcal{M}/h_B \cdot s$. We shall choose A on K_A to be the type IIB theory, B on K_B to be the S -transformed type IIB theory, h_A to be $(-1)^{F_L}$, h_B (the image of h_A under S) to be Ω , \mathcal{M} to be a four dimensional torus T^4 labelled by x^m ($6 \leq m \leq 9$) and s to be the transformation \mathcal{I}_4 that transforms x^m to $-x^m$ for ($6 \leq m \leq 9$). Thus we arrive at the duality:

$$(\text{type IIB on } T^4/(-1)^{F_L} \cdot \mathcal{I}_4) \leftrightarrow (\text{type IIB on } T^4/\Omega \cdot \mathcal{I}_4). \quad (3.5)$$

Let us now perform an $R \rightarrow (1/R)$ duality transformation on the x^6 coordinate in the first theory. This is a T -duality transformation that converts the type IIB theory to type IIA theory. It also maps the transformation $(-1)^{F_L} \cdot \mathcal{I}_4$ in the type IIB theory to the transformation \mathcal{I}_4 in the type IIA theory – this can be checked either by studying the action of various transformations on the massless fields of the theory, or by studying their action on the world-sheet fields. Thus the first theory is T -dual to type IIA on $(T^4)'/\mathcal{I}_4$. This is a special point in the moduli space of type IIA on K3.

Next let us perform $R \rightarrow (1/R)$ duality transformation on all the four circles of the second theory. This gives us back a type IIB theory, but maps the transformation $\Omega \cdot \mathcal{I}_4$ to just Ω . Thus the second theory is related by a T -duality transformation to type IIB on $(T^4)''/\Omega$. But as mentioned in the introduction, type IIB modded out by Ω is the type I theory. Thus the second theory is equivalent to type I on $(T^4)''$. By the duality between type I and SO(32) heterotic string theory in ten dimensions, this theory is equivalent to heterotic string theory on $(T^4)''$.

Thus we have arrived at the duality between heterotic string theory on $(T^4)''$ and type IIA string theory on K3, *i.e.* the string-string duality conjecture. Note that although this relation was ‘derived’ at a special point in the moduli space (orbifold limit of K3), once the exact duality between two theories is established at one point in the moduli space, we can go away from this special point by switching on background fields in both theories without destroying the duality. This was done explicitly in ref.[34] for this case. In fact

this analysis automatically provides an explicit map between the moduli spaces of the two theories.

3.2 Derivation of S-Duality of Type IIB String Theory

The strategy that we shall follow for ‘deriving’ this duality is to first study the duality symmetries of type IIB theory compactified on an orientifold $T^2/(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$ where \mathcal{I}_2 denotes the transformation that reverses both directions on the torus, and then take the limit of large size of T^2 to recover the duality symmetries of the ten dimensional type IIB string theory. This model was studied in detail in ref.[35]. In this case the compact manifold T^2/\mathcal{I}_2 has the structure of a tetrahedron, and each of the four vertices (representing a seven dimensional hyperplane) of the tetrahedron acts as a source of -4 units of RR charge associated with the nine form RR field strength. Since T^2/\mathcal{I}_2 is a compact manifold, this RR charge must be cancelled by putting sixteen Dirichlet 7-branes transverse to T^2/\mathcal{I}_2 . The positions of these seven branes act as extra moduli for this compactification.

It was argued in ref.[35] that non-perturbative quantum corrections modify the geometry of the compact manifold, and also produces a background λ_{IIB} field (defined in eq.(2.2)) that varies on the compact manifold in a complicated manner. However, these complications do not arise in one special case, when each vertex of the tetrahedron contains four seven branes. In this case the geometry of the tetrahedron is flat everywhere except for conical deficit angle π at each of the four vertices, and the field λ_{IIB} is constant on the tetrahedron. This theory has an unbroken gauge group $SO(8)^4$, with one $SO(8)$ factor associated with each vertex. For this configuration, the moduli labelling this compactification are the volume V_{IIB} of T^2 measured in the type IIB metric, the complex structure modulus τ_{IIB} of T^2 , and λ_{IIB} . There is no background $B_{\mu\nu}$ or $B'_{\mu\nu}$ field since they are odd under $(-1)^{F_L} \cdot \Omega$.

We now make the following set of duality transformations:

- We first make an $R \rightarrow (1/R)$ duality transformation on both circles of T^2 . This takes the type IIB theory to a type IIB theory, but converts the transformation $(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$ to Ω [36, 35]. If we denote the new torus by $(T^2)'$, then the resulting theory is type IIB on $(T^2)'/\Omega$, *i.e.* type I theory on $(T^2)'$. Note that the gauge group is broken to $SO(8)^4$ by Wilson lines.

- Using the duality between type I and SO(32) heterotic string theories, we can now regard this theory as heterotic string theory compactified on $(T^2)'$, with the gauge group broken to $SO(8)^4$ by Wilson lines.

In the heterotic description we shall choose as moduli the complex structure τ_{het} of $(T^2)'$, and,

$$\rho_{het} = B_{89}^{(het)} + iV_{het}, \quad (3.6)$$

$$\Psi_{het} = \Phi_{het} - \ln V_{het}, \quad (3.7)$$

where $B_{\mu\nu}^{(het)}$ is the rank two anti-symmetric tensor field of the heterotic string theory, V_{het} is the volume of $(T^2)'$ measured in the heterotic metric, and Φ_{het} is the dilaton of the ten dimensional heterotic string theory. By following the duality relations given in ref.[12] one can find the map between the moduli in the type IIB description and the heterotic description. They are as follows:

$$\begin{aligned} \tau_{het} &= \tau_{IIB}, \\ \rho_{het} &= \lambda_{IIB}, \\ e^{\Psi_{het}} &= (e^{-\frac{\Phi_{IIB}}{4}} V_{IIB})^2. \end{aligned} \quad (3.8)$$

In the heterotic description the full $O(18,2;Z)$ T-duality group of the theory has a subgroup $SL(2,Z) \times SL(2,Z)'$ which acts on the moduli as

$$\tau_{het} \rightarrow \frac{p'\tau_{het} + q'}{r'\tau_{het} + s'}, \quad \rho_{het} \rightarrow \frac{p\rho_{het} + q}{r\rho_{het} + s}, \quad \Psi_{het} \rightarrow \Psi_{het}, \quad (3.9)$$

without affecting the Wilson lines. Here $p, q, r, s, p', q', r', s'$ are integers satisfying $ps - qr = 1$ and $p's' - q'r' = 1$. This implies the following duality transformations on the type IIB moduli:

$$\tau_{IIB} \rightarrow \frac{p'\tau_{IIB} + q'}{r'\tau_{IIB} + s'}, \quad \lambda_{IIB} \rightarrow \frac{p\lambda_{IIB} + q}{r\lambda_{IIB} + s}, \quad e^{-\frac{\Phi_{IIB}}{4}} V_{IIB} \rightarrow e^{-\frac{\Phi_{IIB}}{4}} V_{IIB}. \quad (3.10)$$

We now want to take the limit $V_{IIB} \rightarrow \infty$ keeping τ_{IIB} and λ_{IIB} fixed, and see which part of the duality symmetries survive in this limit.⁴ From eq.(3.10) we see that since $e^{-\Phi_{IIB}/2} \equiv Im(\lambda_{IIB})$ is finite before and after the transformation, an $SL(2,Z) \times SL(2,Z)'$

⁴In this limit we have $SO(8)$ Yang-Mills gauge theory living at each of the four vertices, but the theory in the bulk is just the ten dimensional type IIB string theory. Thus the surviving duality group must reflect a symmetry of the ten dimensional type IIB string theory.

transformation takes large V_{IIB} configuration to large V_{IIB} configuration. Thus the full $SL(2,Z) \times SL(2,Z)'$ duality group survives in this limit. Of these $SL(2,Z)'$ can be identified as the global diffeomorphism group of T^2/\mathcal{I}_2 , and hence becomes part of the general coordinate transformation in the ten dimensional type IIB theory. The only non-trivial duality group in this limit is then $SL(2,Z)$, which can easily be identified as the $SL(2,Z)$ S-duality group of the ten dimensional type IIB theory.

This finishes our ‘derivation’ of the S-duality of type IIB theory from the $SO(32)$ heterotic - type I duality in ten dimensions, and the T-duality symmetries. One can also find out the $SL(2,Z)$ transformation property of the ten dimensional metric from this analysis. From eq.(3.10) we see that $\exp(-\Phi_{IIB}/4)V_{IIB}$ is invariant under this $SL(2,Z)$ transformation. In other words, the volume of T^2 measured in the metric $\exp(-\Phi_{IIB}/4)G_{\mu\nu}^{(IIB)}$ is invariant. Thus $\exp(-\Phi_{IIB}/4)G_{\mu\nu}^{(IIB)}$ remains invariant under the $SL(2,Z)$ transformation.

The net result of the analysis in this section can be stated as follows:

All conjectured non-perturbative duality symmetries involving conventional string theory compactifications can be ‘derived’, according to the rules given in section 2, from the duality between type I and $SO(32)$ heterotic string theories in ten dimensions.

4 Compactifications involving M -theory

M -theory has been proposed as the eleven (10+1) dimensional theory which arises in the strong coupling limit of the type IIA string theory[37, 12]. The low energy limit of M -theory is the 11 dimensional supergravity theory with $N = 1$ supersymmetry. More precisely, M -theory compactified on a circle of radius R , measured in the M -theory metric, is given by type IIA theory at coupling constant

$$e^{\langle\Phi_{IIA}\rangle/2} \equiv g_{IIA} = R^{3/2}. \quad (4.1)$$

The metric $G_{\mu\nu}^{(M)}$ ($0 \leq \mu, \nu \leq 9$) of M -theory is related to the metric $G_{\mu\nu}^{(IIA)}$ of the type IIA theory via the relation

$$G_{\mu\nu}^{(M)} = R^{-1}G_{\mu\nu}^{(IIA)}. \quad (4.2)$$

At present this is the only known way of defining M -theory. Thus we shall take this to be the definition of M -theory and use this to ‘derive’ other duality conjectures involving M -theory. Note however that there is a non-trivial ingredient that has gone into this definition of M -theory, – namely that the theory defined this way has an eleven dimensional

general coordinate invariance. In particular, if we compactify type IIA theory on a circle, then this would correspond to M -theory on a two dimensional torus, and there should be a symmetry that exchanges the two circles of the torus. It has been shown by Schwarz[38] and by Aspinwall[39] that this exchange symmetry can be traced back to the $SL(2,Z)$ self-duality of the ten dimensional type IIB theory. This $SL(2,Z)$ transformation commutes with the decompactification limit in which the size of the two dimensional torus, measured in the M -theory metric, goes to infinity. Thus this definition of M -theory as the strong coupling limit of type IIA theory is consistent with eleven dimensional general coordinate invariance.

Let us now suppose we have M -theory compactified on some manifold K . In order to define this theory, we can compactify the theory further on a circle of radius R , identify this to type IIA theory on K , and then take the limit $R \rightarrow \infty$ in order to recover M -theory on K . More precisely, suppose $G_{mn}^{(IIA)}$ denotes the metric on K in the type IIA metric. Then the limit we want to take is

$$\lim_{R \rightarrow \infty} (\text{IIA on } K \text{ with } g_{IIA} = R^{3/2}, G_{mn}^{(IIA)} = RG_{mn}^{(M)}), \quad (4.3)$$

with $G_{mn}^{(M)}$ fixed.

If K has the structure of S^1 fibered over some other manifold \mathcal{M} , there is a direct approach to finding a dual of this compactification. By fiberwise application of the equivalence between M -theory on S^1 and type IIA string theory, we can replace M -theory on K by type IIA on \mathcal{M} , with the coupling constant of the type IIA theory varying on \mathcal{M} according to the variation of the radius of S^1 on \mathcal{M} in the fibration. Implicitly, this corresponds to applying the U-duality transformation given in (2.1), (2.3) of the type IIA theory on S^1 fiberwise in the definition (4.3).

All conjectured dualities involving M -theory can be ‘derived’ from these relations. I shall illustrate this through two examples.

4.1 M -theory on T^5/Z_2

It has been conjectured by Dasgupta and Mukhi[40] and by Witten[41] that M -theory on T^5/Z_2 is dual to type IIB string theory compactified on $K3$. Here Z_2 is generated by the transformation \mathcal{J}_5 which changes the sign of all the five coordinates of T^5 and at the same time changes the sign of the rank three anti-symmetric tensor field C_{MNP} that appears in

M -theory. We shall see how we can ‘derive’ this conjecture from the principles outlined before. Our discussion will be brief since the details have been given in ref.[42].

By our analysis in section 2, T^5/Z_2 has the structure of S^1 fibered over T^4/Z_2 . Thus we can directly identify this theory to a type IIA compactification on T^4/Z_2 by replacing M -theory on S^1 by type IIA theory fiberwise. By knowing the action of the Z_2 transformation \mathcal{J}_5 on the massless fields of the M -theory, we can find its action on the massless fields in type IIA theory. The net result of this analysis is that it corresponds to the transformation $(-1)^{F_L} \cdot \mathcal{I}_4$, where \mathcal{I}_4 denotes the reversal of sign of all four coordinates on T^4 . Thus M -theory on T^5/\mathcal{J}_5 is dual to type IIA on $T^4/(-1)^{F_L} \cdot \mathcal{I}_4$. An $R \rightarrow (1/R)$ duality transformation on one of the circles of T^4 converts the type IIA theory to type IIB theory, and at the same time converts the transformation $(-1)^{F_L} \cdot \mathcal{I}_4$ to \mathcal{I}_4 . This establishes the duality between M -theory on T^5/Z_2 and type IIB on T^4/\mathcal{I}_4 , which is simply the orbifold limit of type IIB on $K3$. As before, we can go away from the orbifold limit by switching on background fields in both theories; and the duality between these two theories at one point in the moduli space will continue to guarantee their duality at all other points.

This principle can be used to ‘derive’ many other duality conjectures involving M -theory[42]. A notable example is the duality between M -theory on $(K3 \times S^1)/Z_2$ [43] and the Dabholkar-Park orientifold[44].

4.2 M -theory on S^1/Z_2

It has been conjectured by Horava and Witten[45] that this orbifold compactification of M -theory is dual to $E_8 \times E_8$ heterotic string theory. The Z_2 transformation \mathcal{J}_1 changes the sign of the coordinate labelling the circle, and also the sign of the rank three anti-symmetric tensor field C_{MNP} of M -theory. If r is the radius of S^1 and $g_{E_8 \times E_8}$ is the coupling constant of the $E_8 \times E_8$ heterotic string theory, then they are related as

$$g_{E_8 \times E_8} = r^{3/2} . \quad (4.4)$$

Furthermore the M -theory metric $G_{\mu\nu}^{(M)}$ and the $E_8 \times E_8$ metric $G_{\mu\nu}^{(E_8 \times E_8)}$ are related as

$$G_{\mu\nu}^{(M)} = r^{-1} G_{\mu\nu}^{(E_8 \times E_8)} . \quad (4.5)$$

We shall try to ‘derive’ this duality from other known duality conjectures, and the definition of M -theory. Since S^1/Z_2 does not have the structure of S^1 fibered over another manifold, we need to use the original definition of M -theory compactification as a limit

of type IIA compactification. This does not require the manifold of compactification to have S^1 fibration.

The manipulations that we are going to carry out are all given explicitly in ref.[45], we shall only give a slightly different interpretation of the results by running the argument backwards. According to eq.(4.3) we define M -theory on S^1/Z_2 with radius of S^1 given by r as

$$\lim_{R \rightarrow \infty} (\text{IIA on } S^1/Z_2 \text{ with } g_{IIA} = R^{3/2}, r_{IIA} = R^{1/2}r). \quad (4.6)$$

By studying the action of the Z_2 transformation \mathcal{J}_1 on the massless fields of the M -theory, one finds that in the type IIA theory it corresponds to the transformation \mathcal{J}'_1 which simultaneously changes the sign of the coordinate labelling S^1 and induces a world-sheet parity transformation.⁵ This particular compactification of type IIA theory is known as type I' theory, or more generally an orientifold[5, 6, 7, 8]. This has the property that the two fixed points (which are really 8 dimensional hyperplanes) on S^1 , representing boundaries of S^1/Z_2 if we regard S^1/Z_2 as a real line segment, act as source of RR 10-form field strength. In particular, each of these planes carry -8 units of RR charge. This charge needs to be cancelled by putting 16 Dirichlet 8-branes, each carrying $+1$ unit of RR charge, transverse to S^1/Z_2 . The positions of these D -branes on S^1/Z_2 are arbitrary, and constitute extra moduli in this compactification of type IIA theory.

Thus we are now faced with the following question. Where shall we place these D -branes as we take the limit (4.6)? Naively one would think that if θ denotes the coordinate on S^1 normalized so as to have periodicity 2π , then we place the D -branes at arbitrary but fixed values of θ and then take the large radius limit of S^1 . However, this would mean that in the $R \rightarrow \infty$ limit the distance between the D -branes and the boundaries of S^1/Z_2 increases. Following the analysis of Polchinski and Witten[46] one can show that during this process we hit singularities at finite values of R since the dilaton of the type IIA theory blows up at some point on S^1/Z_2 . The only way to avoid this singularity is to keep eight of the D -branes at $\theta = 0$ and eight of the D -branes at $\theta = \pi$. In this case the dilaton is constant on S^1/Z_2 and we can take the $R \rightarrow \infty$ limit in (4.6) without encountering any singularities. Thus we must define M -theory on S^1/Z_2 through this limit.

⁵Note that in this case the S^1 of M -theory compactification, on which Z_2 acts, represents a physical circle on which the type IIA theory is compactified. If on the other hand this had represented the internal circle used in identifying M -theory on S^1 with type IIA string theory, then the same Z_2 transformation \mathcal{J}_1 would have corresponded to the transformation $(-1)^{F_L}$ in type IIA theory as in the example studied in the previous subsection.

For this configuration of D -branes the type I' theory has an $\text{SO}(16)\times\text{SO}(16)$ gauge symmetry, with the two $\text{SO}(16)$ factors living at the two boundaries. We shall now make the following series of duality transformations:

1. First we make an $r_{IIA} \rightarrow r_{IIA}^{-1}$ duality transformation to map the type IIA theory into a type IIB theory. This maps the transformation \mathcal{J}'_1 into the world-sheet parity transformation Ω of the type IIB theory. The resulting theory is type IIB on S^1 modded out by Ω , which is just the type I theory on S^1 . Since the unbroken gauge symmetry is $\text{SO}(16)\times\text{SO}(16)$, the $\text{SO}(32)$ gauge symmetry of type I is broken to $\text{SO}(16)\times\text{SO}(16)$ by the presence of Wilson line.
2. Using the duality between type I and $\text{SO}(32)$ heterotic string theories, we map this theory to $\text{SO}(32)$ heterotic string theory compactified on S^1 , with the gauge group $\text{SO}(32)$ broken down to $\text{SO}(16)\times\text{SO}(16)$ by the presence of Wilson line.
3. We now make an $r_{\text{SO}(32)} \rightarrow (r_{\text{SO}(32)})^{-1}$ T-duality transformation. This transforms the $\text{SO}(32)$ heterotic string theory to the $E_8 \times E_8$ heterotic string theory compactified on S^1 , with the gauge group still broken to $\text{SO}(16)\times\text{SO}(16)$ due to the presence of Wilson line.

Throughout these set of duality transformations we can compute the coupling constant, and the radius of S^1 in various theories according to the formulae given in ref.[12]. This gives the following set of equivalent definitions of M -theory on S^1/Z_2 :

$$\begin{aligned}
& \lim_{R \rightarrow \infty} (\text{type IIA on } S^1/Z_2 \text{ with } g_{IIA} = R^{3/2}, r_{IIA} = R^{1/2}r) \\
\equiv & \lim_{R \rightarrow \infty} (\text{type I on } S^1 \text{ with } g_I = Rr^{-1}, r_I = R^{-1/2}r^{-1}) \\
\equiv & \lim_{R \rightarrow \infty} (\text{SO}(32) \text{ heterotic on } S^1 \text{ with } g_{\text{SO}(32)} = R^{-1}r, r_{\text{SO}(32)} = R^{-1}r^{-1/2}) \\
\equiv & \lim_{R \rightarrow \infty} (E_8 \times E_8 \text{ heterotic on } S^1 \text{ with } g_{E_8 \times E_8} = r^{3/2}, r_{E_8 \times E_8} = Rr^{1/2}). \quad (4.7)
\end{aligned}$$

By examining eq.(4.7) we see for each of the first three theories, either the coupling constant becomes strong, or the radius of the circle becomes small in the $R \rightarrow \infty$ limit. Thus $R \rightarrow \infty$ does not correspond to a simple limit in these theories. However, for the $E_8 \times E_8$ theory, the $R \rightarrow \infty$ limit corresponds to large radius limit of the theory at *fixed value* of the $E_8 \times E_8$ coupling constant. Thus in this limit the theory reduces to ten dimensional $E_8 \times E_8$ heterotic string theory.

Recall that we are sitting at the point in the moduli space where $E_8 \times E_8$ is broken down to $SO(16) \times SO(16)$. Thus we have a constant background gauge field configuration. In order to break $E_8 \times E_8$ to $SO(16) \times SO(16)$ we need a fixed amount of Wilson line, *i.e.* a fixed amount of $A_9^I \cdot r_{E_8 \times E_8}$, where A_μ^I ($1 \leq I \leq 16$) denote the gauge fields in the Cartan subalgebra of $E_8 \times E_8$.⁶ Since in the limit of large R , $r_{E_8 \times E_8}$ also becomes large, we get $A_9^I \rightarrow 0$. In other words in this limit the background gauge field goes to zero, and locally we recover the $E_8 \times E_8$ heterotic string theory without any background gauge field.

This establishes the duality between M -theory on S^1/Z_2 and the $E_8 \times E_8$ heterotic string theory.⁷ By carefully following this argument we can also identify the origin of the gauge symmetry in M -theory. For this we note that in identifying M -theory on S^1/Z_2 with a particular limit of type IIA theory on S^1/Z_2 , we have identified the manifolds S^1/Z_2 in the two theories. In particular the two boundaries of S^1/Z_2 in the M -theory compactification gets mapped to the two boundaries of S^1/Z_2 in the type IIA compactification. Since the two $SO(16)$ factors in the type IIA compactification have their origin at the two boundaries of S^1/Z_2 , the same must be the case in M -theory. Enhancement of $SO(16)$ to E_8 is a non-perturbative phenomenon from the type IIA / M -theory point of view that cannot be understood directly in these theories in a simple manner.

Finally, I would like to mention that the duality between M -theory on T^5/Z_2 and type IIB string theory on K3, discussed in the previous subsection, can also be established this way.

⁶We are using a coordinate system in which $G_{99}^{(E_8 \times E_8)} = 1$, where x^9 is the coordinate on the circle.

⁷In order to avoid misuse of this procedure, I would like to emphasize again that for this procedure to work we must ensure that the large radius limit on the two sides match. As an example let us consider the duality between M -theory on $(K3 \times S^1)/Z_2$ [43] and the Dabholkar-Park orientifold[44] that can be described as type IIB on $K3/Z_2'$. By compactifying the first theory on another circle $(S^1)'$ we get type IIA on $(K3 \times S^1)/Z_2$. An $R \rightarrow (1/R)$ duality transformation on S^1 converts this to type IIB on $(K3 \times (S^1)'')/Z_2''$. It is easy to verify that this Z_2'' does not act on $(S^1)''$, and in fact has the same action as the Z_2' used in the construction of ref.[44]. From this one might be tempted to conclude that this ‘proves’ the duality between M -theory on $(K3 \times S^1)/Z_2$ and type IIB on $K3/Z_2'$. However, by following the duality relations carefully one finds that the limit where the radius of $(S^1)'$ goes to infinity keeping the volume of $K3$ and S^1 measured in the M -theory metric fixed, does not correspond to taking the radius of $(S^1)''$ to infinity keeping the volume of $K3$ measured in the type IIB metric and the type IIB coupling constant fixed. Thus this analysis *does not* prove the duality between these two theories, although there is a more involved argument[42] which does establish this duality.

5 Compactification involving F -theory

In conventional compactification of type IIB theory, the complex field λ_{IIB} is constant on the internal manifold. F -theory is a way of compactifying type IIB theory which does not suffer from this constraint[47]. The starting point in an F -theory compactification is an elliptically fibered manifold \mathcal{M} which is constructed by erecting at every point on a base manifold B a copy of the two dimensional torus T^2 , with the moduli of T^2 varying over the base in general. Thus if \vec{z} denotes the coordinate on the base B and τ denotes the complex structure modulus of the torus, then for a given \mathcal{M} we have a function $\tau(\vec{z})$ specifying the variation of T^2 on B . By definition[47], F -theory on \mathcal{M} is type IIB string theory compactified on B , with,

$$\lambda_{IIB}(\vec{z}) = \tau(\vec{z}). \quad (5.1)$$

An example of such a manifold is elliptically fibered K3, which can be viewed as T^2 fibered over CP^1 with appropriate $\tau(z)$. Thus F -theory compactification on this manifold corresponds to type IIB compactification on CP^1 with appropriately varying $\lambda_{IIB}(z)$. This theory was conjectured to be dual to heterotic string theory compactified on T^2 [47]. By applying this duality conjecture fiberwise on Calabi-Yau manifolds admitting K3 fibration, many new duality conjectures relating F -theory compactification on Calabi-Yau manifolds and different heterotic compactifications have been ‘derived’[48].

We shall now see how the original duality conjecture between heterotic string theory on T^2 and F -theory on elliptically fibered K3 manifold can be ‘derived’ using the methods of section 2. Since this has been discussed in detail in ref.[35] our discussion will be very brief. We go to the T^4/\mathcal{I}_4 orbifold limit of K3 where \mathcal{I}_4 denotes the reversal of sign of all four coordinates on the torus. Reexpressing T^4/\mathcal{I}_4 as $(T^2 \times (T^2)')/\mathcal{I}_2 \cdot \mathcal{I}'_2$, we see that T^4/\mathcal{I}_4 can be regarded as T^2 fibered over $(T^2)'/\mathcal{I}'_2$, with a twist \mathcal{I}_2 on the fiber as we move along a closed cycle around a fixed point on $(T^2)'/\mathcal{I}'_2$. By definition F -theory compactified on T^4/\mathcal{I}_4 can then be regarded as type IIB theory compactified on $(T^2)'/\sigma \cdot \mathcal{I}'_2$ where σ represents the $SL(2,Z)$ transformation $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$. By studying the action of this $SL(2,Z)$ transformation on the massless fields in the theory given in eq.(3.1) one can identify this transformation to $(-1)^{F_L} \cdot \Omega$. Thus F -theory on T^4/\mathcal{I}_4 can be identified to type IIB on $(T^2)' / (-1)^{F_L} \cdot \Omega \cdot \mathcal{I}'_2$. As was shown in section 3.2 this theory in turn is dual to heterotic string theory on T^2 . Thus we see that F -theory on K3 in the orbifold limit is dual to

heterotic string theory on T^2 . Once the duality is established at one point in the moduli space, it is guaranteed to hold at all other points. This has been discussed explicitly in ref.[35]

This procedure of ‘deriving’ duality conjectures involving F -theory has been used to ‘derive’ many other duality conjectures involving F -theory and orientifolds[49, 28, 29, 30]. Indeed, all conjectured dualities involving F -theory can be ‘derived’ either using this procedure, or by fiberwise application of the duality between F -theory on elliptically fibered K3 and heterotic string theory on T^2 .

6 Conclusion

To summarise, we have shown that all conjectured dualities involving string, M - and F - theories and their compactifications can be ‘derived’ from the duality between type I and $SO(32)$ heterotic string theories in ten dimensions and the definitions of M - and F -theories. The set of rules that we have followed during these ‘derivations’ are outlined in section 2. Some of them are intuitively more obvious than others. The least obvious of these rules is the fiberwise application of duality discussed in section 2.5. This is mainly based on the assumption that if we can argue equivalence between two string compactifications in the bulk of the compact manifold, then it automatically forces the two theories to be the same on boundaries / singular submanifolds of codimension ≥ 1 where there is no direct argument for the equivalence between the two theories. Perhaps the success of this procedure itself teaches us something deep about non-perturbative string theory which we have not yet been able to uncover.

Acknowledgement: I wish to thank the organisers of the conference for providing a very stimulating environment. I would like to thank J. Blum, K. Dasgupta, R. Gopakumar, A. Hanany, C. Hull, K. Intriligator, S. Mukhi, M. Ruiz-Altaba, C. Vafa and E. Witten for useful discussions at various stages of this work.

A Minimal set of T-duality conjectures

In our analysis we have used T-duality symmetries of various string theories indiscriminately. We shall now try to identify the minimal set of T-duality conjectures from which all T-duality symmetries can be derived following the principles outlined in section 2.

We shall begin with type II theories. Upon compactification on S^1 , type IIA and type IIB theories transform into each other under the $R \rightarrow (1/R)$ duality transformations[9, 7]. This duality cannot be attributed to any gauge or general coordinate transformation and must be added to the list of input conjectures. Let us now consider type IIA and type IIB theories compactified on T^2 . Since these two theories compactified on S^1 are dual to each other, they will be dual to each other upon compactification on T^2 as well. But besides this, both these theories have $SL(2,Z) \times SL(2,Z)'$ self-duality symmetries. In the type IIA theory $SL(2,Z)$ can be identified to the global diffeomorphism symmetry of the torus and hence is part of the general coordinate transformation of the ten dimensional type IIA theory. $SL(2,Z)'$ on the other hand acts as modular transformation on the (complexified) Kahler modulus of the torus and has no simple geometric interpretation. In type IIB theory their roles get reversed. Now $SL(2,Z)'$ can be identified to the global diffeomorphism symmetry of the torus and hence is part of the general coordinate transformation of the ten dimensional type IIB theory. $SL(2,Z)$ acts as modular transformation on the (complexified) Kahler modulus of the torus and has no simple geometric interpretation. Thus if we believe in general coordinate invariance of type IIA and type IIB theories, then the full T-duality symmetries of type IIA/IIB theories on T^2 follow as a consequence of the duality between type IIA and type IIB theories compactified on S^1 . It can be shown that all the T-duality symmetries of type IIA/IIB theories compactified on T^n can be derived in a similar manner.

Let us now turn to T-dualities involving heterotic string theory. We start in nine dimensions by compactifying both the heterotic string theories on S^1 . In this case there is an $R \rightarrow (1/R)$ duality transformation that relates the $SO(32)$ heterotic string theory with gauge group broken to $SO(16) \times SO(16)$ via Wilson line to $E_8 \times E_8$ heterotic string theory with gauge group broken to $SO(16) \times SO(16)$ via Wilson line. Furthermore, in the absence of Wilson lines each theory has an $R \rightarrow (1/R)$ self-duality symmetry; this transformation gets modified in the presence of Wilson lines. However, it can be shown that the $R \rightarrow (1/R)$ self-duality in the $E_8 \times E_8$ heterotic string theory gets mapped to an $SO(32)$ gauge transformation in the $SO(32)$ heterotic string theory under the map that relates the two heterotic string theories on S^1 . This is done as follows. While viewing the nine dimensional theory as a compactification of $E_8 \times E_8$ ($SO(32)$) heterotic string theory, it is natural to regard the $O(17,1)$ lattice as $\Gamma_8 \oplus \Gamma_8 \oplus \Gamma_{1,1}$ ($\Gamma_{16} \oplus \Gamma'_{1,1}$). The $O(17,1)$ rotation that relates these two lattices was constructed explicitly in ref.[11].

Now, the $R \rightarrow (1/R)$ self-duality of the $E_8 \times E_8$ theory exchanges the two basis vectors of $\Gamma_{1,1}$. By using the $O(17,1)$ rotation constructed in ref.[11] one can show that exchanging the two basis vectors of $\Gamma_{1,1}$ corresponds to a transformation that leaves $\Gamma'_{1,1}$ invariant and acts only on Γ_{16} . Thus it must correspond to a gauge transformation in $SO(32)$. Exactly similar analysis shows that the $R \rightarrow (1/R)$ self-duality of the $SO(32)$ heterotic string theory compactified on S^1 gets mapped to a gauge transformation in the $E_8 \times E_8$ theory. Hence gauge invariance of the two heterotic string theories, together with the duality between them upon compactification on S^1 , automatically implies self-duality of both of them on S^1 . This result is also supported by the fact that this self-duality can be regarded as part of the $SU(2)$ enhanced gauge symmetry that appears at the self-dual point[9].

Compactification of either of these heterotic string theories on T^n has a T-duality group $O(16+n, n; Z)$, which can be similarly derived by combining the duality symmetries of these theories on S^1 , and gauge and general coordinate invariance. Thus the only T-duality that needs to be added to the list of independent duality conjectures is the duality between $SO(32)$ heterotic and $E_8 \times E_8$ heterotic string theories on S^1 .

It has also been argued recently that all mirror symmetries involving compactification on Calabi-Yau and K3 manifolds can be understood as a result of fiberwise application of $R \rightarrow (1/R)$ duality transformations[50, 51]. Thus these can also be traced to the duality between the two type II and the two heterotic string theories compactified on S^1 .

References

- [1] A. Sen, Int. J. Mod. Phys. **A9** (1994) 3707 [hep-th/9402002] and references therein.
- [2] J. Schwarz, hep-th/9607067 and references therein.
- [3] J. Polchinski, hep-th/9607050 and references therein.
- [4] M. Dine, hep-th/9609051.
- [5] A. Sagnotti, 'Open Strings and their Symmetry Groups', Talk at Cargese Summer Inst., 1987;
 G. Pradisi and A. Sagnotti, Phys. Lett. **B216** (1989) 59;
 M. Bianchi, G. Pradisi and A. Sagnotti, Nucl. Phys. **B376** (1992) 365.

- [6] P. Horava, Nucl. Phys. **B327** (1989) 461, Phys. Lett. **B231** (1989) 251.
- [7] J. Dai, R. Leigh, and J. Polchinski, Mod. Phys. Lett. **A4** (1989) 2073;
R. Leigh, Mod. Phys. Lett. **A4** (1989) 2767;
J. Polchinski, Phys. Rev. **D50** (1994) 6041 [hep-th/9407031].
- [8] E. Gimon and J. Polchinski, Phys. Rev. **D54** (1996) 1667 [hep-th/9601038];
J. Polchinski, S. Choudhuri and C. Johnson, hep-th/9602052;
and references therein.
- [9] M. Dine, P. Huet and N. Seiberg, Nucl. Phys. **B322** (1989) 301.
- [10] K. Narain, Phys. Lett. **B169** (1986) 61;
K. Narain, M. Sarmadi and E. Witten, Nucl. Phys. **B279** (1987) 369.
- [11] P. Ginsparg, Phys. Rev. **D35** (1987) 648.
- [12] E. Witten, Nucl. Phys. **B443** (1995) 85 [hep-th/9503124].
- [13] B. deWitt, V. Kaplunovsky, J. Louis and D. Lust, Nucl. Phys. **B451** (1995) 53
[hep-th/9504006].
- [14] I. Antoniadis, S. Ferrara, E. Gava, K. Narain and T. Taylor, Nucl. Phys. **B447** (1995)
35 [hep-th/9504034].
- [15] A. Girardello, M. Porrati and A. Zaffaroni, hep-th/9608056.
- [16] A. Sen, Nucl. Phys. **B434** (1995) 179 [hep-th/9408083].
- [17] C. Hull and P. Townsend, Nucl. Phys. **B438** (1995) 109 [hep-th/9410167].
- [18] M. Duff, Nucl. Phys. **B442** (1995) 47 [hep-th/9501030].
- [19] P. Aspinwall and D. Morrison, Phys. Lett. **B355** (1995) 141 [hep-th/9505025].
- [20] M. Duff, J. Liu and J. Rahmfeld, Nucl. Phys. **B459** (1996) 125 [hep-th/9508094].
- [21] C. Hull, Nucl. Phys. **B468** (1996) 113 [hep-th/9512181].
- [22] C. Vafa and E. Witten, hep-th/9507050.

- [23] S. Kachru and C. Vafa, Nucl. Phys. **B450** (1995) 69 [hep-th/9605105].
- [24] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. **B357** (1995) 313 [hep-th/9506112].
- [25] G. Aldazabal, A. Font, L. Ibanez and F. Quevedo, Nucl. Phys. **B461** (1996) 537.
- [26] B. Hunt and R. Schimmrigk, Phys. Lett. **B381** (1996) 427 [hep-th/9512138];
B. Hunt, M. Lynker and R. Schimmrigk, hep-th/9609082.
- [27] S. Ferrara, J. Harvey, A. Strominger and C. Vafa, Phys. Lett. **B361** (1995) 59 [hep-th/9505162].
- [28] J. Blum and A. Zaffaroni, hep-th/9607019.
- [29] A. Dabholkar and J. Park, hep-th/9607041.
- [30] R. Gopakumar and S. Mukhi, hep-th/9607057.
- [31] J. Blum, hep-th/9608053.
- [32] M. Duff, R. Minasian and E. Witten, Nucl. Phys. **B465** (1996) 413 [hep-th/9601036].
- [33] M. Berkooz, R. Leigh, J. Polchinski, J. Schwarz, N. Seiberg and E. Witten, hep-th/9605184.
- [34] A. Sen, Nucl. Phys. **B474** (1996) 361 [hep-th/9604070].
- [35] A. Sen, hep-th/9605150.
- [36] A. Dabholkar and J. Park, hep-th/9604178.
- [37] P. Townsend, Phys. Lett. **350B** (1995) 184 [hep-th/9501068].
- [38] J. Schwarz, Phys. Lett. **B360** (1995) 13 [hep-th/9508143]; Phys. Lett. **B367** (1996) 97 [hep-th/9510086].
- [39] P. Aspinwall, hep-th/9508154.
- [40] K. Dasgupta and S. Mukhi, Nucl. Phys. **B465** (1996) 399 [hep-th/9512196].
- [41] E. Witten, Nucl. Phys. **B463** (1996) 383 [hep-th/9512219].

- [42] A. Sen, Mod. Phys. Lett. **A11** (1996) 1339 [hep-th/9603113].
- [43] A. Sen, Phys. Rev. **D53** (1996) 6725 [hep-th/9602010].
- [44] A. Dabholkar and J. Park, Nucl. Phys. **B472** (1996) 207 [hep-th/9602030].
- [45] P. Horava and E. Witten, Nucl. Phys. **B460** (1996) 506 [hep-th/9510209]; hep-th/9603142.
- [46] J. Polchinski and E. Witten, Nucl. Phys. **B460** (1996) 525 [hep-th/9510169].
- [47] C. Vafa, Nucl. Phys. **B469** (1996) 403 [hep-th/9602022].
- [48] D. Morrison and C. Vafa, Nucl. Phys. **B473** (1996) 74; [hep-th/9602114]; hep-th/9603161.
- [49] E. Gimon and C. Johnson, hep-th/9606176.
- [50] A. Strominger, S.-T. Yau and Zaslow, hep-th/9606040.
- [51] D. Morrison, alg-geom/9608006.