Adding Charges to N=4 Dyons

Nabamita Banerjee, Dileep P. Jatkar and Ashoke Sen

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, INDIA

E-mail: nabamita, dileep, sen@mri.ernet.in

Abstract

The spectrum of dyons in a class of N=4 supersymmetric string theories has been found for a specific set of electric and magnetic charge vectors. We extend the analysis to more general charge vectors by considering various charge carrying collective excitations of the original system.

Contents

1 Introduction

We now have a good understanding of the spectrum of quarter BPS dyons in a class of $\mathcal{N} = 4$ supersymmetric string theories [1–12], obtained by taking a $\mathbb{Z}_N$ orbifold of type II string theory compactified on $K3 \times T^2$ or T^6. However, in the direct approach to the computation of the spectrum based on counting of states the spectrum has so far been computed only for states carrying a restricted set of charges [7–9]. Our goal in this paper will be to extend this analysis to states carrying a more general set of charges, obtained from collective excitations of the system that has been analyzed earlier. For simplicity we shall restrict our analysis to type II string theory compactified on $K3 \times T^2$. Generalizing this to the case of $\mathcal{N} = 4$ supersymmetric orbifolds of this theory is straightforward, requiring setting to zero some of the charges which are not invariant under the orbifold group. The analysis for $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifolds of type II string theory compactified on T^6 can also be done in an identical manner.

2 Background

We consider the case of type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1$ or equivalently heterotic string theory on $T^4 \times S^1 \times \widehat{S}^1$. The latter description – to be called the second description – is obtained from the first description by first making an S-duality transformation in ten dimensional type IIB string theory, followed by a T-duality along the circle $\widetilde{S}^1$ that converts it to type IIA string theory on $K3 \times S^1 \times \widehat{S}^1$ and then using the six dimensional string-string duality that converts it to heterotic string theory on $T^4 \times S^1 \times \widehat{S}^1$. The coordinates ψ, y and χ along $\widetilde{S}^1$, S^1 and $\widehat{S}^1$ are all normalized to have period $2\pi\sqrt{\alpha'}$. Other normalization and sign conventions have been described in appendix A.

The compactified theory has 28 U(1) gauge fields and hence a given state is characterized by 28 dimensional electric and magnetic charge vectors Q and P. We shall use the second description of the theory to classify charges as electric and magnetic. Since in this description there are no RR fields or D-branes, an electrically charged state will correspond to an elementary string state and a magnetically charged state will correspond to wrapped NS 5-branes and Kaluza-Klein monopoles. The relationship between Q and P and the fields which appear in the supergravity theory underlying the second description will follow the convention of [13] in the $\alpha' = 16$ unit. This has been reviewed in appendix A, eq.(A.5). In this description the theory has an $SO(6, 22;\ \mathbb{Z})$ T-duality symmetry, and the T-duality invariant combination of charges is given by

$$Q^2 = Q^T L Q, \qquad P^2 = P^T L P, \qquad Q \cdot P = Q^T L P\,, \tag{2.1}$$

where L is a symmetric matrix with 22 eigenvalues -1 and 6 eigenvalues $+1$. We shall choose a basis in which L has the form

$$L = \begin{pmatrix} \widehat{L} & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 0_2 & I_2 \\ & & & I_2 & 0_2 \end{pmatrix}, \tag{2.2}$$

where $\widehat{L}$ is a matrix with 3 eigenvalues $+1$ and 19 eigenvalues -1. The charge vectors will be labelled as

$$Q = \begin{pmatrix} \widehat{Q} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{P} \\ l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \end{pmatrix}. \tag{2.3}$$

According to the convention of appendix A, k_3, k_4, $-k_5$ and $-k_6$ label respectively the momenta along $\widehat{S}^1$, S^1 and fundamental string winding along $\widehat{S}^1$ and S^1 in the second description of the theory. On the other hand $-l_3$, l_4, l_5 and l_6 label respectively the number of NS 5-branes wrapped along $S^1 \times T^4$ and $\widehat{S}^1 \times T^4$, and Kaluza-Klein monopole charges associated with $\widehat{S}^1$ and S^1. The rest of the charges label the momentum/winding or monopole charges associated with the other internal directions. By following the duality chain that relates the first and second description of the theory and using the sign convention of appendix A, the different components of P and Q can be given the following interpretation in the first description of the theory. k_3 represents the D-string winding charge along $\widetilde{S}^1$, k_4 is the momentum along S^1, k_5 is the D5-brane charge along $K3 \times \widetilde{S}^1$, k_6 is the number of Kaluza-Klein monopoles associated with the compact circle $\widetilde{S}^1$, l_3 is the D-string winding charge along S^1, $-l_4$ is the momentum along $\widetilde{S}^1$, l_5 is the D5-brane charge along $K3 \times S^1$ and l_6 is the number of Kaluza-Klein monopoles associated with the compact circle S^1. Other components of Q (P) represent various other branes of type IIB string theory wrapped on $\widetilde{S}^1$ (S^1) times various cycles of $K3$. We shall choose a convention in which the 22-dimensional charge vector $\widehat{Q}$ represents 3-branes wrapped along the 22 2-cycles of K3 times $\tilde{S}^1$, k_1 represents fundamental type IIB string winding charge along $\widetilde{S}^1$, k_2 represents the number of NS 5-branes of type IIB wrapped along $K3 \times \widetilde{S}^1$, the 22-dimensional charge vector $\widehat{P}$ represents 3-branes wrapped along the 22 2-cycles of K3 times S^1, l_1 represents fundamental type IIB string winding charge along S^1 and l_2 represents the number of NS 5-branes of type IIB wrapped along $K3 \times S^1$. In this convention $\widehat{L}$ represents the intersection matrix of 2-cycles of $K3$. Using the various sign conventions described in appendix A, and the T-duality transformation laws for the RR fields given in [14] one can verify that the combinations $k_3 k_5 + \widehat{Q}^2/2$, $l_3 l_5 + \widehat{P}^2/2$ and $k_3 l_5 + l_3 k_5 + \widehat{Q} \cdot \widehat{P}$ are invariant under the mirror symmetry transformation on $K3$.

The original configuration studied in [7] has charge vectors of the form:

$$Q = \begin{pmatrix} \widehat{0} \\ 0 \\ 0 \\ 0 \\ -n \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{0} \\ 0 \\ 0 \\ Q_1 - Q_5 = Q_1 - 1 \\ -J \\ Q_5 = 1 \\ 0 \end{pmatrix}. \tag{2.4}$$

Thus in the first description we have $-n$ units of momentum along S^1, J units of momentum along $\widetilde{S}^1$, a single Kaluza-Klein monopole (with negative magnetic charge) associated with $\widetilde{S}^1$, a single D5-brane wrapped on $K3 \times S^1$ and Q_1 D1-branes wrapped on S^1. The D5-brane wrapped on $K3 \times S^1$ also carries -1 units of D1-brane charge along S^1; this is responsible for the shift by -1 of Q_1 as given in (2.4). The associated invariants are

$$Q^2 = 2n, \qquad P^2 = 2(Q_1 - 1), \qquad Q \cdot P = J\,. \tag{2.5}$$

The degeneracy of this system was calculated in [7] as a function of n, Q_1 and J. If we call this function $f(n, Q_1, J)$, then we can express the degeneracy $d(Q, P)$ as a function of Q, P as:

$$d(Q, P) = f\left(\frac{1}{2}Q^2, \frac{1}{2}P^2 + 1, Q \cdot P\right). \tag{2.6}$$

Ref. [7] actually considered a more general charge vector where Q_5, representing the number of D5-branes wrapped along $K3 \times S^1$, was arbitrary and derived the same formula (2.6) for $d(Q,P)$. However, the analysis of dyon spectrum becomes simpler for $Q_5 = 1$. For this reason we have set $Q_5 = 1$. We shall comment on the more general case at the end.

3 Charge Carrying Deformations

Our goal will be to consider charge vectors more general than the ones given in (2.4) and check if the degeneracy is still given by (2.6). We shall do this by adding charges to the existing system by exciting appropriate collective modes of the system. These collective modes come from three sources:

1. The original configuration in the type IIB theory contains a Kaluza-Klein monopole associated with the circle $\widetilde{S}^1$. This solution is given by

$$ds^2 = \left(1 + \frac{K\sqrt{\alpha'}}{2r}\right)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) + K^2\left(1 + \frac{K\sqrt{\alpha'}}{2r}\right)^{-1}\left(d\psi + \frac{\sqrt{\alpha'}}{2}\cos\theta d\phi\right)^2 \tag{3.1}$$

with the identifications:

$$(\theta, \phi, \psi) \equiv (2\pi - \theta, \phi + \pi, \psi + \frac{\pi}{2}\sqrt{\alpha'}) \equiv (\theta, \phi + 2\pi, \psi + \pi\sqrt{\alpha'}) \equiv (\theta, \phi, \psi + 2\pi\sqrt{\alpha'}) \,. \tag{3.2}$$

The coordinate ψ can be regarded as the coordinate of $\widetilde{S}^1$, whereas (r, θ, ϕ) represent spherical polar coordinates of the non-compact space. K is a constant related to the physical radius of $\widetilde{S}^1$. This geometry, also known as the Taub-NUT space, admits a normalizable self-dual harmonic form ω, given by [15, 16]

$$\omega \propto \frac{2}{\sqrt{\alpha'}} \frac{r}{r + \frac{1}{2}K\sqrt{\alpha'}} d\sigma_3 + \frac{K}{(r + \frac{1}{2}K\sqrt{\alpha'})^2} dr \wedge \sigma_3 \,, \qquad \sigma_3 \equiv \left(d\psi + \frac{\sqrt{\alpha'}}{2}\cos\theta d\phi\right) \,. \tag{3.3}$$

(3.1) represents the geometry of the space-time transverse to the Kaluza-Klein monopole. Besides the $K3$ surface, the world-volume of the Kaluza-Klein monopole spans the circle S^1, which we shall label by y, and time t.

Now type IIB string theory compactified on K3 has various 2-form fields, – the original NSNS and RR 2-form fields B and $C^{(2)}$ of the ten dimensional type IIB string theory as well as the components of the 4-form field $C^{(4)}$ along various 2-cycles of $K3$. Given any such 2-form field C_{MN}, we can introduce a scalar mode φ by considering deformations of the form [17]:

$$C = \varphi(y,t)\,\omega \,, \tag{3.4}$$

where y denotes the coordinate along S^1. If the field strength dC associated with C is self-dual or anti-self-dual in six dimensions then the corresponding scalar field φ is chiral in the $y-t$ space; otherwise it represents a non-chiral scalar field. We can now consider configurations which carry momentum conjugate to this scalar field φ or winding number along y of this scalar field φ, represented by a solution where φ is linear in t or y. In the six dimensional language this corresponds to $dC \propto dt \wedge \omega$ or $dy \wedge \omega$. From (3.3) we see that $dC \propto dt \wedge \omega$ will have a component proportional to $r^{-2}\, dt \wedge dr \wedge d\psi$ for large r, and hence the coefficient of this term represents the charge associated with a string, electrically charged under C, wrapped along $\widetilde{S}^1$. On the other hand $dC \propto dy \wedge \omega$ will have a component proportional to $\sin\theta\, dy \wedge d\theta \wedge d\phi$ and the coefficient of this term will represent the charge associated with a string, magnetically charged under C, wrapped along $\widetilde{S}^1$. If the 2-form field C represents the original RR or NSNS 2-form field of type IIB string theory in ten dimensions, then the electrically charged string would correspond to a D-string or a fundamental type IIB string and the magnetically charged string would correspond to a D5-brane or NS 5-brane wrapped on $K3$. On the other hand if the 2-form C represents the component of the 4-form field along a 2-cycle of K3, then the corresponding string represents a D3-brane wrapped on a 2-cycle times $\widetilde{S}^1$. Recalling the interpretation of the charges $\widehat{Q}$ and k_i appearing in (2.3) we now see that the momentum and winding modes of φ correspond to the charges $\widehat{Q}$, k_1, k_2, k_3 and k_5. More specifically, after taking into account the sign conventions described in appendix A, these charges correspond to switching on deformations of the form:

$$
\begin{aligned}
& dB \propto -k_1 dt \wedge \omega, \quad dB \propto k_2 dy \wedge \omega, \quad dC^{(2)} \propto -k_3 dt \wedge \omega, \quad dC^{(2)} \propto k_5 dy \wedge \omega, \\
& dC^{(4)} \propto \sum_\alpha \widehat{Q}_\alpha (1+*)\, \Omega_\alpha \wedge dy \wedge \omega\,,
\end{aligned} \tag{3.5}
$$

where $\{\Omega_\alpha\}$ denote a basis of harmonic 2-forms on $K3$ ($1 \le \alpha \le 22$) satisfying $\int_{K3} \Omega_\alpha \wedge \Omega_\beta = \widehat{L}_{\alpha\beta}$. Thus in the presence of these deformations we have a more general electric charge vector of the form

$$
Q_0 = \begin{pmatrix} \widehat{Q} \\ k_1 \\ k_2 \\ k_3 \\ -n \\ k_5 \\ -1 \end{pmatrix}. \tag{3.6}
$$

As can be easily seen from (3.5), k_2 represents NS 5-brane charge wrapped along $K3 \times \widetilde{S}^1$.

However, for weakly coupled type IIB string theory, the presence of this charge could have large backreaction on the geometry. In order to avoid it we shall choose

$$k_2 = 0 . \tag{3.7}$$

2. The original configuration considered in [7] also contains a D5-brane wrapped around $K3 \times S^1$. We can switch on flux of world-volume gauge field strengths $\mathcal{F}$ on the D5-brane along the various 2-cycles of K3 that it wraps. The net coupling of the RR gauge fields to the D5-brane in the presence of the world-volume gauge fields may be expressed as [14]

$$\int \left[C^{(6)} + C^{(4)} \wedge \mathcal{F} + \frac{1}{2} C^{(2)} \wedge \mathcal{F} \wedge \mathcal{F} + \cdots \right] , \tag{3.8}$$

up to a constant of proportionality. The integral is over the D5-brane world-volume spanned by y, t and the coordinates of $K3$. In order to be compatible with the convention of appendix A that the D5-brane wrapped on $K3 \times S^1$ carries negative $(dC^{(6)})_{(K3)yrt}$ asymptotically, we need to take the integration measure in the yt plane in (3.8) as $dy \wedge dt$, i.e. $\epsilon^{yt} > 0$. Via the coupling

$$\int C^{(4)} \wedge \mathcal{F} , \tag{3.9}$$

the gauge field configuration will produce the charges of a D3-brane wrapped on a 2-cycle of $K3$ times S^1, – i.e. the 22 dimensional magnetic charge vector $\widehat{P}$ appearing in (2.3). More precisely, we find that the gauge field flux required to produce a specific magnetic charge vector $\widehat{P}$ is

$$\mathcal{F} \propto -\sum_\alpha \widehat{P}_\alpha \, \Omega_\alpha . \tag{3.10}$$

3. The D5-brane can also carry electric flux along S^1. This will induce the charge of a fundamental type IIB string wrapped along S^1. According to the physical interpretation of various charges given earlier, this gives the component l_1 of the magnetic charge vector P.

The net result of switching on both the electric and magnetic flux along the D5-brane world-volume is to generate a magnetic charge vector of the form:

$$P_0 = \begin{pmatrix} \widehat{P} \\ l_1 \\ 0 \\ Q_1 - 1 \\ -J \\ 1 \\ 0 \end{pmatrix} . \tag{3.11}$$

4 Additional Shifts in the Charges

This, however, is not the end of the story. So far we have discussed the effect of the various collective mode excitations on the charge vector to first order in the charges. We have not taken into account the effect of the interaction of deformations produced by the collective modes with the background fields already present in the system, or the background fields produced by other collective modes. Taking into account these effects produces further shifts in the charge vector as described below.

1. As seen from (3.8), the D5-brane world-volume theory has a coupling proportional to $\int C^{(2)} \wedge \mathcal{F} \wedge \mathcal{F}$. Thus in the presence of magnetic flux $\mathcal{F}$ the D5-brane wrapped on $K3 \times S^1$ acts as a source of the D1-brane charge wrapped on S^1. The effect is a shift in the magnetic charge quantum number l_3 that is quadratic in $\mathcal{F}$ and hence quadratic in $\widehat{P}$ due to (3.10). A careful calculation, taking into account various signs and normalization factors, shows that the net effect of this term is to give an additional contribution to l_3 of the form:

$$\Delta_1 l_3 = -\widehat{P}^2/2\,. \tag{4.1}$$

2. Let C be a 2-form in the six dimensional theory obtained by compactifying type IIB string theory on $K3$ and $F = dC$ be its field strength. As summarized in (3.5), switching on various components of the electric charge vector Q requires us to switch on F proportional to $dt \wedge \omega$ or $dy \wedge \omega$. The presence of such background induces a coupling proportional to

$$-\int \sqrt{-\det g}\, g^{yt} F_{ymn} F_t^{\ mn} \tag{4.2}$$

with the indices m, n running over the coordinates of the Taub-NUT space. This produces a source for g^{yt}, i.e. momentum along S^1. The effect of such terms is to shift the component k_4 of the charge vector Q. A careful calculation shows that the net change in k_4 induced due to this coupling is given by

$$\Delta_2 k_4 = k_3 k_5 + \widehat{Q}^2/2\,, \tag{4.3}$$

where we have used the fact that k_2 has been set to zero. The $k_3 k_5$ term comes from taking F if (4.2) to be the field strength of the RR 2-form field, and $\widehat{Q}^2/2$ term comes from taking F to be the field strength of the components of the RR 4-form field along various 2-cycles of $K3$.

3. The D5-brane wrapped on $K3 \times S^1$ or the magnetic flux on this brane along any of the 2-cycles of $K3$ produces a magnetic type 2-form field configuration of the form:

$$F \equiv dC \propto \sin\theta\, d\psi \wedge d\theta \wedge d\phi\,, \tag{4.4}$$

where C is any of the RR 2-form fields in six dimensional theory obtained by compactifying type IIB string theory on $K3$. One can verify that the 3-form appearing on the right hand side of (4.4) is both closed and co-closed in the Taub-NUT background and hence F given in (4.4) satisfies both the Bianchi identity and the linearized equations of motion. The coefficients of the term given in (4.4) for various 2-form fields C are determined in terms of $\widehat{P}$ and the D5-brane charge along $K3 \times S^1$ which has been set equal to 1. This together with the term in F proportional to $dt \wedge \omega$ coming from the collective coordinate excitation of the Kaluza-Klein monopole generates a source of the component $g^{\psi t}$ of the metric via the coupling proportional to

$$-\int \sqrt{-\det g} g^{\psi t} F_{\psi mn} F_t{}^{mn} \tag{4.5}$$

This induces a net momentum along $\widetilde{S}^1$ and gives a contribution to the component l_4 of the magnetic charge vector P. A careful calculation shows that the net additional contribution to l_4 due to this coupling is given by

$$\Delta_3 l_4 = k_3 + \widehat{Q} \cdot \widehat{P} \, . \tag{4.6}$$

In this expression the contribution proportional to k_3 comes from taking F in (4.5) to be the field strength associated with the RR 2-form field of IIB, whereas the term proportional to $\widehat{Q} \cdot \widehat{P}$ arises from taking F to be the field strength associated with the components of the RR 4-form field along various 2-cycles of $K3$.

4. Eqs.(3.11) and (4.1) show that we have a net D1-brane charge along S^1 equal to

$$l_3 = Q_1 - 1 - \widehat{P}^2/2 \, . \tag{4.7}$$

If we denote by $C^{(2)}$ the 2-form field of the original ten dimensional type IIB string theory, then the effect of this charge is to produce a background of the form:

$$dC^{(2)} \propto (Q_1 - 1 - \widehat{P}^2/2) \, r^{-2} \, dr \wedge dt \wedge dy \, . \tag{4.8}$$

Again one can verify explicitly that the right hand side of (4.8) is both closed and co-closed in the Taub-NUT background. We also have a component

$$dC^{(2)} \propto k_5 \, dy \wedge \omega \, , \tag{4.9}$$

coming from the excitation of the collective coordinate of the Kaluza-Klein monopole. This gives a source term for $g^{\psi t}$ via the coupling proportional to

$$-\int \sqrt{-\det g} g^{\psi t} F_{\psi ry} F_t{}^{ry} \tag{4.10}$$

producing an additional contribution to the charge l_4 of the form

$$\Delta_4 l_4 = k_5(Q_1 - 1 - \widehat{P}^2/2)\,. \qquad (4.11)$$

So far in our analysis we have taken into account possible additional sources produced by the terms quadratic in the fields. What about higher order terms? It is straightforward to show that the possible effect of the higher order terms on the shift in the charges will involve one or more powers of the type IIB string coupling. Since the shift in the charges must be quantized, they cannot depend on continuous moduli. Thus at least in the weakly coupled type IIB string theory there are no additional corrections to the charges. Incidentally, the same argument can be used to show that the shifts in the charges must also be independent of the other moduli; thus it is in principle sufficient to calculate these shifts at any particular point in the moduli space.

Combining all the results we see that we have a net electric charge vector Q and a magnetic charge vector P of the form:

$$Q = \begin{pmatrix} \widehat{Q} \\ k_1 \\ 0 \\ k_3 \\ -n + k_3 k_5 + \widehat{Q}^2/2 \\ k_5 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{P} \\ l_1 \\ 0 \\ Q_1 - 1 - \widehat{P}^2/2 \\ -J + k_3 + \widehat{Q}\cdot\widehat{P} + k_5(Q_1 - 1 - \widehat{P}^2/2) \\ 1 \\ 0 \end{pmatrix}. \qquad (4.12)$$

This has

$$Q^2 = 2n, \qquad P^2 = 2(Q_1 - 1), \qquad Q\cdot P = J\,. \qquad (4.13)$$

Thus the additional charges do not affect the relationship between the invariants Q^2, P^2, $Q \cdot P$ and the original quantum numbers n, Q_1 and J.

5 Dyon Spectrum

Let us now turn to the analysis of the dyon spectrum in the presence of these charges. For this we recall that in [7] the dyon spectrum was computed from three mutually non-interacting parts, – the dynamics of the Kaluza-Klein monopole, the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole and the motion of the D1-branes relative to the D5-brane. The precise dynamics of the D1-branes relative to the D5-brane is affected by the presence of the gauge field flux on the D5-brane since it changes the non-commutativity parameter describing the dynamics of the gauge field on the D5-brane world-volume [18]. As a result the moduli space of D1-branes, described as non-commutative instantons in this gauge theory [19], gets deformed. However, we do not expect this to change the elliptic genus of the corresponding conformal field theory [20] that enters the degeneracy formula. With the exception of the zero mode associated with the D1-D5 center of mass motion in the Kaluza-Klein monopole background, the rest of the contribution to the degeneracy came from the excitations involving non-zero mode oscillators of the collective coordinates of the Kaluza-Klein monopole and the collective coordinates associated with the overall motion of the D1-D5 system [7]. This is not affected either by switching on gauge field fluxes on the D5-brane world-volume or the momenta or winding number of the collective coordinates of the Kaluza-Klein monopole. On the other hand the dynamics of the D1-D5-brane center of mass motion in the background geometry is also not expected to be modified in the weakly coupled type IIB string theory since in this limit the additional background fields due to the additional charges are small compared to the one due to the Kaluza-Klein monopole. (For this it is important that the additional charges do not involve any other Kaluza-Klein monopole or NS 5-brane charge.) Thus we expect the degeneracy to be given by the same function $f(n, Q_1, J)$ that appeared in the absence of the additional charges. Using (4.13) we can now write

$$d(Q,P) = f\left(\frac{1}{2}Q^2, \frac{1}{2}P^2 + 1, Q \cdot P\right) . \tag{5.1}$$

This is a generalization of (2.6) and shows that for the charge vectors given in (4.12), the degeneracy $d(Q,P)$ depends on the charges only through the combination Q^2, P^2 and $Q \cdot P$.

As was discussed in [11], the formula for the degeneracy for a given charge vector can change across walls of marginal stability in the moduli space. Hence a given formula for the degeneracy makes sense only if we specify how the region of the moduli space in which we are carrying out our analysis is situated with respect to the walls of marginal stability. In the theory under consideration the moduli space is the coset $(SL(2,\ \mathbb{Z})\backslash SL(2,\ \mathbb{R})/U(1)) \times (SO(6,22;\ \mathbb{Z})\backslash SO(6,22;\ \mathbb{R})/SO(6) \times SO(22))$, parametrized by a complex modulus τ and a 28×28 symmetric $SO(6,22)$ matrix M. For fixed M, the walls of marginal stability are either straight lines in the τ plane, intersecting the real axis at an integer, or circles intersecting the real axis at rational points a/c and b/d with $ad - bc = 1$, $a, b, c, d \in\ \mathbb{Z}$. The precise shape of the circles and the slopes of the straight lines depend on the modulus M and the charge vector of the state under consideration. It was shown in [11] that for the charge vector given in (2.4) the region where the type IIB string coupling and the angle between S^1 and $\widetilde{S}^1$ are small and the other moduli are of order 1 can fall into one of the two domains in the upper half τ plane. The first of these domains is bounded by a pair of straight lines in the τ

plane, passing through the points 0 and 1 respectively, and a circle passing through the points 0 and 1. The second domain is bounded by a pair of straight lines passing through the points -1 and 0 respectively and a circle passing through the points -1 and 0. Carrying out a similar analysis for the modified charge vector (4.12) one finds that as long as all the charges are finite, the region of moduli space where type IIB coupling is small falls inside the same domains, i.e. domains bounded by a set of walls of marginal stability which intersect the real τ axis at the same points. This is just as well; had the new charge vector landed us into a different domain in the τ plane, our result (5.1) would be in contradiction with the result of [11] that in different domains bounded by different walls of marginal stability the degeneracies are given by different functions of P^2, Q^2 and $Q \cdot P$.

6 More General Charge Vector

The charge vector given in (4.12), while more general than the one considered in [7], is still not the most general charge vector. Is it possible to extend our analysis to include more general charge vectors? First of all note that l_5, representing the number of D5-branes wrapped on $K3 \times S^1$, was chosen to be an arbitrary integer instead of 1 in [7]. Thus we can certainly take as our starting point the more general charge vector where Q_5 in (2.4) is chosen to be an arbitrary integer instead of 1. Our analysis up to (4.13) proceeds in a straightforward manner (with $Q_1 - 1$ replaced by $Q_5(Q_1 - Q_5)$). The issue, however, is how the additional charges affect the dyon spectrum. In particular one needs to examine carefully the effect of the gauge field flux on the D5-brane on the dynamics of the D1-D5 system, generalizing the analysis given in [20]. However, as long as, we do not switch on gauge field flux on the D5-branes, i.e. consider configurations with $\widehat{P} = 0$, $l_1 = 0$, there is no additional complication and the final degeneracy will still be given by (5.1). On the other hand following the analysis of [11] one can show that the region of the moduli space where the type IIB string coupling and the angle between S^1 and $\widetilde{S}^1$ are small is still bounded by the same set $\mathcal{B}_R$, $\mathcal{B}_L$ of walls of marginal stability.

In (4.12) we have set the component k_2 of the electric charge vector to zero even though we could switch it on by switching on an NSNS sector 3-form field strength of the form $dy \wedge \omega$. The reason for this was that this charge represents the number of NS 5-branes wrapped along $K3 \times \widetilde{S}^1$ and the presence of NS 5-branes could have large backreaction on the geometry thereby invalidating our analysis. We can, however, keep its effect small compared to that of the original background produced by the Kaluza-Klein monopole by taking the radius R of S^1 to be large compared to $\sqrt{\alpha'}$. Since in the string metric the mass of the Kaluza-Klein monopole is proportional to R while the NS

5-brane wrapped along $\widetilde{S}^1$ does not have such a factor, we can expect that for large R the effect of the background produced by the NS 5-brane will be small compared to that of the Kaluza-Klein monopole. We can then analyze the system in the same manner as for the other charges and conclude that the formula for the degeneracy in the presence of this additional charge is still given by (4.13). One also finds that the region of the moduli space where the type IIB string coupling and the angle between S^1 and $\widetilde{S}^1$ are small is still bounded by the same set $\mathcal{B}_R$, $\mathcal{B}_L$ of walls of marginal stability.

Let us now turn to k_6 which has been set equal to -1 in (4.12). This is the number of Kaluza-Klein monopoles associated with the compactification circle $\widetilde{S}^1$. Changing this number would require us to study the dynamics of multiple Kaluza-Klein monopoles. While, in principle, this can be done, this will certainly require a major revision of the analysis done so far. Thus there does not seem to be a minor variation of our analysis that can change the charge k_6 to any other integer.

This leaves us with the components l_2 and l_6 both of which have been set to 0 in (4.12). l_2 represents the number of NS 5-branes wrapped on S^1. Switching this charge on would require us to introduce explicit NS 5-brane background and study the dynamics of D-branes in such a background. This would require techniques quite different from the one used so far. On the other hand, the component l_6 represents the Kaluza-Klein monopole charge associated with the compact circle S^1. This also causes significant change in the background geometry and calculation of the spectrum of such configurations would require fresh analysis.

Acknowledgement: We would like to thank Justin David for many useful discussions and collaboration at early stages of this work.

A Normalization and Sign Conventions

In this appendix we shall describe the various normalization and sign conventions we use during our analysis. We begin by describing the ten dimensional action of type IIB string theory that appears in the first description:

$$\begin{aligned} S &= \frac{1}{(2\pi)^7(\alpha')^4}\int d^{10}x\,\sqrt{-\det g}\bigg[e^{-2\Phi}\bigg(R+4\partial_M\Phi\partial^M\Phi-\frac{1}{2\cdot 3!}H_{MNP}H^{MNP}\bigg) \\ & -\frac{1}{2}F^{(1)}_M F^{(1)M}-\frac{1}{2\cdot 3!}\widetilde{F}^{(3)}_{MNP}\widetilde{F}^{(3)MNP}-\frac{1}{4\cdot 5!}\widetilde{F}^{(5)}_{M_1\cdots M_5}\widetilde{F}^{(5)M_1\cdots M_5}\bigg] \\ & +\frac{1}{2(2\pi)^7(\alpha')^4}\int C^{(4)}\wedge\widetilde{F}^{(3)}\wedge H\,, \end{aligned} \tag{A.1}$$

where

$$H = dB, \quad F^{(1)} = dC^{(0)}, \quad F^{(3)} = dC^{(2)}, \quad F^{(5)} = dC^{(4)}$$
$$\widetilde{F}^{(3)} = F^{(3)} - C^{(0)} H, \quad \widetilde{F}^{(5)} = F^{(5)} - \frac{1}{2} C^{(2)} \wedge H + \frac{1}{2} B \wedge F^{(3)}\,, \tag{A.2}$$

g_{MN} denotes the string metric, B_{MN} denotes the NSNS 2-form fields, Φ denotes the dilaton and $C^{(k)}$ denotes the RR k-form field. The field strengths $dC^{(k)}$ are subject to the relations $*dC^{(k)} = (-1)^{k(k-1)/2} dC^{(8-k)} + \cdots$ where $*$ denotes Hodge dual taken with respect to the string metric and $\cdots$ denote terms quadratic and higher order in the fields. For $k = 4$ this gives a constraint on $C^{(4)}$ whereas for $k > 4$ this defines the field $C^{(k)}$. In computing the Hodge dual in the first description we shall use the convention that on $S^1 \times \widetilde{S}^1 \times \mathbb{R}^{3,1}$ we have $\epsilon^{ty\psi r\theta\phi} > 0$ where r, θ, ϕ and t denote the spherical polar coordinates and the time coordinate of the (3+1) dimensional non-compact space-time and y and ψ denote coordinates of S^1 and $\widetilde{S}^1$ respectively, each normalized to have period $2\pi\sqrt{\alpha'}$. Inside $K3$ we use the standard volume form on K3 to define the ϵ tensor. Our normalization conventions are consistent with that of [14].

As is well known, the moduli space of $K3$ with NSNS 2-form fields switched on, is labelled by elements of the coset $SO(4,20;\ \mathbb{Z})\backslash SO(4,20)/(SO(4)\times SO(20))$. These elements may be parametrized by a symmetric SO(4,20) matrix $\widetilde{M}$ and we choose the coordinate system on this coset in such a way that the identity matrix represents a $K3$ of volume $(2\pi\sqrt{\alpha'})^4$ in string metric, with the NSNS 2-form fields set to zero.

The low energy effective action of heterotic string theory on T^4 that appears in the second description has the form:

$$\frac{1}{(2\pi)^3(\alpha')^2} \int d^6x\, \sqrt{-\det g}\, e^{-2\Phi} \Big[R + 4\partial_\alpha\Phi\partial^\alpha\Phi - \frac{1}{2\cdot 3!} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - \frac{1}{8} Tr(\partial_\alpha \widetilde{M}\widetilde{L}\partial^\alpha \widetilde{M}\widetilde{L}) \\ - \mathcal{F}^{(a)}_{\alpha\beta} (\widetilde{L}\widetilde{M}\widetilde{L})_{ab} \mathcal{F}^{(b)\alpha\beta} \Big] \tag{A.3}$$

where $\widetilde{L}$ is a fixed 24×24 matrix with 4 positive and 20 negative eigenvalues, $\widetilde{M}$ is a 24×24 symmetric matrix valued scalar field satisfying $\widetilde{M}\widetilde{L}\widetilde{M} = \widetilde{L}$, and $\mathcal{F}^{(a)}_{\alpha\beta}$ for $1 \le a \le 24$, $0 \le \alpha, \beta \le 5$ are the field strengths associated with 24 U(1) gauge fields $\mathcal{A}^{(a)}_\alpha$ obtained by heterotic string compactification on T^4. The fields $g_{\alpha\beta}$, $B_{\alpha\beta}$ and Φ are the string metric, NSNS 2-form field and the six dimensional dilaton of the heterotic string theory and should be distinguished from those appearing in (A.1). Upon further compactification on $\widehat{S}^1 \times S^1$ labelled by $x^4 \equiv \chi$ and $x^5 \equiv y$, both normalized to have period $2\pi\sqrt{\alpha'}$, we get four more gauge fields $A^{(i)}_\mu$ $(1 \le i \le 4,\ 0 \le \mu, \nu \le 3)$ and a 4×4 symmetric matrix valued scalar field $\bar{M}$ defined via the relations:

$$\begin{aligned}
&\widehat{G}_{mn} \equiv g_{mn}, \quad \widehat{B}_{mn} \equiv B_{mn}\,, \qquad m,n = 4,5\,, \\
&\bar{M} = \begin{pmatrix} \widehat{G}^{-1} & \widehat{G}^{-1}\widehat{B} \\ -\widehat{B}\widehat{G}^{-1} & \widehat{G} - \widehat{B}\widehat{G}^{-1}\widehat{B} \end{pmatrix} \\
&A^{(m-3)}_\mu = \frac{1}{2}(\widehat{G}^{-1})^{mn} G^{(10)}_{m\mu}, \quad A^{(m-1)}_\mu = \frac{1}{2} B^{(10)}_{m\mu} - \widehat{B}_{mn} A^{(m-3)}_\mu, \\
&\qquad 4 \le m,n \le 5, \quad 0 \le \mu,\nu \le 3\,.
\end{aligned} \tag{A.4}$$

For simplicity we have set the Wilson lines of the gauge fields $\mathcal{A}^{(a)}_\alpha$ along S^1 and $\widehat{S}^1$ to zero. In the $\alpha' = 16$ unit the electric and magnetic charges $(k_3, \cdots k_6, l_3, \cdots l_6)$ appearing in eq.(2.3) are related to the asymptotic values of the gauge field strengths $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$ via the relations [13]

$$(\bar{L}\bar{M}\bar{L})_{ij} F^{(j)}_{rt}\Big|_\infty = \frac{k_{i+2}}{r^2}\,, \qquad \bar{L}_{ij}\, F^{(j)}_{\theta\phi}\Big|_\infty = l_{i+2}\sin\theta\,, \qquad \bar{L} \equiv \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix}. \tag{A.5}$$

The other charges $\widehat{Q}$, k_1, k_2 and $\widehat{P}$, l_1, l_2 appearing in (2.3) can be related to the asymptotic values of the gauge field strengths $\mathcal{F}^{(a)}_{rt}$ and $\mathcal{F}^{(a)}_{\theta\phi}$ in a similar manner.

The chain of duality transformations taking us from the first to the second description are chosen so that at the linearized level the first S-duality transformation of IIB acts as $C^{(2)} \to B$, $B \to -C^{(2)}$, and the next $R \to 1/R$ duality transformations of $\widetilde{S}^1$ acts as $g_{\psi\mu} \to -B_{\chi\mu}$, $B_{\psi\mu} \to -g_{\chi\mu}$ together with appropriate transformations on the various RR gauge fields. The final string string duality transformation acts via a Hodge duality transformation in six dimensions on the NS sector 3-form field strength with $\epsilon^{t\chi yr\theta\phi} > 0$, and maps various four dimensional gauge fields arising from various components of the RR sector fields to the 24 gauge fields in heterotic string theory on T^4.

Finally, we use the following convention for the signs of the charges carried by various branes in the first description. If $F^{(3)} \equiv dC^{(2)}$ denotes the RR 3-form field strength, then asymptotically a D1-brane along S^1 will carry positive $F^{(3)}_{yrt}$, a D5-brane along $\widetilde{S}^1 \times K3$ will carry positive $F^{(3)}_{\theta y\phi}$, a D1-brane along $\widetilde{S}^1$ will carry positive $F^{(3)}_{\psi rt}$ and a D5-brane along $S^1 \times K3$ will carry negative $F^{(3)}_{\theta\psi\phi}$. The same convention is followed for fundamental string and NS 5-brane with $F^{(3)}$ replaced by the NSNS 3-form field strength $H = dB$. A state carrying positive momentum along S^1 or $\widetilde{S}^1$ is defined to be the one which produces positive $\partial_r g_{yt}$ or $\partial_r g_{\psi t}$, and a positively charged Kaluza-Klein monopole associated with the circle S^1 or $\widetilde{S}^1$ is defined to be the one that carries positive $\partial_\theta g_{y\phi}$ or $\partial_\theta g_{\psi\phi}$ asymptotically. Note that in this convention the asymptotic configuration for $F^{(7)} \equiv dC^{(6)}$ around a D5-brane wrapped on $K3 \times S^1$ or $K3 \times \widetilde{S}^1$ will have negative $F^{(7)}_{(K3)yrt}$ or $F^{(7)}_{(K3)\psi rt}$, with the subscript ${}_{(K3)}$ denoting components of $F^{(7)}$ along the volume form of $K3$.

The same conventions are followed for the signs of the charges carried by various states in the second description, with the coordinate ψ of $\widetilde{S}^1$ replaced by the coordinate χ of $\widehat{S}^1$.

References

[1] R. Dijkgraaf, E. P. Verlinde and H. L. Verlinde, "Counting dyons in N = 4 string theory," Nucl. Phys. B **484**, 543 (1997) [arXiv:hep-th/9607026].

[2] D. Shih, A. Strominger and X. Yin, "Recounting dyons in N = 4 string theory," arXiv:hep-th/0505094.

[3] D. Gaiotto, "Re-recounting dyons in N = 4 string theory," arXiv:hep-th/0506249.

[4] D. P. Jatkar and A. Sen, "Dyon spectrum in CHL models," JHEP **0604**, 018 (2006) [arXiv:hep-th/0510147].

[5] J. R. David, D. P. Jatkar and A. Sen, "Product representation of dyon partition function in CHL models," JHEP **0606**, 064 (2006) [arXiv:hep-th/0602254].

[6] A. Dabholkar and S. Nampuri, "Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift," arXiv:hep-th/0603066.

[7] J. R. David and A. Sen, "CHL dyons and statistical entropy function from D1-D5 system," JHEP **0611**, 072 (2006) [arXiv:hep-th/0605210].

[8] J. R. David, D. P. Jatkar and A. Sen, "Dyon spectrum in N = 4 supersymmetric type II string theories," arXiv:hep-th/0607155.

[9] J. R. David, D. P. Jatkar and A. Sen, "Dyon spectrum in generic N = 4 supersymmetric Z(N) orbifolds," arXiv:hep-th/0609109.

[10] A. Dabholkar and D. Gaiotto, "Spectrum of CHL dyons from genus-two partition function," arXiv:hep-th/0612011.

[11] A. Sen, "Walls of marginal stability and dyon spectrum in N = 4 supersymmetric string theories," arXiv:hep-th/0702141.

[12] A. Dabholkar, D. Gaiotto and S. Nampuri, "Comments on the spectrum of CHL dyons," arXiv:hep-th/0702150.

[13] A. Sen, "Entropy function for heterotic black holes," JHEP **0603**, 008 (2006) [arXiv:hep-th/0508042].

[14] R. C. Myers, "Dielectric-branes," JHEP **9912**, 022 (1999) [arXiv:hep-th/9910053].

[15] D. Brill, Phys. Rev. B133 (1964) 845.

[16] C. N. Pope, "Axial Vector Anomalies And The Index Theorem In Charged Schwarzschild And Taub - Nut Spaces," Nucl. Phys. B **141**, 432 (1978).

[17] A. Sen, "Kaluza-Klein dyons in string theory," Phys. Rev. Lett. **79**, 1619 (1997) [arXiv:hep-th/9705212].

[18] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999) [arXiv:hep-th/9908142].

[19] N. Nekrasov and A. S. Schwarz, "Instantons on noncommutative R**4 and (2,0) superconformal six dimensional theory," Commun. Math. Phys. **198**, 689 (1998) [arXiv:hep-th/9802068].

[20] R. Dijkgraaf, G. W. Moore, E. P. Verlinde and H. L. Verlinde, "Elliptic genera of symmetric products and second quantized strings," Commun. Math. Phys. **185**, 197 (1997) [arXiv:hep-th/9608096].