S-duality Action on Discrete T-duality Invariants

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Abstract

In heterotic string theory compactified on T^6, the T-duality orbits of dyons of charge (Q, P) are characterized by $O(6, 22;\ \mathbb{R})$ invariants Q^2, P^2 and $Q \cdot P$ together with a set of invariants of the discrete T-duality group $O(6, 22;\ \mathbb{Z})$. We study the action of S-duality group on the discrete T-duality invariants and study its consequence for the dyon degeneracy formula. In particular we find that for dyons with torsion r, the degeneracy formula, expressed as a function of Q^2, P^2 and $Q \cdot P$, is required to be manifestly invariant under only a subgroup of the S-duality group. This subgroup is isomorphic to $\Gamma^0(r)$. Our analysis also shows that for a given torsion r, all other discrete T-duality invariants are characterized by the elements of the coset $SL(2,\ \mathbb{Z})/\Gamma^0(r)$.

Dyons in heterotic string theory on T^6 are characterized by a pair of charge vectors (Q, P) each taking value on the Narain lattice Λ [1,2]. Given two pairs of charge vectors, an interesting question is: under what condition can they be related via a T-duality transformation? This question was answered in [3] where a complete set of T-duality invariants classifying a pair of charge vectors (Q, P) were constructed. These include the invariants of the continuous T-duality group $O(6, 22;\ \mathbb{R})$

$$Q^2, \quad P^2, \quad Q \cdot P\,, \tag{1}$$

together with a set of invariants of the discrete T-duality group $O(6, 22;\ \mathbb{Z})$. These are defined as follows. We shall assume that the dyon is primitive so that (Q, P) cannot be written as an integer multiple of (Q_0, P_0) with $Q_0, P_0 \in \Lambda$, but we shall not assume that Q and P themselves are primitive. Now consider the intersection of the two dimensional vector space spanned by (Q, P) with the Narain lattice Λ. The result is a two dimensional lattice Λ_0. Let (e_1, e_2) be a pair of basis elements whose integer linear combinations generate this lattice. We can always choose (e_1, e_2) such that in this basis

$$\begin{aligned} Q = r_1 e_1, \qquad P = r_2(u_1 e_1 + r_3 e_2), \qquad r_1, r_2, r_3, u_1 \in\ \mathbb{Z}^+, \\ \gcd(r_1, r_2) = 1, \quad \gcd(u_1, r_3) = 1, \quad 1 \le u_1 \le r_3\,. \end{aligned} \tag{2}$$

It was found in [3] that besides Q^2, P^2 and $Q \cdot P$, the integers r_1, r_2, r_3 and u_1 are T-duality invariants. Furthermore it was found that this is the complete set of T-duality invariants. Thus a pair of charge vectors (Q, P) can be transformed into another pair (Q', P') via a T-duality transformation if and only if all the invariants agree for these two pairs.

Our first goal is to study some aspects of the action of the S-duality transformation

$$Q \to Q' = aQ + bP, \quad P \to P' = cQ + dP, \quad a, b, c, d \in\ \mathbb{Z}, \quad ad - bc = 1\,, \tag{3}$$

on the invariants r_1, r_2, r_3 and u_1. Substituting (2) into (3), and expressing the resulting (Q', P') as $(r_1' e_1', r_2'(u_1' e_1' + r_3' e_2'))$ for some primitive basis (e_1', e_2') of Λ_0 we can determine (r_1', r_2', r_3', u_1'). Since the resuting expressions are somewhat complicated and not very illuminating we shall not describe them here. Instead we shall focus on some salient features of the transformation laws of (r_1, r_2, r_3, u_1). We first note that the torsion $r(Q, P)$ associated with a pair of charges (Q, P), defined as [4, 5]

$$r(Q, P) = Q_1 P_2 - Q_2 P_1\,, \tag{4}$$

with Q_i, P_i being the components of Q and P along e_i, is invariant under the S-duality transformation (3). Furthermore, for the charge vectors (Q, P) given in (2) we have

$$r(Q, P) = r_1 r_2 r_3 \,. \tag{5}$$

We shall now show that one can always find an S-duality transformation that brings the T-duality invariants (r_1, r_2, r_3, u_1) to $(r_1 r_2 r_3, 1, 1, 1)$ together with an appropriate transformation on Q^2, P^2 and $Q \cdot P$ induced by (3). For this we note that under the S-duality transformation (3), (Q, P) given in (2) transforms to

$$Q' = \{ar_1 + br_2(u_1 + kr_3)\}e_1 + br_2 r_3(e_2 - ke_1), \quad P' = \{cr_1 + dr_2(u_1 + kr_3)\}e_1 + dr_2 r_3(e_2 - ke_1)\,, \tag{6}$$

where k is an arbitrary integer. We shall choose

$$k = \prod_i p_i \,, \tag{7}$$

where $\{p_i\}$ represent the collection of primes which are factors of r_1 but not of u_1. Now we know from (2) that $\gcd(r_1, r_2) = 1$. On the other hand it follows from a result derived in appendix E of [6] that for the choice of k given in (7) we have $\gcd(r_1, u_1 + kr_3) = 1$. Thus if we choose

$$b = r_1, \quad a = -r_2(u_1 + kr_3)\,, \tag{8}$$

we have $\gcd(a, b) = 1$ and hence we can always find c, d satisfying $ad - bc = 1$. For this particular choice of $SL(2, \mathbb{Z})$ transformation we have

$$Q' = r_1 r_2 r_3(e_2 - ke_1), \qquad P' = -e_1 + dr_2 r_3(e_2 - ke_1)\,. \tag{9}$$

We now define

$$e_1' = (e_2 - ke_1), \qquad e_2' = -e_1 + (dr_2 r_3 - 1)(e_2 - ke_1)\,. \tag{10}$$

Since the matrix relating (e_1', e_2') to (e_1, e_2) has unit determinant, (e_1', e_2') is a primitive basis of the lattice Λ_0. In this basis (Q', P') can be expressed as

$$Q' = r_1 r_2 r_3 e_1', \qquad P' = e_1' + e_2'\,. \tag{11}$$

Comparing this with (2) we see that for the new charge vector (Q', P') we have

$$r_1' = r_1 r_2 r_3, \quad r_2' = 1, \quad r_3' = 1, \quad u_1' = 1\,. \tag{12}$$

This proves the desired result.

Next we shall study the subgroup of S-duality transformations which takes a configuration with $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ to another configuration with $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$. The initial configuration has

$$Q = re_1, \qquad P = e_1 + e_2 \,. \tag{13}$$

An S-duality transformation (3) takes this to

$$Q' = are_1 + b(e_1 + e_2)\,, \qquad P' = cre_1 + d(e_1 + e_2)\,. \tag{14}$$

In order that Q' is r times a primitive vector, we must demand

$$b = 0 \text{ mod } r\,. \tag{15}$$

Expressing b as $b_0 r$ with $b_0 \in \mathbb{Z}$ we get

$$Q' = re'_1, \qquad P' = e'_1 + e'_2\,, \tag{16}$$

where

$$e'_1 = (a + b_0)e_1 + b_0 e_2, \qquad e'_2 = (cr + d - a - b_0)e_1 + (d - b_0)e_2\,. \tag{17}$$

Since the determinant of the matrix relating (e'_1, e'_2) to (e_1, e_2) is given by

$$(a + b_0)(d - b_0) - b_0(cr + d - a - b_0) = ad - bc = 1\,, \tag{18}$$

we conclude that (e'_1, e'_2) is a primitive basis of Λ_0. Comparison with (2) now shows that (Q', P') has $r'_1 = r$, $r'_2 = r'_3 = u'_1 = 1$ as required. Thus the only condition on the $SL(2, \mathbb{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for preserving the $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ condition is that it must have $b = 0$ mod r, i.e. it must be an element of $\Gamma^0(r)$.

Using this we can now determine the subgroup of $SL(2, \mathbb{Z})$ that takes a pair of charge vectors (Q, P) with invariants (r_1, r_2, r_3, u_1) to another pair of charge vectors with the same invariants. For this we note that any $SL(2, \mathbb{Z})$ transformation matrix $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b given in (8) takes the set (r_1, r_2, r_3, u_1) to the set $(r_1 r_2 r_3, 1, 1, 1)$. Since the latter set is preserved by the $\Gamma^0(r)$ subgroup of $SL(2, \mathbb{Z})$, the original set must be preserved by the subgroup $g_0^{-1}\Gamma^0(r) g_0$. This is isomorphic to the group $\Gamma^0(r)$.

To see an example of this consider the case

$$r_1 = r_2 = 1, \quad r_3 = 2, \quad u_1 = 1\,. \tag{19}$$

In this case the $SL(2,\ \mathbb{Z})$ transformation $g_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ takes a configuration given in (19) to a configuration with $r_1 = 2$, $r_2 = r_3 = u_1 = 1$. Thus the $SL(2,\ \mathbb{Z})$ transformations which take a configuration with $(r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1)$ to a configuration with the same discrete invariants will be of the form:

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 2b_0 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-c & a-c-d+2b_0 \\ c & c+d \end{pmatrix}\,. \tag{20}$$

Since the condition $ad - 2b_0c = 1$ requires a and d to be odd, we have

$$a' + b' \in 2\ \mathbb{Z} + 1, \qquad c' + d' \in 2\ \mathbb{Z} + 1\,. \tag{21}$$

Conversely given any $SL(2,\ \mathbb{Z})$ matrix $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ satisfying (21), it can be written as g_0 conjugate of the $\Gamma^0(2)$ matrix $\begin{pmatrix} a'+c' & -a'-c'+b'+d' \\ c' & -c'+d' \end{pmatrix}$. Thus (21) characterizes the subgroup of S-duality group which preserves the condition (19).

The results derived so far make it clear that for a given torsion r the discrete T-duality invariants are in one to one correspondence with the elements of the coset $SL(2,\ \mathbb{Z})/\Gamma^0(r)$. The representative element for a given set of invariants (r_1, r_2, r_3, u_1) is the element $g_0^{-1} \in SL(2,\ \mathbb{Z})$ that takes a configuration with $(r_1r_2r_3, 1, 1, 1)$ to a configuration with discrete invariants (r_1, r_2, r_3, u_1). Multiplying g_0^{-1} by a $\Gamma^0(r)$ element from the right does not change the final values (r_1, r_2, r_3, u_1) of the discrete invariants since a $\Gamma^0(r)$ transformation does not change the discrete T-duality invariants of the initial configuration.

We shall now examine the consequences of these results for the formula expressing the degeneracy $d(Q, P)$ – or more precisely an appropriate index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets for a given set of charges[1] – of quarter BPS dyons as a function of (Q, P). We note first of all that besides depending on (Q, P), the degeneracy can also depend on the asymptotic values of the moduli fields, collectively denoted as ϕ. We expect the dependence on ϕ to be mild, in the sense that the degeneracy formula should be ϕ independent within a given domain bounded by walls of

[1]Up to a normalization this is equal to the helicity trace $B_6 = Tr(-1)^{2h}h^6$ over all states carrying charge quantum numbers (Q, P). Here h denotes the helicity of the state.

marginal stability. It follows from the analysis of [7,8] that the decays relevant for the walls of marginal stability are of the form

$$(Q,P) \to (\alpha Q + \beta P, \gamma Q + \delta P) + ((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P)\,, \tag{22}$$

where α, β, γ, δ are not necessarily integers, but must be such that $\alpha Q + \beta P$ and $\gamma Q + \delta P$ belong to the Narain lattice Λ. If we denote by $m(Q,P;\phi)$ the BPS mass of a dyon of charge (Q,P) then the wall of marginal stability associated with the set $(\alpha,\beta,\gamma,\delta)$ is given by the solution to the equation

$$m(Q,P;\phi) = m(\alpha Q + \beta P, \gamma Q + \delta P;\phi) + m((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P;\phi)\,. \tag{23}$$

For appropriate choice of $(\alpha,\beta,\gamma,\delta)$ this describes a codimension one subspace of the moduli space labelled by ϕ. Since the BPS mass formula is invariant under a T-duality transformation $Q\to\Omega Q$, $P\to\Omega P$, $\phi\to\phi_\Omega$:

$$m(\Omega Q,\Omega P;\phi_\Omega) = m(Q,P;\phi) \qquad \Omega\in O(6,22;\ \mathbb{Z})\,, \tag{24}$$

eq.(23) may be written as

$$m(\Omega Q,\Omega P;\phi_\Omega) = m(\alpha\Omega Q + \beta\Omega P, \gamma\Omega Q + \delta\Omega P;\phi_\Omega) + m((1-\alpha)\Omega Q - \beta\Omega P, -\gamma\Omega Q + (1-\delta)\Omega P;\phi_\Omega)\,. \tag{25}$$

This is identical to eq.(23) with (Q,P,ϕ) replaced by $(\Omega Q,\Omega P,\phi_\Omega)$. This shows that under a T-duality transformation on charges and moduli, the wall of marginal stability associated with the set $(\alpha,\beta,\gamma,\delta)$ gets mapped to the wall of marginal stability associated with the same $(\alpha,\beta,\gamma,\delta)$. Thus if we consider a domain bounded by the walls of marginal stability associated with the sets $(\alpha_i,\beta_i,\gamma_i,\delta_i)$ for $1\le i\le n$ – collectively denoted by a set of discrete variables $\vec c$ – then under a simultaneous T-duality transformation on the charges and the moduli this domain gets mapped to a domain labelled by the same vector $\vec c$. The precise shape of the domain of course changes since the locations of the walls in the moduli space depends not only on $(\alpha_i,\beta_i,\gamma_i,\delta_i)$ for $1\le i\le n$ but also on the charges (Q,P) which transform to $(\Omega Q,\Omega P)$.

We now use the fact that the dyon degeneracy formula must be invariant under a simultaneous T-duality transformation on the charges and the moduli, and also the fact that the dependence of $d(Q,P;\phi)$ on the moduli ϕ comes only through the domain in which ϕ lies, i.e. the vector $\vec c$. Since $\vec c$ remains unchanged under a T-duality transformation, we have

$$d(Q,P;\vec c) = d(\Omega Q,\Omega P;\vec c)\,, \qquad \Omega\in O(6,22;\ \mathbb{Z})\,. \tag{26}$$

This shows that $d(Q,P;\vec{c})$ must depend only on (Q,P) via the T-duality invariants:

$$d(Q,P;\vec{c}) = f(Q^2, P^2, Q\cdot P, r_1, r_2, r_3, u_1; \vec{c})\,, \tag{27}$$

for some function f.

Let us now study the effect of S-duality transformation on this formula. Typically an S-duality transformation will act on the charges and hence on all the T-duality invariants and also on the vector $\vec{c}$ labelling the domain bounded by the walls of marginal stability [5, 9, 10]. Indeed, as is clear from the condition (23), under an S-duality transformation of the form (3), the wall associated with the parameters $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ gets mapped to the wall associated with

$$\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}. \tag{28}$$

Thus S-duality invariance of the degeneracy formula now gives

$$f(Q^2, P^2, Q\cdot P, r_1, r_2, r_3, u_1; \vec{c}) = f(Q'^2, P'^2, Q'\cdot P', r_1', r_2', r_3', u_1'; \vec{c}\,')\,, \tag{29}$$

where $\vec{c}\,'$ stands for the collection of the sets $\{\alpha_i', \beta_i', \gamma_i', \delta_i'\}$ computed according to (28). We now use the result that there exists a special class of S-duality transformations under which

$$(r_1', r_2', r_3', u_1') = (r_1 r_2 r_3, 1, 1, 1)\,. \tag{30}$$

Using this S-duality transformation we get

$$f(Q^2, P^2, Q\cdot P, r_1, r_2, r_3, u_1; \vec{c}) = f(Q'^2, P'^2, Q'\cdot P', r_1 r_2 r_3, 1, 1, 1; \vec{c}\,')\,. \tag{31}$$

Thus the complete information about the spectrum of quarter BPS dyons is contained in the set of functions

$$g(Q^2, P^2, Q\cdot P, r; \vec{c}) \equiv f(Q^2, P^2, Q\cdot P, r, 1, 1, 1; \vec{c})\,. \tag{32}$$

We shall focus our attention on this function during the rest of our analysis. Using the fact that $\Gamma^0(r)$ transformations leave the set $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ fixed, we see that

$$g(Q^2, P^2, Q{\cdot}P, r; \vec{c}) = g(Q'^2, P'^2, Q'{\cdot}P', r; \vec{c}\,') \quad \text{for} \begin{pmatrix}Q'\\ P'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}Q\\ P\end{pmatrix}, \quad \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma^0(r)\,. \tag{33}$$

In other words, the function $g(Q^2, P^2, Q\cdot P, r; \vec{c})$ is expected to have manifest invariance under the $\Gamma^0(r)$ subgroup of S-duality transformations.

So far our discussion has been independent of any specific formula for the function $g(Q^2, P^2, Q \cdot P, r; \vec{c})$. For $r = 1$ dyons an explicit formula for the function g has been found in a wide class of $\mathcal{N} = 4$ supersymmetric theories [5, 9–25]. In all the known examples the function g is obtained as a contour integral of the inverse of an appropriate modular form of a subgroup of $Sp(2, \mathbb{Z})$. In particular for heterotic string theory on T^6 the modular form is the well known Igusa cusp form of weight 10 of the full $Sp(2, \mathbb{Z})$ group, with the S-duality group $SL(2, \mathbb{Z})$ embedded in $Sp(2, \mathbb{Z})$ in a specific manner. Furthermore the dependence on the domain labelled by $\vec{c}$ is encoded fully in the choice of the integration contour and not in the integrand. If a similar formula exists for $g(Q^2, P^2, Q \cdot P, r; \vec{c})$ for $r > 1$, then our analysis would suggest that the integrand should involve a modular form of a subgroup of $Sp(2, \mathbb{Z})$ that contains $\Gamma^0(r)$ in the same way that the full $Sp(2, \mathbb{Z})$ contains $SL(2, \mathbb{Z})$. It remains to be seen if this constraint together with other physical constraints reviewed in [25] can fix the form of the integrand.

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