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Quantum Fluctuations in Radiation Dominated Anisotropic Cosmology*

Thant Zin Naing[1] & J. V. Narlikar, *Inter-University Centre for Astronomy and Astrophysics, Post Bag 4 , Pune 411 007, India*

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Abstract. Using the metric conformal transformation and simple path integral, Feynman propagator method, for computing its quantum fluctuations, we analyse the radiation dominated anisotropic Bianchi Type I cosmology. We proceed to show that the quantum conformal fluctuations diverge at the classical spacetime singularity, suggesting that a singularity free solution can exist in anisotropic cosmology in the quantum regime.

Key words. Quantum cosmology—quantum fluctuations—spacetime singularity.

1. Introduction

Quantum conformal fluctuations are supposed to play a fundamental role in the evolution and dynamics of the early universe. Narlikar (1978, 1979, 1981, 1984) had shown that the conformal degrees of freedom in metric can be quantized and that the procedure leads to the fluctuations around the solutions of the classical Einstein field equations. As the state of classical singularity is approached the quantum uncertainty diverges and within the range of quantum uncertainty non-singular final states are possible.

One of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe is the Bianchi Type I Cosmology. Unlike the FRW model which has the same scale factor for each of the three spatial directions, the Bianchi Type I Cosmology has a different scale factor in each direction, which produces an anisotropy in expansion (Sahni 1988; Saha & Shikin 1997). Narlikar (op. cit.) had shown that the quantum conformal fluctuations of a dust driven model diverge at the spacetime singularity. The purpose of the present paper is to extend our study to quantum conformal fluctuations of radiation dominated universes which may probably be taken as much more realistic, since like the FRW models the Bianchi model is expected to be radiation driven rather than dust driven, near the classical singularity. We shall also investigate the behavior of conformal fluctuations in a radiation dominated cosmology and use the Feynman path integral method to compute the probability amplitudes.

This paper is organized as follows. In section 2 a brief account of the path integral approach to the study of conformal fluctuations is given. In section 3, the classical

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[1] Permanent Address: Physics Department, Yangon University, Myanmar.

solution of a radiation dominated anisotropic Bianchi Type I cosmology is computed. In section 4 applications of the quantum conformal analysis of anisotropic Bianchi Type I Cosmology are discussed. The last section is devoted to the concluding remarks of our study.

2. The path integral and conformal degrees of freedom

One of the most tractable approaches to quantum gravity is via the Feynman path integral. The classical Einstein field equations are

$$R_{ik} - \frac{1}{2} g_{ik} R = -\frac{8\pi G}{c^4} T_{ik}, \tag{1}$$

where R is the scalar curvature, R_{ik} the Ricci tensor, g_{ik} the metric tensor, G the gravitational constant and c is the speed of light. In the path integral approach a unique geometry $\bar{\mathcal{G}}$ obtained as the solution of (1) is replaced by a variety of geometries $\mathcal{G}$ which should be possible with specified end conditions. One treats each geometry as a path or 'history' with a probability amplitude (Feynman and Hibbs 1965),

$$\exp \frac{iS[\mathcal{G}]}{\hbar}$$

where S = classical action computed for geometry $\mathcal{G}$ $\hbar$ = Planck's constant. In the notation of 3-Geometries $\mathcal{G}$, the probability amplitude is provided by,

$$K[\mathcal{G}_2, \Sigma_2; \mathcal{G}_1, \Sigma_1] = \int \exp\left[\frac{i}{\hbar} S[\mathcal{G}]\right] \mathcal{D}\mathcal{G}, \tag{2}$$

here Σ_1 and Σ_2 are two spacelike hypersurfaces, $\mathcal{G}$ and $\mathcal{G}_2$ are three geometries and $\mathcal{D}\mathcal{G}$ is the measure of the integral on the right hand side. K is the quantum mechanical propagator. As was pointed out by DeWitt (1976), the conformal degrees of freedom provide a simple subset of all possible geometries. Suppose one denotes by an overbar the solution of the classical Einstein equations and writes a 'nonclassical' metric conformal tensor as

$$g_{ik} = (1 + \phi)^2 \bar{g}_{ik}, \tag{3}$$

where ϕ is a scalar function of the spacetime coordinates. The Hilbert action principle is given by,

$$S = \frac{c^3}{16\pi G} \int R\sqrt{-g}\,d^4x + S_m, \tag{4}$$

where R = scalar curvature, g= determinant of g_{ik}, while S_m denotes the action of matter (and radiation) which gives rise to the energy momentum tensor in equation (1). From equations (3) and (4), one has,

$$\int R\sqrt{-g}\,d^4x \sim \int [(1 + \phi)^2 \bar{R} - 6\phi_i \phi^i] \sqrt{-\bar{g}}\,d^4x. \tag{5}$$

(The ~ indicates that the surface terms have been ignored. $\bar{R}$, $\bar{g}$ refer to the classical solution.) The problem of computation of K can now be restated in the following form,

$$K[\phi_2, \Sigma_2; \phi_1, \Sigma_1] = \sum \exp \frac{i}{\hbar}\left(S_m + \frac{c^3}{16\pi G}\int [(1+\phi)^2\bar{R} - 6\phi_i\phi^i]\sqrt{-\bar{g}}\mathrm{d}^4x \right) \tag{6}$$

The action of matter is given by

$$S_m = -\sum_a \int m_a ds_a, \tag{7}$$

where m_a = mass of ath particle and ds_a, the element of its propertime. If we assume that the universe is homogeneous, and continues to be so despite conformal fluctuations, then ϕ has only the time derivative. In the case of radiation dominated anisotropic cosmology, the curvature term vanishes simply because of the cancellation of trace in the stress tensor, and the matter term also vanishes (no massive particles exist). Then, the above equation (6) becomes,

$$K[\phi_2, \Sigma_2; \phi_1, \Sigma_1] = \sum \exp\left(\frac{3ic^3}{8\pi G\hbar}\int -\dot{\phi}^2\sqrt{-\bar{g}}d^4x \right). \tag{8}$$

If we integrate out the spatial part of equation (8),we get the 3-volume $\mathcal{V}$of the space and are left with only a one-dimensional path integral. We next apply these results to the Bianchi Type I model. Since we wish to explore the effect of quantum gravity, the model refers to the very early (near the spacetime singularity) universe when it was radiation dominated. We will consider such a model in the classical region first.

3. Radiation dominated anisotropic Bianchi Type I cosmology

The classical solution in this case is given by the line element,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - X^2(t)(\mathrm{d}x)^2 - Y^2(t)(\mathrm{d}y)^2 - Z^2(t)(\mathrm{d}z)^2 \tag{9}$$

where X, Y and Z are functions of time only. The Einstein equations for $X(t)$, $Y(t)$ and $Z(t)$ corresponding to the metric can be written as

$$\left(T_1^1 - \frac{1}{2}T\right) = \frac{\ddot{X}}{X} + \frac{\dot{X}}{X}\left(\frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right) = -\frac{8\pi\rho}{3}, \tag{10}$$

$$\left(T_2^2 - \frac{1}{2}T\right) = \frac{\ddot{Y}}{Y} + \frac{\dot{Y}}{Y}\left(\frac{\dot{Z}}{Z} + \frac{\dot{X}}{X}\right) = -\frac{8\pi\rho}{3}, \tag{11}$$

$$\left(T_3^3 - \frac{1}{2}T\right) = \frac{\ddot{Z}}{Z} + \frac{\dot{Z}}{Z}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}\right) = -\frac{8\pi\rho}{3}, \tag{12}$$

$$\left(T_0^0 - \frac{1}{2}T\right) = \frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} = 8\pi\rho, \tag{13}$$

since $T_k^i = 8\mathrm{p}\pi(\rho, -\frac{\rho}{3}, -\frac{\rho}{3}, -\frac{\rho}{3})$ and we take $c = G = 1$. From the continuity equations we have $T^i_{k;i} = 0$, hence $\rho\tau^{\frac{4}{3}}$ = constant, where $\tau = XYZ = \sqrt{-\bar{g}}$ From

Equations (10), (11) and (12) we have

$$\frac{X}{Y}=D_1\exp\left(A_1\int\frac{dt}{\tau}\right), \tag{14}$$

$$\frac{Z}{Y}=D_2\exp\left(A_2\int\frac{dt}{\tau}\right), \tag{15}$$

$$\frac{Z}{X}=D_3\exp\left(A_3\int\frac{dt}{\tau}\right), \tag{16}$$

where D_1, D_2, D_3 and A_1, A_2, A_3 are constants . There are functional relations such that $D_2 = D_1D_3, A_2 = A_1 + A_3$. Inserting equations (14), (15) and (16) into (13) gives us

$$\frac{\ddot{\tau}}{\tau}-\frac{2}{3}\left(\frac{\dot{\tau}}{\tau}\right)^2+\frac{2}{3}\frac{C}{\tau^2}=-\kappa\rho, \tag{17}$$

where $C = A^2{}_1 + A_3\, A_3^2$. For simplicity, we change the constant $\frac{2}{3}$ C into B and since $\kappa\rho\tau^{\frac{4}{3}}$ = constant, we denote this constant by A. Equation (17) now becomes,

$$\frac{\ddot{\tau}}{\tau}-\frac{2}{3}\left(\frac{\dot{\tau}}{\tau}\right)^2+\frac{B}{\tau^2}+\frac{A}{\tau^{4/3}}=0. \tag{18}$$

To solve the above differential equation, we use the conformal time coordinate,

$$\eta=\int\frac{dt}{\tau(t)},\ \text{i.e,}\ \dot{\eta}=\frac{d\eta}{dt}=\frac{1}{\tau}. \tag{19}$$

The first and second derivatives of τ with respect to time t are,

$$\dot{\tau}=\frac{d\tau}{d\eta}\frac{d\eta}{dt}=\frac{\tau'}{\tau}, \tag{20}$$

$$\ddot{\tau}=\frac{d\eta}{dt}\frac{d\dot{\tau}}{d\eta}=\frac{1}{\tau}\left(\frac{\tau''}{\tau}-\frac{\tau'^2}{\tau^2}\right), \tag{21}$$

where $'$ denotes the differentiation with respect to η. Putting equations (20) and (21) into equation (18) gives us

$$\frac{\tau''}{\tau^3}-\frac{5}{3}\frac{\tau'^2}{\tau^4}+\frac{B}{\tau^2}+\frac{A}{\tau^{4/3}}=0,\quad \text{or}$$
$$\tau''\tau-\frac{5}{3}\tau'^2+B\tau^2+A\tau^{8/3}=0. \tag{22}$$

Using the simple calculus one can show that the general solution to equation (22) is given by,

$$\tau^{-2/3}=\alpha e^{a\eta}+\beta e^{-a\eta}-\frac{A}{B}, \tag{23}$$

Here $\frac{A}{B}$ and, $a=\sqrt{\frac{2B}{3}}$ are constants. α and β are arbitrary constants. It is clear that whatever we choose for α and β the volume factor τ goes to zero at some stage. Thus singularity is inevitable. We want the asymptotic nature of the solution close to the singularity. Rewrite (23) as

$$\tau^{-2/3} = p\sinh(a\eta) + q\cosh(-a\eta) - \frac{A}{B}, \tag{24}$$

where p, q are arbitrary constants. Now look for the limit as $\eta \to -\infty$ and the asymptotic solution is given by,

$$\tau^{-2/3} \sim \frac{p}{2} e^{-a\eta}. \tag{25}$$

This is the relation between the scale factor and the conformal time coordinate close to the spacetime singularity ($\tau \to 0$) of the classical solution.

Hence, we have

$$a\eta = \text{const.} + \frac{2}{3}\ln\tau. \tag{26}$$

Since = $\dot{\eta} = \frac{1}{\tau}$, we have

$$t = \left(\frac{p}{2}\right)^{-\frac{3}{2}} \int e^{\frac{3}{2}a\eta} \mathrm{d}\eta. \tag{27}$$

Then, the conformal time for radiation dominated Bianchi Type I cosmology is given by,

$$\eta \sim \text{const.} + \frac{2}{3a}\ln t. \tag{28}$$

As described in section 2, the action functional for the radiation dominated cosmology is

$$S = -\frac{c^3}{16\pi G}\int 6\phi_i\phi^i\sqrt{-g}\,\mathrm{d}^4x = -\frac{3c^3\mathcal{V}}{8\pi G}\int \dot{\phi}^2\tau(t)\mathrm{d}t. \tag{29}$$

Using the standard path integral techniques, the probability amplitude for the conformal fluctuations to evolve from ϕ_1 at $t = t_1$ to f$_2$ at $t = t_2$ is given by,

$$K[\phi_2, t_2; \phi_1, t_1] = \int \exp\left(-\frac{3\mathcal{V}i}{8\pi}\int \dot{\phi}^2\tau(t)\mathrm{d}t\right)\mathcal{D}\phi, \tag{30}$$

where we take c= 1, G= 1 and $\hbar$ = 1 and $D\phi$ = measure of the integration (for details see Feynman and Hibbs 1965). Using the fact that $t = t(\eta)$ and ϕ =(dϕ /dη) × (dη/dt) = $\phi'\,\dot{\eta}$, we get

$$\int \dot{\phi}^2\tau(t)\mathrm{d}t = \int \phi'^2\dot{\eta}\tau(t)\mathrm{d}\eta = \int \phi'^2\mathrm{d}\eta \tag{31}$$

Now the propagator (30) can be written as

$$K[\phi_2, \eta_2; \phi_1, \eta_1] = \int \exp\left(-\frac{3\mathcal{V}i}{8\pi}\int_{\eta_1}^{\eta_2} \phi'^2\mathrm{d}\eta\right)\mathcal{D}\phi$$
$$= \frac{8\pi^2(\eta_2 - \eta_1)}{3\mathcal{V}}\exp\left(-\frac{3\mathcal{V}i(\phi_2 - \phi_1)^2}{8\pi(\eta_2 - \eta_1)}\right). \tag{32}$$

It is more interesting to apply the above propagator to study the evolution of the wavefunction of the universe. We have,

$$\Psi(\phi_2, \eta_2) = \int K[\phi_2, \eta_2; \phi_1, \eta_1]\Psi(\phi_1, \eta_1)\mathrm{d}\phi_1, \tag{33}$$

which connects the wavefunction at epoch h1 to the wavefunction at epoch η_2.

Normally this technique is used to compute the probability amplitude for evolution of a wavefunction *forward* in time. Thus we normally have $t_2 > t_1$. Here, however, we use the Feynman propagator *backward* in time to compute the probability amplitude for the wavefunction to have evolved from a state ϕ_2 to a state ϕ_1 i.e. $t_2 < t_1$. This technique was used by Narlikar (1981) for estimating the probable range of states at an earlier epoch t_2 from which the universe could have evolved to its present state ϕ_1 at time t_1. Since the present state is very nearly classical, an appropriate expression for Ψ (ϕ_1, η_1) is a Gaussian wavepacket,

$$\Psi(\phi_1, \eta_1) = (2\pi\Delta_1^2)^{-1/4}\exp\left(-\frac{\phi_1^2}{4\Delta_1^2}\right). \tag{34}$$

Using (32) and (34) one can easily compute $\Psi(\phi_2, \eta_2)$. The early state was also a Gaussian wavepacket but its spread Δ_2 was much larger:

$$\Delta_2{}^2 = \Delta_1{}^2 + \frac{4\pi^2(\eta_2 - \eta_1)^2}{9V^2\Delta_1{}^2}, \tag{35}$$

hence, since $\Delta_1 \ll 1$ and from equation (28), we get

$$\begin{aligned}\Delta_2 &\approx \frac{2\pi}{3V\Delta_1}(\eta_2 - \eta_1)\\ &= \frac{4\pi}{9aV\Delta_1}\ln\left(\frac{t_2}{t_1}\right).\end{aligned} \tag{36}$$

Thus Δ_2 diverges logarithmically as $t_2 \longrightarrow 0$. That is, close to the classical singularity, the quantum uncertainty diverges—a result found for dust dominated Bianchi Type I cosmologies also. This uncertainty therefore permits us to obtain non-classical non-singular solutions around the classical singular epoch, suggesting that a singularity free solution can exist in anisotropic radiation dominated cosmology in the quantum regime.

5. Concluding remarks

In the previous papers (Narlikar 1978, 1979, 1981, 1984), the quantum conformal fluctuations in a collapsing dust ball, the Schwarzschild spacetime and the Bianchi Type I dust dominated cosmology were investigated (see also Narlikar & Padmanabhan 1986). In the present paper we confined ourselves to radiation dominated anisotropic Bianchi Type I cosmology. The classical solution (relation between scale factor and conformal time) has been calculated and using this relation, quantum conformal fluctuations in the radiation dominated Bianchi Type I cosmology are obtained. The quantum dispersion diverges as $t_2 \longrightarrow 0$ (the classical singular epoch) and it indicates that quantum fluctuations dominate as the classical solution approaches singularity. The classical solution thus ceases to be reliable, and non-singular finite solutions can be found if one enlarges the theory to include these conformal effects of quantum gravity. In our future investigations we shall be concerned with non-conformal fluctuations of the classical solution.

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