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Some Consequences of a Spatially Varying Cosmological Constant in a Spherically Symmetric Distribution of Matter

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Abstract. This paper investigates the effects of the spatial variation of the cosmological constant Λ on the spacetime geometry within and outside a massive object. It is seen that the variation of Λ with the radial coordinate introduces non-trivial changes leading to spacetime closing on itself around a massive object. It may also be possible to generate interior solutions that lead to flat rotation curves of galaxies.

Key words: cosmologial constant—spatial variation

1. Introduction

The so called cosmological constant Λ has played numerous roles in the frame-work of general relativity. Introduced by Einstein (1917) in his field equations

$$G_{ik} \equiv R_{ik} - \tfrac{1}{2} g_{ik} R + \Lambda g_{ik} = -\kappa T_{ik}, \tag{1}$$

the part played by the Λ-term was to provide a repulsive counter to the gravitational attraction in a static universe. In de Sitter's model (1917), Λ led to an expanding universe. Subsequently the discovery by Hubble (1929) that the universe is expanding and the realization that the models developed by Friedman (1922, 1924) describe the expanding universe without recourse to the Λ-term led Einstein to discard that term from his field equations.

In recent times the Λ-term has interested theoreticians and observers for various reasons. The nontrivial role of the vacuum in the early universe generates a Λ-term that leads to the inflationary phase. Observationally this term provides an additional parameter to accommodate conflicting data on the values of the Hubble constant, the deceleration parameter, the density parameter and the age of the universe (Gunn & Tinsley 1975; Wampler & Burke 1987).

Assuming that Λ owes its origin to vacuum interactions, as suggested in particular by Sakharov (1988) it follows that it would in general be a function of space and time coordinates, rather than be a strict constant. In a homogeneous universe Λ will be at most time dependent. Such a model was considered recently by Peebles & Ratra

(1988). The aim of this attempt was to reconcile low dynamical estimates of mass density ($\Omega \cong 0.2$) with the $k \simeq 0$ flat model required by inflation. While this approach can generate Λ that varies both with space and time, as mentioned above, only time-variation of Λ was considered by the authors in the cosmological context.

In considering the nature of local massive objects, however, the space dependence of Λ cannot be ignored. In a quasi-static situation where the cosmological time scale is too large, it may in fact be a good approximation to ignore the time dependence of Λ and highlight its space-dependence. Could a spatial variation of Λ produce any appreciable observable effect on the morphology of galaxies? In particular can the flat rotation curves be related to the existence of Λ? Does a variable Λ help us understand the physical behaviour of super massive objects located in the nuclei of galaxies?

In this paper we undertake a preliminary study of these problems by exploring the solutions of (1)for a spherically symmetric situation in which a single massive object is the source of gravity. For reasons outlined earlier, we shall discuss static solutions only.

2. The field equations

The field Equations (1) imply (through the usual Bianchi identities) that

$$\kappa T^{i}_{k;i} = -\Lambda_{;k}. \tag{2}$$

In the normal interpretation, both sides of this equation vanish separately. However, in the present scenario there is a dynamic interaction between matter and the vacuum so that in general $\Lambda_{;k} \neq 0$. Thus there is non-conservation of T^{i}_{k}. In modern particle physics this effect is produced through the non-trivial behaviour of vacuum in a quantum field theory and usually a scalar field is invoked for this purpose. This situation is analogous to the C-field cosmology of Hoyle & Narlikar (1963) where apparent non-conservation of matter and energy was due to the existence of a cosmological creation field C. Needless to add that as in that framework here also, there is an overall conservation of matter and energy as implied by Einstein's field equations. (At the time the C-field theory was proposed, particle physics had not matured to the present day levels and nonconservation of baryons was considered anathema.) It is also worth recalling that it was in the steady state cosmological context that McCrea (1951) had first highlighted the possible dynamical role of a nontrivial vacuum.

There is one respect, however, in which the present analysis is different from the C-field cosmology. In the latter, the effect of the non-conservation of ordinary matter manifested itself only at the instant of creation of a particle. Therefore, the particle moved in a geodesic with the C-field no longer affecting its motion. In the present case the gradient of Λ always affects the motion of matter, as given by the right hand side of (2).

Thus in principle, the Λ-force would enable an observer to measure his motion relative to vacuum. To the extent that there exists a cosmological rest frame, this vacuum has a special status as in standard cosmology. In principle this vacuum would have fluctuations in time as well as in space. While in general one should include both types of fluctuations, we are concerning ourselves with situations in which the time scales (associated with galaxies and cosmology) are long enough to render the temporal fluctuations unimportant. Similarly, the spatial fluctuations are important only over the scale determined by the matter distribution *i.e.,* over the galactic scale or on the scale of a super massive object.

Finally, the extra force introduced by the gradient of Λ in (2) needs to be estimated in the laboratory context. A crude approximation may be made as follows.

It is clear that Λ decreases in magnitude as we move across the Galaxy. Taking the central value as given by the Zone I of Section 3 we get $\Lambda \sim 3M_N/R_N^3$ where M_N is the nuclear mass and R_N the nuclear size. Assuming that Λ falls to the cosmological value over the galactic radius (as per the rotation curve) R_G, the gradient is $\sim 3M_N/R_N^3 R_G$. Restoring G, c this becomes $3GM_N/C^2R_N^3 \; R_G$.

Now, the T^i_k in the left hand side of (2) is $(8\pi G/c^4)\rho u^i u_k$ in the smooth fluid approximation. In the locally flat reference frame in which the fluid is at rest this term gives simply $8\pi G\rho a/c^4$ where a is the measured acceleration. The Equation (2) now gives

$$a \sim \frac{3}{8\pi}\frac{M_N c^2}{R_N^3 R_G \rho}.$$

Thus in principle, the acceleration depends on density ρ, in violation of the weak principle of equivalence. For $M_N \sim 10^9 M_\odot$, R_N= 1 pc, R_G= 30 kpc and $\rho \sim 1$ g cm^{-3}, we get $a \sim 10^{-16}$ cm s^{-2} $\sim 10^{-19}$ g where g is the acceleration due to gravity. This effect; though small, is still an overestimate since, from Fig. 1, the gradient of Λ is expected to be steeper than estimated above near the nuclear region and flatter at the distances where we are located. At present this effect is below the limits set by

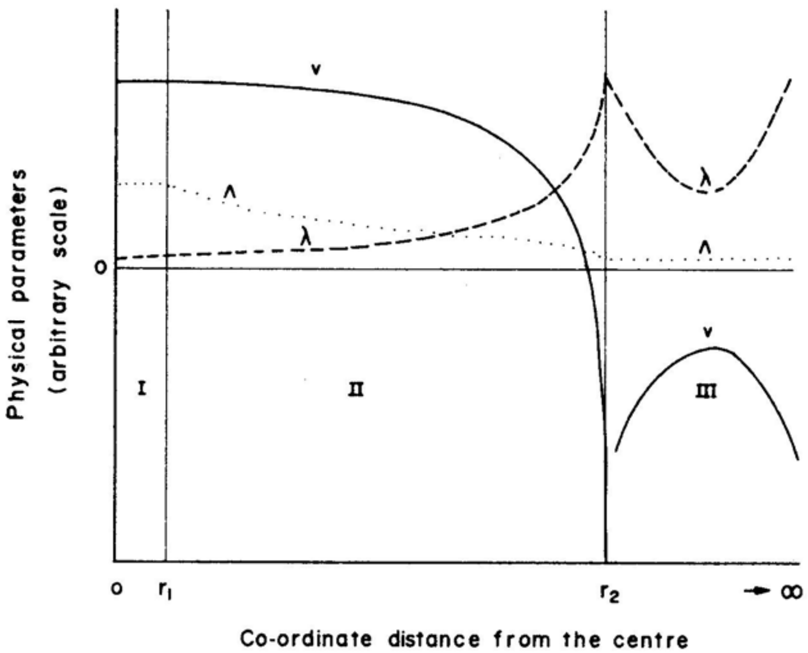

Figure 1. Behaviour of the physical parameters λ, ν and Λ. The generic behaviour of these parameters in the solutions of *Section 3* is shown by these curves.

experiments detecting the so called fifth force. However, a more detailed theory can estimate the effect more precisely.

The Equation (2) uses a smooth fluid approximation of the type commonly used in the equations of standard cosmology. In the Friedman models the uniform density assumption is only justified on a large enough scale, say 10–100 Mpc, in view of the discrete structure of matter. Superimposed on this are the random motions of galaxies in clusters moving under their N-body interaction. Likewise here we will treat ρ as an average density of interstellar medium over scales of $\sim 10^{-2}$ pc. Superimposed on this average motion one has to include motions of discrete objects like stars. Since here we are concerned more with rotation of hydrogen clouds, we will stick to the smooth fluid approximation in what follows.

Thus, to estimate the acceleration of the interstellar medium (ISM) we note that if the right hand side of (2) were zero (*i.e.*, if there were no Λ-force) then in the Newtonian approximation one would arrive at an acceleration of the ISM of the order of 10^{-8} cm s^{-2} at a distance of ~ 10 kpc from the Galactic centre. In an equilibrium situation with the Λ-force, the smooth fluid approximation would introduce small perturbations of the above situation and not significantly alter the Newtonian acceleration. Because of its ρ^{-1} dependence on density the Λ-force will have even less effect on the dynamics of the more dense objects like stars. We will therefore ignore motions of such objects in our analysis.

In the spherically symmetric and static case we will take the line element in the Schwarzschild coordinates:

$$ds^2 = c^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{3}$$

with ν and λ as functions of r only. We take $c = 1$, $G = 1$ so that k = 8π. The T^i_k is given by

$$T^i_k = \rho u^i u_k, \quad \rho = \rho(r), \quad u^i = (u, o, o, o) \tag{4}$$

Thus, the only non-zero component of T^i_k is $T^o_o = \rho$ = density of matter. As implied earlier, we also have $\Lambda = \Lambda(r)$.

The $\binom{1}{1}$ and $\binom{o}{o}$ components of the field equations take the form

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -\Lambda, \tag{5}$$

$$e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2} = -8\pi\rho - \Lambda. \tag{6}$$

The primes, as usual, denote derivatives with respect to r.

The conservation law (2) leads to an extra equation

$$\Lambda' = 4\pi\rho\nu'. \tag{7}$$

There are, however, four unknowns, λ, ν, ρ and Λ to be determined and thus we need an extra equation besides (5)–(7). Normally, this should be provided by the underlying dynamic interaction between matter and the Λ-term (as in the C-field case cited earlier).

Rather than speculate about this interaction on theoretical ground we will adopt a heuristic approach. We will postulate an extra condition that seems indicated by the observations of local massive objects. Two of the possible ways in which this can be done are described in the following sections. We do not claim any uniqueness about our somewhat ad-hoc assumptions. Rather it is to be seen as indicative of the possible new inputs that a variable Λ can bring to the relativistic descriptions of massive objects.

1. A triple-zone solution

Before proceeding with a mathematical model, we outline the physical scenario. Corresponding to a massive galaxy with a nuclear core situated in a tenuous intergalactic space we consider here a massive spherical object of finite extent with a dense core. Within the core itself we ignore spatial variations of physical quantities and take a uniform distribution to represent the actual one over the compact size of the core. Outside the core the spatial variations become important and quantities decrease outwards until the boundary of the object is reached. Beyond lies the cosmological spacetime. Our mathematical model reflects this scenario.

In this solution we have three zones, the first and the innermost one representing the core, the second (in the form of a shell) representing the envelope of the massive object and the third (outer) part describing matter-free space. The specific properties of the three zones are described below.

Zone I ($0< r < r_1$)

This is a miniature section of the Einstein universe with a constant value of $\Lambda(= A)$. The solution is given by

$$e^{\nu}=B=\text{constant}, \tag{8}$$

$$e^{-\lambda}=1-Ar^2 \tag{9}$$

$$\rho=\frac{A}{4\pi}. \tag{10}$$

$$\Lambda=A. \tag{11}$$

If this part were to describe the inner regions of a galaxy, then the Keplerian circular velocity of a gas cloud in this zone would increase in proportion to r.

Zone II ($r_1 \leqslant r < r_2$)

Following the galaxy model, if we wish the rotation curve to be flat, we should have $\rho \propto 1/r^2$ approximately. Taking this as the guiding point we assume that beyond $r = r_1$ the density should drop off as above while maintaining a continuity at $r = r_1$, with the value in Zone I. The Solution in this region is therefore described by the equations

$$\rho=\frac{Ar_1^2}{4\pi r^2}, \tag{12}$$

$$e^{-\lambda}=1-Ar_1^2\left(3-\frac{2r_1}{r}\right)-\frac{x}{r}, \tag{13}$$

$$\nu' = \frac{1}{Ar_1^2}\left(x'' - \frac{2x'}{r}\right), \tag{14}$$

$$\Lambda = \frac{x'}{r^2}, \tag{15}$$

$$\nu' + \lambda' = \frac{\rho}{8\pi} e^{\lambda} r. \tag{16}$$

The unknown function in these equations is $x(r)$ which is to be determined by eliminating ν and λ between the Equations (13), (14) and (16).

Zone III ($r \geqslant r_2$)

This is like the empty de Sitter universe outside a massive object. The solution is described by:

$$\rho = 0. \tag{17}$$

$$\Lambda = C = \text{constant}, \tag{18}$$

$$\nu + \lambda = 0, \tag{19}$$

$$e^{-\lambda} = 1 - \frac{D}{r} - \frac{C}{3} r^2, \quad D = \text{constant}. \tag{20}$$

The constants C and D are to be determined by the continuity of A, ν and Λ at $r = r_2$, between zones II and III. D is related to the gravitational mass of the object and C to the square of Hubble's constant.

The solution to the entire problem therefore essentially involves determining the function $x(r)$ in Zone II. This function satisfies a nonlinear differential equation of second order:

$$\left(1 - 2A - \frac{x}{r}\right)\left(x'' - \frac{2x'}{r}\right) + A\left(\frac{x'}{r} - \frac{x}{r^2}\right) - \frac{2A^2}{r} = 0. \tag{21}$$

This equation cannot be integrated analytically, but its numerical solutions for various values of its parameters are easy to obtain. The equations are integrated with $r_1 = 1$ and $r_2 = n$. The unit of length L can be fixed suitably afterwards. By substituting the values of c and G and expressing distances in L parsecs, density in g cm^{-3} and mass in solar mass units, we can convert the typical numerical solution to its astrophysical counterpart. Thus,

$$\rho \cong 1.2 \times 10^{-10} L^{-2} \mathrm{A} \cdot \mathrm{g\,cm}^{-3}. \tag{22}$$

The gravitational mass of the object is given by

$$M \cong (10^{13} LD) M_{\odot}. \tag{23}$$

The effective Hubble constant $H_o = 100\, h_o$ km s^{-1} Mpc^{-1} is given by

$$h_o = 1.92 \times 10^9\, L^{-1} C^{1/2}. \tag{24}$$

The chosen unit of L parsecs is arbitrary and can be scaled up or down to get more realistic values. From (24) and (23) we get

$$Mh_o = 1.92 \times 10^{22}\, DC^{1/2}\, M_\odot. \tag{25}$$

Thus for h_o=1, M= 10^{12} $M_\odot$ we need to pick a solution that satisfies $DC^{1/2} \simeq 5 \times 10^{-11}$. By adjusting the parameters A and r_2/r_1 we have generated a number of different numerical solutions. The general behaviour of λ, v and Λ is shown in Fig. 1. It is clear that Λ varies by a very small amount on the galactic scale in such solutions. Nevertheless the presence of Λ is necessary because these solutions are totally different in character from the $\Lambda = 0$ case.

It is clear that for $A \ll 1$, one can obtain models of relevance to galactic masses, by adjusting A and the ratio r_2/r_1. However, when A increases to an appreciable fraction of unity we cannot have arbitrarily large r_2/r_1. For example, for A=0.5 we have $r_2/r_1 < 2.5$. In this case B $\cong$ 0.13 C $\cong$ 0.17 and D $\cong$ 1.49. The reason why such cases cannot be extended to large values of r_2/r_1 is because the nonlinear differential equation satisfied by x becomes singular through the vanishing of $e^{-\lambda}$. This latter situation describes, massive objects with strong gravitational fields which make spacetime close onto itself. The Λ-term seems to play a more significant role in such cases, compared to when $A \ll 1$. The above conclusions are based on the assumption about $\rho(r)$ as given by (12). Within the present framework this is an empirical statement. However, Equation (7) indicates qualitatively that the rate of change of Λ is proportional to the magnitude of ρ. For a density range similar to that considered here, we expect the qualitative behaviour of Λ to be the same as obtained here.

It would, of course, be more realistic to consider disc-type systems rather than spherical ones, since the former are more realistic representations of galaxies in which flat rotation curves are observed. Axially symmetric solutions are more difficult in general relativity and an effort in their direction is justified only after we have convinced ourselves that a variable Λ can lead to significantly different solutions. This and the solution of the following section both suggest that such an effort in the future will be worthwhile.

4. A numerical solution, fitted to the exponential decay of density

The mathematical model described in Section 3 is by no means unique. It is given as an indication of what is possible under the assumption of a variable Λ. The somewhat artificial division of the solution into Zones II and III can be obviated by choosing an exponentially decreasing density outside the nuclear region. Thus instead of a sharp boundary the object 'tapers off' into an asymptotical low density background. We describe such a scenario by another mathematical model given below.

In order to solve the Equations (5), (6), (7), plus the "fourth" equation given by the definition of the function $\rho(r)$, we have to eliminate v'. After some manipulation we finally arrive at the equation for Λ'

$$\Lambda' = A(r)\frac{1}{r}\left(\frac{r - F(r) - \Lambda(r)r^3}{F(r)}\right) \tag{26}$$

where the function $F(r)$ is equal to: $r - \int_0^r r'^2(2A(r') + \Lambda(r'))dr'$ and where the function $A(r) = 4\pi\, \rho(r)$.

Equation (26) has to be solved numerically for a specified A (r). The function $A(r)$ can be defined either by some analytical formula, or by some interpolable table of figures. We have assumed, in our numerical procedure, the function $A(r)$ to have the form:

$$A(r)=A=\text{constant for } r\leqslant r_1$$
$$A(r)=(A-B)e^{-C(r-r_1)}+B \text{ for } r\geqslant r_1 \tag{27}$$

A typical example of A $r)$ is shown in Fig. 2, together with an example of the representation of our Section 3 above. At the origin, the two functions coincide.

The numerical integration of (26) runs obviously into some practical difficulty around the value $r = r_c$ defined by

$$F(r_c)=0 \tag{28}$$

Then Λ' apparently becomes infinite; however this is compensated by the following fact. Let us put $r - r_c = \varepsilon$; then to the first order in ε, one has:

$$F(r)\simeq -\varepsilon(2A(r)+\Lambda(r))r^2. \tag{29}$$

Hence:

$$\varepsilon\Lambda'(r)\simeq\frac{-A(r)}{2A(r)+\Lambda(r)}[1-r^2\Lambda(r)] \tag{30}$$

The term in brackets being very small when $\Lambda = \Lambda = 1/r^2$; one can avoid the singularity by the use of suitable cutoffs in the steps of r as one crosses the "barrier". The stability of the results fully justifies confidence in the steps used. It is easy to show that the value $r = r$ lies in a restricted range of values, depending on the behaviour of $A(r)$.

Fig. 3 illustrates the generic behaviour of $\Lambda(r)$. Starting with the value $\Lambda = A$ at r =0, it reaches a very flat minimum at $\Lambda = \Lambda_c$ and $r = r_c$. The fact that the minimum is flat is linked with the stability of the solution when one changes the interval of integration. After its flat minimum $\Lambda(r)$ increases slowly, and reaches a limit at large values of r, as seen in the figure.

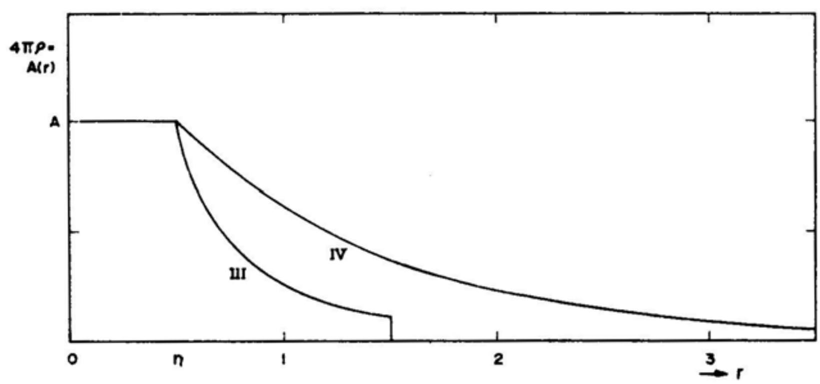

Figure 2. Behaviour of the reduced density $\pi\rho/A$. The curve labelled III corresponds to the definition of *Section 3*. One has here $r_1 = 0.5$ and $r_2 = 1.5$. The curve labelled IV corresponds to the definitions of *Section 4*. One has there $r_1 = 0.5$, $B = 0$, $C = 1$.

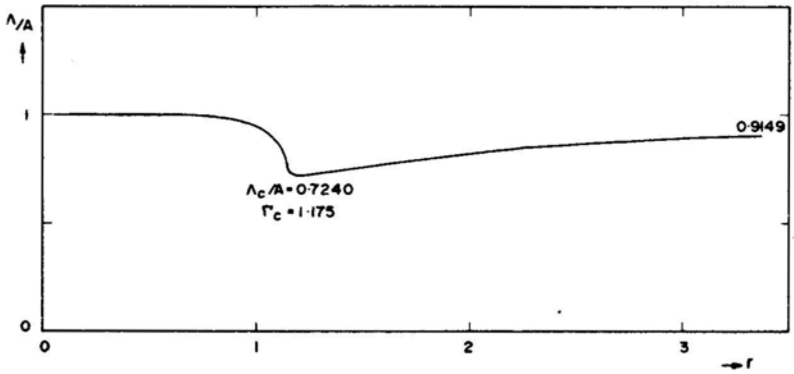

Figure 3. Typical Behaviour of Λ( ). The curve of the figure corresponds to the variation of density described on curve IV, Fig. 2.

The above situation was encountered in Section 3 also and can be interpreted as massive local objects that close the spacetime onto itself. As mentioned there the effect arises from the Λ' equation, *i.e.,* from the spatial variation of Λ.

5. Conclusion

These calculations indicate the broad behaviour of the local Λ-variation in gravitating objects. By and large the variation of Λ is governed by the density of matter. As shown in Section 3, the variation in Λ is very small for matter distribution of galactic interest. The effect may be somewhat more drastic in locations of high density. For example, increasing A to a value close to 0.5 makes it difficult to have a large value for the ratio r_2/r_1. The solution becomes 'singular' as r_2/r_1 approaches values in the range 2.5–3. This would imply space closing in on itself to make $e^{\nu} < 0$. A similar effect arises in the exponential decay from of ρ. It therefore seems a characteristic effect of variable -Λ on the spacetime geometry.

In a cosmological solution, we have to recognize that several such massive objects exist and so the cosmological value of Λ would be the net effect of 'summing' the asymptotic values from individual massive objects. Since a relativistic many body problem is intractable, one may assume a linearized approximation for this purpose. In that case the cosmological Λ may be a function of the total number of such masses.

Further work is needed to see whether the cosmological value of A can be related to the microwave background temperature ~3K as done by Sakharov. Also it is necessary to show that the model is stable. That is, with expansion or collapse, a variable A should introduce counteractive terms.

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