Generalized Fock spaces, new forms of quantum statistics and their algebras

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Abstract. We formulate a theory of generalized Fock spaces which underlies the different forms of quantum statistics such as 'infinite', Bose–Einstein and Fermi–Dirac statistics. Single-indexed systems as well as multi-indexed systems that cannot be mapped into single-indexed systems are studied. Our theory is based on a three-tiered structure consisting of Fock space, statistics and algebra. This general formalism not only unifies the various forms of statistics and algebras, but also allows us to construct many new forms of quantum statistics as well as many algebras of creation and destruction operators. Some of these are: new algebras for infinite statistics, q-statistics and its many avatars, a consistent algebra for fractional statistics, null statistics or statistics of frozen order, 'doubly-infinite' statistics, many representations of orthostatistics, Hubbard statistics and its variations.

1. Introduction

Recently, much effort has been devoted to q-deformations of oscillators. Both single oscillators [1–12] as well as systems of oscillators [13–33] have been studied. However, inspite of the larger literature which now exists, a unified picture of multioscillator systems covering the various q-deformations and algebras that have been proposed, has not emerged.

The aim of the present work is to construct a general formalism which may help one to develop such a unified picture. We construct a theory of generalized Fock spaces which has sufficient flexibility to encompass all types of oscillator algebras that have been proposed in the past as well as those that may be proposed in the future. Using this formalism, we are able to classify and clarify the interconnections that exist between different q-deformations and different algebras.

In another paper [34] we have pointed out that as far as a single-oscillator is concerned, q-deformation does not lead to anything fundamentally new, and is merely equivalent to a change of variable. A q-deformed oscillator is just a different representation of the usual oscillator. On the other hand, when we go to multioscillator systems, new things are possible; these are the new forms of statistics. However, even here, we find that many of the q-deformations for multioscillator systems proposed in the literature again belong to the category of substitution or change of variables and should be regarded only as different representations of the well-known Bose–Einstein or Fermi–Dirac statistics. Particular mention must be made of the work of Greenberg [13, 14] in this context. In contrast to most of the recent work on multioscillator systems which are

nothing but Bose or Fermi statistics in disguise, Greenberg's proposal involves a new statistics, called infinite statistics and this statistics is infact based on a new Fock space which is much larger than the usual bosonic or fermionic Fock spaces.

In order to construct a unified theory, we find it convenient to start with the underlying space of the allowed states of the system. We construct the creation and destruction operators $c^\dagger$ and c as outer products of the state vectors. It is this device of starting with the set of state vectors of the system as the primary concept, that cuts through the jungle of different algebras which, *prima facie* look different, but on closer examination are found to be related. Two sets of creation and destruction operators which are related by substitution, operate on the same space of state vectors and hence describe essentially the same class of systems.

We first formulate the theory of the generalized Fock spaces. The key element is the notion of independence of the permutation-ordered states. The largest linear vector space constructed in this way is the super Fock space. The subsequent specification of a subset of states in this space as null states leads to many reduced Fock spaces. The general theory which applies to the super Fock space as well as to the reduced Fock spaces, all of which are to be called collectively as generalized Fock spaces, then allows us to construct annihilation, creation and number operators. Whereas the annihilation and creation operators and their algebras even for a particular Fock space are not unique and many representations are possible, a universal representation for the number operators valid for all forms of statistics and algebras exists.

We apply the formalism to the super Fock space as well as to the bosonic and fermionic Fock spaces, the latter being the most familiar examples of reduced Fock spaces. Super Fock space is characterized by a unique statistics named 'infinite statistics'. However, the same infinite statistics can be represented by different algebras of c and $c^\dagger$ and we give a number of representations of infinite statistics. In the bosonic and fermionic Fock spaces, many different forms of statistics can be defined and for each form of statistics many different representations of c and $c^\dagger$ are possible.

Although the main aim of the general framework presented here is to develop a unified picture of the various forms of statistics and algebras, the same formalism also allows us to construct a large number of new forms of quantum statistics as well as new algebras of c and $c^\dagger$. In fact there is no limit to the number of possibilities.

An important part of our work deals with two-indexed systems. Our general formalism for the single-indexed system applies to most multi-indexed systems also since in most cases a multiplet of indices can be mapped into a single index. But, we show that there exist certain situations where such a mapping is not possible. Consequently, we develop the general formalism for the two indexed system and discover an enormously rich variety of novel forms of quantum statistics and algebras.

The plan of the paper is as follows. The theory of the generalized Fock spaces is given in § 2. In § 3 we apply the general theory to the super Fock space and in § 4 we apply it to the reduced bosonic and fermionic Fock spaces. Section 4 also contains a new statistics called null statistics. The alternative approach of starting with $cc^\dagger$ algebras and deriving cc relations therefrom is presented in § 5. Two-indexed systems are treated in § 6 and Appendix. Section 7 is devoted to summary and discussion.

A quick overview of the paper may be obtained from the pictorial summaries given in figures 3 and 4 and the tabular information provided in tables 1–4.

Generalized Fock spaces, statistics and algebras

2. Generalized Fock spaces

Given a set of oscillators with indices $g, h, i \cdots m$, we construct the state vector

$$|n_g, n_h \cdots n_m; 1\rangle = |\underbrace{1_g \cdots 1_g}_{n_g} \underbrace{1_h \cdots 1_h}_{n_h} \cdots \underbrace{1_m \cdots 1_m}_{n_m}\rangle \tag{2.1}$$

On the r.h.s. of this equation, $1_g, 1_h \cdots 1_m$ appear $n_g, n_h \cdots n_m$ times respectively and $n_g \cdots n_m$ denote the number of quanta with indices $g \cdots m$ respectively. Together with the state (2.1) we consider the set of all states obtained through all distinct permutations of the entries on the r.h.s. of (2.1). Thus, we have

$$|n_g, n_h \cdots n_m; 2\rangle = |\underbrace{1_g \cdots 1_g}_{n_g - 1} 1_h 1_g \underbrace{1_h \cdots 1_h}_{n_h - 1} \cdots \underbrace{1_m \cdots 1_m}_{n_m}\rangle \tag{2.2}$$

$$\vdots$$

$$|n_g, n_h \cdots n_m; 35\rangle = |1_g 1_g 1_m 1_m \underbrace{1_h \cdots 1_h}_{n_h} \cdots \underbrace{1_m \cdots 1_m}_{n_m - 2}\rangle \tag{2.3}$$

$$\vdots$$

$$|n_g, n_h \cdots n_m; s\rangle = |\underbrace{1_m \cdots 1_m}_{n_m} \cdots \underbrace{1_h \cdots 1_h}_{n_h} \underbrace{1_g \cdots 1_g}_{n_g}\rangle \tag{2.4}$$

Here we have given examples of a few permutations. In (2.2), the positions of one g-quantum and one h-quantum has been interchanged and in (2.3), we have indicated a few more interchanges. Collectively, we shall denote the set of all these states as

$$|n_g, n_h \cdots n_m; \mu\rangle, \quad \mu = 1, 2, \ldots s \tag{2.5}$$

where s is the total number of distinct permutations and μ labels each of these states. It is easy to see that

$$s = \frac{(n_g + n_h + \cdots + n_m)!}{n_g! n_h! \cdots n_m!} \tag{2.6}$$

(In (2.3), μ has been put 35 rather arbitrarily). We assume the existence of a unique vacuum state corresponding to zero occupation number for all the oscillators:

$$|0\rangle \equiv |0, 0, 0 \ldots 0\rangle \tag{2.7}$$

We shall first consider the set of all the s states given in (2.5) as linearly independent. Although they are linearly independent, they may not be orthogonal to each other in general, nor are they normalized. However, a state in one sector characterized by the occupation numbers $(n_g, n_h \ldots n_m)$ is orthogonal to any state in another sector characterized by another set of occupation numbers $(n'_g, n'_h \ldots n'_m)$. We can summarize these statements by the equation:

$$\langle n'_g, n'_h \ldots n'_m; \alpha | n_g, n_h \ldots n_m; \beta \rangle = \delta_{n'_g n_g} \delta_{n'_h n_h} \ldots \delta_{n'_m n_m} M_{\alpha\beta} \tag{2.8}$$

Note in particular that the inner product vanishes even if a single occupation number does not match. Within the same sector, the inner product is given by

$$\langle n_g \ldots n_m; \alpha | n_g \ldots n_m; \beta \rangle = M_{\alpha\beta} \tag{2.8a}$$

where M is a $s \times s$ hermitian matrix. In fact, there is an infinite set of matrices of varying dimensions, one corresponding to each sector $\{n_g \ldots n_m\}$. We choose all these matrices to be positive definite. This set of inner-product matrices M plays an important role in the general formalism.

From the set of linearly independent vectors given in (2.5), it is possible to construct an orthonormal set of vectors which we shall denote by a double-barred ket:

$$\| n_g \ldots n_m; \mu \rangle\!\rangle; \quad \mu = 1 \ldots s. \tag{2.9}$$

These satisfy the orthonormality relation

$$\langle\!\langle n'_g \ldots n'_m; \alpha \| n_g \ldots n_m; \beta \rangle\!\rangle = \delta_{n'_g n_g} \ldots \delta_{n'_m n_m} \delta_{\alpha\beta}. \tag{2.10}$$

There is no unique way of doing this and the resulting orthonormal set is not unique. One may use Gram–Schmidt orthogonalization procedure or calculate the eigenstates of the inner-product matrix M or follow any other method. Whatever may be the method, one can write the relation connecting the two sets of kets:

$$\| n_g \ldots n_m; \mu \rangle\!\rangle = \sum_v X_{v\mu} | n_g \ldots n_m; v \rangle \tag{2.11}$$

and the inverse relation:

$$| n_g \ldots n_m; \alpha \rangle = \sum_\beta (X^{-1})_{\beta\alpha} \| n_g \ldots n_m; \beta \rangle\!\rangle \tag{2.12}$$

where X is a non-singular matrix. Although X is not unique (since it depends on the particular orthogonalization procedure used), it is possible to show, using (2.8), (2.10), (2.11) and (2.12), that $XX^\dagger$ is the inverse of the innerproduct matrix [34a].

$$M^{-1} = XX^\dagger \tag{2.13}$$

Thus, one simple way of ensuring positivity of the inner-product matrix is to choose a non-singular matrix X and then determine M using (2.13). Again it must be kept in mind that we are dealing with an infinite set of matrices, X, one for each sector $\{n_g \ldots n_m\}$. Also the orthonormality relation holds between vectors from two different sectors and we have already used this in writing (2.10).

The completeness relation for the orthonormal set of states $\| n_g \ldots n_m; \mu \rangle\!\rangle$ can be written in the form:

$$I = \sum_{n_g \ldots n_m} \sum_\mu \| n_g \ldots n_m; \mu \rangle\!\rangle \langle\!\langle n_g \ldots n_m; \mu \| \tag{2.14}$$

where I is the identity operator. Substituting from (2.11) into (2.14) and using (2.13), we get the resolution of the identity operator in terms of the non-orthonormal set of states:

$$I = \sum_{n_g \ldots n_m} \sum_{\lambda, v} | n_g \ldots n_m; v \rangle (M^{-1})_{v\lambda} \langle n_g \ldots n_m; \lambda | \tag{2.14a}$$

It is convenient to define the projection operator

$$P(n_g \ldots n_m) = \sum_\mu \| n_g \ldots n_m; \mu \rangle\!\rangle \langle\!\langle n_g \ldots n_m; \mu \| \tag{2.15}$$

$$= \sum_{\lambda, v} | n_g \ldots n_m; v \rangle (M^{-1})_{v\lambda} \langle n_g \ldots n_m; \lambda | \tag{2.16}$$

so that we have

$$I = \sum_{n_g \dots n_m} P(n_g \dots n_m) \tag{2.17}$$

One can easily verify the following properties of the projection operators:

$$P(n_g \dots n_m) \| n_g' \dots n_m'; \mu \rangle\!\rangle = \delta_{n_g n_g'} \dots \delta_{n_m n_m'} \| n_g \dots n_m; \mu \rangle\!\rangle \tag{2.18}$$

$$P(n_g \dots n_m) | n_g' \dots n_m'; \mu \rangle = \delta_{n_g n_g'} \dots \delta_{n_m n_m'} | n_g \dots n_m; \mu \rangle \tag{2.19}$$

It is worth noting that $P(n_g \dots n_m)$ projects out any single state not only from the orthonormal set $\| n_g \dots n_m; \mu \rangle\!\rangle$ but also from the non-orthonormal set $| n_g \dots n_m; \mu \rangle$.

In terms of the above projection operators it is very easy to construct the number operators:

$$N_k = \sum_{n_g \dots n_k \dots n_m} n_k P(n_g \dots n_k \dots n_m) \tag{2.20}$$

which satisfy the following properties

$$N_k \| n_g \dots n_k \dots n_m; \mu \rangle\!\rangle = n_k \| n_g \dots n_k \dots n_m; \mu \rangle\!\rangle \tag{2.21}$$

$$N_k | n_g \dots n_k \dots n_m; \mu \rangle = n_k | n_g \dots n_k \dots n_m; \mu \rangle \tag{2.22}$$

$$[N_k, N_j] = 0 \quad \text{for any } k \text{ and } j \tag{2.23}$$

We now introduce the transition operators which connect states, lying in different sectors. Obviously it is enough to define the so-called annihilation and creation operators c_j and $c_j^\dagger$. We define

$$c_j^\dagger = \sum_{n_g \dots n_j \dots n_m} \sum_{\mu' \nu} A_{\mu' \nu} | n_g \dots (n_j + 1) \dots n_m; \mu' \rangle \langle n_g \dots n_j \dots n_m; \nu | \tag{2.24}$$

and c_j as the hermitian conjugate of $c_j^\dagger$, where $A_{\mu' \nu}$ are a set of arbitrary (complex) numbers. Note that the span of μ' is larger than that of ν and this is the reason for the prime on μ. Specifically,

$$\mu' = 1 \dots s'; \quad s' = \frac{(n_g + \dots (n_j + 1) + \dots n_m)!}{n_g! \dots (n_j + 1)! \dots n_m!} \tag{2.25a}$$

$$\nu = 1 \dots s; \quad s = \frac{(n_g + \dots n_j + \dots n_m)!}{n_g! \dots n_j! \dots n_m!} \tag{2.25b}$$

Hence A is a rectangular matrix. Since A is arbitrary in general, the relation (2.24) provides the most general definition of the creation operator. Even in this general case, it is possible to verify the following commutation relation between the number operator defined in (2.20) and the creation operator defined in (2.24)

$$[c_j^\dagger, N_k] = -c_j^\dagger \delta_{jk} \tag{2.26}$$

The projection property given in (2.19) plays a crucial role in the proof of (2.26).

So far we did not specify how the ordered state vectors $| n_g \dots n_m; \mu \rangle$ are constructed. In fact, in general there is no need to specify any procedure for their explicit construction. The formalism given so far holds whatever may be the explicit form of their construction. However, once annihilation and creation operators c and $c^\dagger$ are

introduced, there exists a natural procedure to construct the set of ordered states using $c^\dagger$ and the vacuum state $|0\rangle$. This procedure has the advantage that the arbitrariness of the matrix A introduced in (2.24) disappears and A in fact gets determined in terms of M. Hence let us do it.

In terms of creation operators, the ordered state (2.2) for instance is constructed as follows:

$$|\underbrace{1_g\ldots 1_g}_{n_g-1}1_h1_g\underbrace{1_h\ldots 1_h}_{n_h-1}\ldots\underbrace{1_m\ldots 1_m}_{n_m}\rangle = (c_g^\dagger)^{n_g-1}c_h^\dagger c_g^\dagger (c_h^\dagger)^{n_h-1}\ldots(c_m^\dagger)^{n_m}|0\rangle \quad (2.27)$$

and other states are constructed in a similar fashion. We may introduce the notation

$$|n_g\ldots n_m;\mu\rangle = (c_g^{\dagger n_g}\ldots c_m^{\dagger n_m};\mu)|0\rangle \quad (2.28)$$

$$\langle n_g\ldots n_m;\mu| = \langle 0|(c_m^{n_m}\ldots c_g^{n_g};\mu) \quad (2.29)$$

where $(c_g^{\dagger n_g}\ldots c_m^{\dagger n_m};\mu)$ is a permutation of the creation operators similar to the permutations defined in (2.1)–(2.5) and $(c_m^{n_m}\ldots c_g^{n_g};\mu)$ is defined as the hermitian conjugate of $(c_g^{\dagger n_g}\ldots c_m^{\dagger n_m},\mu)$. For states constructed in this manner, there exists a simple formula connecting states in 'adjacent' sectors. For instance,

$$|1_j,\underbrace{1_g\ldots 1_g}_{n_g}\ldots\underbrace{1_m\ldots 1_m}_{n_m}\rangle = c_j^\dagger|\underbrace{1_g\ldots 1_g}_{n_g}\ldots\underbrace{1_m\ldots 1_m}_{n_m}\rangle \quad (2.30)$$

More generally, we may write

$$c_j^\dagger|n_g\ldots n_j\ldots;\lambda\rangle = |1_j,n_g\ldots n_j\ldots;\lambda\rangle \quad (2.31)$$

where $|1_j,n_g\ldots n_j\ldots;\lambda\rangle$ on the right is a subset of states in which one j quantum appears on the extreme left. Although the total number of states in the set $|n_g\ldots(n_j+1)\ldots;\lambda'\rangle$ is s' given by (2.25a), the total number of states in the subset $|1_j,n_g\ldots n_j\ldots;\lambda\rangle$ is s given by (2.25b). Further the states in the subset $|1_j,n_g\ldots n_j\ldots;\lambda)$ are given the same ordinal number λ as in the set $|n_g\ldots n_j\ldots;\lambda\rangle$. This is possible since the quanta $\{n_g\ldots n_j\ldots\}$ are permuted among themselves without disturbing the extra j-quantum sitting on the extreme left.

We now substitute the expression for $c_j^\dagger$ given by (2.24) into the l.h.s. of (2.31). We have

$$c_j^\dagger|n_g\ldots n_j\ldots;\lambda\rangle$$

$$= \sum_{n_g'\ldots n_j'\ldots}\sum_{\mu'\nu}A_{\mu'\nu}|n_g'\ldots(n_j'+1)\ldots;\mu'\rangle\langle n_g'\ldots n_j'\ldots;\nu|n_g\ldots n_j\ldots;\lambda\rangle$$

$$= \sum_{\mu'\nu}A_{\mu'\nu}M_{\nu\lambda}|n_g\ldots(n_j+1)\ldots;\mu'\rangle \quad (2.32)$$

where we have used (2.8). On comparing with (2.31), we get

$$\sum_\nu A_{\mu'\nu}M_{\nu\lambda} = \delta_{\mu'\lambda} \quad (2.33)$$

From (2.33) we see that $\mu'=\lambda$. This means that in (2.24), only the subset $|1_j,n_g\ldots n_j\ldots;\lambda\rangle$ contributes and hence A is in fact a square matrix and is equal to the

inverse of M^{-1}:

$$A = M^{-1} \tag{2.34}$$

So, we may rewrite (2.24):

$$c_j^\dagger = \sum_{n_g \ldots n_j \ldots} \sum_{\lambda, v} (M^{-1})_{\lambda v} |1_j, n_g \ldots n_j \ldots; \lambda\rangle \langle n_g \ldots n_j \ldots; v|. \tag{2.35}$$

Thus, A and hence c and $c^\dagger$ are completely determined in terms of the set of M matrices. Next, consider the expression for the number operator (eqs. (2.20), (2.15) and (2.16)):

$$N_k = \sum_{n_g \ldots n_k \ldots n_m} n_k \sum_\mu \|n_g \ldots n_k \ldots n_m; \mu\rangle\!\rangle \langle\!\langle n_g \ldots n_k \ldots n_m; \mu\| \tag{2.36}$$

$$= \sum_{n_g \ldots n_k \ldots n_m} n_k \sum_{\lambda v} |n_g \ldots n_k \ldots n_m; \lambda\rangle (M^{-1})_{\lambda v} \langle n_g \ldots n_k \ldots n_m; v| \tag{2.37}$$

One may try to express this in terms of c and $c^\dagger$. Using (2.28) and (2.29),

$$N_k = \sum_{n_g \ldots n_k \ldots n_m} n_k \sum_{\lambda v} (c_g^{\dagger n_g} \ldots c_k^{\dagger n_k} \ldots c_m^{\dagger n_m}; \lambda)|0\rangle$$

$$\langle 0|(c_m^{n_m} \ldots c_k^{n_k} \ldots c_g^{n_g}; v)(M^{-1})_{\lambda v} \tag{2.38}$$

Apart from $c^\dagger$ and c, the above expression contains the vacuum projector $|0\rangle \langle 0|$. Later we shall give examples where $|0\rangle \langle 0|$ is determined as products of $c^\dagger$ and c so that N_k can be expressed entirely in terms of $c^\dagger$ and c. However, in general, this does not lead to simple results, whereas (2.36) provides us with a universal representation of number operators which is valid in all cases.

So far, we regarded the set of state vectors $|n_g, n_h \ldots; \mu\rangle$; $\mu = 1 \ldots s$ (where s is given by (2.6)) to be linearly independent and the resulting generalized Fock space is the complete Fock space, which we shall call the *super Fock space.*

We shall now show how to construct reduced Fock spaces. The motivation for this is that many Fock spaces of physical interest such as the bosonic Fock space or fermionic Fock space are reduced Fock spaces. There are various ways of doing this. One may postulate relationships between states connected by permutations, or one may disallow certain permutations by equating them to null vectors. Yet another way to achieve this is to use the permutation group S_n acting on the n-particle state. The super Fock space we have constructed consists of all the representations of the permutation group. If we allow only certain representations of S_n, we get a reduced Fock space.

All these possibilities are contained in the statement that in the space of vectors $|n_g, n_h \ldots; \mu\rangle$, there are r null vectors ($r < s$)

$$\sum_\mu B_\mu^p |n_g, n_h \ldots; \mu\rangle = 0; \quad p = 1, 2, \ldots r; \quad r < s. \tag{2.39}$$

where B_μ^p are constants. So, the dimension of the vector space in the sector $\{n_g, n_h \ldots\}$ is reduced to d given by

$$d = s - r. \tag{2.40}$$

An important class of reduced Fock spaces are those for which $d = 1$. Here, all the states connected by permutation of indices will be taken to be related to each other through equations of the type (2.39). In other words, the number of relations r in (2.39) is $s - 1$. We shall call this space as the *bosonic Fock space*. If we impose the additional restriction: $n_g = 0$ or 1 only, the resulting space will be called *fermionic Fock space*. This restriction can also be stated in the form of (2.39):

$$|n_g, n_h \ldots; \mu\rangle = 0 \quad \text{for} \quad n_g, n_h \ldots \geqslant 2. \tag{2.41}$$

We can define a new reduced Fock space, also of dimension $d = 1$ in each sector, by taking the set of all the permuted states as null states except a single state (of a chosen order) which is taken to be the allowed state. We shall call this as the Fock space of frozen order.

Another important class of reduced Fock spaces are those associated with parastatistics [35–37], which we shall call parabosonic and parafermionic Fock spaces. For these, the number of relations r is smaller than for bosonic or fermionic Fock spaces, so that the dimension d satisfies

$$1 < d < s. \tag{2.42}$$

The formalism constructed in the present section is valid for all these reduced Fock spaces also, with the modification that all the summations over μ, ν etc. will now go over the range $1 \ldots d$ and correspondingly; X, M and A become $d \times d$ matrices. There is an arbitrariness in the choice of the d states. Any choice of d states will do, as long as they are non-null states.

All these Fock spaces, the super Fock space as well as the reduced ones will be collectively called generalized Fock spaces. To sum up this discussion we may note that a generalized Fock space is completely defined by stating what are the allowed states of the system.

We shall define statistics by the precise relationship linking states obtained by permutation. In general, many relationships can be envisaged and hence many different forms of statistics can reside within a particular reduced Fock space. However, in super Fock space, all the states obtained by permutation are independent and so there is a unique statistics associated with this Fock space, namely 'infinite statistics'. Similarly, in the Fock space of frozen order too, there exists only a single statistics, named 'null statistics'.

In this section we started with the generalized Fock space consisting of the set of allowed states of the system and constructed the creation, annihilation and number operators in terms of the outer products of state vectors. Do these c and $c^\dagger$ form an operator algebra? In general, c and $c^\dagger$ constructed in this way may not form a simple algebra, or even a closed algebra. Historically, it is the reverse route that has been followed; one postulates an algebra of c and $c^\dagger$ and then deduces the states allowed by the algebra. In this sense, a given relation involving c and $c^\dagger$ implicitly defines an inner product and through it specifies the allowed and the null states of the system. In practice, starting with an algebra is an easier procedure and we shall use it in the latter sections. Actually, it is complementary to the approach described in this section.

Therefore we have two equivalent ways of dealing with the generalized Fock spaces. In the first approach, which we have formulated in this section we construct c and $c^\dagger$ in terms of the allowed states of the system and the algebra of c and $c^\dagger$ is then a derived

consequence. This is the more fundamental approach and is of universal validity. In the second approach, which we have used in the latter sections, we start with an algebra of c and $c^\dagger$ and then determine the states of the system allowed by the $cc^\dagger$ algebra. Since the res···trictions on the allowed states of the system can generally be stated in the form of cc relations, the first approach can be characterized as $cc \to cc^\dagger$ while the second is $cc^\dagger \to cc$. Within the second approach, an elegant method to derive cc relations from $cc^\dagger$ algebras will be described.

For infinite statistics, there is no restriction on the allowed states and cc relations do not exist. Hence, for this statistics, the first approach should be interpreted as 'no cc relation' $\to cc^\dagger$ algebra while in the second approach, starting with any particular $cc^\dagger$ algebra describing infinite statistics one shows that there does not exist any cc relation. In those cases where the $cc^\dagger$ algebra depends on a continuous parameter q, one can determine the values of q where cc relation exists. Generally, these values of q correspond to the boundary of the region in the parameter space where infinite statistics with positive definite M exists. On this boundary, one or more eigenvalues of the M matrices become zero, thus leading to the emergence of the same number of null vectors in the Fock space which can equivalently be interpreted as the emergence of cc relations. Thus the formalism unifies infinite statistics residing on the super Fock space with the various forms of statistics residing on reduced Fock spaces.

It must be noted that the inner product matrices M occurring in (2.35) are quite arbitrary. Consequently, more than one realization or representation of creation and destruction operators is possible. In fact, it is this freedom to select arbitrary M which enables one to construct different algebras involving c and $c^\dagger$, all operating over the same Fock space.

To sum up, we construct a three-tiered structure consisting of Fock space, statistics and algebra. Fock space is specified by the set of the allowed states of the system. Statistics is defined by the nature of the symmetry of the allowed states under permutation. Algebra of the creation and destruction operators is determined by the choice of the inner product matrices M.

Different representations of infinite statistics are constructed in §3. In the bosonic and fermionic Fock spaces, many forms of quantum statistics which include Bose and Fermi statistics are possible. These and the null statistics in the Fock space of frozen order are taken up in §4.

3. Super Fock space and infinite statistics

3.1 *The standard representation* $(M = 1)$

Having taken all the s states to be independent, the simplest choice of the matrix X is the unit matrix which implies M and A also to be unit matrices (of appropriate dimensions) for all the sectors $\{n_g, n_h \ldots\}$:

$$X = M = A = 1. \tag{3.1}$$

For this choice, which we shall call the standard representation of infinite statistics, the ordered states (2.5) themselves form an orthonormal set:

$$\langle n_g, n_h \ldots; \mu | n_g, n_h \ldots; v \rangle = \delta_{\mu v}. \tag{3.2}$$

Further, the creation and annihilation operators are given by

$$c_j^\dagger = \sum_{n_g, n_h \dots} \sum_\mu |1_j, n_g, n_h \dots; \mu\rangle \langle n_g, n_h \dots; \mu| \tag{3.3}$$

$$c_j = \sum_{n_g, n_h \dots} \sum_\mu |n_g, n_h \dots; \mu\rangle \langle 1_j, n_g, n_h \dots; \mu| \tag{3.4}$$

From (3.3) and (3.4), using the orthogonality and completeness relations, one can derive the $cc^\dagger$ algebra:

$$c_i c_j^\dagger = \delta_{ij} \tag{3.5}$$

This algebra was first proposed by Greenberg [13], in an important paper on infinite statistics.

In the standard representation, we also have the useful identity:

$$\sum_i c_i^\dagger c_i = 1 - |0\rangle\langle 0| \tag{3.6}$$

Although this identity can be obtained from (3.3) and (3.4) that define $c^\dagger$ and c, it is important to note that within Fock space it follows from the $cc^\dagger$ algebra (3.5). We shall prove it by showing that (3.6) is valid, when applied on any state in the super Fock space. First applying on $|0\rangle$, we find that both sides are zero. Next we apply on the other states $c_k^\dagger c_l^\dagger \dots |0\rangle$:

$$\left\{ \sum_i c_i^\dagger c_i - 1 + |0\rangle\langle 0| \right\} c_j^\dagger c_k^\dagger \dots |0\rangle = 0 \tag{3.7}$$

Using (3.5) and noting $\langle 0|c_j^\dagger = 0$ we see that the left side of (3.7) infact vanishes, thus completing the proof of (3.6).

In a different context, Cuntz [38] had studied the algebra defined by (3.5) and by the relation:

$$\sum_i c_i^\dagger c_i = 1. \tag{3.8}$$

Cuntz algebra is inconsistent with Fock space, as can be seen by applying both sides of (3.8) on $|0\rangle$. In contrast, our (3.6), because of the inclusion of the vacuum projector term in it, is consistent with Fock space and is infact a consequence of the algebra of c and $c^\dagger$ in the standard representation of infinite statistics.

Putting $M = 1$, the number operator N_k given in (2.38) becomes

$$N_k = \sum_{n_g \dots n_k \dots n_m} n_k \sum_\mu (c_g^{\dagger n_g} \dots c_k^{\dagger n_k} \dots c_m^{\dagger n_m}; \mu)|0\rangle\langle 0|(c_m^{n_m} \dots c_k^{n_k} \dots c_g^{n_g}; \mu) \tag{3.9}$$

Substitution of $|0\rangle\langle 0|$ from (3.6) into this equation and a straightforward but tedious calculation finally leads to

$$N_k = \sum_{n_g \dots n_k \dots n_m} \sum_\mu (c_g^{\dagger n_g} \dots c_k^{\dagger n_k} \dots c_m^{\dagger n_m}; \mu)c_k^\dagger c_k (c_m^{n_m} \dots c_k^{n_k} \dots c_g^{n_g}; \mu) \tag{3.9a}$$

This expression is identical to Greenberg's formula for N_k [13], though it is written in a different form. This illustrates the derivation of the representation of N_k in terms of c and $c^\dagger$. As already mentioned, in general neither the procedure nor the result is simple

and in any case one does not need it. The universal representation of N_k given in (2.36) is sufficient.

3.2 *q-mutators with real q*

Many other choices of M are possible. A particularly interesting choice is the one in which M is given as a function of a real parameter q. Consider the inner product between the n-particle state vectors with all occupation numbers unity: $|1_g, 1_h...; \mu\rangle$. The inner product matrix M in this sector has dimension $n! \times n!$ and its matrix element is taken to be

$$\langle 1_g, 1_h...; \mu| 1_g, 1_h...; \nu\rangle = q^J, \tag{3.10}$$

where q is a real number lying in the range $-1 < q < +1$ and J is the number of inversions required to transform the state $|1_g, 1_h...; \nu\rangle$ into the state $|1_g, 1_h...; \mu\rangle$. Number of inversions is the minimum number of successive interchanges between adjacent quanta that will take the state $|1_g, 1_h...; \nu\rangle$ to $|1_g, 1_h...; \mu\rangle$. For example,

$$\langle 1_h 1_g 1_k| 1_g 1_h 1_k\rangle = q \tag{3.11}$$

$$\langle 1_h 1_k 1_g| 1_g 1_h 1_k\rangle = q^2. \tag{3.12}$$

The positivity of the q-dependent M matrices defined above has been proved by Fivel [15] and Zagier [16]. In particular, Zagier has given the explicit form of the determinant of the $(n! \times n!)$ dimensional M matrix for arbitrary n:

$$\det M = \prod_{k=1}^{n-1} [1 - q^{k(k+1)}]^{(n-k)n!/k(k+1)} \tag{3.13}$$

At $q = 0$, M is the unit matrix and so is positive-definite. Hence, it will remain positive-definite in $-1 < q < 1$, if $\det M$ has no zeroes there. According to (3.13), for real q, zeroes of $\det M$ occur only at $q = \pm 1$, thus proving the positive-definiteness of M in the range $-1 < q < 1$.

The inner product for states with occupation numbers larger than unity (which are the same as states with repeated indices) is obtained from the above inner product for states with distinct indices by symmetrizing with respect to the repeated indices. For example, consider

$$\langle 2_g 1_m| 1_g 1_m 1_g\rangle = \langle 1_g 1_g 1_m| 1_g 1_m 1_g\rangle \tag{3.14}$$

Replace one of the g's by h in both the initial and final states and thus get a matrix element with distinct indices which can be calculated using (3.10). This replacement can be done in $(2!)^2$ ways. The sum of these $(2!)^2$ matrix elements divided by 2! is the required answer. Thus,

$$\langle 1_g 1_g 1_m| 1_g 1_m 1_g\rangle = \tfrac{1}{2}[\langle 1_g 1_h 1_m| 1_g 1_m 1_h\rangle + \langle 1_g 1_h 1_m| 1_h 1_m 1_g\rangle$$
$$+ \langle 1_h 1_g 1_m| 1_g 1_m 1_h\rangle + \langle 1_h 1_g 1_m| 1_h 1_m 1_g\rangle] \tag{3.15}$$

$$= q + q^2 \tag{3.16}$$

It is clear that the matrix M for repeated indices is obtained from the higher dimensional matrix for distinct indices by a process of collapse or reduction. It can be shown that this process retains the positivity of the matrix.

It is worth noting that the norm for n-particle state with all the indices repeated is

$$\langle n_g | n_g \rangle = [n_g!]_q = [n_g]_q [n_g - 1]_q \cdots [2]_q [1]_q \tag{3.17}$$

where the 'q-number' $[n]_q$ is defined by

$$[n]_q \equiv \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots q^{n-1} \tag{3.18}$$

Inverting these M matrices and defining the creation operator through (2.35), one can derive the algebra of c and $c^\dagger$. This is more difficult than the reverse procedure which is the way this subject really developed. Greenberg [14] proposed the q-mutator algebra:

$$c_i c_j^\dagger - q c_j^\dagger c_i = \delta_{ij} \tag{3.19}$$

The q-dependent M matrices defined through (3.10), (3.15) and (3.16) follow from this algebra. Fivel and Zagier then proved the positivity of these M matrices for $-1 < q < +1$. In spite of the fact that we have not derived (3.19) from the M matrices defined here, we can assert its validity because the M matrices completely determine c and $c^\dagger$. It is in order to emphasize this point that we have presented the M matrices first and the algebra of c and $c^\dagger$ as a derived consequence.

3.3 *New representations of infinite statistics*

First we briefly consider the 2-parameter algebra

$$c_i c_j^\dagger - q_1 c_j^\dagger c_i - q_2 \delta_{ij} \sum_k c_k^\dagger c_k = \delta_{ij} \tag{3.20}$$

where q_1 and q_2 are real parameters. This may be regarded as another representation of infinite statistics. Although the determination of the full region in the $\{q_1, q_2\}$ parameter space for which M is positive definite is still an unsolved problem, one can show [19] positivity of M on the straight line defined by $q_1 = 0$; $-1 < q_2 < \infty$. Infact it is possible to map this whole line on to the point $q_1 = 0$; $q_2 = 0$ by a redefinition of c and $c^\dagger$ so that we just get back the standard representation defined by (3.5).

We next present two new algebras:

$$c_i c_j^\dagger - c_j^\dagger c_i = \delta_{ij} p^{2\Sigma_{k<i} N_k} p^{N_i} \tag{3.21}$$

and

$$\left. \begin{array}{l} c_i c_j^\dagger - p^{-1} c_j^\dagger c_i = 0 \quad \text{for} \quad i \neq j \\ c_i c_i^\dagger - c_i^\dagger c_i = p^{N_i} \end{array} \right\} \tag{3.22}$$

where p is a real parameter and N_i are number operators already written down in § 2. Of these two algebras, the first one (3.21) is based on ordered indices, that is, given any two indices i and j, one must be considered larger or smaller than the other. So, we may take the indices to be the natural numbers $1, 2, 3 \ldots$

Equations (3.21) and (3.22) are again representations of the same infinite statistics, for it is possible to map both these algebras on to Greenberg's q-mutator algebra of (3.19) with the following identification of the parameters: $p = q^{-1}$. Temporarily renaming the $(c, c^\dagger)$ of (3.21) and (3.22) by $(b, b^\dagger)$ and $(d, d^\dagger)$ respectively, the mapping or

transformation equations are

$$b_i = p^{\Sigma_{k < i} N_k} p^{(1/2)N_i} c_i \tag{3.23}$$

$$d_i = p^{(1/2)N_i} c_i. \tag{3.24}$$

Hence, it is clear that both the algebras of (3.21) and (3.22) lead to positive definite M matrices for $p > +1$ and $p < -1$ (corresponding to $-1 < q < +1$).

An interesting feature of the algebra (3.21) is the validity of ordinary commutation relation between c_i and $c_j^\dagger$ for $i \neq j$ as in Bose statistics; nevertheless the full algebra (3.21) describes infinite statistics!

In contrast to the situation for the q-mutator algebra of (3.19), where the number operators can be expressed in terms of c and $c^\dagger$ only after considerable algebraic manouvers [16, 17] and the resulting expressions are quite complicated, the corresponding expressions for the algebras (3.21) and (3.22) are simple. For (3.21), we get (taking $p > 0$)

$$\left. \begin{aligned} N_1 &= \frac{1}{\log p} \log(c_1 c_1^\dagger - c_1^\dagger c_1) \\ &\vdots \\ N_i &= \frac{1}{\log p} \log(c_i c_i^\dagger - c_i^\dagger c_i) - 2 \sum_{k < i} N_k \end{aligned} \right\} \tag{3.25}$$

For (3.22), the number operator is even simpler:

$$N_i = \frac{1}{\log p} \log(c_i c_i^\dagger - c_i^\dagger c_i) \quad \text{for all } i. \tag{3.26}$$

In spite of our ability to write down such formal expressions for N_i in terms of c's and $c^\dagger$s, we must also point out that they are not of much use. All that one ever needs of the number operators are the properties contained in (2.21)–(2.23) and (2.26) and as for explicit representation, (2.36) and (2.37) will do.

3.4 q-mutators with complex q

We now generalize Greenberg's q-mutator algebra to complex q. This generalized algebra is based on ordered indices and is defined by the following equations.

$$c_i c_j^\dagger - q c_j^\dagger c_i = 0 \quad \text{for} \quad i < j \tag{3.27}$$

$$c_i c_i^\dagger - p c_i^\dagger c_i = 1 \tag{3.28}$$

where q and p are complex and real parameters respectively and the indices i, j etc. refer to any of the natural numbers $1, 2, 3, \ldots$. The relation for the opposite order $i > j$ is derivable from (3.27) by hermitian conjugation:

$$c_i c_j^\dagger - q^* c_j^\dagger c_i = 0 \quad \text{for} \quad i > j \tag{3.27'}$$

and so it is not an independent relation.

Let us now calculate the inner product matrix for this algebra. For distinct indices, we find

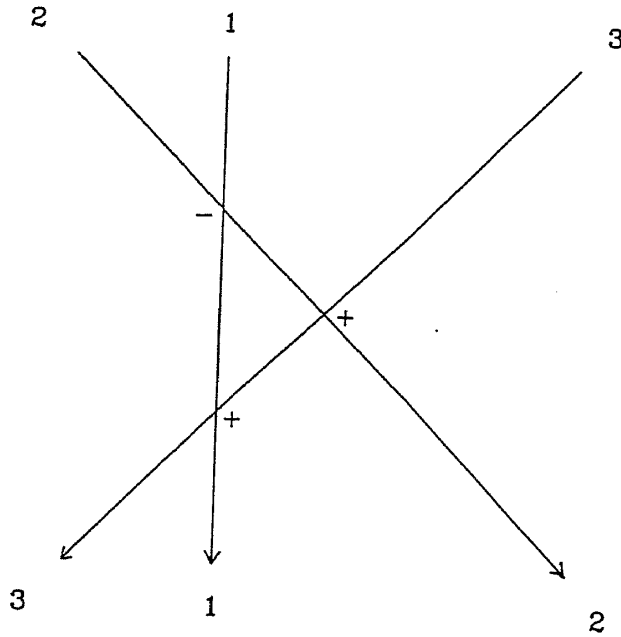$$\langle 1_g 1_h 1_k \ldots ; \mu | 1_g 1_h 1_k \ldots ; \nu \rangle = (q^*)^{J_+} q^{J_-} \tag{3.29}$$

Figure 1. Inversion diagram for the permutation $(213) \to (312)$.
Positive inversions: $(1, 3) \to (3, 1)$ and $(2, 3) \to (3, 2)$.
Negative inversion: $(2, 1) \to (1, 2)$.

where J_+ and J_- are the number of positive and negative inversions in the permutation $v \to \mu$. The total number of inversions (the sum of positive and negative inversions) is the same as the number of inversions already defined below (3.10). We further define an inversion as positive if it is a transposition of indices from the ascending order to the descending order and as negative if it is the reverse transposition. For example, $(1, 2) \to (2, 1)$ is a positive inversion while $(2, 1) \to (1, 2)$ is a negative inversion. Thus we have

$$\langle 1_3 1_1 1_2 | 1_2 1_1 1_3 \rangle = (q^*)^2 q \qquad (3.29')$$

since the permutation $(213) \to (312)$ contains two positive and one negative inversions as shown in figure 1.

The relationship between the algebra defined by (3.27) and (3.28) with complex q (and $p = |q|$) and Greenberg's algebra defined by (3.19) with real q can be given. Calling the creation operators for the former and latter algebras as $c^\dagger(q)$ and $c^\dagger(|q|)$ respectively, the relationship is

$$c_i^\dagger(q) = e^{i\theta \Sigma_{k<i} N_k} c_i^\dagger(|q|) \qquad (3.30)$$

where θ is the phase of q:

$$q = |q| e^{i\theta} \qquad (3.31)$$

and N_k are the number operators defined in (2.37).

As a consequence, the inner product matrices for the two algebras are related by θ-dependent unitary matrices:

$$M(q) = T^\dagger(\theta) M(|q|) T(\theta) \qquad (3.32)$$

where
$$T^\dagger T = T T^\dagger = 1. \tag{3.33}$$

We have already given in (3.13), Zagier's result for the determinant of M for real q. For our algebra with complex q, because of (3.32), the determinant of M for n particles with distinct indices is

$$\det M_n(q) = \prod_{k=1}^{n-1} (1 - |q|^{k(k+1)})^{(n-k)n!/k(k+1)} \tag{3.34}$$

Zeroes of $\det M_n(q)$ occur only on the circle $|q| = 1$. Hence, $M_n(q)$ for complex q is positive for $|q| < 1$, and we have thus extended the Fivel–Zagier result for positivity to complex q (for the case $p = |q|$).

In contrast to the situation for Greenberg's algebra, the inner product matrix M for states with repeated indices cannot be derived from that for states with distinct indices, for the present algebra. For n-particle states with all the indices repeated, the norm is the same as in (3.17) and (3.18), with q replaced by p:

$$\langle n_g | n_g \rangle = (1 + p)(1 + p + p^2)\ldots(1 + p + p^2 + \ldots p^{n_g - 1}) \tag{3.35}$$

which is positive for $p > -1$. For states with only some indices repeated, M is a function of both q and p and the problem is more complicated; but we have verified positivity of the M's up to 3-particle states for $-1 < p < |q|^{-2}$.

If we choose $p = -1$, all states with repeated indices are forbidden. This is just Pauli's exclusion principle. Thus, the algebra defined by (3.27) and (3.28) with $p = -1$ leads to *infinite statistics with exclusion principle*.

Infact, we can restrict both (3.5) and (3.19) to $i \neq j$ and for $i = j$ replace them by

$$c_i c_i^\dagger + c_i^\dagger c_i = 1 \tag{3.36}$$

In all these cases, we have infinite statistics with exclusion principle.

A compilation of the various algebras all representing infinite statistics is given in table 1.

3.5 *Unitary transformations*

We must point out an important difference between the algebras described by (3.5), (3.19) and (3.20) on the one hand and the algebras of (3.21), (3.22), (3.27) and (3.28) on the other hand. It is easy to see that the former are covariant under the unitary transformations on the indices:

$$c_k \rightarrow \sum_m U_{km} c_m; \quad U^\dagger U = U U^\dagger = 1. \tag{3.37}$$

In fact, one can show [19] that under certain conditions (3.20) is the most general bilinear algebra of c_i and $c_j^\dagger$ invariant under this unitary transformation. This property is violated for the complex q-algebra of (3.27) and (3.28) as well as the algebras of (3.21) and (3.22). Covariance under the unitary transformation is desirable in a general context since it is closely connected to the superposition principle in quantum mechanics [3]. Nevertheless, algebras violating this requirement have been proposed in the recent literature, either because of the possibility of applications to specific

Table 1. Representations of infinite statistics.

Statistics	Representation	Algebra
Infinite	Standard representation	$c_i c_j^\dagger = \delta_{ij}$
,,	q-mutator (with real q)	$c_i c_j^\dagger - q c_j^\dagger c_i = \delta_{ij}$
,,	Two-parameter algebra	$\begin{cases} c_i c_j^\dagger - q_1 c_j^\dagger c_i \\ - q_2 \delta_{ij} \Sigma_k c_k^\dagger c_k = \delta_{ij} \end{cases}$
,,	q-mutator, transformed	$c_i c_j^\dagger - c_j^\dagger c_i = \delta_{ij} p^{2\Sigma_{k<i} N_k} p^{N_i}$
,,	q-mutator, transformed	$\begin{cases} c_i c_j^\dagger - p^{-1} c_j^\dagger c_i = 0, \quad \text{for } i \ne j \\ c_i c_i^\dagger - c_i^\dagger c_i = p^{N_i} \end{cases}$
,,	q-mutator (with complex q)	$\begin{cases} c_i c_j^\dagger - q c_j^\dagger c_i = 0, \quad \text{for } i < j \\ c_i c_i^\dagger - p c_i^\dagger c_i = 1 \end{cases}$
Infinite with Pauli principle	Standard representation	$\begin{cases} c_i c_j^\dagger = 0, \quad \text{for } i \ne j \\ c_i c_i^\dagger + c_i^\dagger c_i = 1 \end{cases}$
,,	q-mutator (with real q)	$\begin{cases} c_i c_j^\dagger - q c_j^\dagger c_i = 0, \quad \text{for } i \ne j \\ c_i c_i^\dagger + c_i^\dagger c_i = 1. \end{cases}$
,,	q-mutator (with complex q)	$\begin{cases} c_i c_j^\dagger - q c_j^\dagger c_i = 0, \quad \text{for } i < j \\ c_i c_i^\dagger + c_i^\dagger c_i = 1 \end{cases}$

systems in condensed matter physics or because of mathematical motivation. Hence, we include such algebras in our investigation.

3.6 *Thermodynamics*

A detailed treatment of thermodynamic aspects is outside the scope of the present paper. Nevertheless a brief statement is in order.

Consider the partition function for the canonical ensemble:

$$Z = Tr e^{-\beta H} \tag{3.38}$$

where $\beta = (kT)^{-1}$ and H is the Hamiltonian of the system. If the system consists of non-interacting particles, we have

$$H = \sum_i \varepsilon_i N_i \tag{3.39}$$

where N_i are the number operators and ε_i are the single-particle energies. Hence, using the orthonormal set $\| \ldots n_i \ldots ; \mu \rangle\!\rangle$ in the evaluation of the trace in (3.38), we get

$$Z = \sum_{n_1, n_2 \ldots} d(\ldots n_j \ldots) e^{-\beta \Sigma_i \varepsilon_i n_i} \tag{3.40}$$

where $d(\ldots n_j \ldots)$ is the dimension of the sector $\{\ldots n_j \ldots\}$, defined in (2.40), and the summation is over all the allowed occupation numbers n_i for all i.

All the thermodynamics of the system can be derived from the partition function. Further, the set of $d(\ldots n_j \ldots)$ on the r.h.s. of (3.40) provide invariant characteristics of

the Fock space that are not dependent on the particular statistics or algebra living in that Fock space. Hence, thermodynamics is the same for all forms of statistics and algebras residing in a given Fock space.

For super Fock space and the associated infinite statistics, for which d is equal to s defined by (2.6):

$$d = n!/\Pi_i n_i!; \quad n = \sum_i n_i, \tag{3.41}$$

the thermodynamics is independent of the particular representation or algebra of c and $c^\dagger$. In particular, if one uses the q-mutator algebra (3.19) to describe infinite statistics, the properties of the system in thermodynamic equilibrium are independent of q, provided q lies in the range $-1 < q < 1$. Greenberg [14] had originally envisaged small violations of Fermi or Bose statistics by choosing $q = \mp (1 - \varepsilon)$, ε small and positive. We see that equilibrium thermodynamics will not manifest such small violations.

The expression for d given by (3.41) is the same as Boltzmann counting and hence Greenberg has called infinite statistics as 'quantum Boltzmann statistics'. However, since the Gibbs factor $1/n!$ is missing, the statistical mechanics of a system of free particles obeying infinite statistics with the index i in (3.40) interpreted as momentum will suffer Gibbs paradox [20]. Therefore, the super Fock space and the associated infinite statistics cannot be naively applied to familiar physical systems. In our approach, super Fock space plays the role of a mathematical template which can be used for carving out various physical systems.

As long as one takes the Hamiltonian to be of the form (3.39) which is the only correct form for non-interacting particles, the statistical mechanics and thermodynamics of the system are independent of the algebra of c and $c^\dagger$. One can then construct interaction terms involving c and $c^\dagger$ as in the usual many-body theory and their effect will of course depend on the algebra.

Recent literature contains many calculations on the statistical distributions or other thermodynamic quantities for the Hamiltonian

$$H = \sum_i \varepsilon_i c_i^\dagger c_i \tag{3.42}$$

with c and $c^\dagger$ satisfying some q-deformed algebra. Since $c_i^\dagger c_i$ is not a number operator in general, the physical meaning of the Hamiltonian (3.42) is not a priori clear, although one may suppose that it takes into account some type of interactions.

4. Statistics and algebras in bosonic, fermionic and frozen Fock spaces

We introduce two types of reduced Fock spaces in subsection 4.1. In one, we take any two states obtained by permutation to be related to each other and we allow all occupation numbers: $n_i = 0, 1, 2, \ldots$ for all i. In the other, any two states obtained by permutation are again related to each other but occupation numbers are restricted by exclusion principle: $n_i = 0$ and 1 only. We shall call the former as bosonic Fock space and the latter as fermionic Fock space.

In subsection 4.2, we introduce the Fock space of frozen order, in which permutations are forbidden and this also comes in two varieties, bosonic and fermionic.

4.1 *Bosonic and fermionic Fock spaces*

A consistent way of defining a general relationship among permuted states for both the bosonic and fermionic Fock spaces is to take

$$|n_1, n_2 \ldots; \mu\rangle = q^J |n_1, n_2 \ldots; 1\rangle \tag{4.1}$$

where q is a complex number and J is the number of inversions in the permutation: $1 \to \mu$. The number of inversions was already defined in §3. Although J was defined there for the case in which the occupation numbers were restricted to unity, the same definition can be extended to larger occupation numbers. We assume $q \neq 0$. The case $q = 0$ will be considered separately. Many different forms of statistics as well as various algebras of c and $c^\dagger$ can be shown to be contained as particular cases of (4.1).

We first make some general remarks amplifying the meaning of (4.1). This equation can be obtained by repeated application of the elementary relation:

$$|\ldots 1_i 1_j \ldots\rangle = q|\ldots 1_j 1_i \ldots\rangle, \quad \text{for} \quad i > j \tag{4.2}$$

where, except for the two adjacent quanta of indices i and j which are interchanged, all other quanta are left unchanged. In (4.2), we have used the same notation for the state vectors as on the r.h.s. of (2.1)–(2.4). It is clear that, for any q other than ± 1, (4.2) makes sense only if the indices i and j are ordered through an inequality (to be specific, $i > j$). Hence we have taken the indices to be the natural numbers $1, 2, 3 \ldots$ in writing (4.1).

The relationship among the state vectors given in (4.1) or (4.2) can be equivalently expressed as a quadratic relation between two creation operators or two annihilation operators. For any state $|\ldots\rangle$, using (2.30) or (2.31) we have

$$c_i^\dagger c_j^\dagger |\ldots\rangle = |1_i 1_j \ldots\rangle$$

$$c_j^\dagger c_i^\dagger |\ldots\rangle = |1_j 1_i \ldots\rangle$$

Comparing with (4.2), we get

$$c_i^\dagger c_j^\dagger |\ldots\rangle = q c_j^\dagger c_i^\dagger |\ldots\rangle, \quad \text{for} \quad i > j$$

Since this is valid for any state $|\ldots\rangle$, we can write

$$c_i^\dagger c_j^\dagger - q c_j^\dagger c_i^\dagger = 0 \quad \text{for} \quad i > j$$

or,

$$c_j c_i - q^* c_i c_j = 0 \quad \text{for} \quad i > j. \tag{4.3}$$

The above cc relation (eq. 4.3) is common to both bosonic and fermionic spaces. But for the fermionic space there exists the additional restriction:

$$c_i^\dagger c_i^\dagger = 0, \quad \text{or} \quad c_i c_i = 0. \tag{4.4}$$

All the states $|n_1, n_2 \ldots; \mu\rangle$ for fixed occupation numbers $(n_1, n_2 \ldots)$ but different values of μ being related to each other through (4.1), the dimension of the vector space in any sector $(n_1, n_2 \ldots)$ is reduced to unity. So, for each sector we choose one standard vector $|n_1, n_2 \ldots; 1\rangle$ which we rewrite as $|n_1, n_2 \ldots\rangle$, dispensing with μ completely.

The matrices X and M then become numbers related by

$$M^{-1} = XX^*. \tag{4.5}$$

Equation (2.11) becomes

$$\|n_1, n_2 \ldots \rangle\!\rangle = X(n_1, n_2 \ldots) |n_1, n_2 \ldots \rangle \tag{4.6}$$

and we have

$$\langle\!\langle n_1, n_2 \ldots \| n_1', n_2' \ldots \rangle\!\rangle = \delta_{n_1 n_1'} \delta_{n_2 n_2'} \ldots \tag{4.7}$$

From (2.35), we get

$$c_j^\dagger = \sum_{n_1 \ldots n_j \ldots} |1_j, n_1, n_2 \ldots n_j..\rangle \langle n_1, n_2 \ldots n_j..| M^{-1}(n_1, n_2..)$$

$$= \sum_{n_1 \ldots n_j \ldots} q^{\Sigma_{i<j} n_i} |n_1, n_2 \ldots (n_j + 1)..\rangle \langle n_1, n_2 \ldots n_j..| M^{-1}(n_1, n_2 \ldots)$$

where we have pushed the j quantum to the right of all i quanta for $i < j$ using (4.2). Also using (4.5) and (4.6),

$$c_j^\dagger = \sum_{n_1 \ldots n_j} q^{\Sigma_{i<j} n_i} \frac{X(n_1..n_j..)}{X(n_1..(n_j+1)..)} \|n_1, n_2 \ldots (n_j + 1)..\rangle\!\rangle \langle\!\langle n_1, n_2 \ldots n_j..\| \tag{4.8}$$

Equation (4.8) and its hermitian conjugate define the most general form of creation and destruction operators for the bosonic and fermionic Fock spaces.

If we assume factorization of the norm M as well as that of X:

$$X(n_1, n_2 \ldots) = \phi(n_1) \phi(n_2) \ldots, \tag{4.9}$$

then we get

$$c_j^\dagger = \sum_{n_1 \ldots n_j \ldots} q^{\Sigma_{i<j} n_i} \frac{\phi(n_j)}{\phi(n_j + 1)} \|n_1..n_j + 1..\rangle\!\rangle \langle\!\langle n_1..n_j..\| \tag{4.10}$$

$$c_j = \sum_{n_1 \ldots n_j \ldots} (q^*)^{\Sigma_{i<j} n_i} \frac{\phi^*(n_j)}{\phi^*(n_j + 1)} \|n_1..n_j..\rangle\!\rangle \langle\!\langle n_1..(n_j + 1)..\| \tag{4.11}$$

We now construct the operator algebras for $c^\dagger$ and c given by (4.10) and (4.11). First of all, *for any form of* $\phi(n_i)$, we get

$$\left. \begin{array}{ll} c_i c_j^\dagger - q c_j^\dagger c_i = 0 & \text{for} \quad i < j \\ c_i c_j^\dagger - q^* c_j^\dagger c_i = 0 & \text{for} \quad i > j \end{array} \right\} \tag{4.12}$$

For $i = j$, the algebra depends on the choice of q and $\phi(n_i)$. We find

$$c_j c_j^\dagger = |q|^{2\Sigma_{i<j} N_i} \left| \frac{\phi(N_j)}{\phi(N_j + 1)} \right|^2 \tag{4.13}$$

$$c_j^\dagger c_j = |q|^{2\Sigma_{i<j} N_i} \left| \frac{\phi(N_j - 1)}{\phi(N_j)} \right|^2 \tag{4.14}$$

where $N_i, N_j \ldots$ are number operators and $\phi(N_i)$ is the function introduced in (4.9). So,

we can write the generalised commutation relation

$$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\Sigma_{i<j} N_i} f(N_j)$$

$$f(N_j) = \left| \frac{\phi(N_j)}{\phi(N_j+1)} \right|^2 - p \left| \frac{\phi(N_j-1)}{\phi(N_j)} \right|^2 \Bigg\} \tag{4.15}$$

where p is any (complex) number. Equations (4.12) and (4.15) constitute the $cc^\dagger$ algebra for q-statistics defined by the cc algebra of (4.3).

What we have derived above can be regarded as the most general algebra of creation and annihilation operators for q-statistics, subject to the assumption of the factorization of the normalization factor implied by (4.9). In particular, since the function $\phi(N_j)$ occurring in (4.15) is arbitrary, we may regard our equations as constituting an infinite-parameter deformation of the algebra of c and $c^\dagger$. Exploiting the arbitrariness of $\phi(N_j)$ we can get many simpler forms of the algebra which we now describe.

(i) Choosing $\phi(n_j)$ to be a constant independent of n_j but noting that $\phi(-1)$ must vanish [38a], (4.15) becomes

$$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\Sigma_{i<j} N_i} \{1 - p(1 - \delta_{N_j,0})\} \tag{4.16}$$

where $\delta_{N_j,0}$ is the Kronecker delta, which can also be represented as $\sin \pi N_j / \pi N_j$.

(ii) Putting $p = 0$ in (4.16) we get the simple algebra:

$$c_j c_j^\dagger = |q|^{2\Sigma_{i<j} N_i} \tag{4.17}$$

(iii) As a third possibility, we may choose $f(N_j)$ in (4.15) to be unity and p real so that we have the algebra:

$$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\Sigma_{i<j} N_i} \tag{4.18}$$

This choice requires that the function $\phi(n)$ must satisfy the equation

$$\left| \frac{\phi(n_j)}{\phi(n_j+1)} \right|^2 - p \left| \frac{\phi(n_j-1)}{\phi(n_j)} \right|^2 = 1 \tag{4.19}$$

The solution of this functional equation (with the constraints that $\phi(0)$ is finite and $\phi(-1)$ vanishes) is

$$|\phi(n)|^{-2} = [(n)!]_p \equiv [n]_p [n-1]_p \ldots [1]_p \tag{4.20}$$

where

$$[n]_p = \frac{p^n - 1}{p - 1} = 1 + p + p^2 + \ldots + p^{n-1} \tag{4.21}$$

For $p = 1$, the 'p-numbers' in (4.20) and (4.21) become ordinary numbers.

(iv) As a final choice we put $p = |q|^2$ in (4.18)–(4.21). As a consequence of this choice, the r.h.s. of (4.18) becomes a bilinear function of c and $c^\dagger$. For, from (4.14), we can prove an identity:

$$(|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k = \sum_{k<j} |q|^{2\Sigma_{i<k} N_i} \left| \frac{\phi(N_k-1)}{\phi(N_k)} \right|^2 (|q|^2 - 1)$$

$$= \sum_{k<j} |q|^{2\Sigma_{i<k}N_i}(|q|^{2N_k} - 1)$$

$$= |q|^{2\Sigma_{i<j}N_i} - 1 \tag{4.22}$$

where we have used (4.20) and (4.21) with $p = |q|^2$. Hence (4.18) can be replaced by

$$c_j c_j^\dagger - |q|^2 c_j^\dagger c_j = 1 + (|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k \tag{4.23}$$

Any one of (4.16), (4.17), (4.18) or (4.23) together with (4.12) constitute a $cc^\dagger$ algebra and one can construct many more examples of $cc^\dagger$ algebras, all of which correspond to the same q-statistics specified by (4.3). We have so far assumed that the occupation numbers are unrestricted: $n_j \geq 0$ and so these algebras are valid for the bosonic q-statistics, namely q-statistics in the bosonic space.

The fermionic q-statistics is defined by (4.3) and (4.4). In this case, n_j in (4.10) and (4.11) can take the value 0 only, while all the other occupation numbers will range over 0 and 1. So, the only arbitrary parameter that enters the definition of c and $c^\dagger$ is $r = \phi(0)/\phi(1)$. The $cc^\dagger$ relation for $i \neq j$ is still the same as in (4.12) while the $c_j c_j^\dagger$ relation of (4.15) can be simplified to

$$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\Sigma_{i<j}N_i}|r|^2 \{\delta_{N_j,0} - p\delta_{N_j,1}\} \tag{4.24}$$

Further, we can generally prove the fermionic analogue of the identity which was earlier proved in (4.22) for the bosonic case only for a special choice of $\phi(n)$:

$$(|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k = (|q|^2 - 1) \sum_{k<j} |q|^{2\Sigma_{i<k}N_i}|r|^2 \delta_{N_k,1}$$

$$= \sum_{k<j} |q|^{2\Sigma_{i<k}N_i}(|q|^{2N_k} - 1)|r|^2 \tag{4.25}$$

$$= (|q|^{2\Sigma_{i<j}N_i} - 1)|r|^2$$

Hence, (4.24) can be rewritten as

$$c_j c_j^\dagger - p c_j^\dagger c_j = \{|r|^2 + (|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k\}(\delta_{N_j,0} - p\delta_{N_j,1}) \tag{4.26}$$

Equations (4.12) and (4.26) constitute the general algebra for the fermionic q-statistics. In contrast to (4.15) of the bosonic case, the fermionic case does not have the freedom of infinite-parameter deformation.

On the r.h.s. of (4.26), apart from the curly bracket { } which is bilinear in c and $c^\dagger$, there are Kronecker deltas which depend on the operator N_j. A simpler relation is obtained for the choice $p = -1$, since $\delta_{N_j,0} + \delta_{N_j,1} = 1$. We then have

$$c_j c_j^\dagger + c_j^\dagger c_j = |r|^2 + (|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k \tag{4.27}$$

A further choice $|r|^2 = 1$ gives

$$c_j c_j^\dagger + c_j^\dagger c_j = 1 + (|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k \tag{4.28}$$

which is the fermionic analogue of (4.23).

Table 2. Statistics and algebras in the bosonic Fock space.

Statistics	cc algebra	$c_i c_j^\dagger$ algebra for $i \neq j$	$c_j c_j^\dagger$ algebra	Remarks
q-statistics	$c_i c_j = q^* c_j c_i$ for $i < j$	$c_i c_j^\dagger = q c_j^\dagger c_i$ for $i < j$	$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\Sigma_{i<j} N_i} f(N_j)$	General representation
,,	$c_i c_j = q^* c_j c_i$ for $i < j$	$c_i c_j^\dagger = q c_j^\dagger c_i$ for $i < j$	$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\Sigma_{i<j} N_i}\{1 - p(1 - \delta_{N,0})\}$	
,,	$c_i c_j = q^* c_j c_i$ for $i < j$	$c_i c_j^\dagger = q c_j^\dagger c_i$ for $i < j$	$c_j c_j^\dagger = |q|^{2\Sigma_{i<j} N_i}$	
,,	$c_i c_j = q^* c_j c_i$ for $i < j$	$c_i c_j^\dagger = q c_j^\dagger c_i$ for $i < j$	$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\Sigma_{i<j} N_i}$	
,,	$c_i c_j = q^* c_j c_i$ for $i < j$	$c_i c_j^\dagger = q c_j^\dagger c_i$ for $i < j$	$c_j c_j^\dagger - |q|^2 c_j^\dagger c_j = 1 + (|q|^2 - 1)\sum_{k<j} c_k^\dagger c_k$	Covariant under $SU_q(n)$
bose statistics	$c_i c_j = c_j c_i$	$c_i c_j^\dagger = c_j^\dagger c_i$ for $i \neq j$	$c_j c_j^\dagger - p c_j^\dagger c_j = f(N_j)$	General representation
,,	$c_i c_j = c_j c_i$	$c_i c_j^\dagger = c_j^\dagger c_i$ for $i \neq j$	$c_j c_j^\dagger - p c_j^\dagger c_j = \{1 - p(1 - \delta_{N,0})\}$	
,,	$c_i c_j = c_j c_i$	$c_i c_j^\dagger = c_j^\dagger c_i$ for $i \neq j$	$c_j c_j^\dagger = 1$	Simplest representation

Statistics	c_ic_j relation	$c_ic_j^\dagger$ relation	$c_jc_j^\dagger$ relation	
"	$c_ic_j = c_jc_i$	$c_ic_j^\dagger = c_j^\dagger c_i$ for $i \neq j$	$c_jc_j^\dagger - pc_j^\dagger c_j = 1$; $\ p \neq -1$	Commuting deformed oscillators
"	$c_ic_j = c_jc_i$	$c_ic_j^\dagger = c_j^\dagger c_i$ for $i \neq j$	$c_jc_j^\dagger - c_j^\dagger c_j = 1$	Canonical rep. of bosons
fermi statistics	$c_ic_j = -c_jc_i$ for $i \neq j$	$c_ic_j^\dagger = -c_j^\dagger c_i$ for $i \neq j$	$c_jc_j^\dagger - pc_j^\dagger c_j = f(N_j)$	General representation
"	$c_ic_j = -c_jc_i$ for $i \neq j$	$c_ic_j^\dagger = -c_j^\dagger c_i$ for $i \neq j$	$c_jc_j^\dagger - pc_j^\dagger c_j = \{1 - p(1 - \delta_{N_j,0})\}$	
"	$c_ic_j = -c_jc_i$ for $i \neq j$	$c_ic_j^\dagger = -c_j^\dagger c_i$ for $i \neq j$	$c_jc_j^\dagger = 1$	
"	$c_ic_j = -c_jc_i$ for $i \neq j$	$c_ic_j^\dagger = -c_j^\dagger c_i$ for $i \neq j$	$c_jc_j^\dagger - pc_j^\dagger c_j = 1$; $\ p \neq -1$	Anticommuting deformed oscillators
"	$c_ic_j = -c_jc_i$ for $i \neq j$	$c_ic_j^\dagger = -c_j^\dagger c_i$ for $i \neq j$	$c_jc_j^\dagger - c_j^\dagger c_j = 1$	Anticommuting bosons
fractional statistics	$c_ic_j = e^{-i\theta}c_jc_i$ for $i < j$	$c_ic_j^\dagger = e^{i\theta}c_j^\dagger c_i$ for $i < j$	Any one of the same $c_jc_j^\dagger$ relations given in bose or fermi statistics above	

Table 3. Statistics and algebras in the fermionic Fock space.

Statistics	cc algebra	$c_i c_j^\dagger$ algebra for $i \neq j$	$c_j c_j^\dagger$ algebra	Remarks
q-statistics	$c_i c_j = q^* c_j c_i$ for $i<j$ $c_i c_i = 0$	$c_i c_j^\dagger = q c_j^\dagger c_i$ for $i<j$	$c_j c_j^\dagger - p c_j^\dagger c_j = (\delta_{N_j,0} - p\delta_{N_j,1})$ $\times \left\{ |r|^2 + (|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k \right\}$	General representation
"	$c_i c_j = q^* c_j c_i$ for $i<j$ $c_i c_i = 0$	$c_i c_j^\dagger = q c_j^\dagger c_i$ for $i<j$	$c_j c_j^\dagger + c_j^\dagger c_j = |r|^2 + (|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k$	
"	$c_i c_j = q^* c_j c_i$ for $i<j$ $c_i c_i = 0$	$c_i c_j^\dagger = q c_j^\dagger c_i$ for $i<j$	$c_j c_j^\dagger + c_j^\dagger c_j = 1 + (|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k$	Covariant under $SU_q(n)$
bose-statistics	$c_i c_j = c_j c_i$ $c_i c_i = 0$	$c_i c_j^\dagger = c_j^\dagger c_i$ for $i \neq j$	$c_j c_j^\dagger - p c_j^\dagger c_j = (\delta_{N_j,0} - p\delta_{N_j,1})|r|^2$	
"	$c_i c_j = c_j c_i$ $c_i c_i = 0$	$c_i c_j^\dagger = c_j^\dagger c_i$ for $i \neq j$	$c_j c_j^\dagger + c_j^\dagger c_j = 1$	Commuting fermions

				Canonical rep. of fermions.
fermi-statistics	$c_i c_j = -c_j c_i$ for all i,j	$c_i c_j^\dagger = -c_j^\dagger c_i$ for $i \neq j$	$c_j c_j^\dagger - p c_j^\dagger c_j = (\delta_{N,0} - p\delta_{N_j,1})|r|^2$	
"	$c_i c_j = -c_j c_i$ for all i,j	$c_i c_j^\dagger = -c_j^\dagger c_i$ for $i \neq j$	$c_j c_j^\dagger + c_j^\dagger c_j = 1$	
fractional statistics	$c_i c_j = e^{-i\theta} c_j c_i$ for $i<j$ $\quad c_i c_i = 0$	$c_i c_j^\dagger = e^{i\theta} c_j^\dagger c_i$ for $i<j$	$c_j c_j^\dagger - p c_j^\dagger c_j = (\delta_{N,0} - p\delta_{N_j,1})|r|^2$	
"	$c_i c_j = e^{-i\theta} c_j c_i$ for $i<j$ $\quad c_i c_i = 0$	$c_i c_j^\dagger = e^{i\theta} c_j^\dagger c_i$ for $i<j$	$c_j c_j^\dagger + c_j^\dagger c_j = 1$	

The algebra given by (4.3), (4.12) and (4.23) (for real q) are covariant under *the quantum group* $SU_q(n)$ where n is the total number of indices. This is true of the fermionic algebra of (4.3), (4.12) and (4.28) also. (See refs. [21–31] for this relationship to the theory of quantum groups.) However, our approach based on states related by q-statistics (4.1) does not involve any notion of quantum group *per se* and so we shall not describe it. The formalism of the generalized Fock space appears to be capable of incorporating many general algebraic structures. In particular, the $c_j c_j^\dagger$ algebra of (4.15) constitutes a general infinite-parameter deformation while the quantum-group related algebra of (4.23) is only a particular case of this.

We have constructed the general algebras for arbitrary complex q. The choice of q defines the symmetry of the state under permutation. Hence we call the different choices of q as different forms of statistics. Particular values of q such as $q = \pm 1$ or $q = e^{i\theta}$ are of special interest, although these cases are all contained in the general formulae already given. We shall call the statistics obtained for $q = +1$, -1 and $e^{i\theta}$ as Bose statistics, Fermi statistics or 'fractional' statistics respectively. Either the bosonic or fermionic Fock space can be used to construct any one of the various forms of statistics. Thus, within our terminology, it is perfectly possible for instance to have Fermi statistics residing in bosonic Fock space or vice versa.

Within a particular statistics, different choices of M or ϕ correspond to different representations of c_j and $c_j^\dagger$, which in turn lead to different $cc^\dagger$ algebras. If we assume the factorization given in (4.9), for a given q, only the $i = j$ part of the $cc^\dagger$ algebra depends on the representation.

For the sake of clarity, all these forms of statistics and algebras are exhibited in tables 2 and 3 for the bosonic and fermionic spaces respectively.

Several comments are in order, concerning the contents of these tables. The algebra

$$c_i c_j - c_j c_i = 0, \quad \text{for} \quad i \neq j \tag{4.29}$$

$$c_i c_j^\dagger - c_j^\dagger c_i = 0, \quad \text{for} \quad i \neq j \tag{4.30}$$

$$c_i c_j^\dagger = 1 \tag{4.31}$$

provide the simplest representation for Bose statistics based on the simple choice of M or ϕ as unity. This is analogous to the standard representation (3.5) for infinite statistics but very different from it. In fact they live in entirely different Fock spaces. In recent literature, (4.31) has been sometimes confused with infinite statistics. Equation (4.31) coupled with (4.29) and (4.30) infact describes Bose statistics only, although in a noncanonical representation.

Attention may be drawn to the 'anticommuting bosons' and the 'commuting fermions' shown in tables 2 and 3 respectively. The commuting fermions have been called hard-core bosons in condensed-matter-physics literature. Our terminology seems more appropriate since they are really fermions in disguise, living in the fermionic Fock space.

The precise connection of our 'fractional' statistics to the exchange property of the wavefunctions proposed [39, 40] for one and two-dimensional systems needs further study. In particular, a suitable mapping of the ordered indices to coordinates in one and two-dimensions is necessary.

Among all the different algebras given in tables 2 and 3, only two of them, namely the canonical representations for the bosons and fermions have the distinction of being covariant under the unitary transformations that mix the indices (see (3.37)). All the other algebras violate this requirement, although, as we have already mentioned, two of them, one in the bosonic and the other in the fermionic Fock spaces, are covariant under the quantum groups $SU_q(n)$.

The examples contained in these tables can be called deformed-oscillator algebras on which many papers [1–12, 21–27, 32, 41, 42] have been written recently. In spite of the multiplicity of these deformed algebras, we must not ignore the basic fact that *they are all avatars of just two primary constructions which may be taken to be the canonical bosonic algebra for the bosonic Fock space and the canonical fermionic algebra for the fermionic Fock space.* All the different forms of statistics belonging to the bosonic Fock space as well as their various algebraic representations are related to each other and to the canonical bosonic algebra and similar is the situation for the fermionic Fock space. Given two different forms of statistics within the bosonic Fock space characterized by the statistics parameters q_1 and q_2 respectively, or/and two different representations characterized by the functions $\phi_1(n)$ and $\phi_2(n)$ respectively, it is easy to get from (4.10) the relationship

$$c_j^\dagger(2) = \left(\frac{q_2}{q_1}\right)^{\Sigma_{i<j} N_i} \frac{\phi_2(N_j - 1)}{\phi_2(N_j)} \frac{\phi_1(N_j)}{\phi_1(N_j - 1)} c_j^\dagger(1) \tag{4.32}$$

The corresponding equation for the fermionic Fock space is

$$c_j^\dagger(2) = \left(\frac{q_2}{q_1}\right)^{\Sigma_{i<j} N_i} \left(\frac{r_2}{r_1}\right) c_j^\dagger(1) \tag{4.33}$$

Hence, from the point of view of Fock space, there is nothing new in all these deformed oscillators.

In the above equations, the number operator N_i are given by

$$N_i = \sum_{n_1 \dots n_i \dots} n_i \| n_1 \dots n_i \dots \rangle\!\rangle \langle\!\langle n_1 \dots n_i \dots \| \tag{4.34}$$

and, written in this fashion in terms of the normalized states $\| \rangle\!\rangle$, N_i *is the same in all the different forms of statistics (within the bosonic or fermionic Fock spaces) and in all the different representations.*

Using (4.14) it is also possible to express the number operators in terms of c and $c^\dagger$ in any statistics or any representation within the bosonic or fermionic Fock space. Noting that for the canonical bosonic algebra

$$q = 1; \quad \phi(n) = \frac{1}{(n!)^{1/2}}, \tag{4.35}$$

we get from (4.14)

$$N_i = c_i^\dagger c_i. \tag{4.36}$$

For other statistics and representations also, the set of equations obtained from (4.14) starting with $j = 1$ and successively increasing j, can be seen to be implicit recursion

formulae for all N_j. Thus we have

$$
\left.
\begin{aligned}
c_1^\dagger c_1 &= \left| \frac{\phi(N_1 - 1)}{\phi(N_1)} \right|^2 \\[2mm]
c_2^\dagger c_2 &= |q|^{2N_1} \left| \frac{\phi(N_2 - 1)}{\phi(N_2)} \right|^2 \\
&\ \vdots
\end{aligned}
\right\}
\tag{4.37}
$$

By choosing $\phi(N_j)$ one then gets the desired explicit expressions for N_j. However, as already emphasized, such expressions are not of much use: the universal representation of N_j given in (4.34) is sufficient for all purposes.

The formalism of Fock space developed here has sufficient flexibility so as to allow many straightforward extensions or generalizations. For instance, the relationship between permuted states defined by (4.1) or (4.2) can be generalized further. Instead of (4.3), we may take

$$
c_j c_i - q_{ij}^* c_i c_j = 0 \quad \text{for} \quad i > j,
\tag{4.38}
$$

where q_{ij} are complex parameters. This may be called *multiparameter q-statistics*. The algebras corresponding to such extensions as well as further generalizations can all be constructed essentially by the same procedure as given in this section. Details will be presented elsewhere. Suffices it to say that, once the relationship among the permuted states is specified through the cc relation, the rest of the story follows.

Special cases of such algebras have been reported in the literature on quantum groups [21–33]. We have already referred to the generality of our approach as compared to quantum groups. As further points of comparison between the two approaches we must mention the following. From the point of view of quantum groups, the whole set of relations among c and $c^\dagger$ are taken for granted. In contrast, our analysis based on the underlying Fock space reveals the cc relation as the key to the whole algebraic structure, although the normalization function $\phi(n)$ also plays a role in determining the actual representation of the operators. Hence, depending on the mode of expressing the cc relation and the choice of $\phi(n)$, one can generate any number of algebras of c and $c^\dagger$. Thus, our approach helps to demystify the quantum-group related algebras by reducing them to their basic essentials which are identified to be simple properties of state vectors in the Fock space. Reversing the procedure, one can probably reconstruct the whole edifice of the quantum group itself starting from the more elementary notions relating to states in the Fock space. This of course lies outside the purview of the present paper.

4.2 *Fock space of frozen order and null statistics*

In subsection 4.1 we had taken any two states obtained by permutation to be related to each other. We now consider a limiting situation in which the particles are frozen in a particular order, with no permutation allowed. This is the *Fock space of frozen states* and the associated statistics is *the statistics of frozen states*. Whereas in infinite statistics each permutation leads to a new state, in the statistics of frozen states no permutation is allowed. To emphasize this contrast, the latter may be called *null statistics*. Although

this may be obtained as the limit $q \to 0$ of the q-statistics, more properly, it must be regarded as an independent statistics. For, whereas the statistics for two (non-zero) values of q are related to each other through (4.32) or (4.33), the statistics for $q = 0$ cannot be related to statistics for $q \neq 0$. Hence, we shall construct the algebra of c and $c^\dagger$ for this system of frozen states directly from the definition. Again, depending on whether the particles obey exclusion principle or not, we have two different versions of the system which we shall call the fermionic and the bosonic versions respectively.

Let us first consider the bosonic version. Referring to (4.1) the Fock space of frozen states is obtained by taking $|n_1 n_2 \ldots; 1\rangle$ as the allowed state and requiring

$$|n_1 n_2 \ldots; \mu\rangle = 0 \quad \text{for} \quad \mu \neq 1 \tag{4.39}$$

Or, equivalently

$$|\ldots 1_j 1_i \ldots\rangle = 0 \quad \text{for} \quad i < j \tag{4.40}$$

and hence

$$c_i c_j = 0 \quad \text{for} \quad i < j \tag{4.41}$$

Assuming factorization of the norm as in (4.9), we can represent the creation operator in the form:

$$c_j^\dagger = \sum_{n_j, n_{j+1} \ldots} \frac{\phi(n_j)}{\phi(n_j + 1)} \|0_1 \ldots 0_{j-1}, (n_j + 1), n_{j+1} \ldots \rangle\!\rangle \langle\!\langle 0_1, \ldots 0_{j-1}, n_j, n_{j+1} \ldots \| \tag{4.42}$$

The zeroes in the state vectors arise from the fact that $c_j^\dagger$ can create a quantum with index j only if indices $i < j$ are unoccupied. From (4.41) and the orthogonality of the states, we get

$$c_k c_j^\dagger = 0 \quad \text{for} \quad k \neq j \tag{4.43}$$

The $c_j c_j^\dagger$ algebra will depend on the choice of $\phi(n_j)/\phi(n_j + 1)$. We shall assume $\phi(n_j) = \phi(n_j + 1)$ for simplicity. Then, by using the completeness relation for the states, one can verify

$$c_j c_j^\dagger = 1 - \sum_{k < j} c_k^\dagger c_k \tag{4.44}$$

Equations (4.41), (4.43) and (4.44) together define the algebra for the bosonic version of the statistics of frozen order.

For the fermionic version, (4.41) is replaced by

$$c_i c_j = 0 \quad \text{for} \quad i \leqslant j \tag{4.45}$$

and one again gets

$$c_k c_j^\dagger = 0 \quad \text{for} \quad k \neq j \tag{4.46}$$

but, instead of (4.44), one finds

$$c_j c_j^\dagger = 1 - \sum_{k \leqslant j} c_k^\dagger c_k \tag{4.47}$$

Thus, the fermionic version of the statistics of frozen states is described by the algebra of (4.45)–(4.47). It is interesting to note that the replacement of the sign $<$ in the bosonic algebra by the sign $\leqslant$ yields the fermionic algebra.

Finally, we note that the expression for the number operators in terms of c and $c^\dagger$ for the above algebra of null statistics can be shown to be essentially the same as that for the standard representation of infinite statistics (eq. 3.9a), but because of (4.41) or (4.45), it can be simplified to read as

$$N_k = \sum_{n_1, n_2 \dots n_k} c_1^{\dagger n_1} c_2^{\dagger n_2} \dots c_k^{\dagger n_k} c_k^\dagger c_k c_k^{n_k} \dots c_2^{n_2} c_1^{n_1} \tag{4.48}$$

where the sum over $n_1, n_2 \dots n_k$ is unrestricted ($\geqslant 0$) for the bosonic version, but is restricted for the fermionic version: $n_1, n_2 \dots n_{k-1}$ go over 0 and 1 while $n_k = 0$.

4.3 Thermodynamic aspects

For the bosonic and fermionic Fock spaces, (3.40) for the partition function becomes

$$Z = \sum_{n_1, n_2 \dots} e^{-\beta \Sigma_i \varepsilon_i n_i} \tag{4.49}$$

where $n_i = 0, 1, 2, \dots \infty$ for the bosonic space and $n_i = 0, 1$ for the fermionic space. This is a well-known result for the canonical Bose and Fermi statistics, but we emphasize the fact that *the partition function and the resulting thermodynamics is the same for all the various q-statistics and q-deformed algebras living in the same bosonic Fock space and similarly for the fermionic Fock space.* This point has been missed in much of the current literature and so considerable confusion has been created. Part of this confusion is due to the choice of the inappropriate Hamiltonian (eq. (3.42)) and the wrong emphasis placed on some particular algebra of c and $c^\dagger$.

We further note that the above Z in (4.49) is valid even for the Fock space of frozen order and null statistics since $d = 1$ in this case too.

To sum up, equilibrium thermodynamics of a system of free particles is determined entirely by the dimension d and spectrum of allowed occupation numbers which together characterize the Fock space and does not depend on the permutation properties of the multiparticle states or the algebras of c and $c^\dagger$.

5. Derivation of cc relations from $cc^\dagger$ algebras

In the last section construction of the bilinear algebra of c and $c^\dagger$ starting from states related by (4.1) was shown. In other words, $cc^\dagger$ algebra has been derived from the cc algebra given by (4.3). The converse is also true; the cc algebra can be derived from the $cc^\dagger$ algebra, as is shown in this section. Thus, within the framework of Fock space it is unnecessary to give both cc and $cc^\dagger$ relations. Either the cc relation or the $cc^\dagger$ relation can be used to define the Fock space and the other can be derived. But there are some caveats:

(i) Although the cc relation does define the Fock space, the $cc^\dagger$ algebra does not follow uniquely; as already pointed out, the operators $c, c^\dagger$ and their algebras depend on the choice of M.

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(ii) In order to define the Fock space completely, the $cc^\dagger$ algebra must fulfill certain conditions. The $cc^\dagger$ algebra must be such that an arbitrary inner product $\langle 0|c_i c_j \dots c_g^\dagger c_h^\dagger|0\rangle$ or infact the vacuum matrix element of any polynomial in the c's and $c^\dagger$'s arbitrarily ordered, can be calculated using the $cc^\dagger$ algebra and the definition of the vacuum state $|0\rangle$:

$$c_k|0\rangle = 0 \quad \text{for all } k. \tag{5.1}$$

A general form of the $cc^\dagger$ algebra that satisfies this requirement is:

$$c_i c_j^\dagger = A_{ij} + \sum_{k,m} B_{ijkm} c_k^\dagger c_m \tag{5.2}$$

where A_{ij} and B_{ijkm} can be constants, but more generally they can be functions of the number operators. All the $cc^\dagger$ algebras considered in this paper are of this form.

There exists an elegant method [19] to derive cc relations from the $cc^\dagger$ algebras. Restricting ourselves to relations quadratic in c, we define

$$Q_{ij} = c_i c_j - q' c_j c_i \tag{5.3}$$

where q' may be an arbitrary complex parameter. Suppose that, by using the given $cc^\dagger$ algebra, we are able to prove

$$Q_{ij} c_k^\dagger = \sum_{i'j'k'} F_{ijk;i'j'k'} c_{k'}^\dagger Q_{i'j'} \tag{5.4}$$

for all i,j and k where $F_{ijk;i'j'k'}$ may be a c-number or operator. Then, by applying this equation successively, we get

$$Q_{ij} c_k^\dagger c_m^\dagger \dots = \sum_{i'j'k'} \sum_{i''j''m'} F_{ijk;i'j'k'} F_{i'j'm;i''j''m'} c_{k'}^\dagger c_{m'}^\dagger \dots Q_{i''j''} \tag{5.5}$$

Allowing both sides of this equation to act on $|0\rangle$, the right side vanishes and so we see that Q_{ij} acting on any Fock state $c_k^\dagger c_m^\dagger \dots |0\rangle$ gives zero. Hence we may write the operator identity:

$$Q_{ij} = 0 \tag{5.6}$$

which is the cc relation sought after. It may also be pointed out that often one finds

$$F_{ijk;i'j'k'} = f_{ijk} \delta_{ii'} \delta_{jj'} \delta_{kk'} \tag{5.7}$$

so that (5.4) is simplified to

$$Q_{ij} c_k^\dagger = f_{ijk} c_k^\dagger Q_{ij} \tag{5.8}$$

Thus, the form-invariance of Q_{ij} on being pushed to the right of $c_k^\dagger$, as explicitly given in (5.4) or (5.8) is the necessary and sufficient condition for the existence of the cc relation. We may now apply this method on various $cc^\dagger$ algebras discussed in the previous sections.

5.1 q-mutator algebra with real q:

The $cc^\dagger$ algebra is

$$c_i c_j^\dagger - q c_j^\dagger c_i = \delta_{ij}, \quad \forall i,j. \tag{5.9}$$

We define

$$Q_{ij} = c_i c_j - q' c_j c_i, \quad \forall i,j \tag{5.10}$$

From (5.9),

$$Q_{ij} c_k^\dagger = (1 - qq') c_i \delta_{kj} + (q - q') c_j \delta_{ki} + q^2 c_k^\dagger Q_{ij} \tag{5.11}$$

The form-invariance of Q_{ij} requires

$$q' = q = \pm 1. \tag{5.12}$$

So, we get the cc relations

$$c_i c_j \pm c_j c_i = 0 \tag{5.13}$$

corresponding to Bose and Fermi statistics at $q = \pm 1$. For $-1 < q < 1$, there are no cc relations and we have infinite statistics. The inner product matrix M which remains positive definite for $-1 < q < 1$, develops zero eigenvalues at $q = \pm 1$. (see eq. (3.13)) corresponding to the null states such as $(c_i^\dagger c_j^+ \pm c_j^\dagger c_i^\dagger)|0\rangle$ arising from (5.13).

5.2 *q-mutator algebra with complex q*

The $cc^\dagger$ relations are

$$c_i c_j^\dagger - q c_j^\dagger c_i = 0 \quad \text{for} \quad i < j \tag{5.14}$$

$$c_i c_i^\dagger - p c_i^\dagger c_i = 1 \tag{5.15}$$

We define

$$Q_{ij} = c_i c_j - q' c_j c_i \quad \text{for} \quad i < j \tag{5.16}$$

$$= c_i c_i \quad \text{for} \quad i = j \tag{5.17}$$

Using (5.14) and (5.15), we find

$$
\begin{aligned}
Q_{ij} c_k^\dagger &= q^2 c_k^\dagger Q_{ij} & \text{for} \quad i \leqslant j < k \\
&= q^{*2} c_k^\dagger Q_{ij} & \text{for} \quad k < i \leqslant j \\
&= |q|^2 c_k^\dagger Q_{ij} & \text{for} \quad i < k < j \\
&= (q^* - q') c_j + pq^* c_i^\dagger Q_{ij} & \text{for} \quad k = i < j \\
&= (1 - qq') c_i + pq c_j^\dagger Q_{ij} & \text{for} \quad i < j = k \\
&= (1 + p) c_i + p^2 c_i^\dagger Q_{ii} & \text{for} \quad i = j = k.
\end{aligned}
\tag{5.18}
$$

The form-invariance of Q_{ij} for $i < j$ requires

$$q' = q^* = e^{-i\theta} \tag{5.19}$$

where θ is real. We thus get the cc relation corresponding to fractional statistics at the boundary $|q| = 1$ of the disc $|q| < 1$ in the complex plane.

We already know that the disc $|q| < 1$ corresponds to the region of positive-definite inner product matrix M for the complex q-mutator algebra of infinite statistics. At the boundary of this positivity region, we have fractional statistics living in a reduced Fock
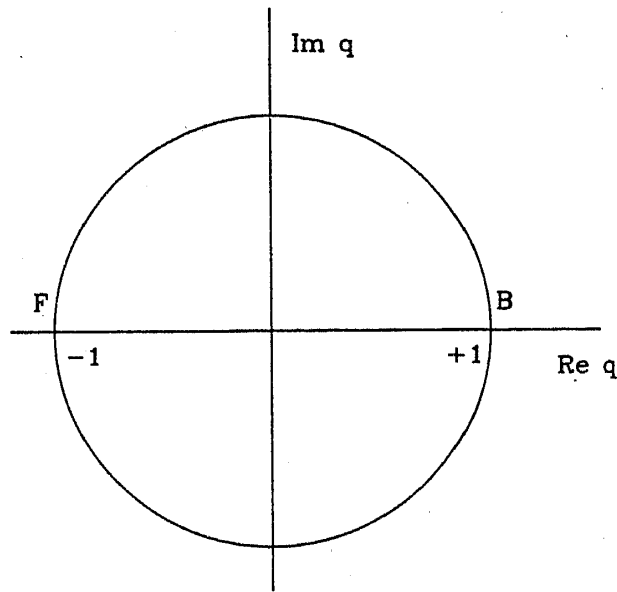
Figure 2. The complex q-plane of the q-mutator algebra. The disc $|q| < 1$ corresponds to infinite statistics and the circle $|q| = 1$ corresponds to fractional statistics. F and B are the Fermi–Dirac and Bose–Einstein points.

space (see figure 2). It is the states of vanishing norm of the form

$$(\ldots c_i^\dagger c_j^\dagger \ldots |0\rangle) \quad - e^{i\theta}(\ldots c_j^\dagger c_i^\dagger \ldots |0\rangle) \quad \text{for} \quad i < j \tag{5.20}$$

which contribute to the zeroes of $\det M_n$ in (3.34) at $|q| = 1$. Once we remove these states, we get the reduced Fock space of positive-definite norm.

The form-invariance of Q_{ij} for $i = j$ requires

$$p = -1 \tag{5.21}$$

with the corresponding cc relation

$$c_i c_i = 0 \tag{5.22}$$

This leads to the exclusion principle namely $n_i = 0$ or 1 only; equivalently, the norms of states with repeated indices are zero as seen by putting $p = -1$ in (3.35).

To sum up, we may note four possibilities all contained in the algebra of (5.14) and (5.15):

(a) Infinite statistics with multiple occupation

$$|q| \neq 1; \quad p \neq -1. \tag{5.23}$$

(b) Infinite statistics with exclusion principle

$$|q| \neq 1; \quad p = -1 \tag{5.24}$$

(c) Fractional statistics with multiple occupation

$$q = e^{i\theta}; \quad p \neq -1 \tag{5.25}$$

(d) Fractional statistics with exclusion principle

$$q = e^{i\theta}; \quad p = -1. \tag{5.26}$$

5.3 *q-statistics*

The $cc^\dagger$ relations are

$$c_i c_j^\dagger - q c_j^\dagger c_i = 0 \quad \text{for} \quad i < j \tag{5.27}$$

$$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\Sigma_{i<j} N_i} f(N_j) \tag{5.28}$$

We define

$$Q_{ij} = c_i c_j - q' c_j c_i \quad \text{for} \quad i > j \tag{5.29}$$

Using (5.27), (5.28) we get

$$
\begin{aligned}
Q_{ij} c_k^\dagger &= q^2 c_k^\dagger Q_{ij} \quad \text{for} \quad i < j < k \\
&= q^{*2} c_k^\dagger Q_{ij} \quad \text{for} \quad k < i < j \\
&= |q|^2 c_k^\dagger Q_{ij} \quad \text{for} \quad i < k < j \\
&= p q^* c_j^\dagger Q_{ij} + c_i |q|^{2\Sigma_{m<j} N_m} f(N_j)(1 - q' q^*) \quad \text{for} \quad i > j = k \\
&= p q c_i^\dagger Q_{ij} + |q|^{2\Sigma_{m<i} N_m} f(N_i) c_j (q - q' |q|^2) \quad \text{for} \quad k = i > j
\end{aligned}
\tag{5.30}
$$

Repeating the argument of form-invariance of Q_{ij}, we conclude

$$c_i c_j - q^{*-1} c_j c_i = 0 \quad \text{for} \quad i > j \tag{5.31}$$

In this proof we have not used any particular form of $f(N_j)$, but have used only the standard commutation relations among N_k and c_i (eqs. (2.23) and (2.26)).

This derivation of the $c_i c_j$ relation for $i \neq j$ is valid for the bosonic as well as the fermionic Fock spaces. However, for the fermionic space, there exists the additional relation:

$$c_i c_i = 0 \tag{5.32}$$

To derive this, we now commute $c_k^\dagger$ through $c_i c_i$ using (5.27–5.28), we get

$$
\begin{aligned}
c_i c_i c_k^\dagger &= q^{*2} c_k^\dagger c_i c_i \quad \text{for} \quad i > k \\
&= q^2 c_k^\dagger c_i c_i \quad \text{for} \quad k > i \\
&= p^2 c_i^\dagger c_i c_i + |q|^{2\Sigma_{m<i} N_m} c_i \{ p f(N_i - 1) + f(N_i) \} \quad \text{for} \quad i = k
\end{aligned}
\tag{5.33}
$$

So, for the validity of (5.32), we require

$$c_i \{ p f(N_i - 1) + f(N_i) \} = 0 \tag{5.34}$$

Substituting the form of $f(N_i)$ for the fermionic Fock space (from eq. (4.24))

$$f(N_i) = |r|^2 (\delta_{N_i,0} - p \delta_{N_i,1}), \tag{5.35}$$

we see that (5.34) is satisfied, since in the fermionic Fock space, $\delta_{N_i,2} = 0$ and $c_i \delta_{N_i,0} = 0$.

One may also note that (5.34) is also satisfied for $p = -1$ and $f(N_i) = $ constant (which may be chosen to be unity). So, for bosonic Fock space, i.e. $c_i c_i \neq 0$, we must avoid $p = -1$ in (5.28) and (4.15).

The above derivations of cc relations have used the general $cc^\dagger$ algebra and hence includes the cases of quantum-group-covariant algebras, independent deformed oscillators, commuting fermions, anticommuting bosons etc. Further, it must be noted that the presence of the term with the factor $|q|^{2\Sigma_{i<j}N_i}$ in (5.28) is crucial for the validity of the cc relations; without this factor, there will be no cc relation and we will get infinite statistics.

6. Two-indexed systems

We have so far considered generalized Fock spaces consisting of states $|n_g, n_h, \ldots, ; \mu\rangle$ where the indices $g, h, \ldots$ may refer either to a single quantum number, or to a collection of quantum numbers, specifying the space, spin and other internal degrees of freedom. In the latter case, it may be supposed that one has mapped a collection of indices to a single index. What is envisaged in this section are situations where such mapping is not possible. This can happen in various ways. To be specific, let us consider oscillators with a pair of indices, a latin index $(g, h, \ldots)$ and a greek index $(\alpha, \beta \ldots)$. There exists a class of systems in which the occupation numbers with a single index n_g or n_α are defined, but occupation numbers with both the indices $n_{g\alpha}$ are not defined. Such systems cannot be mapped into single-indexed systems. In another class of systems, $n_{g\alpha}$ do exist, but the subsidiary conditions that define the reduced Fock space depend on the two indices g and α in such a way that prevents mapping of (g, α) into a single index. We consider these two classes of systems in subsections 6.1 and 6.2 respectively.

In may also be mentioned that we first encountered such systems in the study of the Hubbard model in the limit of infinite Coulomb repulsion [43–46]. Although this was our original motivation, this has now opened the door to a more general framework encompassing novel forms of statistics and algebras.

6.1 *Systems in which $n_{g\alpha}$ do not exist*

We specify the states as $|n_g, n_h \ldots; n_\alpha, n_\beta \ldots; \mu\rangle$, where $n_g, n_h \ldots$ are the numbers of quanta with indices $g, h \ldots$ respectively, while $n_\alpha, n_\beta, \ldots$ are the numbers of quanta with indices $\alpha, \beta \ldots$ respectively and we have the constraint:

$$n_g + n_h + \ldots = n_\alpha + n_\beta + \ldots \tag{6.1}$$

We may regard n_g as the total number of quanta with index g whatever may be their greek index and similarly for n_α. In such a state, the occupancy number with both indices such as $n_{k\alpha}$ is not defined. The latin indices and the greek indices are independently permutated and this leads to a much enlarged space in each sector. Now μ goes over $1 \ldots s'$ where

$$s' = \frac{(n_g + n_h + \ldots)!}{n_g! n_h! \ldots} \times \frac{(n_\alpha + n_\beta + \ldots)!}{n_\alpha! n_\beta! \ldots} \tag{6.2}$$

This is to be compared to the size of the space of the states $|n_{g\alpha}, n_{h\beta} \ldots; \mu\rangle$ for which the range of μ is

$$s = \frac{(n_{g\alpha} + n_{h\beta} + \ldots)!}{n_{g\alpha}! n_{h\beta}! \ldots} \tag{6.3}$$

In general, s' is larger than s.

Let us consider the two-particle sector as an example. If we specify occupation numbers with both indices, we have the two states in the sector $(1_{k\alpha}, 1_{m\beta}):|1_{k\alpha}, 1_{m\beta}; \mu\rangle$ with $\mu = 1$ or 2 which correspond to $|1_{k\alpha}, 1_{m\beta}\rangle$ and $|1_{m\beta}, 1_{k\alpha}\rangle$ and another two states in the sector $(1_{m\alpha}, 1_{k\beta}):|1_{m\alpha}, 1_{k\beta}; \mu\rangle$ with $\mu = 1$ or 2 corresponding to $|1_{m\alpha}, 1_{k\beta}\rangle$ and $|1_{k\beta}, 1_{m\alpha}\rangle$. (Here we are not using $k\alpha, m\beta$ etc. in their generic sense; they are used to denote specific values of the indices). On the other hand, with the new type of states with decoupled indices for which occupation number with both indices do not exist, we have the two-particle states $|1_k, 1_m; 1_\alpha, 1_\beta; \mu\rangle$ and now μ goes over 1 to 4 corresponding to the four state vectors $|1_{k\alpha}, 1_{m\beta}\rangle$, $|1_{m\beta}, 1_{k\alpha}\rangle, |1_{m\alpha}, 1_{k\beta}\rangle, |1_{k\beta}, 1_{m\alpha}\rangle$. Thus, both the sectors considered earlier are combined to form a single enlarged sector. Such a regrouping of sectors with consequent enlargement occurs throughout the Fock space, in the case of the decoupled indices.

The construction of the orthonormal set as well as the other properties of the generalized Fock spaces given in §2 goes through for the present case of decoupled indices also except that the matrices X, M etc. will be of higher dimensions. The creation and destruction operators $c^\dagger_{k\alpha}, c_{k\alpha}$ also can be constructed in the same way as in §2. The relevant equations and formulae with the appropriate changes incorporated are given in Appendix A. They are self-explanatory.

Just as in the case of single-indexed systems (cf. §2), we can again have a super Fock space in which all the states connected by independent permutation of the latin and greek indices are taken to be independent. We shall call the associated statistics as 'doubly-infinite' statistics since it is infinite statistics in latin and greek indices separately. By imposing relations among the permutated states, one can get many kinds of reduced Fock spaces. Because of the larger number of available states in each sector, many new types of statistics become possible. These can be discussed and the associated algebras can be constructed by the same procedure as in §3, 4 and 5. However, we shall be brief and restrict ourselves to presenting some of the important results only. Some of these algebras and the new kinds of statistics implied by them have been discussed by us in greater detail in the earlier papers [18, 19, 46].

Doubly-infinite statistics

Consider the $cc^\dagger$ algebra described by

$$c_{k\alpha} c^\dagger_{m\beta} - q\delta_{\alpha\beta} \sum_\gamma c^\dagger_{m\gamma} c_{k\gamma} = \delta_{km} \delta_{\alpha\beta} \tag{6.4}$$

where q is a real parameter lying in the range

$$-1 < q < 1. \tag{6.5}$$

It is the second term on the left of (6.4) in which α and β have been dissociated from k and m respectively that leads to the decoupling of the latin and greek indices and

prevents the mapping of latin and greek indices to a single index. One can show [19] that the inner-product matrices following from this algebra are all positive-definite for $-1 < q < 1$. Further, one can show using arguments similar to those in § 5 that there is no cc relation for this algebra in the same parameter range and so all the states connected by *independent* permutation of the latin and greek indices are independent. The underlying Fock space is the full super Fock space of dimension given by (6.2) and we have infinite statistics in latin and greek indices separately, or doubly-infinite statistics.

For $q = 0$ in (6.4), the two indices can be mapped into a single index and the algebra reduces to the standard representation of single-indexed infinite statistics described in § 3.1.

The algebra of (6.4) is covariant under the unitary transformations on the latin indices:

$$d_{k\alpha} = \sum_m U_{km} c_{m\alpha}; \quad U^\dagger U = UU^\dagger = 1 \tag{6.6}$$

as well as under separate unitary transformations on the greek indices:

$$e_{k\alpha} = \sum_\lambda V_{\alpha\lambda} c_{k\lambda}; \quad V^\dagger V = VV^\dagger = 1. \tag{6.7}$$

But the algebra is not covariant under the enlarged unitary transformations involving both the latin and greek indices (for $q \neq 0$). A special case of the unitary transformations of (6.6) and (6.7) is the phase transformation. Equation (6.4) is covariant under either of the following phase transformations:

$$d_{k\alpha} = e^{i\phi_k} c_{k\alpha} \tag{6.8}$$

$$e_{k\alpha} = e^{i\phi_\alpha} c_{k\alpha} \tag{6.9}$$

As a consequence, the number operators N_k and N_α exist. However, (6.4) is not covariant under the transformation:

$$f_{k\alpha} = e^{i\phi_{k\alpha}} c_{k\alpha} \tag{6.10}$$

and correspondingly, $N_{k\alpha}$ does not exist.

We can make q in (6.4) complex provided we order the latin indices and thus we get an algebra which is the analogue of the q-mutator algebra with complex q for the single-indexed systems (§ 3):

$$c_{i\alpha} c_{j\beta}^\dagger - q\delta_{\alpha\beta} \sum_\gamma c_{j\gamma}^\dagger c_{i\gamma} = 0 \quad \text{for} \quad i < j \tag{6.11}$$

$$c_{j\alpha} c_{j\beta}^\dagger - p\delta_{\alpha\beta} \sum_\gamma c_{j\gamma}^\dagger c_{j\gamma} = \delta_{\alpha\beta} \tag{6.12}$$

where q is complex, but p is real. This again describes the same doubly-infinite statistics; only the representation and algebra are different. This algebra is no longer covariant under the unitary transformations of (6.6) and (6.7), but is still covariant under the phase transformations of (6.8) and (6.9). Positivity of the inner-product matrices M requires $|q| < 1$. As for p, similar statements as in § 3 can be made.

Coming back to (6.4), at the boundary of the range of q given in (6.5), namely at $q = \pm 1$, we get two new forms of statistics called orthobose and orthofermi statistics [18, 19, 46] which reside in reduced Fock spaces defined by cc relations. The algebras for these are given below:

Orthobose statistics:

$$c_{k\alpha}c_{m\beta}^{\dagger} - \delta_{\alpha\beta} \sum_{\gamma} c_{m\gamma}^{\dagger}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta} \qquad (6.13)$$

$$c_{k\alpha}c_{m\beta} - c_{m\alpha}c_{k\beta} = 0. \qquad (6.14)$$

Orthofermi statistics:

$$c_{k\alpha}c_{m\beta}^{\dagger} + \delta_{\alpha\beta} \sum_{\gamma} c_{m\gamma}^{\dagger}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta} \qquad (61.5)$$

$$c_{k\alpha}c_{m\beta} + c_{m\alpha}c_{k\beta} = 0. \qquad (6.16)$$

The inner product matrices M for the reduced Fock spaces generated by these algebras (eqs. (6.13)–(6.16)) can be shown to be positive. In both these statistics, any two states obtained by permuting the greek indices are independent. On the other hand, states obtained by permuting the latin indices are either equal to each other, or related by $(-1)^J$ where J is the number of inversions in the permutation of the latin indices, for the orthobose or orthofermi statistics respectively. Thus, we have infinite statistics in the greek indices and Bose or Fermi statistics in the latin indices. For orthofermi statistics, we have the further condition:

$$n_k = 0 \text{ or } 1 \text{ only.} \qquad (6.17)$$

It must be noted that the exclusion implied by (6.17) is stronger than the usual Pauli exclusion principle; (6.17) requires that there cannot be more than one particle with index k whatever may be its greek index.

We have constructed a local relativistic quantum field theory [47] based on orthostatistics. It may be worth mentioning that although orthostatistics does involve infinite statistics, the problems faced by finite statistics are avoided by associating the latin index (pertaining to Fermi or Bose statistics) with the conventional degrees of freedom such as momentum and spin and assigning the greek index (of infinite statistics) to a new degree of freedom.

Orthostatistics can be generalized to q-orthostatistics. We keep the infinite statistics in the greek indices but have q-bose or q-fermi statistics in the latin indices. More precisely, we must regard these as q-orthostatistics lying in orthobosonic and orthofermionic Fock spaces. We can call them q-orthobose or q-orthofermi statistics.

q-Orthobose statistics:

$$c_{i\alpha}c_{j\beta}^{\dagger} + q\delta_{\alpha\beta} \sum_{\gamma} c_{j\gamma}^{\dagger}c_{i\gamma} = 0 \quad \text{for} \quad i < j \qquad (6.18)$$

$$c_{j\alpha}c_{j\beta}^{\dagger} - |q|^2 \delta_{\alpha\beta} \sum_{\gamma} c_{j\gamma}^{\dagger}c_{j\gamma} = \delta_{\alpha\beta} + \delta_{\alpha\beta}(|q|^2 - 1) \sum_{k<j} \sum_{\gamma} c_{k\gamma}^{\dagger}c_{k\gamma} \qquad (6.19)$$

$$c_{i\alpha}c_{j\beta} + q^* c_{j\alpha}c_{i\beta} = 0 \quad \text{for} \quad i < j \qquad (6.20)$$

q-Orthofermi statistics:

$$c_{i\alpha}c^{\dagger}_{j\beta} + q\delta_{\alpha\beta}\sum_{\gamma}c^{\dagger}_{j\gamma}c_{i\gamma} = 0 \quad \text{for} \quad i < j \tag{6.21}$$

$$c_{j\alpha}c^{\dagger}_{j\beta} + \delta_{\alpha\beta}\sum_{\gamma}c^{\dagger}_{j\gamma}c_{j\gamma} = \delta_{\alpha\beta} + \delta_{\alpha\beta}(|q|^2 - 1)\sum_{k<j}\sum_{\gamma}c^{\dagger}_{k\gamma}c_{k\gamma} \tag{6.22}$$

$$c_{i\alpha}c_{j\beta} + q^{*}c_{j\alpha}c_{i\beta} = 0 \quad \text{for} \quad i < j \tag{6.23}$$

$$c_{j\alpha}c_{j\beta} = 0 \tag{6.24}$$

In (6.18)–(6.24), q is an arbitrary complex parameter. For $q = e^{i\theta}$, and $q = 0$ we shall have fractional statistics and statistics of frozen order respectively, but in the latin indices only. Of course one can construct many other algebras corresponding to the same statistics, just as in the case of the single-indexed systems.

Further, these equations are analogous to (4.3), (4.12), (4.23) and (4.28) which are covariant under quantum groups. So, another direction is indicated here for the further generalization of quantum-group-theoretic structures to two-indexed systems.

6.2 *Systems in which $n_{g\alpha}$ exist*

We shall now consider a different kind of double-indexed systems. Here, the occupation numbers $n_{g\alpha}$ exist, nevertheless mapping to single-indexed systems is not possible because of the subsidiary conditions that define the reduced Fock space. We give below three examples of such double-indexed systems, described by the algebras:

(a)

$$c_{k\alpha}c^{\dagger}_{m\beta} + (1 - \delta_{km})c^{\dagger}_{m\beta}c_{k\alpha} = \delta_{km}\delta_{\alpha\beta}\left(1 - \sum_{\gamma}c^{\dagger}_{k\gamma}c_{k\gamma}\right) \tag{6.25}$$

$$c_{k\alpha}c_{m\beta} + (1 - \delta_{km})c_{m\beta}c_{k\alpha} = 0. \tag{6.26}$$

(b)

$$c_{k\alpha}c^{\dagger}_{m\beta} - (1 - \delta_{km})c^{\dagger}_{m\beta}c_{k\alpha} = \delta_{km}\delta_{\alpha\beta}\left(1 + \sum_{\gamma}c^{\dagger}_{k\gamma}c_{k\gamma}\right) \tag{6.27}$$

$$(1 - \delta_{km})(c_{k\alpha}c_{m\beta} - c_{m\beta}c_{k\alpha}) = 0. \tag{6.28}$$

(c)

$$c_{k\alpha}c^{\dagger}_{m\beta} - (1 - \delta_{km})c^{\dagger}_{m\beta}c_{k\alpha} = \delta_{km}\delta_{\alpha\beta}\left(1 - \sum_{\gamma}c^{\dagger}_{k\gamma}c_{k\gamma}\right) \tag{6.29}$$

$$c_{k\alpha}c_{m\beta} - (1 - \delta_{km})c_{m\beta}c_{k\alpha} = 0. \tag{6.30}$$

Each of these algebras is covariant under the unitary transformation on the greek indices defined by (6.7), but not covariant under the unitary transformation on the latin indices defined by (6.6). However, all of them are covariant under not only the phase transformations in (6.8) and (6.9) but also the $k\alpha$-dependent phase transformation of (6.10), hence all the occupation numbers n_k, n_α as well as $n_{k\alpha}$ exist.

Algebra (a) leads to states which are antisymmetric for simultaneous interchange of latin and greek indices as in Fermi–Dirac statistics, but the usual Pauli exclusion principle is replaced by the stronger or more exclusive exclusion principle, as in

orthofermi statistics (see eq. (6.17)):

$$n_k = 0 \text{ or } 1 \text{ only,}$$

where

$$(6.31)$$

$$n_k \equiv \sum_{\alpha} n_{k\alpha}$$

$$(6.32)$$

These are precisely the states that are allowed in the Hubbard model of strongly correlated electrons in the limit of infinite intrasite Coulomb repulsion if we interpret the latin index as the site and the greek index as the spin. Hence, this algebra and the statistics can be called Hubbard algebra and Hubbard statistics respectively. For more details, the reader is referred to [46]. Since $n_{k\alpha}$ exists, the states of the system can be

Table 4. Statistics and algebras for 2-indexed systems.

$N_{k\alpha}$	Statistics in latin indices	Statistics in greek indices	$cc^{\dagger}$ algebra
Do not exist	infinite	infinite	$c_{k\alpha}c_{m\beta}^{\dagger} - q\delta_{\alpha\beta}\sum_{\gamma} c_{m\gamma}^{\dagger}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta}$ (real q)
"	"	"	$\begin{cases} c_{k\alpha}c_{m\beta}^{\dagger} - q\delta_{\alpha\beta}\sum_{\gamma} c_{m\gamma}^{\dagger}c_{k\gamma} = 0, & \text{for } k < m \\ c_{k\alpha}c_{k\beta}^{\dagger} - p\delta_{\alpha\beta}\sum_{\gamma} c_{k\gamma}^{\dagger}c_{k\gamma} = \delta_{\alpha\beta} & \text{(complex } q \\ & \text{and real } p) \end{cases}$
"	Fermi/Bose	"	$c_{k\alpha}c_{m\beta}^{\dagger} \pm \delta_{\alpha\beta}\sum_{\gamma} c_{m\gamma}^{\dagger}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta}$
"	q-fermi/ q-bose	"	$\begin{cases} c_{i\alpha}c_{j\beta}^{\dagger} - q\delta_{\alpha\beta}\sum_{\gamma} c_{j\gamma}^{\dagger}c_{i\gamma} = 0 \quad \text{for } i < j \\ c_{j\alpha}c_{j\beta}^{\dagger} - x\delta_{\alpha\beta}\sum_{\gamma} c_{j\gamma}^{\dagger}c_{j\gamma} \\ \quad = \delta_{\alpha\beta} + \delta_{\alpha\beta}(|q|^2 - 1)\sum_{k<j}\sum_{\gamma} c_{k\gamma}^{\dagger}c_{k\gamma} \\ x = |q|^2 \text{ for } q\text{-bose; } x = -1 \text{ for } q\text{-fermi} \end{cases}$
Exist	Antisymmetric for total exchange, but $n_k \leqslant 1$		$c_{k\alpha}c_{m\beta}^{\dagger} + (1 - \delta_{km})c_{m\beta}^{\dagger}c_{k\alpha}$ $= \delta_{km}\delta_{\alpha\beta}\left(1 - \sum_{\gamma} c_{k\gamma}^{\dagger}c_{k\gamma}\right)$
"	Symmetric for total exchange for $k \neq m$, but infinite statistics in greek indices for $k = m$		$c_{k\alpha}c_{m\beta}^{\dagger} - (1 - \delta_{km})c_{m\beta}^{\dagger}c_{k\alpha}$ $= \delta_{km}\delta_{\alpha\beta}\left(1 + \sum_{\gamma} c_{k\gamma}^{\dagger}c_{k\gamma}\right)$
"	Symmetric for total exchange but $n_k \leqslant 1$		$c_{k\alpha}c_{m\beta}^{\dagger} - (1 - \delta_{km})c_{m\beta}^{\dagger}c_{k\alpha}$ $= \delta_{km}\delta_{\alpha\beta}\left(1 - \sum_{\gamma} c_{k\gamma}^{\dagger}c_{k\gamma}\right)$

characterized as $|n_{k\alpha}, n_{m\beta} \ldots; \mu\rangle$. However, because of the above constraint (eq. (6.31)) which can be rewritten in the form:

$$\sum_{\alpha} n_{k\alpha} = 0 \text{ or } 1 \text{ only} \qquad (6.33)$$

this system cannot be mapped into a single-indexed system.

Algebra (b) can be regarded as the bosonic 'counterpoint' of algebra (a). States are symmetric under simultaneous exchange of latin and greek indices for quanta with different latin indices $(k \neq m)$. But for quanta with identical latin indices $(k = m)$, there is no restriction on the symmetry with respect to the greek indices. In other words, there is infinite statistics in greek indices if the corresponding latin indices are identical. Whereas algebra (a) leads to more exclusive states than allowed by Pauli, algebra (b) leads to 'more inclusive' states than allowed by Bose. For this reason we may call algebra (b) as the 'inclusive counterpoint' to algebra (a). Such a restriction on the allowed states of a two-indexed system which distinguishes $k = m$ from $k \neq m$ cannot be mapped into a condition for a single-indexed system.

Finally, algebra (c) leads to states that are symmetric for simultaneous exchange of latin and greek indices, but the stronger exclusion principle of (6.33) is also valid as in algebra (a). Hence, algebras (a) and (c) represent two forms of statistics which may be called antisymmetric and symmetric Hubbard statistics respectively both residing within the same reduced Fock space. On the other hand, algebra (b) and its statistics lie in a different reduced Fock space which is a Fock space with the new 'inclusion' principle.

A compilation of the algebras and statistics for two-indexed systems is given in table 4. The cc relations are not included since they can be shown to follow from the $cc^\dagger$ algebra, whenever they exist.

7. Summary and discussion

We have formulated a theory of generalized Fock spaces which is sufficiently general so as to encompass the well-known Fock spaces and many newer ones. We have shown that such a theory can be constructed without introducing creation and annihilation operators. The only requirements for constructing a generalized Fock space are to specify the set of allowed states, and to make it an inner product space. By freeing the notion of the underlying state space from c and $c^\dagger$, we are able to define different forms of quantum statistics in a representation independent manner. Subsequently, one can construct c and $c^\dagger$ and their algebras in any desired representation.

Our general formalism not only unifies the various forms of statistics and algebras proposed so far but also allows one to construct many new forms of quantum statistics as well as algebras of c and $c^\dagger$ in a systematic manner. Some of these are the following:

(a) Many new algebras for infinite statistics
(b) Complex q-statistics and a number of $cc^\dagger$ algebras representing them
(c) A consistent algebra of c and $c^\dagger$ for 'fractional' statistics
(d) Null statistics or statistics of frozen order
(e) 'Doubly-infinite' statistics and its representations
(f) q-orthobose and q-orthofermi statistics

(g) A statistics for two-indexed systems with a new 'inclusion principle'.
(h) A symmetric version of Hubbard statistics.

Our primary concept is that of generalized Fock space, of which many categories have been introduced in this paper. Next comes the notion of statistics which is defined by the type of symmetry or relationship among the state vectors residing in the particular type of Fock space. In a given Fock space, more than one type of symmetry can be postulated, the prime example of this being the symmetry, antisymmetry or q-symmetry in the bosonic and fermionic Fock spaces. For a given statistics, there can exist different representations of c and $c^\dagger$, leading to different $cc^\dagger$ relations. To summarize, a particular Fock space can admit different statistics, and a particular statistics can be represented by more than one $cc^\dagger$ algebra. But *the important point is that various statistics and algebras residing in a given Fock space are all interelated.*
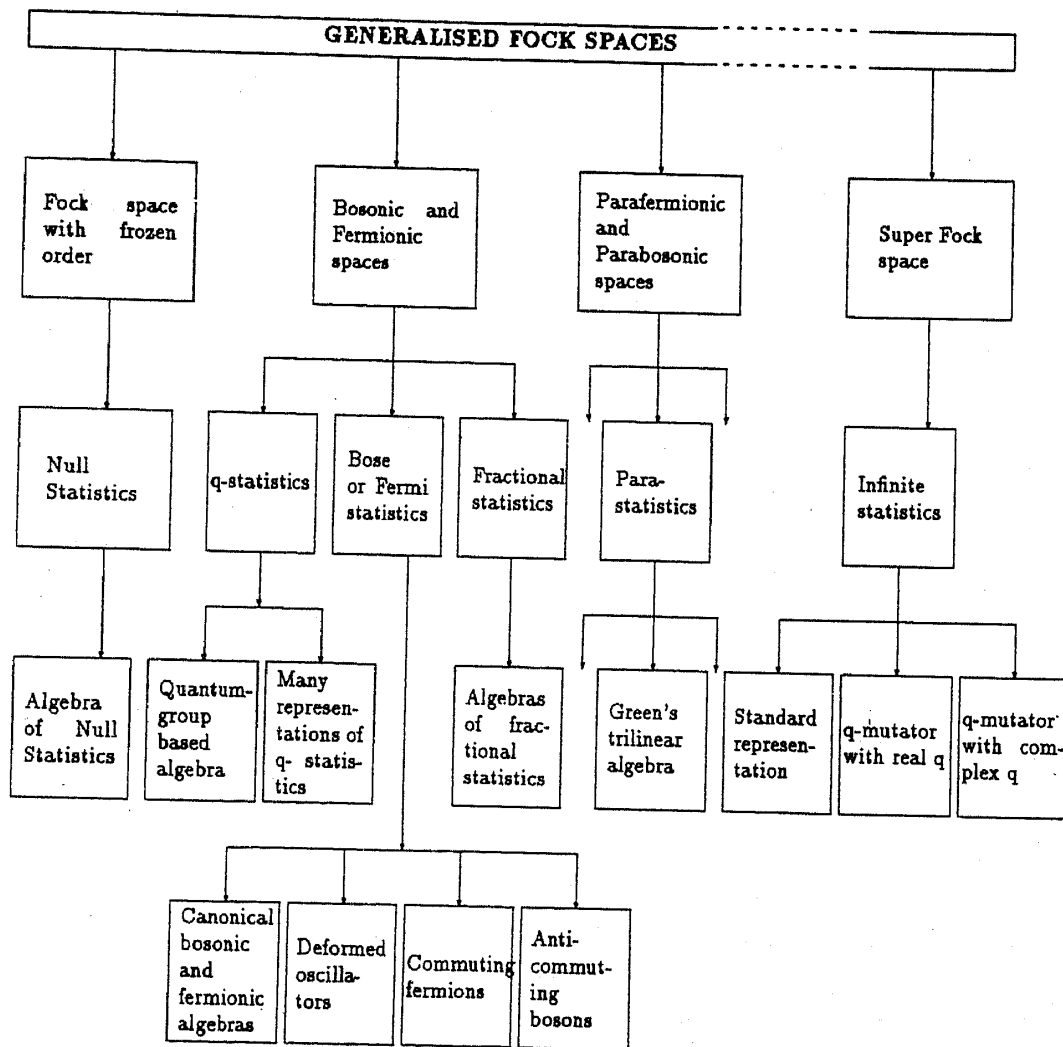
Figure 3. Generalized Fock spaces, quantum statistics, algebras and their interconnections.

Generalized Fock spaces, statistics and algebras

These interconnections are given by generalized versions of the well-known Jordan–Wigner–Klein transformations. No such interconnections exist among statistics and algebras belonging to distinct Fock spaces. We must further add that equilibrium thermodynamics of a system of free particles is the same for all the different statistics and algebras within the same Fock space.

For the sake of clarity, the above-described logical order of concepts as well as their logical interconnections are presented in the form of flow charts or block diagrams in figures 3 and 4. The single-indexed systems are considered in figure 3. The Fock spaces of higher dimension are shown to the right of those of lower dimension. The Fock space of frozen order as well as the bosonic and fermionic Fock spaces have the lowest dimension $d = 1$ in any sector $\{n_g, n_h...\}$. Next come the parafermionic and parabosonic Fock spaces which have $d > 1$. At the extreme right, we have the super Fock space which has the largest dimension $d = s$ in each sector with s given by (2.6). Null statistics and infinite statistics can be regarded as the opposite limiting cases of generalized statistics and hence these two forms of statistics along with their Fock spaces occupy the opposite ends of the diagram. Although not shown separately
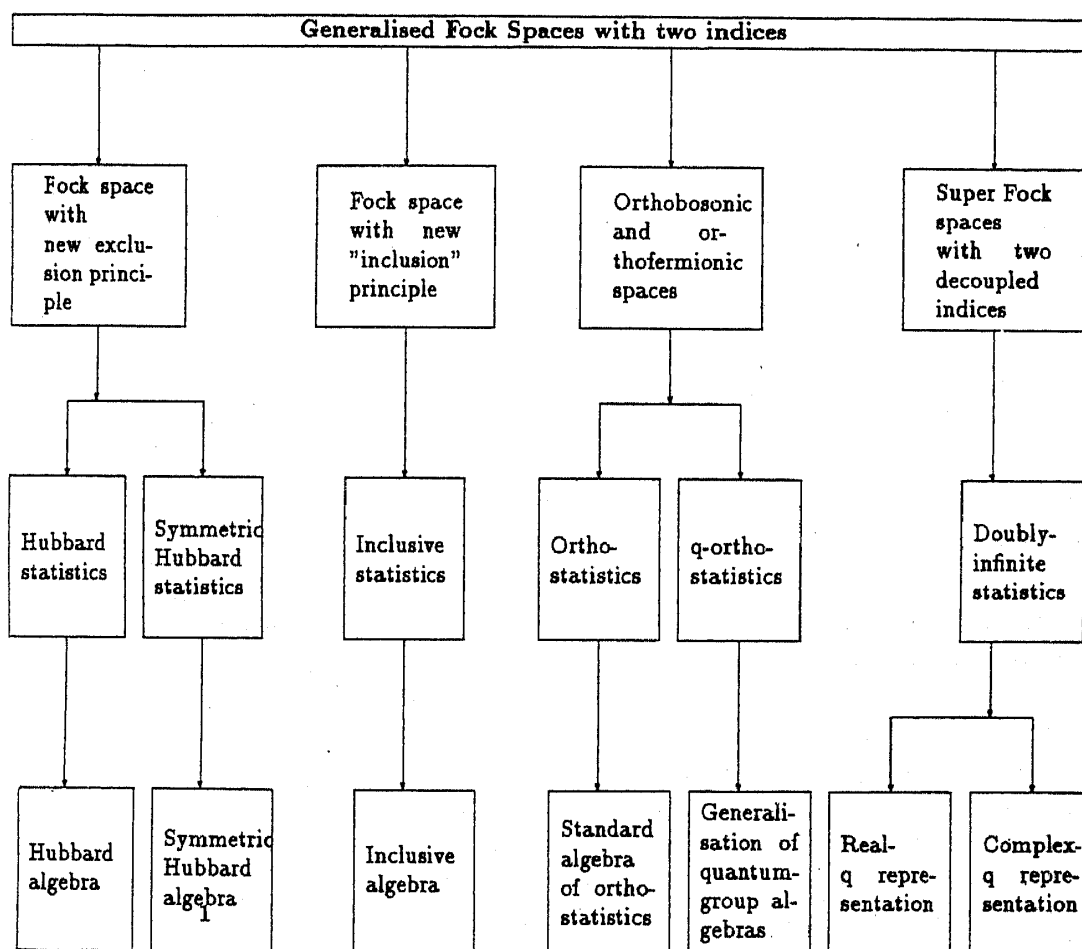
Figure 4. Same as figure 3 for systems with two indices that cannot be mapped into a single index.

in figure 3 because of lack of space, the bosonic and fermionic Fock spaces are distinct and each must be separately associated with the complete set of statistics and algebras shown. Same is true of the parabosonic and parafermionic Fock spaces. Further, there are two Fock spaces of frozen order, the bosonic and fermionic type. And finally, there exists another super Fock space with exclusion principle, which is not shown separately.

Within the parafermionic and parabosonic Fock spaces many 'deformations' of parastatistics and many other representations and algebras apart from Green's trilinear algebra [35] are possible. These are indicated by the hanging arrows in figure 3. Further, as shown by the dotted lines, there is enough room for many new varieties of Fock spaces and associated statistics and algebras. These possibilities may be pursued in the future.

Coming to figure 4 depicting the systems with two indices, here again Fock spaces of higher dimension generally lie to the right. Although shown together, the orthobosonic and orthofermionic Fock spaces must be regarded distinct. Here, one can envisage a richer harvest of new Fock spaces, statistics and algebras because of the two indices and this again is for the future.

We now conclude with some general comments:

1. We must once again repeat and emphasize the point that most of the q-deformations on oscillators discussed in the literature amount to only a change of variable and hence must be regarded as different avatars of bosonic or fermionic systems. However, one must clearly distinguish those deformations such as the q-mutator algebra of Greenberg that require the construction of new types of Fock spaces. Obviously, Greenberg-type of deformations can never be reduced to change of variables living within the bosonic or fermionic Fock space. Some degree of confusion prevails in recent literature since this distinction is not kept in mind. (see for instance [7, 32, 41, 42, 48]).

2. In § 4, we have shown that algebras that are covariant under quantum groups are only a particular case of the more general class of algebras that can be derived from the formalism of generalized Fock spaces. This formalism is based on linear vector space and linear operators acting on this space; mathematically, no more sophistication is required. And yet it is capable of handling quantum-group related structures in a self-contained manner. It would seem that the basic concepts of quantum groups are contained in the theory of generalized Fock spaces and it must be possible to construct quantum group itself starting from this theory.

3. We have already referred to the desirability of covariance under unitary transformations that mix the indices as a requirement for the algebras of creation and annihilation operators. We shall call the algebras that statisfy this requirement as covariant algebras. This property stems from the superposition principle in quantum mechanics. Since the indices describe quantum states of a single particle, if we demand that, for any orthonormal set of quantum states obtained by superposition of the original set of quantum states, the algebra should retain the same form, then covariance under unitary transformations follows. Many of the algebras presented in this paper violate this requirement. Nevertheless, these algebras may be useful to describe specific systems in specific states such as those encountered in condensed matter physics.

Some of these noncovariant algebras do have other nice physical and mathematical properties. Under this category, we may include the algebras that are covariant under quantum groups or the algebras representing braid group statistics [49, 50].

Among the algebras for single-indexed systems that have been discussed, Greenberg's q-mutator algebra is the only q-deformation that is covariant under unitary transformations, but then one has to pay the price of the enlarged Fock space. Every other known q-deformation leads to a noncovariant algebra.

Greenberg's q-mutator algebra (including the case $q = 0$ which is the standard representation), the canonical bosonic and fermionic algebras and Green's trilinear algebras for parabosons and parafermions are the covariant representatives living respectively in the super Fock space, bosonic and fermionic Fock spaces and the parabosonic and parafermionic Fock spaces. All the other algebras living in these three categories of Fock spaces, although noncovariant, can be transformed to these covariant algebras through equations such as (4.32). This is not the case for the algebra of null statistics or the algebra of infinite statistics with Pauli principle living respectively in the Fock space of frozen order and the super Fock space with Pauli principle. In these Fock spaces, covariant algebras do not exist.

4. Quantum mechanics is sometimes viewed as a deformation of classical mechanics since the commutator bracket of quantum mechanics can be related to the deformation of the classical Poisson bracket, the Planck's constant playing the role of the deformation parameter. Relying on similar reasoning it has been proposed that a deformation of canonical commutation relations will lead to fundamentally new mathematical or physical structures [51–53]. The analysis presented in this paper shows that nothing of this sort happens, if viewed within the framework of Fock space. The transition from classical to quantum mechanics requires the replacement of the notion of the phase space by that of the Hilbert space or Fock space. In contrast, we have seen that all the deformations of commutation relations can be formulated within the framework of Fock space. In fact most of the deformed structures proposed in the literature exist within the time-honoured bosonic and fermionic Fock spaces only. Even Greenberg's infinite statistics lives within a Fock space, although an enlarged one.

5. While remaining within the framework of quantum mechanics, the general theory of Fock spaces presented here throws light on the enlarged framework within which the familiar quantum field theory and statistical mechanics reside and hence may lead to newer forms of quantum field theory and statistical mechanics. This is infact the main motivation behind our work. Apart from earlier work on parastatistics [37], we may mention as examples of new forms of quantum field theories, Greenberg's construction [13, 14, 54] of a non-relativistic quantum field theory based on infinite statistics and our construction [47] of a local relativistic quantum field theory based on orthostatistics. Many other forms of quantum field theories based on the generalized Fock spaces may be possible. Their formulation and study is an agenda for the future.

6. One may not be able to construct local relativistic quantum field theories corresponding to many of the newer forms of statistics and algebras, since admissible statistics in relativistic systems is severely restricted by the axioms of local quantum physics [55]. However, nonrelativistic quantum field theories based on such 'inadmissible' statistics are still possible. Condensed matter physics is a rich field where

applications of such theories may be relevant. In fact there is no reason why the quasiparticles encountered in condensed matter systems should be bosons or fermions only. We have shown that any of the generalized Fock spaces provides a perfectly valid quantum-mechanical framework for many-particle systems. Hence, quasiparticles living in a generalized Fock space offer an important field of study.

Appendix A

Generalised Fock Spaces for Two-indexed Systems

Here we consider only those two-indexed systems in which $n_{g\alpha}$ do not exist. See §6.1. *The state vectors, inner products and orthonormal sets:*

$$\langle n'_g, n'_h \ldots; n'_\alpha, n'_\beta \ldots; \mu | n_g, n_h \ldots; n_\alpha, n_\beta \ldots; v \rangle$$

$$= \delta_{n'_g n_g} \delta_{n'_h n_h} \ldots \delta_{n'_\alpha n_\alpha} \delta_{n'_\beta n_\beta} \ldots M_{\mu v} \tag{A1}$$

$$\| n_g, n_h \ldots; n_\alpha, n_\beta \ldots; \mu \rangle\rangle = \sum_v X_{v\mu} | n_g, n_h \ldots; n_\alpha, n_\beta \ldots; v \rangle \tag{A2}$$

$$\langle\langle n'_g, n'_h \ldots; n'_\alpha, n'_\beta \ldots; \mu \| n_g, n_h \ldots; n_\alpha, n_\beta \ldots; v \rangle\rangle$$

$$= \delta_{n'_g n_g} \delta_{n'_h n_h} \ldots \delta_{n'_\alpha n_\alpha} \delta_{n'_\beta n_\beta} \ldots \delta_{\mu v} \tag{A3}$$

$$M^{-1} = XX^+ \tag{A4}$$

$$I = \sum_{\substack{n_g, n_h \ldots \\ n_\alpha, n_\beta \ldots}} \sum_\mu \| n_g, n_h \ldots; n_\alpha, n_\beta \ldots; \mu \rangle\rangle \langle\langle n_g, n_h \ldots; n_\alpha, n_\beta \ldots; \mu \| \tag{A5}$$

$$= \sum_{\substack{n_g, n_h \ldots \\ n_\alpha, n_\beta \ldots}} \sum_{\lambda, v} | n_g, n_h \ldots; n_\alpha, n_\beta \ldots; v \rangle (M^{-1})_{v\lambda} \langle n_g \ldots; n_\alpha \ldots; \lambda | \tag{A6}$$

Projection operators:

$$P(n_g, n_h \ldots; n_\alpha, n_\beta \ldots)$$

$$= \sum_\mu \| n_g, n_h \ldots; n_\alpha, n_\beta \ldots; \mu \rangle\rangle \langle\langle n_g, n_h \ldots; n_\alpha, n_\beta \ldots; \mu \| \tag{A7}$$

$$= \sum_{\lambda, v} | n_g, n_h \ldots; n_\alpha, n_\beta \ldots; v \rangle (M^{-1})_{v\lambda} \langle n_g \ldots; n_\alpha \ldots; \lambda | \tag{A8}$$

$$I = \sum_{\substack{n_g, n_h \ldots \\ n_\alpha, n_\beta \ldots}} P(n_g, n_h \ldots; n_\alpha, n_\beta \ldots) \tag{A9}$$

$$P(n_g, n_h \ldots; n_\alpha, n_\beta \ldots) \| n'_g, n'_h \ldots; n'_\alpha, n'_\beta \ldots; \mu \rangle\rangle$$

$$= \delta_{n_g n'_g} \ldots \delta_{n_\alpha n'_\alpha} \ldots \| n_g \ldots; n_\alpha \ldots; \mu \rangle\rangle \tag{A10}$$

$$P(n_g \ldots; n_\alpha \ldots) | n'_g \ldots; n'_\alpha \ldots; \mu \rangle$$

$$= \delta_{n_g n'_g} \ldots \delta_{n_\alpha n'_\alpha} \ldots | n_g \ldots; n_\alpha \ldots; \mu \rangle \tag{A11}$$

Number operators:

$$N_k = \sum_{\substack{n_g...n_k... \\ n_\alpha...}} n_k P(n_g...n_k...;n_\alpha...) \tag{A12}$$

$$N_\beta = \sum_{\substack{n_g... \\ n_\alpha,n_\beta...}} n_\beta P(n_g...;n_\alpha,n_\beta...) \tag{A13}$$

$$N_k \| n_g...n_k...;n_\alpha,n_\beta...;\mu \rangle\!\rangle = n_k \| n_g...n_k...;n_{\alpha},n_\beta...;\mu \rangle\!\rangle \tag{A14}$$

$$N_k | n_g...n_k...;n_\alpha,n_\beta...;\mu \rangle = n_k | n_g...n_k...;n_\alpha,n_\beta...;\mu \rangle \tag{A15}$$

$$N_\beta \| n_g...n_k...;n_\alpha,n_\beta...;\mu \rangle\!\rangle = n_\beta \| n_g...n_k...;n_\alpha,n_\beta...;\mu \rangle\!\rangle \tag{A16}$$

$$N_\beta | n_g...n_k...;n_\alpha,n_\beta...;\mu \rangle = n_\beta | n_g...n_k...;n_\alpha,n_\beta...;\mu \rangle \tag{A17}$$

$$[N_k,N_j] = [N_\alpha,N_\beta] = [N_k,N_\alpha] = 0 \quad \text{for any } k,j \text{ and any } \alpha,\beta \tag{A18}$$

Total number operator is

$$N = \sum_k N_k = \sum_\alpha N_\alpha \tag{A19}$$

Creation and destruction operators:

$$c_{j\beta}^\dagger = \sum_{\substack{n_g,n_j... \\ n_\alpha,n_\beta...}} \sum_{\mu'\nu} A_{\mu'\nu} | n_g...(n_j+1)...;n_\alpha,n_\beta+1...;\mu' \rangle$$

$$\otimes \langle n_g...n_j...;n_\alpha,n_\beta...\nu | \tag{A20}$$

$$[c_{j\beta}^\dagger,N_k] = -c_{j\beta}^\dagger \delta_{jk} \tag{A21}$$

$$[c_{j\beta}^\dagger,N_\alpha] = -c_{j\beta}^\dagger \delta_{\alpha\beta} \tag{A22}$$

For some particular μ,

$$|3_g,2_h;4_\alpha,1_\beta;\mu \rangle = |1_{g\alpha}1_{g\alpha}1_{g\beta}1_{h\alpha}1_{h\alpha} \rangle = (c_{g\alpha}^\dagger)^2 c_{g\beta}^\dagger (c_{h\alpha}^\dagger)^2 |0 \rangle \tag{A23}$$

$$c_{j\beta}^\dagger | n_g...n_j...;n_\alpha,n_\beta...;\lambda \rangle = | \underbrace{1_{j\beta};n_g...n_j...;n_\alpha,n_\beta...;}_{n_g...n_j+1...;n_\alpha,n_\beta+1} \lambda \rangle \tag{A24}$$

$$c_{j\sigma}^\dagger | n_g...n_j...;n_\alpha...n_\sigma...;\lambda \rangle$$

$$= \sum_{\substack{n_g'...n_j'... \\ n_\alpha'...n_\sigma'...}} \sum_{\mu'\nu} A_{\mu'\nu} | n_g'...(n_j'+1)...;n_\alpha'...n_\sigma'+1...;\mu' \rangle \tag{A25}$$

$$\otimes \langle n_g'...n_j'...;n_\alpha'...n_\sigma'...;\nu | n_g...n_j...;n_\alpha...n_\sigma...;\lambda \rangle$$

$$= \sum_{\mu'\nu} A_{\mu'\nu} M_{\nu\lambda} | n_g...(n_j+1)...;n_\alpha...n_\sigma+1...;\mu' \rangle \tag{A26}$$

$$\sum_\nu A_{\mu'\nu} M_{\nu\lambda} = \delta_{\mu'\lambda} \tag{A27}$$

$$A = M^{-1} \tag{A28}$$

$$c_{j\sigma}^\dagger = \sum_{\substack{n_g...n_j.. \\ n_\alpha...n_\sigma..}} \sum_{\lambda\nu} (M^{-1})_{\lambda\nu} | 1_{j\sigma};n_g...n_j...;n_\alpha...n_\sigma...;\lambda \rangle$$

$$\otimes \langle n_g...n_j...;n_\alpha...n_\sigma...;\nu | \tag{A29}$$

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