# Travelling waves in a drifting flux lattice

R.Aditi Simha[a] and Sriram Ramaswamy[b,c]
*Department of Physics, Indian Institute of Science, Bangalore 560 012, India*
(final version published in Phys. Rev. Lett. **83** (1999) 3285)

Starting from the time-dependent Ginzburg-Landau (TDGL) equations for a type II superconductor, we derive the equations of motion for the displacement field of a moving vortex lattice ignoring pinning and inertia. We show that it is linearly stable and, surprisingly, that it supports *wavelike* long-wavelength excitations arising *not* from inertia or elasticity but from the strain-dependent mobility of the moving lattice. It should be possible to image these waves, whose speeds are a few $\mu$m/s, using fast scanning tunnelling microscopy.

PACS numbers: 74.60.Ge, 74.20.De



It was shown in [1], on general symmetry grounds, that an ordered array of particles moving through a dissipative medium (*e.g.*, a steadily sedimenting colloidal crystal or a flux-point lattice drifting through a type II superconductor) is governed by dynamical equations qualitatively different from those for a lattice at thermal equilibrium. Even the long-wavelength dynamical stability of such a drifting lattice was shown to rest not on its elasticity but on the signs of certain phenomenological parameters [see eqns. (1) and (2) below] governing the dependence of the local mobility on the lattice strain. A microscopic calculation [2] showed (see [1]) that for a sedimenting colloidal crystal the signs were such as to lead to an *instability*. We know of no analogous calculation for driven flux lattices. In this Letter we ask: are drifting flux lattices stable? We answer this question of fundamental importance starting from a time-dependent Ginzburg-Landau (TDGL) treatment *without quenched disorder*. We find that the moving lattice state is *stable*, with small-amplitude, long-wavelength disturbances propagating as *underdamped waves* whose speed, we emphasise, is determined by the *strain -dependent mobility* and the imposed current, and *not* by inertia and flux-lattice elasticity. We calculate the wavespeed (a few $\mu$m/s) in terms of independently measurable parameters arising in the TDGL equations.

We begin by summarising the derivation of the coarse-grained dynamical equations for a drifting lattice [1] and defining the quantities we are going to calculate. Consider a slab of type II superconductor of thickness much larger than the magnetic penetration depth $\lambda_H$, lying in the $xy$ plane, threaded by a flux lattice (spacing $\lesssim \lambda_H$) with magnetic field along the $z$ direction. An applied spatially uniform transport current density $\mathbf{J}_t = J_t\hat{\mathbf{x}}$, gives a Lorentz force $-J_t\phi\hat{\mathbf{y}}/c$ per unit length on a vortex carrying flux $\phi\hat{\mathbf{z}}$, $c$ being the speed of light. The perfect flux-point lattice will then acquire a constant, spatially uniform drift speed $v_L = MJ_t\phi/c$. Here $M$, the macroscopic mobility of the lattice, is determined by dissipative processes in the normal core as well as by the relaxation of the electromagnetic and order-parameter fields in the region between the vortices. Any perturbation of the perfect moving lattice will result in inhomogeneities in the local electromagnetic and order parameter fields, and thus to a spatially varying flux-point velocity. The mobility is thus a tensor which depends on the local state of distortion of the flux lattice. For a lattice drifting along $-\hat{\mathbf{y}}$, ignoring Hall effects, pinning, inertia [3], and the effects of lattice periodicity, the displacement field $\mathbf{u} = (u_x, u_y)$ as a function of position $\mathbf{r}$ and time $t$, defined with respect to a perfectly ordered crystal, in a frame co-moving on average with the flux lattice, must then obey [1]

$$\partial_t u_x = v_1\partial_y u_x + v_2\partial_x u_y + D_T\nabla^2 u_x + D_L\partial_x^2 u_x + D_L\partial_x\partial_y u_y + O(\nabla u\nabla u); \quad (1)$$

$$\partial_t u_y = v_3\partial_x u_x + v_4\partial_y u_y + D_T\nabla^2 u_y + D_L\partial_y^2 u_y + D_L\partial_x\partial_y u_x + O(\nabla u\nabla u), \quad (2)$$

where the terms containing the phenomenological coefficients [4] $v_i \propto v_L$ arise from the "hydrodynamic" interaction of the moving vortices, $D_L = M(\lambda + 2\mu)$, and $D_T = M\mu$, $\lambda$ and $\mu$ being the Lamé coefficients of the flux lattice [5]. These equations are constructed using general symmetry arguments and hold for any steadily drifting lattice at large length scales ($\gg \lambda_H$, for a flux lattice). In this Letter we calculate the coefficients $v_i$ for the specific case of a drifting flux lattice, from a time-dependent Ginzburg-Landau (TDGL) description to which we turn next. The importance of the $\{v_i\}$ for the long-wavelength behaviour of the drifting flux lattice is clear: $v_2v_3 > 0$ yields a wavelike dispersion whereas $v_2v_3 < 0$ a linear instability.

Scaling lengths by $\lambda_H$, energies by the condensation energy $E_c$ in a volume $\lambda_H^3$, the order parameter by its bulk mean-field value in the superconducting phase, times by $\hbar/E_c$, the magnetic field $\mathbf{H}$ by $\sqrt{2}H_c$ where $H_c$ is the thermodynamic critical field, the total electrochemical potential by $E_c/e^*$ where $e^* = 2e$ is the charge of the Cooper pair, and defining the Ginzburg-Landau parameter $\kappa = \lambda_H/\xi$ where $\xi$ is the bare coherence length, we obtain the dimensionless TDGL equations [6–8] for the dynamics of the superconducting order parameter $\psi(\mathbf{r}, t)$:

$$(\partial_t + i\Phi)\psi = \Gamma\left[\left(\frac{\nabla}{\kappa} - i\mathbf{A}\right)^2 \psi + \psi - \mid \psi \mid^2 \psi\right] , \quad (3)$$

where the phenomenological kinetic coefficient $\Gamma$ is in general complex, with real and imaginary parts $\Gamma_1$ and $\Gamma_2$ respectively.

The equation of motion for the vector potential is given by Ampère's law,

$$\nabla \times \nabla \times \mathbf{A} = \mathbf{J}_n + \mathbf{J}_s , \quad (4)$$

where the normal and super currents are, respectively,

$$\mathbf{J}_n = \sigma.\left[-\frac{\nabla\Phi}{\kappa} - \partial_t \mathbf{A}\right] ,$$
$$\mathbf{J}_s = \frac{1}{2\kappa i}(\psi^* \nabla\psi - \psi\nabla\psi^*) - \mid \psi \mid^2 \mathbf{A} , \quad (5)$$

$\sigma$ being the normal-state conductivity tensor.

We work in the large $\kappa$ limit, where a phase-only approximation of the TDGL equations (3) applies, and, for simplicity, we set the normal state Hall conductivity $\sigma_{xy} = 0$. We begin by writing $\psi$ in terms of an amplitude $f$ and a phase $\chi$:

$$\psi(\mathbf{r}, t) = f(\mathbf{r}, t)\exp[i\chi(\mathbf{r}, t)]. \quad (6)$$

In terms of the gauge-invariant vector and scalar potentials, $\mathbf{Q} = \mathbf{A} - \nabla\chi/\kappa$ and $P = \Phi + \partial_t \chi$, the magnetic and electric fields are then, respectively,

$$\mathbf{h} = \nabla \times \mathbf{Q} , \qquad \mathbf{E} = -\frac{\nabla P}{\kappa} - \partial_t \mathbf{Q} . \quad (7)$$

For large $\kappa$, deep in the superconducting phase, the amplitude relaxes rapidly to a value determined by the phase. We can thus solve for $f$ in terms of $\chi$ from (3) yielding, to leading order in $1/\kappa$, the effective phase-only TDGL equation

$$\partial_t \chi + \Phi = P = -\nabla.\mathbf{Q}/\gamma_1 \kappa . \quad (8)$$

with $\gamma_1 = \text{Re}\left[\Gamma^{-1}\right]$ and

$$\nabla \times \nabla \times \mathbf{Q} = \sigma.\left[\frac{-\nabla P}{\kappa} - \partial_t \mathbf{Q}\right] - \mathbf{Q} . \quad (9)$$

Lastly, charge conservation — $\nabla.(\mathbf{J}_n + \mathbf{J}_s) = 0$ – with (5) and (7) leads to

$$\nabla.(\sigma.\mathbf{E}) - \nabla.\mathbf{Q} = 0 \quad (10)$$

which, with (8) and (9) implies

$$\sigma_{xx}\frac{\nabla.}{\kappa}\left[-\frac{\nabla P}{\kappa} - \partial_t \mathbf{Q}\right] + \gamma_1 P = 0 . \quad (11)$$

The $\{v_i\}$ in (1), (2), which encode the change in the mobility of a region of the flux lattice when it is compressed or tilted, arise primarily from electromagnetic field disturbances, screened on the scale $\lambda_H$ [9]. Ideally, therefore, we should calculate the mobility of distorted regions on a scale $\lambda_H$. However, our main concern is the *signs* of the $\{v_i\}$, i.e., in the *direction* of drift of a tilted region and in whether a denser region drifts *faster* or *slower* than a rarer region. To this end, we take the simplest compressions/rarefactions and tilts, namely, those taking place at the level of a pair of particles. This should give a qualitatively correct assessment of the stability and a reasonable estimate of the wavespeed. Indeed, our calculation shows that the $v_i$s decrease by a factor of 10 as the flux-lattice spacing varies from .25 $\lambda_H$ to $\lambda_H$, justifying *post facto* this nearest neighbour approximation. We work, therefore, with a *pair* of flux points moving rigidly with a velocity $\mathbf{v}_L$, as a function of their fixed separation vector $\mathbf{a}$. For such rigid motion, time-derivatives can be replaced by $-\mathbf{v}_L.\nabla$. Expanding (8), (9), and (11) in powers of $v_L$, we obtain at O(1) the equilibrium, time-independent Ginzburg-Landau equations, and at O($\mathbf{v}_L$) a set of linear inhomogenous differential equations.

Exploiting [10,8] the invariance of the time-*independent* Ginzburg-Landau equations under an arbitrary virtual displacement $\mathbf{d}$, the requirement of compatibility between $\mathbf{v}_L$ and the imposed transport current $\mathbf{J}_t$ leads, for large $\kappa$ and within the phase only approximation, to the "solvability condition" for the inhomogeneous O($v_L$) equations:

$$\frac{1}{\kappa}\int d\mathbf{S}.(\mathbf{J}_{1s}\chi_d - \mathbf{J}_d\chi_1) = \gamma_1 \int (\chi_d P) d\mathbf{r}; \quad (12)$$

the integral on the left-hand side is over the boundary of the sample, $\mathbf{J}_d \equiv \mathbf{d}.\nabla\mathbf{J}_0$, $\chi_d \equiv \mathbf{d}.\nabla\chi_0$, $J_0$ and $\chi_0$ being respectively the supercurrent and phase field at *equilibrium*, and the subscripts 0,1 denoting the O(1) and O($\mathbf{v}_L$) parts respectively of the term in question. Eq. (12) will yield the relation between $\mathbf{J}_t = J_t\hat{\mathbf{x}}$ and $\mathbf{v}_L$, *i.e.* the vortex equation of motion.

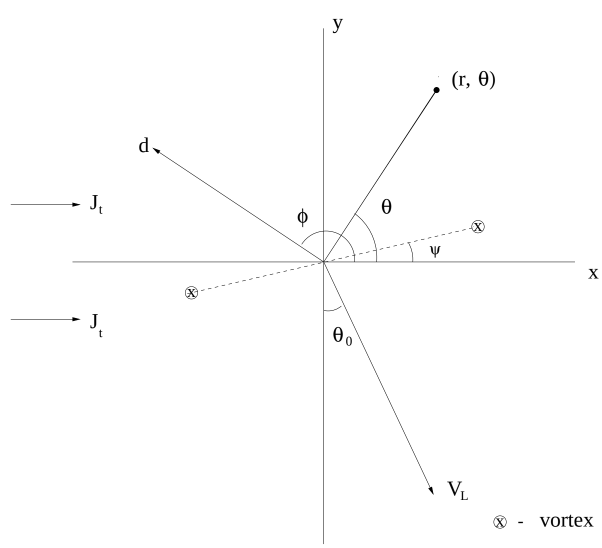


FIG. 1. Coordinate system $(r, \theta)$ for two-vortex case

Consider a pair of identical unit vortices, in a geometry defined in Fig.1 (in cylindrical polar coordinates $(r, \theta, z)$). $\psi$ and $\phi$ are the angles made by $\mathbf{J}_t$ with $\mathbf{a}$ and the virtual displacement $\mathbf{d}$ respectively, and $\theta_0$ that between $\mathbf{v}_L$ and the negative $y$-axis. We assume the flux lines to be parallel to the $z$-axis and ignore the effects of line wandering. We also define cylindrical coordinates $(r_1, \theta_1, z)$ and $(r_2, \theta_2, z)$ with their origins at the two vortices. The surface integral on the left-hand side of (12) can be expressed in terms of the applied transport current. At the boundaries the fields are effectively those of a single vortex at the origin, with twice the winding number. Therefore $\mathbf{J}_{1s}(r = \infty, \theta) = \mathbf{J}_t$, $\mathbf{J}_d.\hat{\mathbf{e}}_r = 2d\sin(\theta - \phi)/(\kappa r^2)$, $\chi_d = 2\mathbf{d}.\nabla\theta = -2d\sin(\theta - \phi)/r$ and $\chi_1 = \kappa J_t r\cos\theta$. Substituting these expressions into the left-hand side of (12), and performing the angular integration we find

$$\frac{1}{\kappa}\int d\mathbf{S}.[\mathbf{J}_{1s}\chi_d - \mathbf{J}_d\chi_1] = -2\frac{2\pi}{\kappa}(\mathbf{J}_t \times \hat{\mathbf{z}}).\mathbf{d}\ . \qquad (13)$$

Evaluation of the right-hand side of (12) requires solving for $P(\mathbf{r}, t)$ from (11) which, at $\mathrm{O}(v_L)$, is simply

$$\frac{\sigma_{xx}}{\kappa^2}\nabla^2 P - \gamma_1 P = 0 \qquad (14)$$

Near the centre of each vortex $P \approx -\mathbf{v}_L.\nabla\chi$ and $\chi$ is equal to the angular variable $\theta_1$ or $\theta_2$ around that vortex. Therefore $P \approx v_L\cos(\theta_1 - \theta_0)/r_1$ as $r_1 \to 0$ and $P \approx v_L\cos(\theta_2 - \theta_0)/r_2$ as $r_2 \to 0$. The solution to eqn.(14) for the vortex pair with these boundary conditions is

$$P(\mathbf{r}) = \tilde{v}[K_1(\alpha r_1)\cos(\theta_1 - \theta_0) + K_1(\alpha r_2)\cos(\theta_2 - \theta_0)] \qquad (15)$$

where $\tilde{v} = v_L\alpha$ and $\alpha = \kappa\sqrt{\gamma_1/\sigma_{xx}}$. Also,

$$\chi_d = \mathbf{d}.\nabla\chi = d[\sin(\phi - \theta_1)/r_1 + \sin(\phi - \theta_2)/r_2]\ . \qquad (16)$$

Using (15) and (16) on the right hand side of (12), and noting that $\mathbf{d}$ is arbitrary, we obtain the vortex-pair equation of motion in the form

$$2\frac{2\pi}{\kappa}(\mathbf{J_t} \times \hat{\mathbf{z}}) = A\mathbf{v}_L + B(\mathbf{v}_L \times \hat{\mathbf{z}}) + C(\mathbf{v}_L.\hat{\mathbf{a}})\hat{\mathbf{a}} - D\hat{\mathbf{a}}.(\mathbf{v}_L \times \hat{\mathbf{z}})\hat{\mathbf{a}} \qquad (17)$$

where $A, B, C$, and $D$ are functions of $\alpha$ and $a = |\mathbf{a}|$ only. All dependence on the angle of tilt $\psi$ is in the scalar and vector products in (17). Evaluating the integrals, we find that $B = 0 = D$ (a consequence of the phase-only approximation *and* the assumption $\sigma_{xy} = 0$), and $A, C > 0$. Inverting (17) we see that

$$v_{Li} = M\left[\delta_{ij} - \frac{N}{1+N}\frac{a_i a_j}{a^2}\right]F_j \qquad (18)$$

where $\mathbf{F} = -\frac{4\pi}{\kappa}J_t\hat{\mathbf{y}}$ is the Lorentz force, $M = 1/A$, and $N = C/A$. (18) differs from that for a single vortex [10,8] in the $N$ term: for $\psi \neq 0$ or $\pi/2$ the centre-of-mass velocity is not parallel to the driving force.

Now consider a steadily drifting undistorted flux-point lattice, and focus on a nearest neighbour pair of flux points with initial separation vector $\mathbf{a}_0$. Perturb it slightly: $\mathbf{a}_0 \to \mathbf{a}_0 + \delta\mathbf{a}$, thus causing a velocity perturbation $\delta\mathbf{v}$. Then we can extract the $\{v_i\}$ by differentiating our two-vortex result (18) as follows: if $\mathbf{a}_0||\hat{\mathbf{y}}$ and $\delta\mathbf{a}||\hat{\mathbf{x}}$, then $\frac{\delta v_x}{\delta a/a_0} = v_1$; if $\mathbf{a}_0||\hat{\mathbf{x}}$ and $\delta\mathbf{a}||\hat{\mathbf{y}}$, then $\frac{\delta v_x}{\delta a/a_0} = v_2$; if $\mathbf{a}_0||\hat{\mathbf{x}}$ and $\delta\mathbf{a}||\hat{\mathbf{x}}$, then $\frac{\delta v_y}{\delta a/a_0} = v_3$; if $\mathbf{a}_0||\hat{\mathbf{y}}$ and $\delta\mathbf{a}||\hat{\mathbf{y}}$, then $\frac{\delta v_y}{\delta a/a_0} = v_4$; We have assumed, as justified early in the paper, that changes in vortex velocity due to a local distortion are local. We find: (a) $v_2 > 0$, so that a lattice moving in the $-\hat{\mathbf{y}}$ direction will veer to the right(left) if the horizontal crystal planes are tilted up to the right (left); and (b) $v_3 > 0$, so that a local $x$-compression of the lattice increases the velocity in the direction of the force. From (1) and (2), this means the moving flux-lattice is *stable*. Also, $v_1 > 0$ and $v_4 < 0$ for the coefficients controlling the wavespeeds along the drift [11]. The resulting mode structure is summarised in Fig.2 and, from (1) and (2), in the small-wavenumber dispersion relation

$$2\omega = -(v_1 + v_4)k\sin\theta \pm v_o k - i(2D_T + D_L)k^2 \pm ik^2 D_L\sin\theta\left[\frac{v_1 - v_4}{v_o}\cos 2\theta + 2\frac{v_2 + v_3}{v_o}\cos^2\theta\right] \qquad (19)$$

between frequency $\omega$ and wavenumber $k$, where

$$v_o = \sqrt{(v_1 - v_4)^2\sin^2\theta + 4v_2 v_3\cos^2\theta}, \qquad (20)$$

and $\theta$ is the circular polar angle.

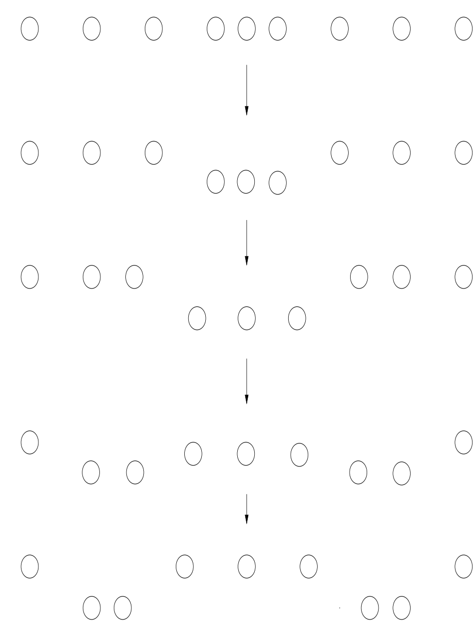

FIG. 2. The wave travelling along $\pm\hat{\mathbf{x}}$ that follows a local compression of an array of vortices moving along $-\hat{\mathbf{y}}$

We estimate the resulting wave-speeds for $NbSe_2$ in the mixed phase [12]. Its TDGL parameters are: $\lambda_H \sim 700$ Å, $\xi \sim 80\mathring{A}$, $T_c \approx 7$ K , $\rho^{(n)} = 5\,\mu\Omega$ cm. For an applied transport current $\mathbf{J}_t = 1$ A/cm$^2$ and inter-vortex separation $a \sim \lambda_H$, the wave-speeds $c_\pm \sim 1\mu$m/sec.

The most obvious physical consequence of these waves is that the dynamic structure factor of a drifting flux lattice should display peaks at nonzero frequency (see (19)). More dramatically, if a region of the flux-lattice moves past an impurity site, the impurity will "pluck" the flux lattice, and the effect will propagate *along* and *transverse* to the axis of drift, shaking up the lattice globally, through the sequence of events depicted in Fig.2. This wave propagation in the absence of inertia is remarkable, and could well be a mechanism for nonthermal noise in drifting flux lattices. In addition, time-dependent external disturbances could excite resonances with the wave-like normal modes.

Let us estimate the length scale $\ell_c$ above which these modes are actually propagative in character. For wavevectors $\mathbf{k} = (k_x, 0)$ we see that

$$\ell_c \sim \frac{\pi D_L}{\sqrt{v_2 v_3}} \tag{21}$$

$D_L \sim M\lambda$ [see after (1), (2)] and $v_i \sim MF$ where $F = J_t \phi_0 / c$ is the Lorentz force per unit length on a vortex. Then

$$\frac{\ell_c}{a} \sim \frac{\lambda}{F}. \tag{22}$$

$\lambda \approx aH^2/8\pi$ [13], $a$ being the flux-lattice spacing, so for $a \sim 10\mu m$ and applied currents $J_t \sim 100$ A/cm$^2$, $\ell_c/a \sim 1$, and the propagating modes should dominate. However, if $a \sim 10^3\mathring{A}$, $\ell_c/a \sim 10^6$.

In closing, we remark that our work settles an important issue in the *theory* of the dynamics of moving flux lattices, namely their stability [14]. We have shown that dynamic interactions between vortices in a drifting flux lattice without inertia or pinning lead to a steady state with *stable* linear-response properties. Small disturbances about the drifting state travel as *waves* with a direction-dependent speed which, when calculated in terms of the parameters in the TDGL equations, turns out to be a few $\mu$m/s. These waves should be observable in systems with large flux-lattice spacing, at large imposed transport currents. The fast scanning tunneling microscopy approach of Troyanovskii et al. [15] seems to be the ideal way to observe these waves directly.

We thank D. Gaitonde, T.V. Ramakrishnan, C. Dasgupta, and S. Bhattacharya for useful discussions.

---

[a] aditi@physics.iisc.ernet.in
[b] Also at Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore 560 064 INDIA.
[c] sriram@physics.iisc.ernet.in

[1] R. Lahiri and S. Ramaswamy, Phys. Rev. Lett. **79**, 1150 (1997); R. Lahiri, Ph D Thesis, Indian Institute of Science, 1997.

[2] J.M. Crowley, J.Fluid Mech., **45**, 151 (1971); Phys. Fluids, **19**, 1296 (1976).

[3] The inertial mass of a vortex is calculated by D.M. Gaitonde and T.V. Ramakrishnan, *Phys. Rev.* B **56**, 11951 (1997)

[4] See also L. Balents, M.C Marchetti and L. Radzihovsky, Phys. Rev. B, **57**, 7705 (1998) where they come from interaction with a structured, static medium, and where only the case $v_2 = v_3 = 0$ is emphasised, and M. Kardar, in *Dynamics of fluctuating interfaces and related phenomena*, ed. D.Kim, H.Park, and B. Kahng (World Scientific, Singapore, 1997) where they are mentioned in passing.

[5] These damping terms are correct to zeroth order in the drift speed.

[6] we adopt the TDGL equations originally proposed by Schmid but with a complex relaxation rate. These equations are known to be valid for Type II superconductors with paramagnetic impurities. See also [7,8].

[7] A. Schmid, Phys. kondens. Materie., **5**, 302 (1966).

[8] Alan T. Dorsey, Phys. Rev.B, **46**, 13, 8376 (1992)

[9] In a suspension of particles drifting through a viscous fluid the $\{v_i\}$ are determined by the *hydrodynamic* interaction, which is screened only on the scale of the sample thickness. In the present problem, that interaction is negligible because the ions are a sink for the momentum of the charged superfluid.

[10] L. P. Gor'kov and N. B. Kopnin, Sov. Phys. -Usp., **18**, 496 (1975).

[11] Similar arguments can be used to deduce the coefficients of nonlinear terms in (1) and (2).

[12] M.J. Higgins and S. Bhattacharya, Physica C, **257**, 232(1996).

[13] E.H. Brandt, Phys. Rev. B, **56**, 9071(1997).

[14] Experimentally, moving flux lattices do appear to be stable: U.Yaron et al., Phys. Rev. Lett. **73**, 2748 (1994).

[15] A.M. Troyanovskii *et al.*, http://rulgm5.leidenuniv.nl/˜msm/msmain.