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Development of robust finite elements for general purpose structural analysis

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Abstract. The finite element method emerged out of the old work and energy methods and matrix structural analysis to become a numerical procedure to solve practical stress analysis problems in solid and structural mechanics. With the impetus given by the rapid development of computer technology, it became the most overwhelmingly popular analysis and design computational tool for a very wide spectrum of engineering science, e.g. fluid mechanics, heat transfer and electro-magnetics. Today, there are very powerful general-purpose software codes that make analyses and design tasks that were once considered to be intractable, routinely simple. Many of these are closely held proprietary codes owned and used in-house by large engineering firms or sold or licensed and supported by specialist companies. (Recent estimates indicate that the market for these codes has reached a turnover of a billion dollars and that industries and institutions spend several tens of billions of dollars in running such codes.) These codes are rarely given out in source code. In order to have an in-house code that could be continuously up-graded and enhanced, NAL initiated some work to develop a medium-sized general purpose code (about 20,000 lines of FORTRAN code) for the analysis of laminated composite structures (FEPACS – finite element package for analysis of composite structures), recognising the importance that laminated composites were assuming in aerospace structural technology.

Several key elements commonly found in general purpose packages (GPP) used by the aerospace, automobile and mechanical engineering industries were identified. These were re-designed incorporating anisotropic composite capabilities and validated. Many hurdles were faced during this task and required an examination of the basic issues at a paradigmatic level. Concepts such as consistency and variational correctness were introduced and studied critically. These guidelines played a critical role in developing robust versions of the elements and are briefly covered in this review. The paradigms also helped to identify procedures to perfo *a priori* error estimates for the quality of approximation and this allowed the elements being developed to be critically validated.

The article concludes with a summary of what has been achieved and also suggests areas where the concepts can be applied fruitfully in the study of the displacement type finite element method.

Keywords. Finite element method; composite structures; structural analysis; element technology; general purpose packages; FEPACS.

1. Introduction – from C^1 to C^0 elements

We shall examine the subject as seen from our own practical viewpoint and as it must have been seen by the larger finite element community as well, as we undertook the task of designing a library of simple, accurate and efficient elements for general purpose finite element structural analysis.

At the time we began our work, around 1978, it was slowly becoming accepted that the element libraries of major general purpose packages (GPP) were replacing what were called the C^1 elements with what were known as the C^0 elements. The former were based on well-known classical theories of beams, plates and shells (i.e. the Euler–Bernoulli beam theory, the Kirchhoff–Love plate theory and the equivalent shell theories), reflecting the confidence that structural analysts have had in such theories for over two centuries. It is indicative that the early history of the finite element technology was almost entirely confined to the use of elements based on such theories. These theories did not allow for transverse shear strain and permitted the modelling of such structures by defining deformation in terms of a single field, w, the transverse deflection of a point on what is called the neutral axis (in a beam) and neutral surface of a plate or shell. The strains could then be computed quite simply from the assumption that normals to the neutral surface remained normal after deformation. One single governing differential equation resulted, although of a higher order (in comparison to other theories we shall discuss shortly), and this was considered to be an advantage.

There were some consequences arising from such an assumption both for the mathematical modelling aspect as well as for the finite element (discretisation) aspect. In the former, it turned out that to capture the physics of deformation of thick or moderately thick structures, or the behaviour of plates and shells made of newly emerging materials such as high performance laminated composites, it was necessary to turn to more general theories accounting for transverse shear deformation as well – these required the definition of rotations of normals which were different from the slopes of the neutral surface. Some of the contradictions that arose as a result of the old C^1 theories – e.g. the use of the fiction of the Kirchhoff effective shear reactions could now be removed, restoring the more physically meaningful set of three boundary conditions on the edge of a plate or shell (the Poisson boundary conditions as they are called) to be used. The orders of the governing equations were correspondingly reduced. A salutary effect that carried over to finite element modelling was that the elements could be designed to have nodal degrees of freedom which were the six basic engineering degrees of freedom – the three translations and the three rotations at a point. This was ideal from the point of view of the organization of a general purpose package. Also, elements needed only simple basis functions requiring only the continuity of the fields across element boundaries – these are called the C^0 requirements. In the older C^1 formulations, continuity of slope was also required and to achieve this in arbitrarily oriented edges, as would be found in triangular or quadrilateral planforms of a plate bending element, it was necessary to retain curvature degrees of freedom (w_{xx}, w_{xy}, w_{yy}) at the nodes and rely on quintic polynomials for the element shape or basis functions. So, as general purpose packages ideal for production run analyses and design increasingly found favour in industry, the C^0 beam, plate and shell elements slowly began to replace the older C^1 equivalents. It may be instructive to note that the general two-dimensional (i.e. plane stress, plane strain and axisymmetric) elements and three-dimensional (solid or brick as they are called) elements were in any case based on C^0 shape functions – thus this development was welcome in that universally valid C^0 shape functions and their derivatives could be used for a very wide range of structural applications.

However, surprisingly dramatic failures came to be noticed when C^0 elements were formulated. The greater part of academic activity in the late seventies, most of the eighties and even in the nineties was spent in understanding and eliminating what were called the locking problems. A good idea of the challenges involved can be seen in two recent reviews – a bibliography of the finite element formulation of constrained media elasticity (Prathap & Nirmala 1990) – about 500 papers in thirty years, and a review of the quest for a reliable degenerate shell element (Gilewski & Radwanska 1991) – over 350 papers in about three decades of activity.

These spectacular failures were called the 'locking' problems in C^0 finite elements. It was not clear why the displacement type method, as it was understood around 1977, should produce for such problems, answers that were only a fraction of a percent of the correct answer with a practical level of discretisation. Studies in recent years have established that an aspect known as consistency must be taken into account.

The consistency paradigm requires that the interpolation functions chosen to initiate the discretisation process must also ensure that any special constraints that are anticipated must be allowed for in a consistent way. Failure to do so causes solutions to lock to erroneous answers. The paradigm showed how elements can be designed to be free of these errors. It also enabled error-analysis procedures that allowed errors to be traced to the inconsistencies in the representation to be developed. The authors have now developed a family of such error-free robust elements for application in structural mechanics and these are now available in a package, FEPACS (finite element package for analysis of composite structures), developed at the National Aerospace Laboratories.

This article would therefore review the understanding of such errors on a paradigmatic basis and the arrival at the end, of a family of robust elements of acceptable accuracy for use in typical GPP.

2. Difficulties with C^0 elements: Locking and stress oscillations

With the wide-spread acceptance of the C^0 family of elements, instances where the finite elements models of practical structures under certain physical conditions produced very erroneous solutions in spite of satisfying the continuity and completeness requirements came to be noticed (Doherty *et al* 1969; Pawsey & Clough 1971; Zienkiewicz *et al* 1971). Such errors are today classified as locking – where errors in solutions grow indefinitely as the physical limits are approached (Prathap & Bhashyam 1982) and delayed convergence – where the convergence rate of the solutions is much lower than that assured by the conventional continuity and completeness requirements of the finite element method (Prathap & Babu 1986b). Such errors in displacement solution are always associated with violent stress oscillations (Prathap & Babu 1987).

Walz *et al* (1970) classified errors into two categories – errors of first kind (errors due to the discretisation process but which disappear rapidly as mesh is improved) and errors of the second kind (discretisation errors which disappear very slowly and which get exaggerated when some structural parameter is changed). At the time this classification appeared (Walz *et al* 1970) it was not known that the latter class of errors were due to incorrect representation of the constrained strain energy components and arise purely from the way the finite element fields are expressed. Recent work shows that such errors become very serious in a particular class of problem – constrained media elasticity (Babu 1985; Prathap 1986, 1993; Naganarayana 1991). These problems span a wide range of structural phenomena – shear-flexible beams using Timoshenko theory (Prathap & Bhashyam 1982) and plates/shells (Mindlin theory) suffering from shear locking (Hughes *et al* 1977; Prathap & Viswanath 1983; Bathe & Dvorkin 1985; Hinton & Huang 1986; Donea & Lamain 1987; Prathap & Somashekar 1988), curved beam/shell structures suffering from membrane locking (Stolarski & Belytschko 1981; Prathap 1985a, b; Prathap & Babu 1986a; Jang & Pinsky 1988), 2-D plane-stress, plane-strain and 3-D elasticity suffering from parasitic shear (Cook 1975; Prathap 1985c; Prathap *et al* 1986) and/or near incompressibility locking (Chandra & Prathap 1989; Naganarayana & Prathap 1991) etc. These are described in detail in table 1.

Many *ad hoc* techniques have been suggested to overcome such difficulties. Reduced/

Table 1. Some constrained-multi-strain problems in structural mechanics.

Class of structural problem	Strain fields		Type of constraints and the associated penalty limits
	Unconstrained	Constrained	
Plane stress Plane strain 3-D elasticity in modes of flexure	Normal ε	Shear γ	$\gamma \to 0$ as $(b/l) \to 0$ where b and l form the section on which γ is acting s.t. $b < 1$
Plane stress Plane strain 3-D elasticity in modes of near incompressibility	Distortional ε_d	Dilatational ε_v	$\varepsilon_v \to 0$ as $\mu \to 0{\cdot}5$ for isotropic materials where μ is Poisson's ratio
Shear-flexible beam (Timoshenko) plates (Mindlin)	Bending χ	Transverse shear γ	$\gamma \to 0$ as $(t/l) \to 0$ where t = thickness and l = "element length"
Curved beams and shells	Bending χ	Membrane ε	$\varepsilon \to 0$ as $(Rt/l^2) \to 0$ where t = thickness and R = radius of curvature l = "element length"

selective integration (Pawsey & Clough 1971; Zienkiewicz *et al* 1971; Zienkiewicz & Hinton 1976; Hughes *et al* 1977, 1978), assumed strain methods (MacNeal 1982; Oleson 1983), addition of bubble modes (Wilson *et al* 1973), residual energy balancing (Fried 1974, 1975), spurious mode decomposition (Belytschko *et al* 1984, 1985), discontinuous force-field mixed methods (Noor & Hartley 1977; Noor & Anderson 1982; Noor & Peters 1981), symbolic Fourier synthesis (Park 1984), unequal field interpolations with condensation of constraints (Tessler & Dong 1981), Kirchhoff mode method (Stolarski *et al* 1985), quasi-conforming techniques (Tang *et al* 1984), use of trignometric basis functions (Hepper & Hansen 1987), using shear constraints (Crisfield 1984) etc. represent a broad coverage of the various artifices used with varying degrees of success to resolve these issues. Often these procedures lacked an explanation for their success. Sometimes they were successful in one context and failed in some other. Again the reason for such behaviour was not clear.

Most explanations for the locking behaviour available at the time we started our work – singularity of shear stiffness (Zienkiewicz 1977), constraint counting and rank of the shear stiffness matrix (Cook *et al* 1981; Hughes 1987) – lacked a rigorous scientific basis. Zienkiewicz (1977) argued that the elements which lock have non-singular shear stiffness matrices while the shear stiffness matrices of elements after reduced integration are singular. It was argued therefore that locking is due to non-singularity of the shear stiffness matrix and a reduced integration order that induces singularity in the shear stiffness matrix is recommended. In other words, the locking behaviour of an element is due to the *high rank* of the penalty-linked stiffness matrices. But, it was soon realised that an arbitrary reduction of the rank of the penalty-linked stiffness matrices may lead to the undesirable spurious zero energy mechanisms (Hughes 1987). The optimal rank for the shear stiffness matrix is often determined by a technique known as constraint counting (Malkus & Hughes 1978). The method of constraint counting makes an attempt to determine optimal integration order based on number of constraints given in a problem. It is based on the ratio r of the total active degrees of freedom in a given mesh n to the total number of penalty constraints $m(r = n/m)$. Locking occurs if $r \leqslant 1$. The mesh does not lock if r is *slightly* greater than unity and may have spurious zero energy modes if it is *too high*. Heuristically it argued that the near-optimal ratios are $r = 2/1$ for 2-D problems and $r = 3/1$ for 3-D problems (Cook *et al* 1981).

It should be observed here that such explanations are heuristic and lack a rigorous scientific basis i.e. the validity of the explanations is not numerically verifiable (falsifiable) since a causal relationship between the locking errors and the rank of the 'constrained' stiffness matrix is not established. Such arguments often follow from the given mesh for a structural problem rather than the basis of discretisation adopted in the finite element formulation. Such an explanation, hence, cannot be generally applied in developing a finite element. These arguments cannot identify the milder problems of delayed convergence which are observed in the case of higher order elements. Finally, they attempt explanations based on the symptoms accompanying the problem of locking (i.e. the high rank of non-singularity of the shear stiffness matrix is a symptom of an inconsistent formulation) rather than exploring the cause for the errors. There is no established procedure for obtaining the optimal rank of the penalty-linked stiffness matrices in literature. Here, we review some of the work done to provide a scientific basis for the origin of such errors. Very recently, it was possible to relate the consistency paradigm and the requirements that follow from it to the rank of the penality-linked stiffness matrix, showing that there is a link between the cause of such errors and the symptoms (i.e. high rank, non-singularity of matrices etc.) associated with the locking problem (Prathap 1994).

In the next sections, we shall present a scheme for error-free displacement type finite element formulation based on the consistency and correctness paradigms. The elements developed thus are now incorporated in an in-house finite element package called FEPACS.

3. Field-consistency

Prathap & Bhashyam (1982) demonstrated that it is the *inconsistent* finite element representation of the constrained state of the strain energy in the respective penalty limits that causes problems like locking and stress oscillations. It considered a Timoshenko beam element formulation and showed that the constraint of vanishing transverse shear strain energy near vanishing beam thickness imposed two types of constraints. The constraints which had contributions from all the displacement fields appearing in the respective strain field definitions were classified as true constraints and the constraints that do not have contributions from at least one of the constituent displacement fields were classified as spurious constraints. It was also demonstrated that the latter (i.e. spurious constraints) disturbed the bending strain energy in the penality limits. Using this fact, analytical *a priori* error estimates were constructed to prove that it was the spurious constraints that caused locking and the associated stress oscillations (Prathap & Babu 1986b, 1987). This was confirmed by conducting appropriate numerical experiments with the *inconsistent* finite element formulations which contain spurious constraints.

Eventually, the field-consistency paradigm emerged (Prathap 1986) to provide an explanation as well as a remedy for difficulties like locking and stress oscillations in the finite element formulation of the problems in constrained media elasticity. It holds the inconsistent representation of the constrained state of the corresponding strain energy components, giving rise to the spurious constraints in the penalty limits, responsible for such difficulties. It also suggests that the energy components associated with the spurious constraints have to be eliminated from the formulation for removal of the above-mentioned difficulties from the element formulation. The paradigm can be broadly stated as follows.

In a constrained media problem, some strains will have to vanish under certain conditions. Strain fields derived from displacement shape functions cannot always do this in a meaningful manner – spurious constraints are generated which cause locking. The consistency condition demands that the discretised strain field interpolations must be so constituted that it will enforce only physically true constraints when the discretised functionals for the strain energy of a finite element are constrained.

In the development of a finite element, the field variables are interpolated using interpolations of a certain order. From these definitions, one can compute the strain fields using the strain-displacement relations. These are obtained as interpolations associated with the constants that were introduced in the field variable interpolations. Depending on the order of the derivatives of each field variable appearing in the definition of that strain field (e.g. the shear strain in a Timoshenko theory will have θ and the first derivative of w), the coefficients of the strain field interpolations may have constants from all the contributing field variable interpolations or from only one or some of these. In some limiting cases of physical behaviour, these strain fields can be constrained to be zero values, e.g. the vanishing shear strain in a thin Timoshenko beam. Where the strain-field is such that all the terms in it (i.e. constant, linear, quadratic etc.) have, associated with it, coefficients from all the independent interpolations of the field variables that appear in the definition of that strain-field, the constraint that appears in the limit can be correctly enforced. We shall call such a representation *field-consistent*. The constraints thus enforced are *true constraints*. Where the strain-field has coefficients in which the contributions from some of the field variables are absent, the constraints may incorrectly constrain some of these terms. This *field-inconsistent* formulation is said to enforce additional *spurious constraints*.

We shall also determine procedures that can modify the element characteristics so that the consistency requirements are met. We shall call such elements the *field-consistent* elements as opposed to the *field-inconsistent* elements which do not take into account such requirements. There is a unique manner in which the field-consistent elements have to be generated – they have to satisfy a condition we shall call the *correctness* condition which will ensure that the variational theorems are not violated in the process of modifying the element stiffness matrix.

Scientific evidence and analytical proof for this paradigm have been provided and verified through several practical examples of finite element formulations over the years (Babu 1985; Prathap 1986, 1993; Naganarayana 1991). Using the so-called field-reconstitution technique, analytical *a priori* estimates of the errors in displacement as well as stress recovery arising due to violation of various field-consistency requirements are derived for a family of elements and are digitally verified using appropriate computational models, see Prathap & Nirmala (1990) for a bibliography.

Most of the *ad hoc* techniques mentioned before offer procedures for achieving field-consistency with varied degrees of success. The assumed strain methods appear to be the most versatile among these since they have the capability of isolating the spurious constraints in the formulation. Thus the behaviour of the inconsistent terms in the constrained strain fields and the associated spurious constraints becomes apparent and the construction of *a priori* error estimates, with reference to displacement as well as stress recovery, becomes simplified. The assumed strain methods essentially try to eliminate the inconsistent terms in the original strain field (derived by the gradient operations on the kinematically admissible displacement fields) for the constrained strain component. The popular procedures used in the literature for achieving this are reduced/selective integration (Zienkiewicz & Hinton 1976; Hughes *et al* 1978), least squares method (Bose & Kirkhop 1984), mean value method (Donea & Lamain 1987), collocation of the strain fields at certain standard points (Huang & Hinton 1984; Bathe & Dvorkin 1985) etc. and all these belong to assumed strain methods in a broad sense of definition.

Various aspects of the field-consistency paradigm are briefly illustrated using the simple example of a 2-noded C^0-continuous Timoshenko beam element. The field-reconstitution technique is used to derive *a priori* analytical error estimates for the additional stiffness parameter and stress oscillations which can be digitally verified. The technique is used to derive *a priori* error norms for many other useful elements in a series of publications.

3.1 *Example–linear Timoshenko beam element* (BEAM2)

A 2-noded beam element with two degrees of freedom (deflection w and section rotation θ) per node, shown in figure 1, is considered.

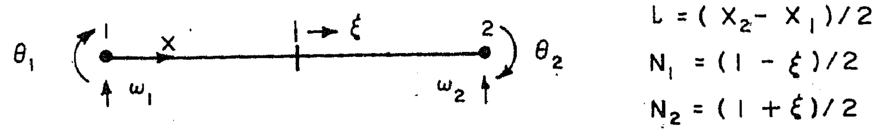

Figure 1. 2-noded beam element.

Using the linear shape functions N_1 and N_2 the displacement fields can be expressed as,

$$\theta = a_0 + a_1\zeta \quad \text{and} \quad w = b_0 + b_1\zeta \tag{1}$$

Now, the two strain fields, flexure χ and the transverse shear γ, can be derived using (1) as

$$\begin{Bmatrix} \chi \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \mathrm{d}\theta/\mathrm{dx} \\ \theta - \mathrm{d}w/\mathrm{dx} \end{Bmatrix} = \begin{Bmatrix} a_1/l \\ (a_0 - b_1/l) + a_1\zeta \end{Bmatrix}. \tag{2}$$

The strain energy can be considered as the sum of bending and shear strain energy components as follows.

$$\begin{aligned} U = U_B + U_S &= \int (EI/2)(\chi)^2 \mathrm{dx} + \int (kGA/2)(\gamma)^2 \mathrm{dx}, \\ &= (EIl)(a_1/l)^2 + (kGAl)\{(a_0 - b_1/l)^2 + (a_1)^2/3\}. \end{aligned} \tag{3}$$

In the penalty limits of vanishing beam thickness, the shear strain energy U_S vanishes resulting in two constraints,

$$a_0 = b_1 l \equiv \theta = \mathrm{d}w/\mathrm{dx}, \tag{4}$$

$$a_1 = 0 \equiv \mathrm{d}\theta/\mathrm{dx} = 0. \tag{5}$$

It is apparent that (4) is the true constraint representing the Euler–Bernoulli condition in the thin limits. But (5) is the spurious constraint which directly disturbs the bending strain energy. It can be noted that the spurious constraint corresponds to a situation where the beam is locked against rotation, thus explaining the locking behaviour of the problem. The field-consistency paradigm now suggests that the linear term in the shear strain field be eliminated for removal of locking and violent stress oscillations. The method of Legendre polynomial expansion can be used here so that the inconsistent linear term in shear strain is to be simply dropped. The procedure and variational justification for it are discussed in detail, in the next section.

It is also possible to obtain *a priori* error estimates by considering the strain energy after discretisation by a procedure called the functional reconstitution technique. For this example, it is possible to show that the effect of retaining the spurious constraint leads to an artificial stiffening of the bending action by a factor $(kGAl^2/3EI)$; i.e. the discretised beam behaves as if the moment of inertia of the cross section has changed from I to I' such that,

$$I'/I = 1 + kGA\, l^2/3EI. \tag{6}$$

Thus an analytical error norm for the additional stiffness parameter e_{th} can be written as,

$$e_{th} = I'/I - 1 = kGAl^2/3EI. \tag{7}$$

This can be compared with results obtained from a actual finite element computation as,

$$e_{fem} = \omega(th)/\omega(fem) - 1 \tag{8}$$

The agreement of the computed errors (8) with the predicted errors (7) is very good and has been documented in Prathap & Bhashyam (1982) and Babu (1985).

Through a similar exercise, in an inconsistent element with severe shear locking (when $e \gg 1$), the shear force resultant can be expressed as,

$$Q = \bar{Q} + (3M_0/l)\zeta,$$

where $\bar{Q}$ represents the correct shear force contributed from the consistent part of the shear strain field and M_0 is the constant part of the exact bending moment distribution. Thus the shear force recovered from the inconsistent element will have violent linear oscillations which are proportional to the constant part of the bending moment M_0. Again, the analytical predictions of the stress oscillations are well-matched by the finite element results (Babu 1985).

Such behaviour, i.e. locking and stress oscillations, is typical of inconsistently formulated beam, plate/shell, plane stress/plane strain and brick elements, the work horse elements of all current general purpose packages. It is imperative therefore that the design of a family of such elements must be carefully done to ensure that during formulation, the inconsistencies are systematically identified and removed so that the elements are free of such errors. The field-consistency paradigm offers us a conceptual scheme which enables this to be done.

4. Variational correctness

We now turn our attention to the task of formulating a consistent element without violating any of the norms required from energy and variational principles. It has become clear to us from the previous section that the field-consistency paradigm suggests the form of the interpolation functions for the constrained strain fields. However, the reconstituted *assumed* strain field interpolations cannot be chosen arbitrarily (as done, for example, in Mohr 1982). A variational basis for the *correct* field-redistribution (that is to find the coefficients of the consistent field from those of the original field) can be obtained by determining the conditions for exact equivalence of the assumed strain displacement approach to the mixed approaches based on the Hellinger–Reissner or Hu–Washizu variational theorems (Simo & Hughes 1986; Prathap 1988). The coefficients of the *field-consistent* assumed strain fields should now be determined from an orthogonality condition that arises when the equivalence of the minimum total potential energy principle with respect to the mixed variational principles is sought. This leads to another fundamental requirement in certain finite element formulations involving redistributed (or assumed) fields (e.g. strain or stress) – *variational correctness* – that the redistributed field should be orthogonal to the error introduced because of the field-redistribution (Prathap 1988; Prathap & Naganarayana

1988; Naganarayana & Prathap 1989a):

$$\int \delta\bar{\gamma}(\bar{\gamma}-\gamma)\,\mathrm{d}x = 0,$$

where γ represents the constrained strain field kinematically derived from the constituent displacement fields while $\bar{\gamma}$ is the assumed field to be determined from the original field such that $\bar{\gamma}$ is field-consistent and the resulting element formulation is free of locking and stress oscillations.

If the orthogonality condition that represents this equivalence is violated while designing the strain field to satisfy the field-consistency criteria alone, the resulting elements have poorer efficiency and are also plagued by undesirable strain and stress oscillations (Prathap & Naganarayana 1988; Naganarayana & Prathap 1989a).

In Prathap & Naganarayana (1988), the variationally correct (or orthogonal) and incorrect (or non-orthogonal) field-consistent assumed strain forms of the quadratic and cubic shear deformable beam elements are used to explore these aspects in detail. It is shown that the non-orthogonal formulations can lead to reasonably accurate displacement solutions but have spurious stress oscillations. Using the field-reconstitution technique, *a priori* analytical estimates were derived for the magnitude and pattern of these stress oscillations and tested digitally using computational results in Prathap & Naganarayana (1988). These stress oscillations are related to the presence of artificially created spurious load mechanisms which are self-equilibrating.

These spurious load mechanisms lead to additional spurious linear oscillations in the shear force and bending moment in a quadratic Timoshenko beam element and spurious quadratic oscillations in bending moment (but no change in the shear forces) in cubic Timoshenko beam element as demonstrated in Prathap & Naganarayana (1988). The resulting extraneous oscillations in the stress fields may often lead to difficulty in identifying points for optimal stress recovery in the element domain e.g. line-consistent 8-noded plate element (Naganarayana & Prathap 1989a).

In many problems, it is possible to determine the variationally correct strain field by expanding the inconsistent field in terms of Legendre polynomials; it is easy then to identify and eleminate the inconsistent terms. The logic is simple and direct; the coefficients associated with each Legendre polynomial represents a discretised constraint in the penality limits; identify the inconsistent Legendre term applying the field-consistency paradigm and simply drop it to get the correct and consistent field. Since the Legendre polynomials are orthogonal in the domain of integration, it follows that this procedure satisfies the orthogonality condition arising from the equivalence of the minimum total potential energy principles with reference to the mixed principles and hence the method is variationally correct. The same result can often be achieved quite simply by using reduced integration if γ is *one order* higher than $\bar{\gamma}$. It will be variationally incorrect otherwise. Reduced integration is universally popular since it is very easy to implement it on a computer. We shall see later that reduced integration cannot satisfy the edge-consistency requirements and hence fails when used in a distorted mesh. Hence the methods like Legendre polynomial expansion and truncation become very important in development of general purpose finite elements.

5. Consistent non-uniform mapping and edge-consistency

So far, we have looked at the consistency requirements in a simplified form that can be directly applied only to straight line elements or elements of rectangular form. It is not practical to have such a restriction for the plate/shell elements in a general purpose application. The development of an efficient and robust quadrilateral plate bending element has therefore been a formidable challenge. The point here is to ensure that consistency is maintained in curved and arbitrary quadrilateral forms of an element. The current practice in the development of elements is to use what is called the isoparametric concept – this requires the use of a covariant or natural co-ordinate system for all interpolations – i.e. the same interpolations serving to map both geometry (from the natural system to the Cartesian system) and displacements. The strains are based on the Cartesian system but interpolated using natural co-ordinates. It is therefore necessary to allow for changes from one system to another without loss of consistency, especially as far as the mapping of the constrained strain fields are concerned. It turned out that one crucial factor was the way tangential strain components on each edge of an element had to be defined under situations of non-uniform mapping.

That small errors in data preparation leading to distortion of the element can cause large errors in the solutions was reported by Hoppe (Hoppe 1984, 1985). This is due to the non-uniform mapping from the covariant system to the Cartesian system. The effect in the case of a constrained media problem is much more severe. Since curved beam, and quadrilateral plate and shell elements will intrinsically have such mapping conditions, there can be very large errors on this account. It is therefore necessary to take care to design an element so that accuracy and efficiency when used in a distorted mesh do not go down rapidly. There have been several attempts in literature to develop elements that are free of locking in their general form (Hughes *et al* 1977; MacNeal 1982; Hinton & Huang 1986; Donea & Lamain 1987) – most of them using the sampling points on the element edges. However, they lack an explanation for their success.

It was observed in Prathap & Naganarayana (1992) that it is not sufficient if field-consistency is achieved in a variationally correct sense in the covariant natural system alone in such cases. It was demonstrated that the difficulties are with the non-uniform mapping of the strain fields because of which the consistency and correctness conditions achieved in the natural covariant system are not preserved over the element domain and across its boundary after transformation into the Cartesian system. Several methods of achieving *consistent mapping* of the constrained strain fields from the natural system to the Cartesian system were discussed from two popular points of view – Cartesian base formulation and covariant base formulation.

The additional requirement for an element to be free of errors in a *patch* is the *edge-consistency* requirement (Prathap & Somashekar 1988). It was shown that the *tangential* strain components which are continuous across the element boundary in the undistorted covariant natural system should remain continuous even after the necessary transformations, and that the *tangential* strain components should be built from their corresponding *tangential* displacement components only. Though the Cartesian base formulation appeared to be very accurate in Prathap & Naganarayana (1992), identification of *tangential strain* components and hence achieving edge-consistency becomes a formidable job. Thus a covariant base formulation becomes desirable for developing general purpose finite elements.

We should note here that, in case of covariant base formulations, consistent mapping preserves the field-consistency over the element domain in the Cartesian system. However, special methods have to be used if edge-consistency has to be satisfied. An effective (from both accuracy and computational points of view) procedure namely, *nodal Jacobian transformation*, is developed and employed to achieve both consistent mapping and edge-consistency while developing several general purpose finite elements (Prathap & Somashekar 1988; Prathap *et al* 1988; Naganarayana & Prathap 1989a, 1989b; Naganarayana *et al* 1992; Prathap & Naganarayana 1992). Such elements which satisfy both field- and edge-consistency requirements are subjected to several severe patch tests and are found to be very accurate.

It is interesting to observe here that the methods of sampling the constrained *tangential* strain components at the error-free points on the element boundary (Hughes *et al* 1977; MacNeal 1982; Hinton & Huang 1986; Donea & Lamain 1987) in fact, try to achieve field- and edge-consistency requirements in the Cartesian system in a similar fashion. This explains their success when used in a distorted mesh.

6. Problems with initial strain/stress and varying moduli

In problems with initial stress/strain or with varying moduli, the discretised stress-resultant and strain fields will be of different order – consequently there is a loss of consistency in the formulation that results in oscillations in the stress-resultant fields (Prathap & Naganarayana 1990a, and to be published; Naganarayana 1991). For example, it has been known for some time that thermal stresses computed directly from stress-strain and strain-displacement matrices in a finite element analysis with simple finite elements can show large errors (Ojalvo 1974; Pittr & Hartl 1980). The conventional wisdom to tackle such problems is to sample stresses/strains at the Gauss points corresponding to a reduced integration rule. However, cases exist, e.g. the tapered quadratic bar element (Prathap & Naganarayana 1990a), where the oscillations are such that easily identifiable points of accurate stress do not exist.

In the computation of the stiffness matrix, energy terms of the form $U = \int \sigma^T \epsilon \, d\Omega$ exist and if the order of σ (σ may be stress or stress-resultant depending on the problem) is higher than that of the strain ε, the higher order term may not "do work" on the strain terms and are not recognised in the stiffness matrix. Thus, stress-resultants computed from the displacements recovered from such a formulation show extraneous oscillations.

It is necessary to define a consistent stress or stress-resultant field to ensure that the stress recovery reflects correctly the order of strain interpolations used. The orthogonality conditions, required for reconstituting the assumed fields for both strain and stress functions (from their original fields) simultaneously, can be obtained from the Hu-Washizu variational principle as,

$$\int \delta\bar{Q}(\bar{\gamma} - \gamma)\, d\Omega = 0; \quad \int \delta\bar{\gamma}(D\bar{\gamma} - \bar{Q})\, d\Omega = 0,$$

where, $\gamma, \bar{\gamma}, D\bar{\gamma}$ and $\bar{Q}$ represent the original inconsistent strain field, consistent strain field, stress-resultant field derived from the consistent strain field through constitutive laws and the consistent stress resultant field respectively.

In a varying moduli problem, due to varying D, the strain and stress-resultant fields will be of different interpolation order. In a thermal stress problem, the initial strain field in the element ε_0 which will be of the same order as the temperature field in the element, and the total strain field, which is obtained by differentiating the displacement fields will be of different order, especially if the temperature fields vary significantly over the domain and are interpolated by the same isoparametric functions as the displacement fields. Consequently, the total stress will be of an order higher than the strain field. Therefore, in recovering stresses in such problems, care must be taken to maintain consistency. In our work, the definitions of stress-resultants, or initial strains/stresses are *a priori* made consistent with the strain fields by invoking the orthogonality conditions seen earlier in this section.

7. A complete consistent and correct procedure for displacement type finite element formulation

So far, we have reviewed the important consistency and correctness paradigms necessary for finite element formulation free of locking and stress oscillations. Here, we sum up all these concepts and the additional considerations that are necessary for developing a general-purpose finite-element library for analysis of composite structures. For continuity, we recapitulate our understanding of the sources for the locking and stress oscillations and the required consistency and correctness paradigms to eliminate the same (table 2).

Continuity: The displacement function must be continuous over the element domain and across the element boundary in a given arbitrary finite element mesh.

Table 2. Types of errors, their sources in finite element analysis and the associated paradigms.

Source	Symptoms	Paradigm/concepts
	Errors of first kind	
Finite element discretisation	Discretisation errors (errors of first kind)	Continuity and Completeness
	Errors of second kind	
Constrained media elasticity	Locking, delayed convergence, stress oscillations	Field-consistency
Element distortion (nonuniform mapping)	Locking, delayed convergence, stress oscillations	Edge-consistency and consistent mapping
Varying moduli	Stress oscillations	Stress field-consistency
Initial strain & stress field representation	Strain and stress oscillations	Stress field-consistency
Redistribution of strain/stress fields	Poorer convergence spurious load mechanisms and stress oscillations	Variational correctness
Modelling warped Surface with linear elements	Erroneous displacements and stress recovery	Warping correction

Completeness: The strain/stress fields should be able to model strain-free rigid body motion of the element and the constant strain state of the element deformation.
Field-consistency: The terms in a constrained strain field that have partial contribution from the constituent displacement fields, leading to the spurious constraints in the penalty limits for the corresponding strain energy components, should be eliminated from the formulation for assuring convergence of results from a finite element model of the structural problems belonging to the class of constrained media elasticity.
Consistent mapping: Mapping of the strain/stress fields from the covariant natural system (where the element configuration is always undistorted) to the Cartesian system should not introduce any additional spurious constraints.
Edge-consistency: The *tangential* strain components which are continuous across the element boundary in the undistorted covariant natural system would remain continuous even after the necessary transformations; and the *tangential* strain components should be built from their corresponding *tangential* displacement components only.
Stress field-consistency: The terms in the strain and/or stress fields that do not participate in the strain energy computations (and hence in the displacement recovery) should be eliminated while recovering the corresponding strain and/or stresses in a displacement type formulation.
Variational correctness: The redistributed field should be orthogonal to the error introduced because of the field-redistribution with reference to the original field.

The ideal characteristics of a finite element formulation for general purpose applications and the paradigms/concepts required to achieve the same in a scientific manner are briefly shown in table 3.

Based on the consistency and correctness principles several linear and quadratic elements, free of locking and stress oscillation, are developed for 1-, 2- and 3-dimensional applications.

8. FEPACS – finite element package for analysis of composite structures

FEPACS is a medium-sized general purpose package (i.e. about 20,000 lines of FORTRAN code) for the finite element analysis of isotropic, anisotropic and layered-composite structures. It has a family of simple, accurate and robust field-consistent elements. The package was initially built around the data and program organisation of SAP-IV, but recently, these have been modified and new solution capabilities are being introduced, to give FEPACS a character of its own.

The structural analysis program (SAP) is one of the earliest general purpose programs used for structural analysis through the finite element method. It was developed by Bathe *et al* (1974) under the sponsorship of many international organisations. The first version of SAP was released in September 1970 (Wilson 1970). The improved version, SAP-IV, which can be used for linear static and dynamic analysis of 3-dimensional structures was released in 1974 (Bathe *et al* 1974). As it had been released with its source code in the public domain, it has served as the spring board for many other finite element packages which were improved versions of the original package. It is well known that SAP-IV has reasonably efficient solution capabilities and data-handling procedures that can solve large 3-dimensional systems. However, its main weakness is its very old, outdated element library based on mainly linear finite elements for isotropic structures.

Table 3. Ideal characteristics of a finite element formulation.

Characteristics	Paradigms/concepts/procedures
Discretisation errors should *vanish* as the mesh is refined	Continuity and completeness
The element should be free of locking and/or delayed convergence	Field-consistency and Variational correctness
Element should be able to give variationally correct stress distribution without any oscillations	Field-consistency, Stress field-consistency, Variational correctness
Distorted geometry should not disturb the element geometry	Consistent mapping and edge-consistency
The element should be free of spurious zero energy mechanisms	Consistency and correctness Integration rules and Spectral analysis
Element formulation should be scientific & should provide *a priori* methods of error analysis (i.e. should not be based on numerically adjusted factors, heuristic arguments etc.)	Consistency and correctness paradigms supported by the functional reconsistution techniques
Element performance should be free from its geometry and position in space	Edge-consistency, appropriate local coordinate system and tensorial transformations

8.1 *Finite element library*

FEPACS has replaced this with the state-of-the-art field-consistent linear and quadratic 1-D, 2-D and 3-D element formulations – thus making the element library *complete* (Prathap *et al* 1989). The emphasis is on laminated anisotropic material structure (any other material constitution becomes a sub-set of this). A complete flow diagram representing the package is shown in figure 2.

Each element in FEPACS utilises field- and edge-consistent stiffness matrices derived in a variationally correct fashion; consistent initial (thermal) strain representation; consistent stress resultant field representation; diagonal or consistent mass matrices; and geometric stiffness matrices which can be used for linear buckling analysis; material constitutive laws are derived assuming general orthotropic layers. The finite elements included in the library are given below.

(1) SPRING – 2-noded constant stiffness spring element,
(2) TRUSS – 2-noded linear truss element,
(3) BEAM2-T – 2-noded laminated linear Timoshenko beam element,
(4) BEAM2-E – 2-noded laminated linear classical beam element,
(5) BEAM3 – 3-noded laminated quadratic curved Timoshenko beam element (with taper and twist)
(6) PLAXTQ – family of plane strain, plane stress & axisymmetric elements of triangular or quadrilater shapes (orthotropic),
(7) SHEL4 – 4-noded shear deformable laminated anisotropic Mindlin plate/plane-shell element of quadrilateral shape with warping corrections,

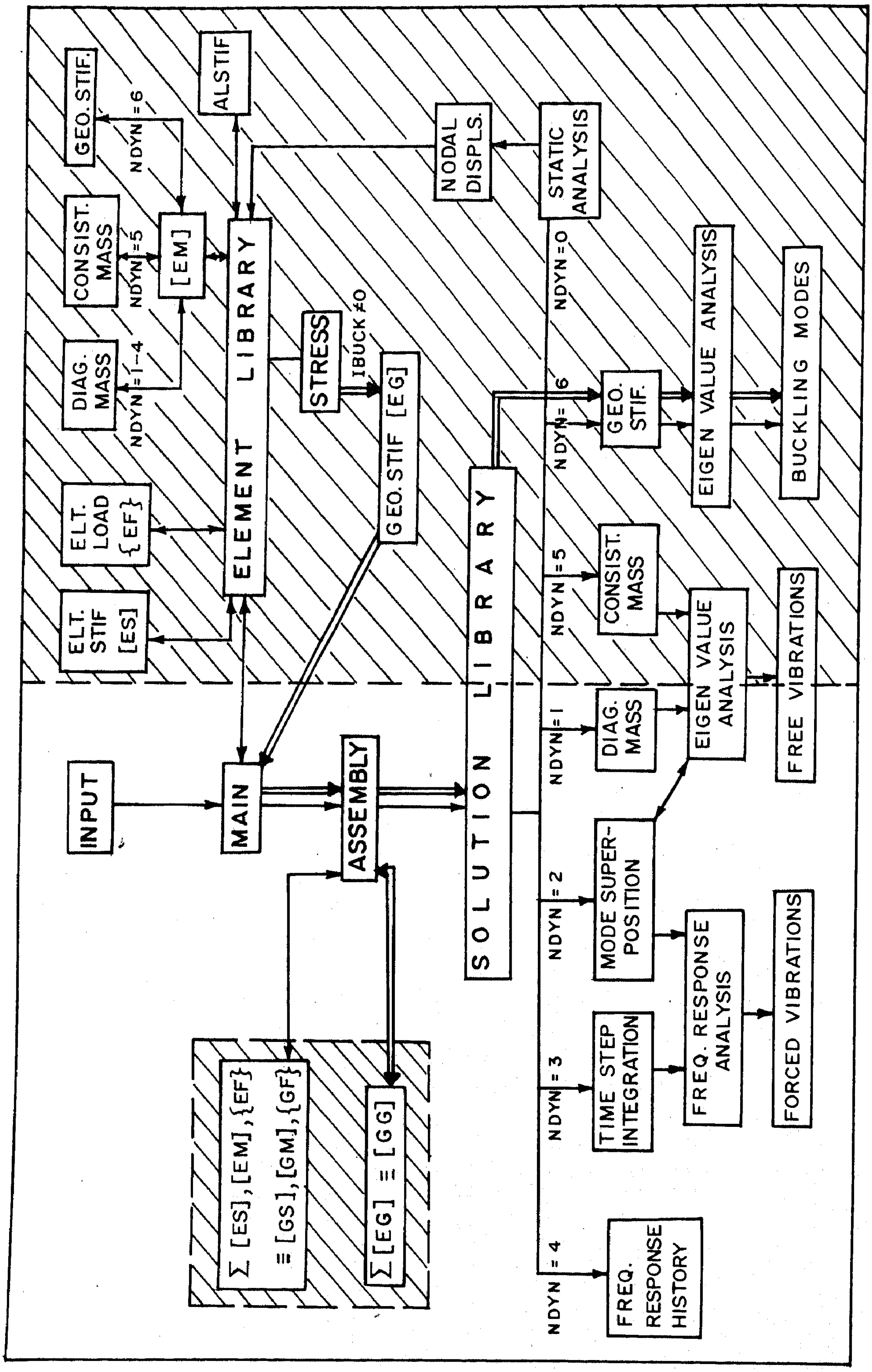

Figure 2. Finite element package for analysis of composite structures (FEPACS).

(8) SHEL8 – 8-noded quadrilateral laminated degenerated shell element,
(9) HEXA8 – 8-noded linear solid brick element (orthotropic),
(10) HEXA27 – 27-noded quadratic brick element (isotropic),

8.2 *Library of solution capabilities*

An eigenvalue solution routine based on the determinant search method has been added to FEPACS so that dynamic eigenvalue analysis using consistent mass matrices and buckling analysis of the linear structures can be carried out with the software. The solution routines can be efficiently used for both small-scale as well as large-scale problems using the respective in-core and out-of-core solution algorithms that are built in the program. The solution capabilities of FEPACS now include the following.

(1) Static analysis under thermomechanical loads – Gauss elimination solution routines based on banded assembled matrices are used;
(2) Natural frequency analysis – Both diagonal as well as consistent mass matrices can be used. Two eigenvalue solution routines, the determinant search method which is optimal for in-core solution and the subspace iteration method which is optimal for out-of-core solution, are available.
(3) Natural frequency analysis followed by response history analysis – Only diagonal mass matrices need to be used. Method of superposition of the natural modes and the forced modes is utilised to get the structural response to dynamic loads.
(4) Response history analysis using direct integration – Explicit time-step integration routines are used to get the structural response to dynamic loads without going through eigenvalue solution.
(5) Natural frequency analysis followed by response spectrum analysis.
(6) Buckling analysis – If the in-plane stresses are known *a priori* for each element, the analysis is done in a single run – construct the geometric stiffness matrices from the element in-plane stresses and solve for buckling loads and modes using the new eigenvalue solution routines. If they are not known (as in general structural analysis) the analysis is done in double runs – the in-plane stresses for each element are calculated from the first run of static analysis and the geometric stiffness matrices calculated from this static solution are then used for the second eigenvalue analysis.

8.3 *Other features*

Some of the other features specially incorporated in FEPACS to enhance its general purpose nature and its accessibility to users – for both regular analysis as well as for enhancement of the package with other capabilities include the following.

(a) Generalised data input and output structures so that integration of pre- and post-processors can be efficiently achieved.
(b) The program is organised in a highly modular fashion so that any new element or solution capability can be included with least difficulty.
(c) The whole program is modified to use only *general* FORTRAN-77 commands such that the package is highly portable except for the scratch file operations, while are kept in a separate module which is compiler-dependent. The package is currently available on a variety of platforms like PC 386/486, workstations, super-mini

computers etc. with UNIX operating system, minimum 2 MB RAM and a FORTRAN compiler supporting scratch files of unlimited record length (only the scratch file allocations have to be modified). It can be loaded in any other operating system with similar support.

9. Conclusions

In this paper, many difficulties arising in displacement-type finite element formulation and the remedial measures offered for these have been reviewed. A general, complete and scientifically based procedure for formulating robust finite elements is provided. A general purpose finite element package (FEPACS) using a library of robust elements, developed along the lines presented here, is also reviewed. Though the emphasis is on the displacement-type finite formulations for structural analysis, the concepts discussed here are equally applicable to other types of finite element formulations applied to many other fields of engineering.

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