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Geometric construction of cohomology for arithmetic groups I

By

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1. Introduction

In this paper we give a geometric construction of cohomology classes for uniform arithmetic subgroups of Lie groups and prove that our classes are not the restrictions of continuous classes from the Lie group. We apply our theory to obtain the following results:

(1) a proof that every unitary representation with non-zero cohomology, see Borel [2], occurs in $L^2(\Gamma\backslash SO(n, 1))$ for a suitable uniform Γ, thereby proving for the group $SO(n, 1)$ a conjecture which seems to be widely believed, though the group $SO(n, 1)$ represents the only case where it has been proved. In terms of differential geometry we show that the n dimensional manifolds of constant negative curvature constructed at the end of Borel [4] admit finite covers with all Betti numbers strictly between 0 and n not equal to zero—and hence arbitrarily large by Borel [2];

(2) a proof that the recent vanishing theorem of Borel-Wallach [5], Casselman-Schmid [6] and Zuckerman [13] is the best possible for the orthogonal groups $SO(p, q)$ for $p \geqslant q$ and q even;

(3) the construction of algebraic cycles in compact quotients of the unit ball in $\mathbf{C}^n$ which are dual to linear subspaces of CP^n but correspond to non-trivial automorphic representations. We construct similar examples for some other bounded symmetric domains, namely, those associated to $SU(p, q)$ and $SO(n, 2)$.

We remark that the identification of the local factors in the automorphic representations of (3) appears to be an important problem.

We now describe our construction.

If Γ is a torsion-free discrete subgroup of a semi-simple Lie group G having no compact simple factors and K is a maximal compact subgroup of G then Γ acts freely on the Riemannian symmetric space $X = G/K$ and consequently the cohomology of the quotient space $\Gamma\backslash X$ coincides with the cohomology of Γ. We deduce the existence of new cohomology for Γ by constructing non-bounding cycles in $\Gamma\backslash X$.

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These cycles will be the projections under $\pi : X \to \Gamma\backslash X$ of the sub-symmetric spaces of X which are fixed point sets of involutive isometries of X which normalise Γ. It is important to observe that the class of sub-symmetric spaces of X which are fixed point sets of involutions is a proper sub-class of the totally geodesic sub-symmetric spaces of X. To try to prove that for suitable Γ the projection of an arbitrary sub-symmetric space of X gives rise to a non-bounding cycle would be unsuccessful as it would contradict the above vanishing theorem. In fact, the class of sub-symmetric spaces of X which arise as the fixed point sets of isometric involutions is characterised as the class of sub-symmetric spaces admitting complements. By this we mean the following: Suppose 0 is the origin of X and X_1 a totally geodesic sub-symmetric space of X passing through 0. Let $\mathfrak{p}_1$ be the tangent space to X_1 at 0 and $\mathfrak{p}_2$ the orthogonal complement of $\mathfrak{p}_1$ in $\mathfrak{p}$, which is identified with the tangent space of X at 0. Let X_2 be the image of $\mathfrak{p}_2$ under the Riemannian exponential map $\exp : T(X, 0) \to X$. We say X_2 is a complement to X_1 if X_2 is a sub-symmetric space of X. In case X_1 is the fixed point set of an involutive isometry σ_1 this is always the case, for if ι is the geodesic reflection at 0 then X_2 is the fixed point set of the involutive isometry $\sigma_2 = \iota\sigma_1$. Projecting X_1 and X_2 down to $\Gamma\backslash X$ we obtain two complementary dimensional closed cycles. By passing to a subgroup Γ' of finite index in Γ we obtain a new projection $\Pi' : X \to \Gamma'\backslash X$. We try to find Γ' so that the cycles $\Pi'(X_1)$ and $\Pi'(X_2)$ intersect transversally with *all* intersection multiplicities positive. In this case the intersection product of the two cycles is non-zero and consequently neither is a boundary. One of our most important theorems, theorem 2.1, then asserts that if one can find such a sub-group then there exists a further subgroup Γ'' of finite index in Γ so that $\Pi''(X_1)$ is not dual to an invariant form, where $\Pi'' : X \to \Gamma''\backslash X$ is the projection.

We would like to express our appreciation to W. Dwyer for the observation at the beginning of the proof of theorem 2.1.

1. Orthogonal decompositions of symmetric spaces

We begin by establishing some notation which will be used in the rest of the paper. X will denote a symmetric space of non-compact type which is assumed to be connected. G will denote the connected component of the identity of the sometry group of X, G will be assumed to be linear and semi-simple. The symmetric space X will be equipped with an origin 0. The isotropy at 0 will be denoted by K; hence, X will be isometric to G/K equipped with a metric in the usual way. $\mathfrak{g}$ will denote the Lie algebra of G, k the Lie algebra of K and $\mathfrak{p}$ the orthogonal complement of k in $\mathfrak{g}$ relative to the Cartan-Killing form. ι will denote the geodesic reflection at 0.

The key notion in this paper is the following. Let X_1 and X_2 be two totally geodesic sub-symmetric spaces of X whose intersection is precisely 0. Let $\mathfrak{p}_1$ denote the tangent suace to X_1 at 0 and $\mathfrak{p}_2$ the tangent space to X_2 at 0. We say that the pair $\{X_1, X_2\}$ is an orthogonal decomposition of X if

(i) $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are perpendicular

(ii) $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, We say that two orthogonal decompositions $\{X_1, X_2\}$ and $\{X_1', X_1'\}$ are equivalent if there is an isometry ϕ of X so that $\phi(X_1) = X_1'$ and $\phi(X_2) = X_2'$.

Remark : Since X is homogeneous there is no point in considering decompositions $\{X_1, X_2\}$ so that $X_1 \cap X_2 = \{x\}$ where $x \neq 0$.

Now we see orthogonal decompositions may be characterised infinitesimally by the following proposition.

Proposition 1.1.

Let $\eta \subseteqq \mathfrak{p}$ be a sub-vector space. Then the following are equivalent.

(1) $Y = \exp \eta$ is a totally geodesic sub-symmetric space

(2) η is a Lie triple system, that is if $u, v, w \in \eta$, then $[[u, v], w] \in \eta$.

Proof : The proof may be found in Helgason [7], theorem 7–2, page 189.

Thus the infinitesimal equivalent of an orthogonal decomposition of X is an orthogonal decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ with $\mathfrak{p}_1$ and $\mathfrak{p}_2$ both Lie triple systems.

We now show how all such arise.

Theorem 1.1

There is a one to one correspondence between equivalence classes of orthogonal decompositions of X and conjugacy classes of involutive isometries of X. This correspondence is obtained as follows: Given an involutive isometry σ_1 of X fixing 0 put $\sigma_2 = \sigma_1 \iota$ (note that σ_1 and ι commute so that σ_2 is an involution). Then denoting by X_1 the fixed point set of σ_1 and by X_2 the fixed point set of σ_2 we find that $\{X_1, X_2\}$ is an orthogonal decomposition.

Proof : Suppose that we are given σ_1 as above. Then defining σ_2, X_1, X_2 as above it is clear that $\{X_1, X_2\}$ is an orthogonal decomposition of X.

To prove the converse consider the infinitesimal decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Then $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are both triple systems. We define a linear map T on $\mathfrak{p}$ by $T|\mathfrak{p}^1 = +1$, $T|\mathfrak{p}_2 = -I$ and T preserves $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

If we can show that T preserves the curvature tensor R at 0 then by Cartan's theorem T will be the derivative of a global isometry σ_1 evaluated at 0. Thus we must show that for all $x, y, z, w \in \mathfrak{p}$ we have

$$\langle R_{Tx, Ty}\, Tw, Tz \rangle = \langle R_{x,y}\, w, z \rangle.$$

But using the well-known formula for the curvature tensor of a symmetric space, see Helgason [7], theorem 4.2, page 180.

$$R_{x,y} z = -[[x, y], z],$$

decomposing x, y, z, w into $\mathfrak{p}_1$ and $\mathfrak{p}_2$ components, using that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are triple systems and that the Cartan-Killing form is invariant the result is immediately verified and theorem 1.1 is proved.

We will often denote the decomposition $\{X_1, X_2\}$ of X by σ_1, σ_2. Here σ_1, σ_2 are a pair of commuting involutive isometries fixing 0 and satisfying

$$\sigma_1 \sigma_2 = \iota.$$

Lemma 1.1

Two orthogonal decompositions (σ_1, σ_2), (σ_1', σ_2') are equivalent if and only if there exists an isometry ϕ of X fixing 0 and satisfying

$$\phi\sigma_1\phi^{-1} = \sigma_1',$$
$$\phi\sigma_2\phi^{-1} = \sigma_2'.$$

Proof: The sufficiency is clear. Suppose then that there is an isometry ϕ of X satisfying

$$\phi(X_1) = X_1',$$
$$\phi(X_2) = X_2',$$

where X_1' and X_2' are the fixed point sets of σ_1' and σ_2'. ϕ map $X_1 \cap X_2$ to $X_1' \cap X_2'$ hence $\phi(0) = 0$. Denoting the tangent spaces of X_1' and X_2' at 0 by $\mathfrak{P}_1'$ and $\mathfrak{P}_2'$ respectively we have :

$$d\phi(\mathfrak{P}_1) = \mathfrak{P}_1',$$
$$d\phi(\mathfrak{P}_2) = \mathfrak{P}_2',$$
$$d(\phi\sigma_1\phi^{-1})\,|\mathfrak{P}_1' = +1,$$
$$d(\phi\sigma_1\phi^{-1})\,|\mathfrak{P}_2' = -1.$$

But $\phi\sigma_1\phi^{-1}$ and σ_1' both leave 0 fixed and have the same derivative at 0. Since they are isometries they must be equal. A similar argument establishes that $\phi\sigma_2\phi^{-1} = \sigma_2'$. This proves lemma 1.1.

We define two subgroups of G

$$G_1 = \{g \in G \mid \sigma_1 g \sigma_1 = g\},$$
$$G_2 = \{g \in G \mid \sigma_2 g \sigma_2 = g\}.$$

We will denote by $\overline{G_1}$ and $\overline{G_2}$ the subgroups of G_1 and G_2 which preserve the orientation of X_1 and X_2 respectively. It is also convenient to define involutions

$$\Sigma_1(g) = \sigma_1 g \sigma_1,$$
$$\Sigma_2(g) = \sigma_2 g \sigma_2,$$
$$\theta(g) = \iota g \iota.$$

Then $\theta = \Sigma_1\Sigma_2$ and is the Cartan involution associated to K. We will sometimes denote the decomposition by $\{\Sigma_1, \Sigma_2\}$.

Lemma 1.2

$\overline{G_1}$ acts transitively on X_1 and $\overline{G_2}$ acts transitively on X_2.

Proof: $P_1 = \exp \mathfrak{P}_1$ is contained in $\overline{G_1}$ and acts transitively on X_1. Similarly $P_2 = \exp \mathfrak{P}_2$ acts transitively on X_2. (Here exp is the exponential $\mathfrak{g} \to G$).

Remark. In our applications G_1 and G_2 will be replaced by smaller groups H_1 and H_2. These groups will be defined in Chapter 4. For the moment we let H_1 and H_2 be subgroups of G_1 and G_2 respectively which act transitively on X_1 and X_2 respectively. We let H_1^0 and H_2^0 denote the identity components of H_1 and H_2 respectively.

Lemma 1.3

Put $\hat{K} = G_1 \cap G_2$. Then $\hat{K} = G_1 \cap K = G_2 \cap K$ and is a maximal compact subgroup of G_1 and G_2. $\hat{K}$ is reductive but it is frequently not semi-simple.

Proof: $K \cap G_1$ is a maximal compact subgroup of G_1 because it is the isotropy of 0. We have $G_1 \cap G_2 \subseteqq K \cap G_1$ because $G_1 \cap G_2$ must leave 0 fixed and consequently is contained in K. But suppose $k \in G_1 \cap K$. Then $\theta(k) = k$ because $k \in K$ and $\Sigma_1(k) = k$ because $k \in G_1$, hence $\Sigma_2(k) = k$, hence $k \in G_2$. This proves lemma 1.3.

Remark. Putting lemmas 1.2 and 1.3 together we see

$$X_1 = G_1/\hat{K}, \quad X_2 = G_2/\hat{K}.$$

Proposition 1.2

Let G be a connected simply-connected semi-simple real Lie group and let σ be an involutive automorphism. Then the fixed point set of σ is a connected subgroup.

Proof: In Helgason [7], theorem 7.2, page 272, the theorem is proved in case G is compact. Now assume G is not compact. σ leaves invariant some maximal compact subgroup K. Then we obtain a σ-equivariant decomposition $G = KP$. But the fixed point set of σ in P is clearly connected because if a point $x \in P$ is fixed, since 0 is also fixed, the minimal geodesic joining 0 to x is pointwise fixed. By the theorem referred to in the first line of this proof the fixed point set of σ in K is connected (K is simple-connected because G is) hence the fixed point set of σ in G is the product of two connected sets. This proves proposition 1.2.

2. Discrete uniform subgroups compatible with an orthogonal decomposition

Definition

We say a discrete subgroup $\Gamma \subseteqq G$ is compatible with a decomposition $\{\sigma_1, \sigma_2\}$ if σ_1 and σ_2 both preserve Γ; that is,

$$\sigma_1 \Gamma \sigma_1 \subseteqq \Gamma,$$

$$\sigma_2 \Gamma \sigma_2 \subseteqq \Gamma.$$

We define subgroups Γ_1 and Γ_2 of Γ by

$$\Gamma_1 = \{\gamma \in \Gamma \mid \sigma_1 \gamma \sigma_1 = \gamma\},$$

$$\Gamma_2 = \{\gamma \in \Gamma \mid \sigma_2 \gamma \sigma_2 = \gamma\}.$$

Remark : σ_1 and σ_2 induce involutions of $\Gamma \backslash X$ which we also denote by σ_1 and σ_2.

Proposition 2.1

Suppose that $\Gamma \subseteq G$ is uniform; then Γ_1 is uniform in G_1 and Γ_2 is uniform in G_2.

Proof : We will need two lemmas.

Lemma 2.1 (Selberg)

Let G be a locally compact group and let Γ be a discrete subgroup. If $\Gamma \backslash G$ is compact then for all $\gamma \in \Gamma$.

$Z(\gamma) \cap \Gamma \backslash Z(\gamma)$ is compact where $Z(\gamma)$ denotes the centraliser of γ in G.

Proof : Let $\gamma \in \Gamma$. Consider the map $k : G \to G$ given by $k(g) = g\gamma g^{-1}$. Then k is continuous. Since $k(\Gamma) \subset \Gamma$, $k(\Gamma)$ is closed and accordingly $k^{-1}(k(\Gamma))$ is closed, that is, $\Gamma Z(\gamma)$ is closed in G. Therefore $\Gamma \backslash \Gamma Z(\gamma)$ is a closed subset of the compact space G and is compact. Hence $Z(\gamma) \cap \Gamma \backslash Z(\gamma)$ is compact.

Remark : The proof of this lemma is taken from Mostow [11], page 62.

The next lemma is hardly more than an observation.

Lemma 2.2

Let G be a topological group and Γ a discrete subgroup. Let G_0 be a subgroup of G of finite index in G. Then Γ is uniform in G if and only if $\Gamma \cap G_0$ is uniform in G_0.

Proof : It is clear that if $\Gamma \cap G_0$ is uniform in G_0 then Γ is uniform in G. To prove the converse it is sufficient to show that $\Gamma \cap G_0 \backslash G_0$ is closed in $\Gamma \backslash G$. This is equivalent to showing ΓG_0 is closed in G. But ΓG_0 is just a union of a finite number of copies of G_0 hence a finite union of closed sets proving lemma 2.2.

Now we are ready to prove proposition 2.1. Consider $G = \text{Aut } G$. We have a natural projection $\Pi : G \to \text{Ad } G$ and $\text{Ad } G$ is a subgroup of finite index of $\text{Aut } G$. $\Pi(\Gamma)$ is a uniform discrete subgroup of $\text{Ad } G$ hence of $\text{Aut } G$. Consider the subgroup Γ' of $\text{Aut } G$ generated by $\Pi(\Gamma)$ together with σ_1 and σ_2. Clearly Γ' is uniform but it is also easily seen to be discrete as σ_1 and σ_2 normalise Γ. Thus we may apply lemma 2.1 to conclude that the centraliser of σ_1 in Γ' is uniform in the centraliser of σ_1 in $\text{Aut } G$ which we denote $Z(\sigma_1)$. But by lemma 2.2, $Z(\sigma_1) \cap \Gamma' \cap \text{Ad } G$ is uniform in $Z(\sigma_1) \cap \text{Ad } G$ since $Z(\sigma_1) \cap \text{Ad } G$ is of finite index in $Z(\sigma_1)$ (here we use that $\text{Ad } G$ is normal in $\text{Aut } G$). From this we see that the inverse image of $Z(\sigma_1) \cap \Gamma' \cap \operatorname{Ad} G$ is uniform in the inverse image of $Z(\sigma_1) \cap \operatorname{Ad} G$ which we denote $(G_1)_*$. But G_1 has finite index in $(G_1)_*$ and

$$\Gamma_1 = G_1 \cap \Pi^{-1}(Z(\sigma_1) \cap \Gamma' \cap \operatorname{Ad} G).$$

We apply lemma 2.2 and obtain that Γ_1 is uniform in G_1 and of course the same proof applies to Γ_2 in G_2.

It is not generally the case that $Y_1 = \Gamma_1 \backslash X_1$ and $Y_2 = \Gamma_2 \backslash X_2$ will be the images of X_1 and X_2 under the quotient projection $\Pi : X \to \Gamma \backslash X$. However, somewhat surprisingly, this is always the case if Γ acts freely on X.

Proposition 2.2 (H Jaffee)

With the above notation if Γ operates freely on X :

$$\Pi(X_1) = \Gamma_1 \backslash X_1,$$

$$\Pi(X_2) = \Gamma_2 \backslash X_2.$$

Proof : Suppose there exist $x_2 \in X_2$, $x_2' \in X$, $\gamma \in \Gamma$ so that $\gamma x_2 = x_2'$. Then let $\nu = \gamma \sigma_2 \gamma^{-1} \sigma_2$. Then since $\sigma_2 \gamma^{-1} \sigma_2 \in \Gamma$, $\nu \in \Gamma$. But $\nu x_2' = x_2'$ hence $\nu = 1$. From this it follows $\sigma_2 \gamma^{-1} \sigma_2 = \gamma^{-1}$ and accordingly $\gamma \in \Gamma_2$. A similar argument establishes $\Pi(X_1) = \Gamma_1 \backslash X_1$.

We receive then (assuming Γ is compatible with $\{\sigma_1, \sigma_2\}$ and acts freely) a compact locally symmetric space Y containing two totally geodesic complementary dimensional totally geodesic submanifolds Y_1 and Y_2. It will be very important to us that Y_1, Y_2 and Y are orientable.

Lemma 2.2

(i) Y is orientable.

(ii) If $\Gamma_1 \subseteqq \bar{G}_1$ then Y_1 is orientable.

Proof : (i) is obvious since Γ is assumed to be contained in G which is connected. (ii) follows immediately from proposition 2.2. We note then if G is simply-connected then from proposition 1.2 it follows that Y_1 and Y_2 are orientable.

In fact we may generalise this last remark considerably.

Proposition 2.3

Suppose G has a finite linear covering group H with the property that σ_1 lifts to an involution $\tilde{\sigma}_1$ of H with a connected centraliser H_1. Then there exists an arithmetic subgroup Γ' of Γ so that $\Gamma_1' \subseteqq \bar{G}_1$ where Γ_1' is the centraliser of σ_1 in Γ'.

Proof : Let $\triangle$ be the kernel of the projection $\Pi : H \to G$. The finite central extension $\triangle \to H \to G$ induces an extension $\triangle \to \tilde{\Gamma} \to \Gamma$. We claim that this extension splits over a subgroup Γ' of Γ of finite index. Indeed, since H is linear we can find a congruence subgroup $\tilde{\Gamma}'$ of Γ containing no elements of finite order and normalised by $\tilde{\sigma}_1$ hence $\triangle \cap \tilde{\Gamma}' = \{1\}$. Put $\Gamma' = \Pi(\tilde{\Gamma}')$. Clearly Γ' is an arithmetic subgroup of Γ. Noting that $\tilde{\Pi} : \Gamma' \to \Gamma'$ is an isomorphism we see that $\triangle \to \Pi^{-1}(\Gamma') \to \Gamma'$ is split. Now we claim that Π maps the centraliser of $\tilde{\sigma}_1$ in $\tilde{\Gamma}'$ onto the centraliser of σ_1 in Π'. Since the first centraliser is contained in the connected group H_1, this will complete the proof of the proposition. Suppose $\gamma \in \Gamma'$ satisfies $\sigma_1 \gamma \sigma_1 = \gamma$. Let $\tilde{\gamma}$ be the element in $\tilde{\Gamma}'$ lying over γ. Then $\Pi(\tilde{\sigma}_1 \tilde{\gamma} \tilde{\sigma}_1 \tilde{\gamma}^{-1}) = 1$ and since $\tilde{\sigma}_1 \tilde{\gamma} \tilde{\sigma}_1 \tilde{\gamma}^{-1} \in \tilde{\Gamma}'$ we must have $\tilde{\sigma}_1 \tilde{\gamma} \tilde{\sigma}_1 = \tilde{\gamma}$ and the proposition is proved.

Corollary. Suppose the universal cover $\tilde{G}$ of G is linear. Then there exists an arithmetic subgroup Γ' of Γ so that $\tilde{\Gamma}'_1 \subseteq G_1$ and $\Gamma'_2 \subseteq \overline{G}_2$.

Remark : In case G is the group of analytic automorphisms of a bounded symmetric domain then $\tilde{G}$ is not linear. However if σ_1 is complex linear (and consequently $\sigma_2 = i\sigma_1$ is also complex linear) then G_1 is a group of analytic automorphisms of the sub-domain $G_1/\hat{K}$ and consequently every element of G_1 is orientation preserving hence $G_1 = \overline{G}_1$ (and $G_2 = \overline{G}_2$).

Lemma 2.3

There is a one to one correspondence between $Y_1 \cap Y_2$ and the set of $\Gamma_1 \times \Gamma_2$ equivalence classes of triples

$$T = \{(x_1, x_2) \in X_1 \times X_2 \times \Gamma \mid \gamma\, x_1 = x_2\}.$$

$\Gamma_1 \times \Gamma_2$ acts on this set by

$$(\gamma_1, \gamma_2) \circ (x_1, x_2, \gamma) = (\gamma_1 x_1, \gamma_2 x_2, \gamma_2 \gamma \gamma_1^{-1}).$$

Proof : Let (x_1, x_2, γ) be an equivalence class representative then $\gamma x_1 = x_2$ hence $\Pi(x_1) = \Pi(x_2) \in Y_1 \cap Y_2$. Now suppose $y \in Y_1 \cap Y_2$, then there exists $x_1 \in X_1$, $x_2 \in X_2$ so that $\Pi(x_1) = y = \Pi(x_2)$ hence there exists $\gamma \in \Gamma$ so that $\gamma x_1 = x_2$ hence y corresponds to the triple (x_1, x_2, γ). Now let $x_1' \in X_1$, $x_2' \in X_2$ be two other choices so that $\Pi(x_1') = y = \Pi(x_2')$. Then there exists γ' so that $\gamma' x_1' = x_2'$. By proposition 2.2 there exists γ_1, γ_2 so that $\gamma_1 x_1' = x_1$, $\gamma_2 x_2' = x_2$ hence $\gamma_2^{-1} \gamma \gamma_1 (x_1') = \gamma' (x_1')$ hence $\gamma_2 \gamma' \gamma_1^{-1} = \gamma$ establishing lemma 2.3.

Proposition 2.3

Let $(x_1, x_2, \gamma) \in T$, then the connected component of $Y_1 \cap Y_2$ passing through $y = \Pi(x_1)$ is just the image of $\gamma\,(X_1) \cap X_2$ under Π. Let $\sigma_1' = \gamma \sigma_1 \gamma^{-1}$. Then the intersection component containing $\Pi\,(x_1)$ is a locally symmetric space $\Lambda \backslash A/B$ where A is the centraliser of σ_1' in G_2, B is the centraliser of σ_1' in $\hat{K}$ and Λ is the centraliser of σ_1' in Γ_2.

Proof : Let y' be in the connected component of $Y_1 \cap Y_2$ containing y. Then according to the correspondence of lemma 2.3 there exist $x_1 \in X_1$, $x_2' \in X_2$ and

$\gamma' \in \Gamma$ so that $\gamma' x_1' = x_2'$ and $\Pi(x_1') = \Pi(x_2') = y'$. But there is a path $a \in Y_1 \cap Y_2$ joining y to y', hence under the correspondence of lemma 2.3 there must exist paths $a_1(t) \in X_1$, $a_2(t) \in X_2$ and $\gamma(t)$ in Γ so that $\Pi \circ a_1 = \Pi \circ a_2 = a$. Since Γ is discrete we must have $\gamma(t) = \gamma$ all t; hence $\gamma x_1' = x_2'$ proving that the connected component of $Y_1 \cap Y_2$ containing y is contained in $\Pi(\gamma(X_1) \cap X_2)$.

To prove the converse it is enough to show that $\gamma(X_1) \cap X_2$ is connected. To this end let $x_2, x_2' \in \gamma(X_1) \cap X_2$, then there exists $x_1, x_1' \in X_1$ with $\gamma(x_1) = x_2$, $\gamma(x_1') = x_2'$. But then the minimising geodesic joining x_1 to x_1' is sent to the minimising geodesic joining x_2 to x_2' and accordingly the latter geodesic is entirely contained in the intersection.

The above description of the intersection component containing $\Pi(x_1)$ follows immediately from the Jaffee lemma.

Remark : Every isolated intersection has multiplicity $+1$ or -1 since $Y_1 \cap Y_2$ is totally geodesic.

We see that the connected components of $Y_1 \cap Y_2$ are indexed by the Γ_2, Γ_1 double cosets of those γ which appear in the third component of the triples in T. Choose a set of representatives for the Γ_2, Γ_1 double cosets of these elements of Γ, $\gamma_1, \gamma_2, \ldots, \gamma_r$ and put $I(\Gamma) = \{\gamma_1, \gamma_2, \ldots, \gamma_r : \gamma_1 = 1\}$. (We will sometimes write I for $I(\Gamma)$.)

Lemma **2.4**

There is a one to one correspondence between I and the $\Gamma_2 \times \Gamma_1$ orbits of $\Gamma \cap H_2^0 K H_1^0$.

Proof : Suppose $\gamma \in I$, then there exists $x_1 \in X_1$ and $x_2 \in X_2$ so that $\gamma x_1 = x_2$. Choose $h_1 \in H_1^0$ so that $h_1 \cdot 0 = x_1$ and $h_2 \in H_2^0$ so that $h_2 \cdot 0 = x_2$, then $\delta(\gamma) = h_2^{-1} \gamma h_1 \in K$ hence $\gamma \in H_2^0 K H_1^0$.

Conversely, given $\gamma \in H_2^0 K H_1^0$ we may choose h_2, h_1 so that $h_2^{-1} \gamma h_1 \in K$ hence $\gamma h_1 \cdot 0 = h_2 \cdot 0$ and $x_2 = h_2 \cdot 0$ we have $\gamma x_1 = x_2$. This proves lemma **2.4**.

We now establish a formula for the multiplicity of an intersection corresponding to $\gamma \in I$. Given $\gamma \in I$ we may associate $\delta(\gamma) \in K$ as above. In what follows we will write δ for $\delta(\gamma)$.

Proposition 2.3

Let $e_1, e_2, \ldots, e_k$ be a frame for $T(X_1, 0)$ and $e_{k+1}, e_{k+2}, \ldots, e_n$ be a frame for $T(X_2, 0)$. Consider the real number $\epsilon(\gamma)$ defined by

$$\epsilon(\gamma) = \langle \delta_*(e_1 \wedge e_2 \wedge \ldots \wedge e_k),\ e_1 \wedge e_2 \wedge \ldots \wedge e_k \rangle$$

where $\langle\ ,\ \rangle_x$ is the Riemannian inner product on k-vectors at x. $\epsilon(\gamma)$ satisfies then $\delta_*(e_1 \wedge e_2 \wedge \ldots \wedge e_k) \wedge e_{k+1} \wedge \ldots \wedge e_n = \epsilon(\gamma)\, e_1 \wedge e_2 \wedge \ldots \wedge e_n$. Then

(a) $\epsilon(\gamma) = 0$ if and only if the intersection is degenerate,

(b) $\epsilon(\gamma)$ is positive if and only if the intersection multiplicity is $+1$,

(c) $\epsilon(\gamma)$ is negative if and only if the intersection multiplicity is -1.

Notation : If $f: X \to X$ is a smooth map and $v_1 \wedge v_2 \wedge \ldots \wedge v_p$ is a p-vector tangent to x then $f_* (v_1 \wedge v_2 \wedge \ldots \wedge v_p) = f_* v_1 \wedge f_* v_2 \wedge \ldots \wedge f_* v_p$ is a p-vector tangent to $f(x)$.

Proof : $\omega = (h_1)_* (e_1 \wedge e_2 \wedge \ldots \wedge e_k)$ is the positive orientation of X_1 at x_1 and $\tau = (h_2)_* (e_{k+1} \wedge e_{k+2} \wedge \ldots \wedge e_n)$ is the positive orientation of X_2 at x_2. Thus the intersection will be positive, negative or degenerate according to whether the n vector $\gamma_* (\omega) \wedge \tau$ is a positive, negative or zero multiple of the volume element of X at x_2. But up to a positive scalar multiple this volume element is just $(h_2)_* (e_1 \wedge e_2 \wedge \ldots \wedge e_n)$; thus, we must evaluate the sign of the inner product

$$\langle \gamma_* (\omega) \wedge \tau, (h_2)_* (e_1 \wedge e_2 \wedge \ldots \wedge e_n \rangle x_2.$$

But h_2 is an isometry of the Riemannian metric on n-vectors; hence, the above inner product is the same as

$$\begin{aligned}
&\langle (h_2^{-1})_* (\gamma_* (\omega) \wedge \tau), e_1 \wedge e_2 \wedge \ldots \wedge e_n \rangle_0 \\
&= \langle (h_2^{-1})_* ((\gamma h_1)_* (e_1 \wedge \ldots \wedge e_k) \wedge (h_2)_* (e_{k+} \wedge \ldots \wedge e)), e_1 \wedge \ldots \wedge e_n \rangle_0 \\
&= \langle (h_2 \gamma h_1)_* (e_1 \wedge \ldots \wedge e_k) \wedge e_{k+} \wedge \ldots \wedge e_n, \ e_1 \wedge \ldots \wedge e_n \rangle_0 \\
&= \langle (\delta)_* (e_1 \wedge \ldots \wedge e_k) \wedge e_{k+} \wedge \ldots \wedge e_n, \ e_1 \wedge \ldots \wedge e_n \rangle_0 \\
&= \epsilon (\gamma).
\end{aligned}$$

This proves proposition 2.3.

We warn the reader that in the degenerate case $\epsilon (\gamma) = 0$ is not necessarily the intersection multiplicity.

Note that the value of $\epsilon (\gamma)$ does not depend on the choice of h_1 and h_2 because any two choices of h_1 differ by multiplication by an clement of $\hat{K} \cap \bar{G}_1$ (which does not change $\epsilon (\gamma)$ as $K \cap G_1$ preserves $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and their orientations) and the same is true for h_2. Now we prove a lemma which is critical for us.

Lemma 2.5

If $\gamma, \gamma' \in I$ are in the same $\bar{G}_2, \bar{G}_1$ double coset in G then $\epsilon (\gamma) = \epsilon (\gamma')$.

Proof : Suppose $\gamma' = h_2^* \gamma h_1^*$ and we choose h_2', h_1' so that $h_2'^{-1} \gamma' h_1' = \delta (\gamma') \in K$. Then putting $h_2 = h_2^{*-1} h_2'$ and $h_1 = h_1^* h_1'$ we have $h_2^{-1} \gamma h_1 = h_2'^{-1} \gamma h_1^* h_1 = \delta (\gamma') \in K$.

Case 1. The intersection corresponding to γ is nondegenerate. In this case there is a unique x_1 (up to action by Γ_1) and a unique x_2 (up to action by Γ_2) so that $\gamma x_1 = x_2$, for if there were x_1' and x_2' inequivalent under Γ_1 and x_2 and x_2' corresponding then the minimal geodesic joining x_1 to x_1' in X_1 would be carried onto the minimal geodesic joining x_2 and x_2' in X_2 and consequently the intersection would be degenerate. Now we have $h_2^{-1} \gamma h_1 \circ 0 = 0$ consequently the choice of $h_2 = h_2^{*-1} h_2'$ and $h_1 = h_1^* h_1'$ is a correct choice to calculate $\epsilon (\gamma)$ as in proposition 2.3. But for this choice we have $\delta (\gamma) = \delta (\gamma')$ hence $\epsilon (\gamma) = \epsilon (\gamma')$ in the non-degenerate case.

Case 2. The intersection corresponding to γ is degenerate. In this case there is an entire geodesic in X_1 which is carried into a geodesic in X_2. But there γ' will carry h_1^{*-1} of this geodesic into h_2^* of the image geodesic in X_2. Thus $\epsilon(\gamma)=0$ if and only if $\epsilon(\gamma')=0$. This proves the lemma.

Corollary

Suppose γ is in the trivial $\bar{G}_2$, $\bar{G}_1$ double coset; that is, $\gamma \in \bar{G}_2\bar{G}_1$; then $\epsilon(\gamma)=+1$.

We shall use the following notation in the rest of this paper. If G is an algebraic group over a number field E and F is a field containing E then $G(F)$ will denote the group of F-rational points of G. More generally, if R is any ring, then $G(R)$ or $(G)_R$ will denote the group of R rational points of G, assuming this makes sense. Also if O_E is the integers in E we will often denote $G(O_E)$ by Φ.

We assume now that we are given an orthogonal decomposition $\{\sigma_1, \sigma_2\}$ of G/K and a compatible uniform torsion-free arithmetic subgroup Γ of G. We assume $\Gamma \subseteq SL(N, O_E)$ where E is a finite algebraic extension of Q. For any ideal a in O_E we define the congruence subgroups.

$$\Gamma(a) = \{\gamma \in \Gamma : \gamma \equiv 1 \bmod a\},$$

$$\Gamma_1(a) = \{\gamma \in \Gamma_1 : \gamma \equiv 1 \bmod a\},$$

$$\Gamma_2(a) = \{\gamma \in \Gamma_2 : \gamma \equiv 1 \bmod a\}.$$

We define the manifolds

$$Y(a) = \Gamma(a)\backslash X,$$

$$Y_1(a) = \Gamma_1(a)\backslash X_1,$$

$$Y_2(a) = \Gamma_2(a)\backslash X_2.$$

We also define finite groups

$$\psi = \Gamma/\Gamma(a),$$

$$\psi_1 = \Gamma_1/\Gamma_1(a),$$

$$\psi_2 = \Gamma_2/\Gamma_2(a).$$

We now prove one of our main theorems.

Theorem 2.1

Assume that all intersections of Y_1 and Y_2 are of positive multiplicity. Then there exists a prime p in E so that the cohomology class corresponding to $Y_1(p)$ under Poincaré duality cannot be represented by an invariant form.

Proof: We begin with the observation due to W. Dwyer that if the cohomology class dual to $Y_1(p)$ could be represented by an invariant form then the homology class carried by $Y_1(p)$ is invariant under the deck transformations of the covering $\Pi : Y(p) \to Y$. This follows immediately from the naturality of the Poincaré duality map combined with the fact that the deck transformations preserve the orientation class of $Y(p)$.

The following lemmas then complete the proof of the theorem.

Lemma 2.6

There exists p a prime ideal in E and $\gamma \in \Gamma$ so that $\sigma_2\gamma\sigma_2\gamma^{-1}$ is not congruent to an element of Γ_1 modulo p.

Proof: Suppose first that we can choose $\gamma \in \Gamma$ so that $\mu = \sigma_2\gamma\sigma_2\gamma^{-1} \notin \Gamma_1$. Then we claim that there exists p so that $\sigma_2\gamma\sigma_2\gamma^{-1}$ is not congruent modulo p to any element of Γ_1. Indeed choose a rational representation ρ of G on a vector space V so that G_1 is the isotropy of the line through $v \in V$. Then $\rho(\mu)v$ is not on the line through v because $\mu \notin G_1$. We may then complete $\{v, \rho(\mu)v\}$ to a rational basis for V. Now let p be any prime so that p does not divide either the numerator or the denominator of this discriminant.

We now prove that there exists $\gamma \in \Gamma$ so that $\sigma_2\gamma\sigma_2\gamma^{-1} \notin \Gamma_1$. Suppose no such γ exists. We define a map $\phi : G_{\mathbf{C}} \to G_{\mathbf{C}}$ by $\phi(g) = \sigma_2 g\sigma_2 g^{-1}$. Since $\phi(\Gamma) \subseteqq \Gamma_1$ and Γ is Zariski dense in $G_{\mathbf{C}}$ we must have $\phi(G_{\mathbf{C}}) \subseteqq (G_1)_{\mathbf{C}}$ and hence $d\phi(g) \subseteqq g_1$ where for $x \in g$ we have $d\phi(x) = \sigma_2 x\sigma_2 - x$. Now we note that $\mathrm{Ad}\,\sigma_2$ induces the identity on k_1 since $\mathrm{Ad}\,\sigma_2$ is the Cartan involution of g_1. But if $v_1 \in \mathfrak{P}_1$, $v_2 \in \mathfrak{P}_2$ and $x = [v_1, v_2]$ then $d\phi(x) = -2x \notin g_1$. With this last statement we obtain a contradiction and the lemma is proved.

Lemma 2.7

Let γ be as in lemma 2.5 and let η be the deck transformation of the covering $\Pi : Y(p) \to Y$ represented by γ. Then $\eta Y_1(p)$ and $Y_1(p)$ are not homologous.

Proof: All intersections of $Y_1(p)$ and $Y_2(p)$ are positive since every intersection of $Y_1(p)$ and $Y_2(p)$ lies over an intersection of Y_1 and Y_2. Also $\eta Y_1(p)$ projects to Y_1, which does not bound, hence $\eta Y_1(p)$ does not bound. Thus to show that $Y_1(p)$ and $\eta Y_1(p)$ are homologically independent it is sufficient to show that the intersection number of $\eta Y_1(p)$ and $Y_2(p)$ is zero. We show in fact that $\sigma_2(\eta Y_1(p)) \cap \eta Y_1(p) = \phi$. Since any point in $\eta Y_1(p) \cap Y_2(p)$ is fixed by σ_2 this will show that this latter set is also empty.

Let $a : \psi \to \psi$ denote the involution of ψ induced by σ_2. Thus $\sigma(\eta) = \overline{\sigma_2\gamma\sigma_2}$ where $\overline{}$ denotes reduction modulo p. Let ν denote the class of η in ψ/ψ_1 and β the involution of ψ/ψ_1 induced by a. By the choice of γ we have $\beta(\nu) \neq \nu$. But we claim $\sigma_2(\eta Y_1(p)) = a(\eta) Y_1(p)$. Indeed let $x \in Y_1(p)$, then $\sigma_2\eta x = \sigma_2\eta\sigma_2\sigma_2 x = a(\eta)\sigma_2 x \in a(\eta) Y_1(p)$ since σ_2 stabilises $Y_1(p)$. But $a(\eta) \neq \eta$ and hence $\eta Y_1(p)$ and $a(\eta) Y_1(p)$ are disjoint and lemma 2.7 is proved. With this the proof of theorem 2.1 is completed.

3. The Galois cocycle associated to an intersection

We assume in this chapter that we are given an orthogonal decomposition $\{X_1, X_2\}$ of the symmetric space X and a compatible uniform torsion-free arithmetic subgroup $\Gamma \subset G$. We assume $\Gamma \subset GL(N, E)$ where E is a totally real number field and that the groups G, H_1 and H_2 of Chapter 1 are the real rational points of groups M, M_1, M_2 defined over E.

Let p be a prime ideal in E. We have defined the congruence subgroup $\Gamma(p)$ of Γ of level p. We have also defined a finite set, $I(\Gamma(p))$, in chapter 2. We will abbreviate $I(\Gamma(p))$ to $I(p)$ and, as before, we will abbreviate $I(\Gamma)$ to I. It is false that $I(p) \subseteqq I$. However, let us define δI, $\delta I(p)$, again following notation established in chapter 2 (above proposition 2.3) by:

$$\delta I = \{\delta\gamma_1, \ldots, \delta\gamma_r \mid \gamma_i \in I\},$$

$$\delta I(p) = \{\delta\gamma_1, \ldots, \delta\gamma_s \mid \gamma_i \in I(p)\}.$$

Lemma 3.1

$$\delta I(p) \subseteqq \delta I.$$

Proof: Let $\gamma \in I(p)$. Then there exists $\gamma_2 \in \Gamma_2$, but not necessarily in $\Gamma_2(p)$, and $\gamma_1 \in \Gamma_1$ so that $\gamma_2\gamma\gamma_1 \in I$. But by lemma 3.1 $\delta(\gamma_2\gamma\gamma_1) = \delta(\gamma)$ hence $\delta(\gamma) \in \delta I$.

Our strategy in the rest of the paper is to try to pass to a congruence subgroup of Γ, $\Gamma(p)$ so that $\delta I(p)$ is contained in the connected component of the normaliser of $\hat{K}$. We now begin our programme of trying to decrease δI.

Theorem 3.1

For all but finitely many primes p we have $\gamma \in I(p)$ implies $\gamma \in (H_2)_{\mathbf{C}}(H_1)_{\mathbf{C}}$—recall $(H_2)_{\mathbf{C}}$ is the complexification of H_2 and $(H_1)_{\mathbf{C}}$ is the complexification of H_1.

Proof: The theorem will follow immediately from the next two lemmas.

Lemma 3.2

The double cosets $(H_2)_{\mathbf{C}}\gamma_i(H_1)_{\mathbf{C}}$ $(\gamma_i \in I)$ are closed in $(G)_{\mathbf{C}}$.

Proof: By a theorem of Birkes [1] it is sufficient to show that the double cosets $H_2\gamma_iH_1$ are closed in G. Note that $H_2\gamma_iH_1$ is closed in G if and only if $H_2\gamma_iH_1\gamma_i^{-1}$ is. Now we will prove H_2H_1 is closed in G and the proof will be valid for $H_2\gamma_iH_1\gamma_i^{-1}$. Let v be an integral vector in V (a representation space for M) so that M_2 is precisely the isotropy of v. Consider the orbit of v in $V_{\mathbf{R}}$. It is enough to prove that the image of H_1 is closed in Gv. But $\Gamma_1 v$ is discrete, hence closed in V_R by Borel [3], 7.13 and $H_1 = \Omega\Gamma_1$ with Ω compact hence $H_1v = \Omega\Gamma_1v$ is closed in $V_{\mathbf{R}}$ proving lemma 3.3.

Lemma 3.3

There exists an $(M_2)_{\mathbf{C}}\times(M_1)_{\mathbf{C}}$ invariant element f of $E[G]$ (the ring of regular functions on M) so that

$$f(x) = 0 \qquad x \in (H_2)_{\mathbf{C}}(H_1)_{\mathbf{C}},$$

$$f(x) = 1 \qquad (Hx \in_2)_{\mathbf{C}}\, \gamma_j\, (H_1)_{\mathbf{C}} \quad \gamma_j \in I',$$

where I' is the set of elements $\gamma_j \in I$ so that

$$(H_2)_{\mathbf{C}}\, \gamma_j\, (H_1)_{\mathbf{C}} \neq (H_2)_{\mathbf{C}}\, (H_1)_{\mathbf{C}}.$$

Proof: It follows from Borel [3], 7.6, that such an f exists with $f \in \mathbf{C}[G]$. However, the regular functions in $E[G]$ invariant under $(M_2)_E \times (M_1)_E$ span the regular functions in $\mathbf{C}[G]$ invariant under $(H_2)_{\mathbf{C}} \times (H_1)_{\mathbf{C}}$. Thus for each $V_j\,(j \in I')$ we may find f_j so that

(i) $f_j \in E[G]$,

(ii) $f_j(x) = 0 \qquad x \in (H_2)_{\mathbf{C}}\gamma\, (H_1)_{\mathbf{C}} \neq (H_2)_{\mathbf{C}}\gamma_j\, (H_1)_{\mathbf{C}},\ \gamma \in I$

(iii) $f_j(x) = 1 \qquad x \in (H_2)_{\mathbf{C}}\, \gamma_j\, (H_1)_{\mathbf{C}}$.

Then $f = \sum_{I'} f_j$ satisfies the conclusions of lemma 3.4.

Theorem 3.1 now follows easily. If p is a prime which does not divide the denominators of f then we claim that if $\gamma \in I(p)$ then $\gamma \in (H)_{\mathbf{C}}\,(H_1)_{\mathbf{C}}$. Suppose that $\gamma \in I(p)$ and $\gamma \notin (H_2)_{\mathbf{C}}\,(H_1)_{\mathbf{C}}$. Since $\gamma \in \Gamma$ there exist $\gamma_2 \in \Gamma_2$ and $\gamma_1 \in \Gamma_1$ so that $\gamma_2\gamma\gamma_1 \in I$, hence $\gamma \in (H_2)_{\mathbf{C}}\,\gamma_j\,(H_1)_{\mathbf{C}}$ for some $j \in I'$. But then we have $f(\gamma) = 1$. However, $\gamma \equiv 1 \mod p$ so we must have $f(\gamma) \equiv 0 \mod p$ which contradicts $f(\gamma) = 1$. Thus theorem 3.1 is proved.

We will now rename $\Gamma(p)$ as Γ and $I(p)$ as I for a chosen so that the conclusions of theorem 3.1 are satisfied. Thus $\gamma \in I$ implies $\gamma \in (H_2)_{\mathbf{C}}\,(H_1)_{\mathbf{C}}$.

If $\gamma \in I$ and $\gamma \in H_2^0 H_1^0$ then the intersection number corresponding to γ has multiplicity $+1$ according to lemma 3.1. Thus we must study the problem. Given $\gamma \in \Gamma$, satisfying $\gamma \in (H_2)_{\mathbf{C}}\,(H_1)_{\mathbf{C}}$ can we obtain $\gamma \in H_2H_1$. This problem is a problem in Galois cohomology which we now treat.

Theorem 3.1 allows us to associate a Galois cohomology class $a(\gamma) \in H^1(E, M_1 \cap M_2)$ to every intersection $\gamma \in I$. Indeed, by theorem 3.1 we may write $\gamma = g_2 g_1$ with $g_1 \in M_1(\mathbf{C})$ and $g_2 \in M_2(\mathbf{C})$. Now if $\sigma \in \mathrm{Gal}(\mathbf{C}, E)$ then $\sigma(\gamma) = \gamma$ and consequently

$$g_1 \sigma(g_1)^{-1} = g_2^{-1} \sigma(g_2).$$

We define the value of the Galois cocycle $a(\gamma)$ at σ, to be denoted a_σ, to be this common value. We then have the following standard lemma.

Lemma 3.4 A

We may write $\gamma = \mu_2\mu_1$ with $\mu_1 \in M_1(E)$ and $\mu_2 \in M_2(E)$ if and only if the class of the cocycle $a(\gamma)$ in $H^1(E, M_1 \cap M_2)$ is trivial.

Since we have identified E with a subfield of $\mathbf{R}$ we have a map

$$H^1(E, M_1 \cap M_2) \to H^1(\mathbf{R}, M_1 \cap M_2).$$

We let $\alpha \in H^1(\mathbf{R}, M_1 \cap M_2)$ be the image of a. Then we have the following standard lemma.

Lemma 3.4 B

We may write $\gamma = h_2 h_1$ with $h_1 \in M_1(\mathbf{R})$ and $h_2 \in M_2(\mathbf{R})$ if and only if the class of a in $H^1(\mathbf{R}, M_1 \cap M_2)$ is trivial.

Remark : In this case the intersection corresponding to γ is isolated. We note that a is in the kernel of the mapping

$$H^1(\mathbf{R}, M_1 \cap M_2) \to H^1(\mathbf{R}, M_1) \times H^1(\mathbf{R}, M_2),$$

induced by the inclusions.

Proposition 3.1

Let $\mathfrak{P}$ denote the set of finite primes of E and for $p \in \mathfrak{P}$ let Ep denote the completion of E at p. Then we may assume a is in the kernel of the map

$$H^1(E, M_1 \cap M_2) \to \prod_{p \in \mathfrak{P}} H^1(Ep, M_1 \cap M_2).$$

Moreover, if all elements of Γ satisfy the congruence $\gamma \equiv 1 \bmod \mathfrak{a}$ for some ideal $\mathfrak{a}$ then for every finite prime p we may write $\gamma = \mu_2\mu_1$ with μ_2 and μ_1 in the set of p-adic integral points of M_2 and M_1 respectively and satisfying $\mu_2 \equiv \mu_1 \equiv 1 \bmod \mathfrak{a}$.

Proof : Since the second part of the statement of the theorem implies the first we prove this part. Note first that if we may solve $\gamma \equiv \mu_2^{(n)} \mu_1^{(n)} \bmod p^n$ with $\mu_1^{(n)}$ and $\mu_2^{(n)}$ p-adic integral matrices congruent to 1 modulo $\mathfrak{a}^1$ (depending on n) for every n, then, since the set of p-adic integral matrices is compact we may find (by taking a limit) p-adic integral matrices μ_1, μ_2 congruent to 1 modulo $\mathfrak{a}$ so that $\gamma = \mu_2\mu_1$. Hence if the second statement does not hold for some finite prime p then there exists n so that $\gamma \not\equiv \mu_2\mu_1 \bmod p^n$ for any p-adic integral matrices μ_2, μ_1 congruent to 1 modulo $\mathfrak{a}$. Now consider the congruence subgroup $\Gamma(p^n)$ of Γ. We claim that there is no $\gamma' \in I(p^n)$ which lies over γ. Indeed suppose such a γ' existed. Then $\gamma' = \gamma_2\gamma\gamma_1$ with $\gamma_2 \in \Gamma_2$, $\gamma_1 \in \Gamma_1$ and $\gamma_2 \equiv 1 \bmod \mathfrak{a}$, $\gamma_1 \equiv 1 \bmod \mathfrak{a}$ and $\gamma' \equiv 1 \bmod p^n$. Consequently $\gamma \equiv \gamma_2^{-1}\gamma_1^{-1} \bmod p^n$ but taking $\mu_2 = \gamma_2^{-1}$ and $\mu_1 = \gamma_1^{-1}$ we obtain a contradiction. In this way we eliminate all intersections not satisfying the conclusion of proposition 3.1.

Theorem 3.2

Suppose the following two hypotheses are satisfied:

(1) $H^1(\mathbf{R}, M_1 \cap M_2) \to H^1(\mathbf{R}, H_1) \times H^1(\mathbf{R}, H_2)$,

is injective—under this assumption we obtain that a is trivial.

(2) The Hasse principle holds for $M_1 \cap M_2$; that is, the map

$$H^1(E, M_1 \cap M_2) \to \prod_{V} H^1(E_V, M_1 \cap M_2)$$

induced by the various completions is injective (the inverse image of the base-point consists of a single point) then we may write each γ in I as a product $\gamma = m_2 m_1$ with $m_1 \in M_1(E)$ and $m_2 \in M_2(E)$.

Proof : Since a is trivial we may write $\gamma = h_2h_1$ with $h_2 \in H_2$ and $h_1 \in H_1$. But if $M_1^{\tau_j}$ is a conjugate of M_1 then $M_1^{\tau_j}(\mathbf{R})$ is compact and consequently the mapping $H^1(\mathbf{R}, M_1^{\tau_j} \cap M_2^{\tau_j}) \to H^1(\mathbf{R}, M_1^{\tau_j})$ is injective, see Serre [12], page III–36, theorem 6. Consequently a is trivial at all the infinite completions of E. But by proposition 3.1 we know a is trivial at all the finite completions of E; hence, by the Hasse principle a is trivial and we may write $\gamma = m_2 m_1$ with $m_2 \in M_2(E)$ and $m_1 \in M_1(E)$.

Remark : In the special case that both H_1 and H_2 are connected then (1) is sufficient to guarantee that all intersections of Y_1 and Y_2 are isolated of multiplicity $+1$.

4. Applications

We will apply our results to three cases. In all three cases we begin with a real quadratic field $E = \mathbf{Q}(\sqrt{m})$ where m is a rational prime.

In the first case we let D be a division quaternion algebra over E so that $D \otimes_E \mathbf{R}$ is the Hamilton quaternions H. We define a hermitian form H on D^n by the formula

$$H(q_1, q_2, \ldots, q_n) = q_1\bar{q}_1 + \ldots + q_1\bar{q}_p - \sqrt{m}\,(q_{p+1}\,\bar{q}_{p+1} + \ldots + q_n\bar{q}_n),$$

where $\bar{q}$ denotes the conjugate of the quaternion q. We assume $p \geq q$ and choose an integer k satisfying $0 < k < p$. We let M be the special unitary group of H; we let M_1 be the subgroup of M fixing the first k coordinates and we let M_2 be subgroup of M fixing the next $p - k$ coordinates. Then M, M_1 and M_2 are algebraic groups defined over E and

$$M(\mathbf{R}) = Sp(p, q),$$

$$M_1(\mathbf{R}) = Sp(p - k, q),$$

$$M_2(\mathbf{R}) = Sp(k, q).$$

Since H is anisotropic with a positive definite conjugate it follows by a standard argument, see for example Millson [10], page 239, that $M(O_E)$ is a uniform discrete subgroup of $\mathrm{Sp}(p, q)$.

In the second case we let F be a quadratic extension of E so that $F \otimes_E \mathbf{R} = \mathbf{C}$. We define a hermitian form H on F^n by the formula

$$H(z_1, z_2, \ldots, z_n) = z_1\bar{z}_1 + \ldots + z_p\bar{z}_p - \sqrt{m}\,(z_{p+1}\,\bar{z}_{p+1} + \ldots + z_n\bar{z}_n),$$

where $\bar{z}$ denotes the conjugate of z by the non-trivial element of the Galois group of F over E. The definitions of M, M_1 and M_2 are now as in the previous case. We now have

$$M(\mathbf{R}) = SU(p, q),$$

$$M_1(\mathbf{R}) = SU(p - k, q),$$

$$M_2(\mathbf{R}) = SU(k, q).$$

Again we find that $M(O_E)$ is a uniform discrete subgroup of $SU(p, q)$.

In the third case we let M be the orthogonal group of the quadratic form f over E defined by

$$f(x_1, x_2, \ldots, x_n) = x_1^2 + \ldots + x_p^2 - \sqrt{m}\,(x_{p+1}^2 + \ldots + x_n^2).$$

The groups M_1' and M_2' are defined to be the subgroups of M_1 and M_2 leaving fixed the first k coordinates and next $p-k$ coordinates respectively. In this case we have

$$M(\mathbf{R}) = SO(p, q),$$
$$M_1(\mathbf{R}) = SO(p-k, q),$$
$$M_2(\mathbf{R}) = SO(k, q).$$

We find that $M(O_E)$ is a uniform discrete subgroup of $SO(p, q)$.

The first two cases can be settled by the remark following theorem 3.3. The mapping $H_1 \cap H_2 \to H_1$ is the inclusion of one unitary group into a larger one, consequently by Witt's theorem the mapping $H^1(\mathbf{R}, M_1 \cap M_2) \to H^1(\mathbf{R}, M_1)$ is injective. Since H_1 and H_2 are connected we obtain the following applications (note that our construction may be repeated interchanging the roles of p and q).

Theorem 4.1

There exists a uniform discrete arithmetic subgroup $\Gamma \subset Sp(p, q)$ so that $H^k(\Gamma, \mathbf{R})$ contains a cohomology class which is not the restriction of a continuous class from $Sp(p, q)$ for any integer k strictly between 0 and $4pq$ and divisible by either $4p$ or $4q$.

Theorem 4.1 (*bis*)

There exists a uniform discrete arithmetic subgroup $\Gamma \subset SU(p, q)$ so that $H^k(\Gamma, \mathbf{R})$ contains a cohomology class which is not the restriction of a continuous class of $SU(p, q)$ for any integer k strictly between 0 and $2pq$ and divisible by either $2p$ or $2q$.

Remark : In the case $q = 1$ the symmetric space we are looking at is the unit ball in $\mathbf{C}^n$, the cycles are quotients of sub-balls dual to linear subspaces of $\mathbf{C}P^n$. This justifies the statement (3) of the introduction.

The remaining case of $M(\mathbf{R}) = SO(p, q)$ is considerably more difficult. The difficulty is that we cannot find a simple algebraic subgroup N acting transitively on X_1 so that $N(\mathbf{R})$ is connected—note that the group $SO(p, q)$ has two components. Our first problem is to ensure that the manifolds Y, Y_1 and Y_2 are orientable. To do this we must ensure that Γ_1 and Γ_2 are contained in the connected components of the identity of G_1 and G_2 respectively. We will show in fact that there exists an ideal $\mathfrak{a}$ in E so that $\gamma \equiv 1 \bmod \mathfrak{a}$ implies that $\theta(\gamma) = 1$ where θ is the spinorial norm which we now define as it will play a major role in what follows.

Definition : Given a quadratic form f on a finite dimensional vector space over a field K we define a homomorphism $\theta : O(f) \to [K^*/(K^*)^2]$ as follows. Given $u \in O(f)$ write u as a product of reflections $u = r_{x_1} r_{x_2} \ldots r_{x_n}$ where r_{x_i} is reflection in the vector x_i. This can always be done—see O'Meara [9], 43.3. Then

$$\theta(u) = f(x_1) f(x_2) \ldots f(x_n) \bmod (K^*)^2.$$

We define $O'(f)$ to be the subgroup of $SO(f)$ consisting of elements of spinorial norm 1. We remark that if $SO(p, q)$ is the group of orientation preserving isometries of a quadratic form over $\mathbf{R}$ of signature (p, q) then the connected component of the identity of $SO(p, q)$ is just $O'(p, q)$. We note $u \in O'(p, q)$ if and only if we may write $u = r_{x_1} r_{x_2} \ldots r_{x_n}$ with $f(x_1) f(x_2) \ldots f(x_n) > 0$. If $q = 1$ we see that $u \in O'(p, 1)$ if and only if u maps the half-space $x_q > 0$ into itself.

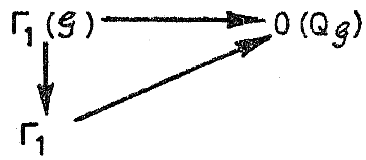
We will use the following notation. If γ is a p-adic integral matrix and b is an ideal in O then $\gamma \equiv 1 \bmod b$ means $\gamma \equiv 1 \bmod p^{v(b)}$ where v is the p-adic valuation.

Proposition 4.1

Let Q be a quadratic form over a number field E. Let $S = \{p_1, p_2, \ldots, p_m\}$ be a finite set of primes of E and let $G(O_S)$ be the group of S integral matrices over E that preserve Q. Then there exist infinitely many relatively prime ideals O in O_S with the property that every element of $G(O_S)$ congruent to 1 modulo a has spinorial norm 1 in E.

Proof: We denote $G(O_S)$ by Γ.

We first consider a prime p in E so that p is not a divisor of 2 and so that p does not divide the discriminant of Q. In this case if E denotes the completion of E at p we obtain a form Qp defined over Ep with orthogonal group $O(Qp)$. We have inclusions



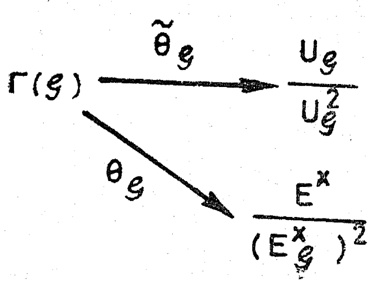
We denote the p-adic spinorial norm by θp. The following lemma was kindly provided for us by T. Tamagawa.

Lemma 5.3

If $\gamma \in \Gamma_1(p)$ then in $O(Qp)$ we may write $\gamma = r_{x_1} o r_{x_2} o \ldots o r_{x_m}$ with $Q(x_j)$ a p-adic unit, $1 \leqq j \leqq m$.

Proof: The proof is by induction. We establish the case $n = 2$ and the induction step with the same argument. Let $\gamma \in \Gamma_1(p)$ and suppose $x \in Ep^n$ with $Q(x)$ a p-adic unit. Let $y = \gamma x$. Let $\rho = r_{x+y} o r_x$, then $\rho x = y$ and hence $\rho^{-1} \gamma(x) = x$. But $Q(x + y) \equiv 4Q(x) \bmod p$ and accordingly is a unit.

Let Up denote the p-adic units. We have established that there exists a homomorphism $\tilde{\theta} p$ satisfying



But since $\gamma \equiv 1 \bmod p$ we have $\theta(\gamma)$ is a square modulo p hence a square in Up by Hensel's lemma. We obtain then

Lemma 5.4

The restriction of p-adic spinorial norm to $\Gamma_1(p)$ is trivial.

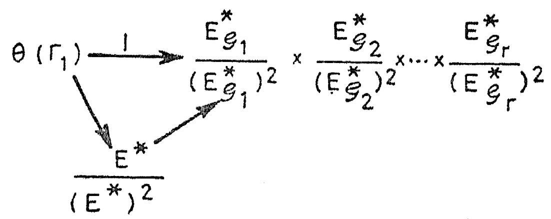
We are now ready to finish the proof of proposition 4.1. We know

$$\theta(\Gamma_1) \subseteq \frac{E^*}{(E^*)^2}$$

is a finitely-generated abelian group of exponent 2. Let $u_1, u_2, \ldots, u_r$ be a set of generators. Then $u \in \theta(\Gamma_1)$ has the form

$$u = u_1^{e_1} u_2^{e_2} \ldots u_r^{e_r}$$

with e_j either 0 or 1 for $1 \leqq j \leqq r$. Since u_j is not a square in E there exist infinitely many p-adic fields Ep so that u_j is not a square in Ep. Let Ep_j be such and satisfying further that u_j is not a square mod Ep_k for $k \neq j$, one of these for each j between 1 and r. Then by construction we have an injection i

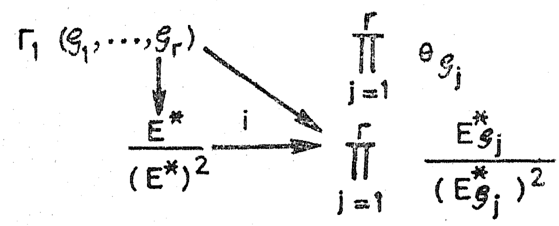


Indeed $u = u_1^{e_1} \ldots u_r^{e_r} \equiv x^2 \bmod p_1 \ldots p_j \Leftrightarrow u_j^{e_j} \equiv x^2 \bmod p_j$ for all j.

Lemma 5.5

$$\theta(\Gamma_1(p_1, p_2, \ldots, p_r)) = \frac{(E^*)^2}{(E^*)^2}.$$

Proof: Suppose that $u \in \theta(\Gamma_1(p_1, p_2, \ldots, p_r))$. To show u is 1 it is sufficient to show $i(u)$ is 1. But we have a commutative diagram



and by lemma 5.2 each $\theta p_j \mid \Gamma(p_1, \ldots, p_r)$ is trivial.

Corollary 1

There exist infinitely many relatively prime ideals $\mathfrak{a}$ in O so that $\gamma \equiv 1 \bmod \mathfrak{a}$ implies $\theta(\gamma) = 1$.

We will choose an ideal $\mathfrak{a}$ satisfying the conditions: $\gamma \equiv 1 \bmod \mathfrak{a}$ implies γ is not of finite order, $\theta(\gamma) = 1$ and so that theorem 3.1 is satisfied. $\mathfrak{a}$ will henceforth denote such an ideal and $\Gamma = \{\gamma \in \Phi : \gamma \equiv 1 \bmod \mathfrak{a}\}$.

Corollary **2**

Y, Y_1, and Y_2 are orientable.

We are now ready to apply our general theory. $M_1 \cap M_2$ is an orthogonal group and consequently satisfies the Hasse principle. The mapping $H_1 \cap H_2 \to H_1$ is the inclusion of an orthogonal group into a bigger orthogonal group, consequently, by Witt's theorem the mapping $H^1(\mathbf{R}, M_1 \cap M_2) \to H^1(\mathbf{R}, M_1)$ is injective. We apply theorem 3.2 and find that by passing to a deeper congruence subgroup which we will continue to denote by Γ we obtain, if $\gamma \in I(\Gamma)$ then $\gamma = m_2 m_1$ with $m_1 \in M_1(E)$ and $m_2 \in M_2(E)$.

We now prove the first claim of the introduction. For this we use the groups discussed under case 3 at the beginning of this chapter with $q = 1$ and k any integer strictly between 0 and p. We will verify that $Y_1 \cap Y_2$ is a single point. Indeed let γ, m_2, m_1 be as above. We will show that the equation $\gamma = m_2 m_1$ determines all the entries of m_2. Of course, $m_2 e_j = e_j$ for $k < j < p+1$. Moreover

$$\gamma e_j = m_2 m_1 e_j = m_2 e_j \text{ for } 1 \leqq j \leqq k.$$

Thus it is immediate that m_2 is determined by γ on all basis vectors with the possible exception of the last vector e_{p+1}. We suppose that $m_2 e_{p+1} = u$. Then u is in the orthogonal complement of the vectors $\gamma e_1, \gamma e_2, \ldots, \gamma e_k$ in the space spanned by $e_1, e_2, \ldots, e_k, e_{p+1}$. Consequently u is determined up to a scalar multiple which is either ± 1 because

$$f(u) = f(m_2 e_{p+1}) = f(e_{p+1}) = -\sqrt{m}.$$

In other words u is determined up to multiplication by ± 1 by the above k linear equations and the quadratic equation $f(u) = -\sqrt{m}$. We denote this set of $k+1$ equations by *. Now given any prime p we have the decomposition $\gamma = \nu_p \mu_p$ of proposition 3.1. But an argument identical to the above shows that $\nu_p e_{p+1}$ is determined up to sign by the same system *. Consequently $u = \pm \nu_G e_{p+1}$ and consequently u is integral. We have found then that m_2 is in fact an integral matrix in $M_2(O)$. We reduce m_2 modulo $\mathfrak{a}$ and find that since $\gamma e_j \equiv e_j \bmod \mathfrak{a}$ $1 \leqq j \leqq p$ we must have $m_2 e_{p+1} \equiv \pm e_{p+1} \bmod \mathfrak{a}$ but the determinant of m_2 is 1; hence, $m_2 \equiv 1 \bmod \mathfrak{a}$. This result combined with results in Borel-Wallach [5] proves the first claim of the introduction.

We now prove the second claim of the introduction. We use the groups M_1, M_2, M discussed under case 3 at the beginning of this chapter with q arbitrary and $k = 1$. From our general theory, given $\gamma \in I$ we may write $\gamma = m_2 m_1$ with $m_2 \in M_2(E)$ and $m_1 \in M_1(E)$. Since $\theta(\gamma) = 1$ we have $\theta(m_2) = \theta(m_1)$. Thus either $\theta(m_2)$ and $\theta(m_1)$ are totally positive or we may rewrite $\gamma = m_2' \tau_2 \tau_1 m_1'$ with $\theta(m_2')$ and $\theta(m_1')$ totally positive, τ_2 the diagonal matrix with first entry -1, $p+q$th entry -1 and the rest $+1$ and τ_1 the diagonal matrix with second diagonal entry -1, $p+q$th entry -1 and the rest $+1$, We now examine under what conditions γ corresponds to a positive intersection. In case $\theta(m_1)$ and $\theta(m_2)$ are both positive then $\gamma \in H_2^0 H_1^0$ and of course $\epsilon(\gamma) = 1$. To compute $\epsilon(\gamma)$ in the second case we have only to compute determinant of the action of $\tau_2\tau_1$ on Φ_1. But it is easily seen that $\tau_2\tau_1 = -1$ Φ_1 and consequently has determinant $(-1)^q$. This completes the proof of second claim in the introduction.

n connection with the third claim of the introduction we note that for $(n, 2)$, for any choice of k, the cycles Y_1 and Y_2 are projective varieties. Since using Witt's theorem) we have shown that all intersections are isolated, we ain algebraic cycles in all even dimensions which correspond to non-trivial omorphic representations.

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