THE WORD AND RIEMANNIAN METRICS ON LATTICES OF SEMISIMPLE GROUPS

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ABSTRACT

Let G be a semisimple Lie group of rank $\geqslant 2$ and Γ an irreducible lattice. Γ has two natural metrics: a metric inherited from a Riemannian metric on the ambient Lie group and a word metric defined with respect to some finite set of generators. Confirming a conjecture of D. Kazhdan (cf. Gromov [Gr2]) we show that these metrics are Lipschitz equivalent. It is shown that a cyclic subgroup of Γ is virtually unipotent if and only if it has exponential growth with respect to the generators of Γ.

1. Introduction

Let G be a semi-simple group. By this we mean that $G = \Pi_{i=1}^{l} \mathbf{G}_i(k_i)$ where for $i = 1, \ldots, l$, k_i is a locally compact non discrete field and $\mathbf{G}_i$ is a connected (almost) simple k_i-group. Denote $\operatorname{rank} G = \sum_{i=1}^{l} \operatorname{rank}_{k_i} \mathbf{G}_i$. Each factor $G_i = \mathbf{G}_i(k_i)$ has a left invariant metric d_i obtained in the following way: If k_i is archimedean then there is a G_i-invariant Riemannian metric defined on the symmetric space G_i/K_i, where K_i is a maximal compact subgroup of G_i and we can lift it to obtain a left invariant Riemannian metric on G_i. Similarly if k_i is non-archimedean the natural (combinatorial) metric on the vertices of the Bruhat-Tits building associated with G_i, can be lifted to a left invariant metric d_i on G_i. We denote $d_R((g_i), (h_i)) = \sum_{i=1}^{l} d_i(g_i, h_i)$, d_R is a left invariant metric on G. In this procedure d_i and d_R are not unique but d_R is determined up to Lipschitz equivalence (coarse). We will refer to d_R as a Riemannian metric of G and sometimes by abuse of language as the Riemannian metric of G. The metric d_R is Lipschitz equivalent to $\sum_{i=1}^{l} \log(1 + \|g - I\|_i)$ where each $\|\cdot\|_i$ is the norm with respect to a fixed embedding of $G_i = \mathbf{G}_i(k_i)$ in $GL_{n_i}(k_i)$ for some n_i. See (3.5) below.

Let Γ be an irreducible lattice of G, i.e., Γ is a discrete subgroup and $\Gamma\backslash G$ carries a finite G-invariant measure. Γ is called a uniform lattice if $\Gamma\backslash G$ is compact. Assume Γ is finitely generated. (This is always the case unless $\operatorname{rank} G = 1$, Γ is non-uniform and $\operatorname{char}(k_{i_0}) > 0$ for the unique $1 \leqslant i_0 \leqslant l$ for which $\mathbf{G}_i(k_i)$ is not compact – cf. [Ma], [Ve], [Ra2], [Lu] and the reference therein). Fixing a finite set Σ of generators of Γ determines a metric d_W on Γ – a word metric. This is the metric induced on Γ from the Cayley graph $X(\Gamma; \Sigma)$ of Γ with respect to Σ, i.e., for $\gamma, \gamma' \in \Gamma$, $d_W(\gamma, \gamma') = n$ if n is the minimal integer so that $\gamma^{-1}\gamma'$ can be written as a word of length n in $\Sigma \cup \Sigma^{-1}$. Again, a different choice of generators leads to a different word metric but any two such metrics are Lipschitz equivalent. By abuse of notation we will refer to d_W as the word metric of Γ.

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It is not difficult to see (see e.g. 3.2 below) and very well known that if Γ is uniform in G, then d_R restricted to Γ is Lipschitz equivalent to d_W. This is not in general the case if Γ is non-uniform. For example for $G = SL_2(\mathbf{R})$, $\Gamma = SL_2(\mathbf{Z})$ and $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, one may check that $d_W(\gamma^n, 1)$ grows linearly while $d_R(\gamma^n, 1) = O(\log n)$.

Our main result confirms a conjecture of Kazhdan (stated by Gromov in [Gr2]) asserting that for higher rank groups the situation is different.

Theorem **A**. — *Let* G *be a semi-simple group and* Γ *an irreducible lattice. Then* d_R *restricted to* Γ *is Lipschitz equivalent to* d_W *provided* $\operatorname{rank} G \geqslant 2$.

By Margulis' arithmeticity theorem ([Ma, Chap. IX (1.11), p. 298]), Γ is an S-arithmetic group in G and G is locally isomorphic to $\prod_{v \in S} \mathbf{G}(k_v)$ where $\mathbf{G}$ is a connected almost simple group defined over a global field k and S is a finite set of places of k containing all the archimedean ones. Our proof makes an essential use of the arithmeticity of Γ. It will be interesting to find a purely geometric proof of Theorem A. We learnt recently that Margulis found a different proof of Theorem A which is more geometric – but still uses the arithmeticity of Γ.

In [Gr2], Gromov proved the special case of Theorem A, when $G = \mathbf{G}(\mathbf{R})$, $\Gamma = \mathbf{G}(\mathbf{Z})$, $\mathbf{G}$ is a $\mathbf{Q}$-group of $\mathbf{Q}$-rank one and $\mathbf{R}$-rank $\geqslant 2$. Gromov studied these type of problems in the broader context of distortion of metric spaces. In this terminology Theorem A says that (Γ, d_W) is undistorted in (G, d_R).

Let $u \in \Gamma$ be a unipotent element of infinite order. The entries of u^n (embedded in a product of metric groups) are polynomials in n and hence $d_R(u^n, 1) = O(\log n)$. By Theorem A, we also have $d_W(u^n, 1) = O(\log n)$, namely, u^n can be written as a word of length $O(\log n)$ using the generators of Γ. This in particular implies that the cyclic group $\langle u \rangle$ has exponential growth with respect to the generators of Γ. An element of Γ with this last property will be called a U-element of Γ.

Theorem **B**. — *Let* $G = \prod_{i=1}^{l} \mathbf{G}_i(k_i)$ *be a semi-simple group,* Γ *an irreducible lattice in* G *and* $\gamma \in \Gamma$. *Then* γ *is a U-element of* Γ *if and only if the following four conditions are satisfied:*

(a) *For every* $i = 1, \ldots, l$, $\operatorname{char}(k_i) = 0$

(b) *For every* $i = 1, \ldots, l$, $\operatorname{rank}_{k_i}(\mathbf{G}_i) \geqslant 1$

(c) $\operatorname{rank} G = \sum_{i=1}^{l} \operatorname{rank}_{k_i} \mathbf{G}_i \geqslant 2$

(d) γ *is virtually unipotent (i.e.,* γ^m *is unipotent for some* $m > 0$*) of infinite order.*

One direction of Theorem B, i.e., that lattices in rank one groups do not contain U-elements is due to Gromov ([Gr2], see 2.18 below). The other direction is essentially a corollary of Theorem A, but our method of proof is different: We first prove a stronger version of Theorem B and use it to prove Theorem A.

The paper is organized as follows: Section 2 is devoted to the definition and examples of U-elements in various groups. We show in a constructive way how some unipotent elements are U-elements and we also begin the proof of Theorem B. At the end of Section 2 we reproduce Gromov's proof that the four conditions are necessary. In Section 3 we complete the proof of Theorem B. While section 4 contains the proof of Theorem A.

The results of this paper (for characteristic zero) were announced in [LMR] where a complete proof was given for the special case $G = SL_n(\mathbf{R})$ and $\Gamma = SL_n(\mathbf{Z})$. The reader is encouraged to consult [LMR] first, as it avoids some of the technicalities which appear especially in Section 4 of the current paper.

We would like to thank G. A. Margulis and H. Abels for pointing to us that establishing Theorems A and B in the cases of characteristic 2 or 3 requires a more careful argument than the one we gave in an earlier version of the paper.

2. U-elements

(2.1). — *Let $\Gamma = \langle \Sigma \rangle$ be a finitely generated group generated by a finite set Σ. For $\gamma \in \Gamma$ denote by $l_\Sigma(\gamma)$ the length of γ as a word in $\Sigma \cup \Sigma^{-1}$. It is equal to the distance from γ to 1 in the Cayley graph $X(\Gamma; \Sigma)$ of Γ with respect to Σ.*

Assume henceforth that $\gamma \in \Gamma$ is an *element of infinite order*. Consider the following three properties of γ in Γ:

(U1) $l_\Sigma(\gamma^n) = O(\log n)$.

(U2) $|\langle\gamma\rangle \cap B_\Sigma(n)|$ grows exponentially with n, i.e., there exists $c > 1$ such that for all large enough n, the ball of radius n around the identity in $X(\Gamma; \Sigma)$ contains at least c^n elements from the cyclic group generated by γ.

(U3) $\liminf \dfrac{\log(l_\Sigma(\gamma^n))}{\log n} = 0$.

It is easy to see that for $j = 1, 2, 3$, Property Uj depends only on Γ and γ but not on Σ. We say that $\gamma \in \Gamma$ is a **U**j*-element of* Γ if it has property Uj. It is said to be a **U**-*element of* Γ if it has at least one of these properties.

We collect here, without proofs, some easy observations on these properties:

(2.2) *Proposition.* — *For $j = 1, 2, 3$.*

(i) *For every $0 \neq r \in \mathbf{Z}$, γ is a Uj-element of Γ if and only if γ^r is.*

(ii) *Let Δ be a finitely generated subgroup of Γ. If $\gamma \in \Delta$ is a Uj-element of Δ then it is a Uj-element of Γ. If $(\Gamma : \Delta) < \infty$, the converse is also true.*

(iii) *Let $r : \Gamma \to \Delta$ be a homomorphism from Γ to a finitely generated group Δ. If $\gamma \in \Gamma$ is a Uj-element of Γ then $r(\gamma)$ is a Uj-element of Δ provided it has infinite order.*

(2.3) *Proposition.* — *(U1)* $\Rightarrow$ *(U2)* $\Rightarrow$ *(U3)*

It seems plausible that the three properties are not equivalent for general finitely generated groups, but we actually do not know any example. It is more likely that the three properties are equivalent in linear groups. It follows from Theorem 2.15 below that this is indeed the case for arithmetic groups.

(2.4). — *If* F *is a field, an element* $g \in \mathrm{GL}_n(\mathrm{F})$ *is called* virtually unipotent *if some power of it is unipotent, i.e., if all its eigenvalues are roots of unity.*

Proposition. — *If* γ *is a* U*-element of* Γ *then for every field* F *and every representation* $\rho : \Gamma \to \mathrm{GL}_n(\mathrm{F})$, $\rho(\gamma)$ *is virtually unipotent.*

Proof. — If $\rho(\gamma)$ is of finite order there is nothing to prove, if not we can, using (2.3 iii), replace Γ by $\rho(\Gamma)$ to assume that Γ is a subgroup of $\mathrm{GL}_n(\mathrm{F})$. As Γ is finitely generated we can assume F to be finitely generated. If λ is an eigenvalue of γ of infinite order then it belongs to some finitely generated field k containing F. By [Til, Lemma 4.1] we can embed k in a locally compact field k' endowed with an absolute value ω so that $\omega(\lambda) \neq 1$.

By replacing γ by γ^{-1} if necessary, we can assume $\omega(\lambda) > 1$. For $\delta \in \Gamma \subset \mathrm{GL}_n(\mathrm{F})$ let $\|\delta\| = \max_{0 \neq v \in k'^n} |\delta v|/|v|$ where for $v = (x_1, \ldots, x_r) \in k'^n$, $|v| = \max_{1 \leqslant i \leqslant n} \omega(x_i)$. It follows that $\|\gamma^n\| \geqslant \omega(\lambda)^n$. Let $a = \max_{\delta \in \Sigma} \{\|\delta\|, \|\delta^{-1}\|\}$. For $\delta \in \Gamma$ we have $\|\delta\| \leqslant a^{l_\Sigma(\delta)}$. Hence we conclude that $l_\Sigma(\gamma^n) \geqslant n \log \omega(\lambda)$. Thus γ does not have property (U3) and by (2.3) it is not a U-element.

(2.5) *Corollary.* — *A finitely generated subgroup of* $\mathrm{GL}(k)$ *where* k *is a field of positive characteristic does not contain any* U*-element. In particular a finitely generated group having a* U*-element cannot be embedded in a linear group over a field of positive characteristic.*

Since uniform lattices in semisimple Lie groups do not contain unipotent elements we have:

(2.6) *Corollary.* — *If* Γ *is a uniform lattice in a semisimple Lie group then* Γ *does not contain a* U*-element.*

Examples. — Let $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We check whether γ is a U-element in various different groups:

(2.7). — *For* $\Gamma = SL_2(\mathbf{Z})$, γ *is not a* U*-element. Indeed, take* $\delta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *then* δ^2 *and* $\gamma^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ *generate a free finite index subgroup* Δ *of* Γ. *Thus for* $\Sigma = \{\delta^2, \gamma^2\}$ *as a set of generators of* Δ, $l_\Sigma(\gamma^{2n}) = n$. *So* γ^2 *is not a U3-element of* Δ *and hence* γ *is not a U3-element of* Γ.

Another way to see this is the following simple lemma:

(2.8) *Lemma.* — *Let* Γ *be a finitely generated group acting isometrically on a metric space* X *with a metric d. Assume there exists* $x_0 \in X$ *and* $c > 0$ *such that* $d(\gamma^n x_0, x_0) \geqslant n \cdot c$ *for every* $n \in \mathbf{N}$ *then* γ *is not a* U*-element of* Γ.

Proof. — Let Σ be a set of generators for Γ, for $c_1 = \max\{d(\sigma x_0, x_0) \mid \sigma \in \Sigma \cup \Sigma^{-1}\}$ we have $d(\gamma^n x_0, x_0) \leqslant l_\Sigma(\gamma^n) \cdot c_1$. Hence γ cannot have property U3.

Now, a free group F acts on its Cayley graph which is a tree in such a way that every non-trivial element is hyperbolic and satisfies the hypothesis of Lemma 2.8, hence F has no U-element.

A similar argument shows:

(2.9) *Proposition.* — *If* Γ *is a hyperbolic group (in the sense of Gromov) then it contains no* U*-element.*

Proof. — By [Gr1] every cyclic subgroup of Γ is quasi-convex which exactly means that with Γ acting on its Cayley graph and $\gamma \in \Gamma$, the assumptions of (2.8) are satisfied with $x_0 = 1$.

Back to our $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For a square-free integer, $0 \leqslant d \in \mathbf{Z}$, the group $SL_2(\mathbf{Z}[\sqrt{-d}])$ is a non-uniform lattice in $SL_2(\mathbf{C})$ which is not a hyperbolic group in the strict sense of Gromov (e.g., it contains $\mathbf{Z} \times \mathbf{Z}$). It is however a special case of lattices considered in Theorem 2.18. In particular we have:

(2.10). $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is not a* U*-element in* $SL_2(\mathbf{Z}[\sqrt{-d}])$.

On the other hand:

(2.11). $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is a U1-element of* $SL_2(\mathbf{Z}[1/p])$ *for every prime p.*

Indeed, let $n \in \mathbf{N}$ write n in base p^2 as:

$$n = \Sigma_{i=0}^{r} a_i p^{2i} \text{ where } r = O(\log n), \text{ and } 0 \leqslant a_i \leqslant p^2 - 1.$$

In "Hörner expression" it is written as:

$$(*) \qquad n = p^2 \left(\ldots p^2(p^2(a_r p^2 + a_{r-1}) + a_{r-2}) + \ldots + a_1\right) + a_0.$$

Let $\beta = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \in \Gamma$, then $\beta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \beta^{-1} = \begin{pmatrix} 1 & p^2x \\ 0 & 1 \end{pmatrix}$ and so by (*):

$$\gamma^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = {}^{\beta}\left(\dots{}^{\beta}\left(\left({}^{\beta}({}^{\beta}(\gamma^{a_r})\gamma^{a_{r-1}})\gamma^{a_{r-2}}\right)\dots\gamma^{a_1}\right)\cdot\gamma^{a_0}\right)$$

where ${}^{\beta}w$ denotes $\beta w \beta^{-1}$. This shows that γ^n can be written as a word of length $O(\log n)$ using γ and β. Hence γ is a U1-element of $SL_2(\mathbf{Z}[1/p])$.

(2.12). $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is a U1-element of* $SL_2(\mathbf{Z}[\sqrt{d}])$ *when* $2 \leqslant d \in \mathbf{Z}$ *and square free.*

The proof is similar to (2.11) with one difficulty: The diagonal subgroup of $SL_2(\mathbf{Z}[\sqrt{d}])$ contains, by Dirichlet unit theorem (cf. [Ja]), an element of infinite order, say $\beta = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$, $b \in \mathbf{Z}[\sqrt{d}]^*$ with $|b| > 1$. This element does not normalize the cyclic group generated by γ – but rather the upper unipotent rank two free abelian group A containing it.

Embed Γ into $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$ by sending $\gamma \in \Gamma$ to (γ, γ^{τ}) where τ is the non-trivial element of the Galois group $Gal(\mathbf{Q}(\sqrt{d})/\mathbf{Q})$. The abelian group A is now a discrete cocompact subgroup in the two dimensional real vector space:

$$V = \left\{\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) \,|\, x, y \in \mathbf{R}\right\} \simeq \mathbf{R} \times \mathbf{R}.$$

A subset W of a metric space (Z, d) is called a *syndetic subset* if there is a constant C such that for every $z \in Z$ there is a $w \in W$ with $d(z, w) < C$. We will show that by using a finite subset Σ' of Γ we can find a syndetic subset W of $\mathbf{R}^2$ contained in A and such that every $w \in W$ can be expressed as a word in Σ' of length $O(\log d(w, 0))$ where d is the euclidean distance in $\mathbf{R}^2$. This will suffice since the discreteness of A in $\mathbf{R}^2$ implies that there exists a finite subset $a_1, \dots, a_r \in A$ such that every $a \in A$ there is $1 \leqslant i \leqslant r$ such that $a a_i^{-1} \in W$. Hence every $a \in A$ can be written as a word of length $O(\log d(a, 0))$ in $\Sigma = \Sigma' \cup \{a_1, \dots, a_r\}$. Since $d(\gamma^n, 0) = O(n)$. This will prove the desired result.

To get the syndetic set W: fix $y_0 \in A$, let

$$W_1 = \left\{\pm\Sigma_{i=0}^{k} r_i \beta^i \cdot y_0 \,|\, r_i \in \mathbf{N},\ 0 \leqslant r_i < |b|^2\right\}$$
$$W_2 = \left\{\pm\Sigma_{i-0}^{k} r_i \beta^{-i} \cdot y_0 \,|\, r_i \in \mathbf{N},\ 0 \leqslant r_i < |b|^2\right\}$$

and $W = W_1 + W_2$ (where $\beta^i \cdot y_0$ denotes the action of β^i on y_0 – this is done by a conjugation within the group Γ). This is indeed a syndetic set: β acts on V with two real eigen-values λ and λ^{-1} with say $|\lambda| > 1$. Let V_1(resp : V_2) be the eigen-space corresponding to λ(resp : λ^{-1}). V_1 and V_2 do not contain non-trivial points from A

– the integral lattice – but W_1(resp : W_2) is contained in $N_c(V_1)$ (resp : $N_c(V_2)$) – the c-neighborhood of V_1 (i.e., $N_c(V_i) = \{y \in V \mid \mathrm{dist}(y, V_i) \leqslant c\}$, for some $c > 0$). Moreover W_i is a syndetic subset of $N_c(V_i)$ and using "Hörner expression" as in (2.11) we see that every element w of W_i can be expressed as a word of length $O(\mathrm{dist}(w, 0))$. As $V = V_1 + V_2$, we deduce that $W = W_1 + W_2$ is syndetic in V and also its elements can be expressed efficiently.

More generally:

(2.13). — *Let $\mathcal{O}_S$ be a ring of* S*-integers in a number field k, i.e.,* S *is a finite set of valuations containing all the archimedean ones. Then* $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathcal{O}_S)$ *is a* U*-element, if and only if* $|S| > 1$ *(i.e., if and only if $\mathcal{O}_S$ has infinitely many units).*

Note that $|S| = 1$ if and only if either $\mathcal{O}_S = \mathbf{Z}$ or $\mathcal{O}_S$ is the ring of integers in the quadratic imaginary field $\mathbf{Q}(\sqrt{-d})$. (2.7) and (2.10) covered these cases. The proof of (2.13) follows the pattern of (2.11) and (2.12) – but one remark is in order:

The action of diagonal subgroup of $SL_2(\mathcal{O}_S)$ on the upper unipotent group is expressed by the action of $\mathcal{O}_S^*$ on $\mathcal{O}_S$ where $\mathcal{O}_S$ is embedded as a lattice in $V = \prod_{v \in S} k_v$ where k_v is the completion of k with respect to v. As in (2.11) and (2.12), we want a syndetic subset of V of elements of $\mathcal{O}_S$ which are efficiently generated. There is however one difference: V decomposes into eigenspaces isomorphic to the fields k_v. k_v can be either a p-adic field, $\mathbf{R}$ or $\mathbf{C}$.

The first two cases are treated as in (2.11) and (2.12). For the last case, a crucial observation (implicitly in [Th]) is that for every $\lambda \in \mathbf{C}$, with $|\lambda| > 1$, there exists a finite set $D = \{0, 1, 2, \dots, N\}$ such that the set of sums $\Sigma_{i=0}^{n} d_i \lambda^i$, $d_i \in D$ is a syndetic set in $\mathbf{C}$. Using this, (2.13) is proved in a similar way to (2.11) and (2.12). We omit the proof as this is a special case of Theorem 3.7 below.

For the last example of this section, think of $SL_2(\mathbf{Z})$ as embedded in the upper left corner of $SL_k(\mathbf{Z})$.

(2.14). $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is a U1-element of* $SL_k(\mathbf{Z})$ *for* $k \geqslant 3$.

To prove this it clearly suffices to show it for $SL_3(\mathbf{Z})$. Now, γ is inside a two dimensional space $A = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ which is acted upon via conjugation by another copy of

$$SL_2(\mathbf{Z}) \cong \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbf{Z},\ ad - bc = 1 \right\}.$$

By taking β (and β^{-1}) in $SL_2(\mathbf{Z})$ with eigenvalues λ and λ^{-1}, – with $|\lambda| > 1$, we can get an efficiently generated syndetic subset of A. As before this makes γ a U1-element.

The next theorem, which is the main result of this section, generalizes all the above examples.

Let G be a semi-simple group. By this we mean that $G = \prod_{i=1}^{l} \mathbf{G}_i(k_i)$, where for every $i = 1, \ldots, l$, k_i is a local field and $\mathbf{G}_i$ is an almost simple k_i-group. Rank G is defined as $\sum_{i=1}^{l} \operatorname{rank}_{k_i} \mathbf{G}_i$ where $\operatorname{rank}_{k_i} \mathbf{G}_i$ is the dimension of the maximal k_i-split torus of $\mathbf{G}_i$. An element of G is *unipotent* if all its components are unipotent, i.e., act as unipotent elements on the Lie algebras associated with the $\mathbf{G}_i$-s. A discrete subgroup Γ of G is called a *lattice* if G/Γ carries a finite G-invariant measure. It is an *irreducible lattice* if, for every i, the projection of Γ to $G_i = \mathbf{G}_i(k_i)$ is dense there.

(2.15) *Theorem. — Let* $G = \prod_{i=1}^{l} \mathbf{G}_i(k_i)$ *be a semi-simple group,* Γ *an irreducible lattice in* G *and* $\gamma \in \Gamma$. *Then* γ *is a U-element in* Γ *if and only if the following four conditions hold:*

(a) *For every* $i = 1, \ldots, l$, $\operatorname{char}(k_i) = 0$.

(b) *For every* $i = 1, \ldots, l$, $\operatorname{rank}_{k_i}(\mathbf{G}_i) \geqslant 1$.

(c) $\operatorname{rank} G = \sum \operatorname{rank}_{k_i}(\mathbf{G}_i) \geqslant 2$.

(d) γ *is a virtually unipotent element of infinite order.*

(2.16) *Remarks.*

(i) We are actually proving in the theorem that $\gamma \in \Gamma$ is a U1-element iff U2-element iff U3-element.

(ii) In fact, if (c) and (d) of the theorem hold then γ is a U-element from which one can easily deduce that (a) and (b) also hold.

(iii) Note that the existence of a non-trivial unipotent (or U-element) in Γ implies that Γ is a non-uniform lattice in G.

(2.17) *Proof of (2.15).* — Denote by r_i the projection from G to $\mathbf{G}_i(k_i)$.

Let $\gamma \in \Gamma$ be a U3-element. Assume $\operatorname{rank}(G) = 1$. This means that except for one factor, say $\mathbf{G}_1(k_1)$, all other factors are compact. The projection $r_1(\Gamma)$ is therefore still a lattice in $\mathbf{G}_1(k_1)$ and $\operatorname{Ker} r_1 \cap \Gamma$ is finite. Thus $r_1(\gamma)$ is a U3-element there. Assume k_1 is a non-archimedean field, then $r_1(\Gamma)$ acts discretely on the Bruhat-Tits tree T associated with $\mathbf{G}_1(k_1)$. The element $r_1(\gamma)$ is of infinite order and hence acts on T as an hyperbolic element (cf. [Se, Proposition 24, p. 63]), i.e., there exists a vertex $x \in T$ for which $d(r_1(\gamma)^n x, x) = mn$ for some fixed $m \in \mathbf{N}$ and every $n \in \mathbf{Z}$. By Lemma 2.8, $r_1(\gamma)$, and hence γ, is not a U3-element. Thus k_1 must be archimedean. This means that $\mathbf{G}_1(k_1)$ is a simple rank one real Lie group. Theorem 2.18 below shows that in this case also there is no U-element. We therefore conclude that rank $G \geqslant 2$. This proves (c). Now, let $J \subset \{1, \ldots, l\} = L$ be the subset of indices for which $G_j(k_j)$ is compact. Then the projection of Γ to $\prod_{i\in L\setminus J} \mathbf{G}_i(k_i)$ is a lattice there and $\Gamma \cap \prod_{j\in J} \mathbf{G}_j(k_j)$ is finite. From a well known theorem of Margulis [Ma, Theorem 4.10] we deduce that every normal subgroup of Γ is either finite or of finite index in Γ. For $j \in J$, $\mathbf{G}_j(k_j)$ does not contain unipotent element of infinite order hence by Proposition 2.4, Ker r_j is of finite index, so $r_j(\Gamma)$ cannot be dense in $G_j(k_j)$. This proves that $J = \emptyset$, i.e., (b) is proved. A similar argument now proves also (a) using Corollary 2.5. (Note that a lattice in a higher rank group is always finitely generated – see [Ra2] and [Ma]). (d) follows immediately from (2.4).

Assume now (a) – (d). By Margulis arithmeticity theorem [Ma, p. 298] Γ is an S-arithmetic group, i.e., there exists a number field k, an almost simple k-group $\mathbf{G}$, a finite set S of valuations of k containing S_∞ – the archimedean ones, such that Γ is commensurable with $\mathbf{G}(\mathcal{O}_S)$ where

$$\mathcal{O}_S = \{x \in k \mid |x|_v \leqslant 1 \text{ for every } v \notin S\}.$$

Theorem 3.7 below (whose proof occupies Section 3) proves that every unipotent element in such an S-arithmetic group is a U1-element. This will complete the proof of (2.15).

We close this section reproducing the proof of the following theorem of Gromov [Gr2, §3.G].

(2.18) *Theorem.* — *Let* $G = \mathbf{G}(F)$, *where* F *is a local field of characteristic zero,* $\mathbf{G}$ *is a* F*-rank one semi-simple group. Let* Γ *be a lattice in* G *and* $\gamma \in \Gamma$. *Then* γ *is not a U-element of* Γ.

Proof. — If F is non-archimedean, Γ has a non-abelian free subgroup Γ' of finite index, Γ' can be realized as a lattice in $SL(2, \mathbf{R})$. If $F = \mathbf{C}$, $\mathbf{G}$ is locally isomorphic SL(2) so that $G \simeq SO(3,1)(\mathbf{R})$ locally. Thus we can assume $F = R$. Let $X = G/K$ be the symmetric space associated with G, where K is a maximal compact subgroup of G. If Γ is cocompact, (2.6) gives the result. So assume Γ is non-uniform. By [GR] (see also [Ra1]), $\Gamma\backslash X$ has finitely many cusps and we can choose in a Γ-equivariant way disjoint open horoballs B_α in X, such that $X_0 = X \setminus \bigcup_\alpha B_\alpha$ is Γ-invariant and $\Gamma\backslash X_0$ is compact. Let $\tilde{d}$ be the path metric of X_0, i.e., for $a, b \in X_0$, $\tilde{d}(a,b)$ is the length of the shortest (with respect to the original metric d of X) path in X_0 between a and b. Γ preserves $\tilde{d}$. Fix some $x_0 \in X_0$. For any $\gamma \in \Gamma$ of infinite order we have $\tilde{d}(\gamma^n x_0, x_0) \geqslant \sqrt{n}c$ for some $c > 0$. Indeed either γ is hyperbolic and the assertion follows for c equal the minimal translation of γ in X (here we get $\geqslant nc$) or γ preserves one of the horospheres, ∂B_α, forming the boundary of X_0. By the Iwasawa decomposition, $G = NAK$, there exists a retraction $\varphi : X_0 \to \partial B_\alpha$, geometrically we map $x \in X_0$ to the point of intersection of ∂B_α with the geodesic ray from x to the point of infinity of the horosphere ∂B_α. It follows, from the negative curvature of X that $\bar{d}(\varphi(x), \varphi(y)) \leqslant \tilde{d}(x,y)$ where $\bar{d}$ is the path metric of ∂B_α. Observe that the retraction φ is Γ_α-equivariant where $\Gamma_\alpha < \Gamma$ is the subgroup preserving ∂B_α. As $\gamma \in \Gamma_\alpha$ it follows that $\tilde{d}(\gamma^n x_0, x_0) \geqslant \bar{d}(\varphi(\gamma^n x_0), \varphi(x_0)) = \bar{d}(\gamma^n\varphi(x_0), \varphi(x_0)) \geqslant \sqrt{n}c$. The last inequality holds since the metric $\bar{d}$ induces on (a torsion free finite index subgroup of) Γ_α via the map $\Gamma_\alpha \to \Gamma_\alpha\varphi(x_0)$ is equivalent to the word metric of Γ_α. Γ_α is a virtually nilpotent group of class $\leqslant 2$. In such a group, for γ of infinite order $d_W(\gamma^n, id) \geqslant \sqrt{n}c$ for some $c > 0$. It follows from an obvious variant of 2.8 that γ is not a U element.

(2.19) *Remark.* — When F is not of characteristic 0, non uniform lattices in G are not finitely generated and hence the notion of a U*j* element has no meaning.

3. Unipotent Subgroups

(3.1). — The goal of this section is to establish Theorem 3.7. This theorem completes the proof of Theorem 2.15 above. It also plays a central role in the proof of Theorem 4.1 below.

We start with some generalities concerning the relation between the word metric on a group Γ acting on a space X and metrics on this space.

Definition. — *A metric space* (Y, d) *is called a* coarse path metric space *if there exists a constant* K_0 *such that for every pair of points* $x, y \in Y$ *we have*

$$d(x,y) = \inf\left\{\sum_{i=0}^{n-1} d(x_i, x_{i+1}) \mid n \in \mathbf{N},\ x_0 = x,\ x_n = y,\ x_i \in Y,\ d(x_i, x_{i+1}) \leqslant K_0\right\}.$$

In what follows (Y, d) *will be called a* path space *if it is a coarse path metric space and closed balls of finite radius are compact.*

The following proposition – at least in a weaker form – is part of folklore.

(3.2) *Proposition.* — *Let* Λ *be a finitely generated group acting properly discontinuously via isometries on a path space* (Y, d). *Assume that* $\Lambda\backslash Y$ *is compact. Let* d_Λ *be a fixed left invariant word metric on* Λ. *Let* $y_0 \in Y$ *be such that* $\mathrm{Stab}_\Lambda(y_0) = \{e\}$. *We can embed* Λ *in* Y *via the map* $\Lambda \to \Lambda y_0$. *Then the pullback of the restriction of d to* Λy_0 *is Lipschitz equivalent to* d_Λ.

Proof. — Let $\Sigma \subset \Lambda$ be a finite symmetric set of generators. Define $C_1 = max\{d(y_0, \sigma y_0) \mid \sigma \in \Sigma\}$. Clearly for every $\lambda \in \Lambda$ we have $d(y_0, \lambda y_0) \leqslant C_1 \ell_\Sigma(\lambda)$ where $\ell_\Sigma(\lambda)$ is the length of λ with respect to the generators Σ. Note that $\ell_\Sigma(\lambda)$ is equivalent to $d_\Lambda(\lambda, 1)$. Let $Y = \bigcup_{\lambda\in\Lambda} \mathcal{F}_\lambda$ be a tessellation of Y by fundamental domains such that $y_0 \in \mathcal{F}_e$ and for every $\lambda \in \Lambda$, $\lambda\mathcal{F}_e = \mathcal{F}_\lambda$. Since the action of Λ is properly discontinuous and $\Lambda\backslash Y$ is compact any compact subset of Y is contained in the union of finitely many fundamental domains. Since $\overline{\mathscr{F}_e}$ is compact it follows that any ball of radius $2K_0$ in Y is contained in a translate by some $\alpha \in \Lambda$ of the union of a fixed set of N_0 fundamental domains for some fixed $N_0 \in \mathbf{N}$. (K_0 is the constant used in the definition of a coarse path metric space). Given $e \neq \lambda \in \Lambda$, let $y_1, y_2, \ldots, y_n = \lambda y_0$ be a sequence of points in Y such that $\sum_{i=0}^{n-1} d(y_i, y_{i+1}) \leqslant 2d(y_0, \lambda y_0)$ and $K_0/2 \leqslant d(y_i, y_{i+1}) \leqslant 2K_0$ for $0 \leqslant i \leqslant n-1$. (Without loss of generality $d(y_0, \lambda y_0) \geqslant K_0/2$). Each y_i, $0 \leqslant i \leqslant n$ belongs to some $\mathscr{F}_{\theta_i}$, $\theta_i \in \Lambda$. Since $d(y_{i-1}, y_i) < 2K_0$ it follows from the observation above that $r_i = \theta_{i-1}^{-1}\theta_i$, $1 \leqslant i \leqslant n$, belongs to a fixed finite collection of elements of Λ. It follows that $\lambda = r_1 r_2 \ldots r_n$ gives a word of length $\leqslant C_2\, d(y_0, \lambda y_0)$ representing λ.

(3.3) It is well known that a metrizable locally compact group G carries a left translation invariant metric d. In general (G, d) need not be a coarse path-space. However if G is compactly generated, G does carry a metric δ such that (G, δ) is a path space. This is seen as follows: Let $\Omega \subset G$ be a symmetric compact neighbourhood of e in G which generates G. An Ω-coarse path in G joining x, y in G is a finite sequence $\mathbf{g} = (g_0, g_1, \ldots, g_{n-1}, g_n)$ in G with $g_0 = x$, $g_n = y$ and $g_i^{-1} g_{i+1} \in \Omega$. Since Ω generates G, there is a Ω-coarse path joining any two points of G. For $x, y \in G$, set $d_\Omega(x, y) = \inf\{\sum_{0 \leqslant i < n} d(g_i, g_{i+1}) \mid \mathbf{g} = (g_0, \ldots, g_n)$, a Ω-coarse path joining x and $y\}$. We assert that closed balls of finite radius for the metric d_Ω are compact (that d_Ω is a metric compatible with the topology is easily seen). This is seen as follows. Since d_Ω is left translation invariant, we need only consider balls of finite radius around e. Let then $d_\Omega(e, x) \leqslant M$ for some $x \in G$. There exists a coarse Ω-path $\mathbf{g} = (g_0, g_1, \ldots, g_n)$ joining e and x such that

$$\sum_{0 \leqslant i < n} d(g_i, g_{i+1}) \leqslant M + 1.$$

Let $a > 0$ be such that the open ball of radius a around e is contained in Ω. Now we can find a subsequence $h_i = g_{r_i}$, $0 \leqslant i \leqslant m$, of $(g_0, \ldots, g_n)$ such that the following holds: let $h'_{i+1} = g_{r_{i+1}-1}$; then $d(h_i, h'_{i+1}) < a$ for $0 \leqslant i < m$ while for $0 \leqslant i < m-1$, $d(h_i, h_{i+1}) \geqslant a$. Evidently then

$$\sum_{0 \leqslant i < m} d(h_i, h_{i+1}) \leqslant \sum_{0 \leqslant i < n} d(g_i, g_{i+1}) \leqslant M + 1.$$

It follows then that $h_i^{-1} h_{i+1} \in B_d(e; a)\overline{\Omega}$, where $B_d(e; a)$ is the closed (compact) ball of radius a with respect to the metric d around e and $\overline{\Omega}$ is the closure of Ω. It follows in particular that $(m-1)a \leqslant (M+1)$ so that $(m-1) \leqslant (M+1)/a$. If we now set $N = [(M+1)/a] + 1$, we see that x belongs to $(B_d(e; a)\overline{\Omega})^N$, a compact set. Since x is an arbitrary element of the ball of radius M in the metric d_Ω, we see that this last ball is compact. We thus conclude that G carries a left invariant metric such that closed balls of finite radius are compact. Since $d_\Omega(x, y) = d(x, y)$ for $x^{-1}y \in \Omega$ one has

$d_\Omega(x,y) = \inf\{\sum_{0\leqslant i<n} d_\Omega(x_i, x_{i+1}) \mid x_0 = x,\ x_n = y,\ x_i \in G,\ x_i^{-1}x_{i+1} \in \Omega\}$. Let $K_0 > 0$ be such that $d_\Omega(1, x^{-1}y) = d(1, x^{-1}y) \leqslant K_0$ for $x^{-1}y \in \Omega$. Then since $d_\Omega(x,y) \leqslant \sum_{0\leqslant i<n} d_\Omega(x_i, x_{i+1})$, for $x_0 = x$, $x_n = y$, $d_\Omega(x_i, x_{i+1}) \leqslant K$, we see that

$$\begin{aligned} d_\Omega(x,y) &\leqslant \inf\{\sum_{0\leqslant i<n} d(x_i, x_{i+1}) \mid x_0 = x,\ x_n = y,\ d_\Omega(x_i, x_{i+1}) \leqslant K_0\} \\ &\leqslant \inf\{\sum_{0\leqslant i<n} d(x_i, x_{i+1}) \mid x_0 = x,\ x_n = y,\ x_i^{-1}x_{i+1} \in \Omega\} = d_\Omega(x,y)\} \end{aligned}$$

it follows that d_Ω is a path space metric.

(3.4). — Next we turn to show that a metric d, such that (G, d) is a path space, is unique up to "coarse Lipschitz equivalence", i.e. suppose that d_1, d_2 are metrics on G such that (G, d_i), $i = 1, 2$, are coarse path spaces. Then we claim that for any open neighbourhood V of e in G, there is a constant $C > 0$ such that for $x \notin V$

$$d_1(e, x)/d_2(e, x) < C.$$

This is seen as follows. Let B_2 be a ball for the metric d_2 such that the following holds: $d_2(x,y) = \inf\{\sum_{0\leqslant i<n} d_2(g_i, g_{i+1}) \mid \mathbf{g} = (g_0, g_1, \ldots, g_n)$ *a* B_2-coarse path in G from x to $y\}$. Let $c > 0$ be a constant such that the closure of the open ball B_2' of radius c around e is contained in the interior of B_2. Let $(g_0, \ldots, g_n)$ be a B_2-coarse path from e to x such that $d_2(e, x) \geqslant \sum_{0\leqslant 1<n} d_2(g_i, g_{i+1}) - 1$. Passing to a subsequence we may assume that $d_2(g_i, g_{i+1}) \geqslant c$ for $0 \leqslant i < n-1$, while for $0 \leqslant i \leqslant n-1$ we have $g_i^{-1}g_{i+1}$ belongs to $\overline{B}_2'B_2$. By compactness and continuity there exists a constant $m < 1$ such that for all $z \in B_2\backslash B_2'$, we have $d_2(e, z) \geqslant m\, d_1(e, z)$. Let $A = \mathrm{diam}_{d_1}(\overline{B'}_2 B_2)$. We have

$$\begin{aligned} d_1(e, x) &\leqslant \sum_{0\leqslant i<n} d_1(g_i, g_{i+1}) \\ &\leqslant m^{-1} \sum_{0\leqslant i<n-1} d_2(g_i, g_{i+1}) + d_1(g_{n-1}, g_n) \\ &\leqslant m^{-1} \sum_{0\leqslant i<n} d_2(g_i, g_{i+1}) + A \\ &\leqslant m^{-1} d_2(e, x) + m^{-1} + A. \end{aligned}$$

It follows that for some large enough b there exists $b' > 0$ so that if $d_1(e, x) > b$, then $d_2(e, x) \geqslant b' d_1(e, x)$. Using the compactness of $B_{d_1}(e; b)\backslash V$ one concludes then that

$$d_1(e, x) \leqslant C d_2(e, x) \text{ for all } x \notin V$$

proving our contention.

The uniqueness up to coarse Lipschitz equivalence leads us to denote any path space metric on the metrizable compactly generated locally compact group by d_G.

Observe that when G is discrete, it is finitely generated and our notation is consistent with the one used for the word metric on discrete groups.

(3.5). — Consider now the case $G = \prod_{1 \leqslant i \leqslant n} G_i$ where each G_i is the group of k_i-rational points $\mathbf{G}_i(k_i)$ of a *reductive* algebraic group $\mathbf{G}_i$ over the local field k_i. Then one sees that the product metric $\prod_{1 \leqslant i \leqslant \ell} d_{G_i}$ on G is a path space metric. Suppose now that we have a realization $\mathbf{G}_i \hookrightarrow SL(n_i)$ of $\mathbf{G}_i$ as a k_i-subgroup of $SL(n_i)$ so that $G_i \subset SL(n_i, k)$. On $SL(n_i, k)$, we have a natural left-translation invariant metric δ_i defined by $\delta_i(x, y)$ $\log(1 + \|x^{-1}y - 1\|)$ where for a matrix $A = \{A_{rs}\}_{1 \leqslant r, s \leqslant n_i}$, $\|A\| = \max\{|A_{rs}| \mid 1 \leqslant r, s \leqslant n_i\}$, with $|\cdot|$ denoting the absolute value in k_i. We assert that $\delta|_G$ is coarse Lipschitz equivalent to d_G (in particular we see that δ is coarse Lipschitz equivalent to $d_{\prod SL(n_i, k_i)}$). In other words given a neighbourhood U of 1 in G_i there is a constant $C > 1$ depending on U such that for all $x \in G_i \backslash U$, one has

$$(*) \qquad C^{-1} \log(1 + \|(x - 1)\|) \leqslant d_G(1, x) \leqslant C \log(1 + \|(x - 1)\|).$$

This is seen as follows. It is well known that if $\mathbf{D}$ is a maximal k_i-split torus in $\mathbf{G}_i$ and $D = \mathbf{D}(k_i)$, then there is a compact subgroup $K \subset G_i$ such that $G_i = K.D.K$ [BT]. It is immediate from this that the problem is reduced to the case when $G_i = D$, a case which is checked easily $-D$ is a direct product of copies of k_i^*; note also that we have assumed that $\mathbf{G}_i \subset SL(n_i)$ – (*) does not for instance hold for $\mathbf{G} = GL(1)$ in GL(1). One may also reformulate the inequality (*) to say that if $G = \prod_{1 \leqslant i \leqslant \ell} G_i$, $G' = \prod_{1 \leqslant i \leqslant \ell} G'_i$ are two groups with $\mathbf{G}_i \subseteq \mathbf{G}'_i$ reductive algebraic groups over k_i and $G_i = \mathbf{G}_i(k_i)$ (resp. $G'_i = \mathbf{G}'_i(k_i)$) then $d_{G'}|_G$ is coarse Lipschitz equivalent to d_G. If k_i is archimedean (resp. non archimedean) let X_i denote the symmetric space (resp. Bruhat-Tits building) associated to G_i. Let δ_i denote the symmetric Riemannian (resp. the combinatorial) metric on X_i. Suppose now that k_i is non-archimedean and $x \in X_i$ is any point and $f : G_i \to X_i$ is the orbit map $f(g) = gx$. If $D \subset G_i$ is a maximal diagonalizable group and x is in the apartment determined by D, it follows from the definition of the metric on X_i, that there are positive constants C_1, C_2 such that

$$(**) \qquad C_1 d_i(e, g) \leqslant \delta_i(x, gx) \leqslant C_2 d_i(e, g)$$

for all $g \in D$ with $d_i(e, g)$ sufficiently large, d_i being a path space metric on G_i. Using the decomposition $G = KDK$ with K a compact group one sees that (**) holds (with perhaps different C_1, C_2) for all $g \in G_i$ with $d_i(e, g)$ sufficiently large. When k_i is archimedean a maximal connected diagonal group under the orbit map for a suitable $x \in X_i$ maps diffeomorphically onto a totally geodesic flat space and then the metric induced by this diffeomorphism is up to a scalar the same as the Riemannian metric induced from G_i. Thus the path space metric distance in G_i is closely related to the distance in the Bruhat Tits building or the symmetric space as the case may be. In the sequel, we will always consider only left invariant metrics on G that make it a path space.

(3.6). — Suppose now that G is as in (3.5), i.e., $G = \prod_{1 \leqslant i \leqslant \ell} G_i$ with $G_i = \mathbf{G}_i(k_i)$ where for $1 \leqslant i \leqslant \ell$, k_i are local fields and $\mathbf{G}_i$ are almost k_i-simple linear algebraic groups over k_i. Let Γ be an irreducible lattice in G, i.e., Γ is a lattice such that the image of Γ in G/H is not discrete for any closed non-compact normal subgroup H. Then if $\sum_{1 \leqslant i \leqslant \ell} k_i$-rank $\mathbf{G} \geqslant 2$, according to a theorem of Margulis [Ma, Chapter IX], Γ is necessarily arithmetic. More precisely, there is a global field k, a finite set S of valuations of k which contains all the archimedean valuations of k, an absolutely almost simple (simply connected) algebraic group $\mathbf{G}$ over k and a homomorphism $f : \prod_{v \in S} \mathbf{G}(k_v) \to G$ such that Kernel f is compact, image of f is a closed normal cocompact subgroup of G and $f(\mathbf{G}(\mathcal{O}_S))$ and Γ are commensurable : here $\mathcal{O}_S$ is the ring of S integers in k and $\mathbf{G}(\mathcal{O}_S)(= G(k) \cap GL(n, \mathcal{O}_S)$ for some realism of $\mathbf{G}$ as a k-subgroup of $GL(n)$ for some integer $n > 0$). Because of this theorem, one sees that one needs only to deal with S-arithmetic groups in absolutely almost simple groups over global fields. We now formulate the central result of this section in the framework of S-arithmetic groups.

(3.7) *Theorem.* — *Let* $\mathbf{G}$ *be a connected simply connected absolutely almost simple linear algebraic groups over a global field k. Let* S *be a finite set of valuations of k including all the archimedean valuations. Let* $\Gamma \subset G(k)$ *be an* S*-arithmetic subgroup and* $\mathbf{U}$ *the unipotent radical of a (proper) k-parabolic subgroup of* $\mathbf{G}$. *Assume that* $\sum_{v \in S} k_v$*-rank* $\mathbf{G} \geqslant 2$. *Let* $G = \prod_{v \in S} G(k_v)$; *then* $d_G|_{U \cap \Gamma}$ *and* $d_\Gamma|_{U \cap \Gamma}$ *are Lipschitz equivalent.*

(Note: $\mathbf{U} \neq \{1\}$ can happen only if k-rank $\mathbf{G} > 0$; also for $v \in S$, k_v is the completion of k at v).

Terminology. — Given a discrete subgroup Θ of a group G, a metric d_1 on Θ and a metric d_2 on G we shall say that Θ *is* (d_1, d_2)*-undistorted* if d_1 and d_2 restricted to Θ are Lipschitz equivalent.

(3.8) *Corollary.* — *Every unipotent element of infinite order in* Γ *is a* U1*-element.*

Proof. — Let $u \in \Gamma$ be a unipotent element of infinite order. Then char $k=0$ and u belongs to the unipotent radical of some k-parabolic subgroup of $\mathbf{G}$. According to (3.7) we then have $d_\Gamma(1, u^n) \approx d_G(1, u^n)$. On the other hand the matrix entries of u^n are of the form $P_{ij}(n)$ where P_{ij} are polynomials with coefficients in k. It follows now from (3.5) that $d(1, u^n) = O(\log n)$. Hence the corollary.

(3.9). We fix the following notation for the rest of this section. $\mathbf{G}$ will be a reductive algebraic group over k. We will always consider $\mathbf{G}$ as a k-subgroup of a fixed $GL(n)$. $\mathcal{O}_S$ will be the ring of S-integers in k and $\mathbf{G}(\mathcal{O}_S) = \mathbf{G} \cap GL(n, \mathcal{O}_S)$. The "standard" norm $\| \ \|_v$ on $M(n, k_v)$ is defined as $\|g\|_v = \sup\{|g_{ij}|_v \mid 1 \leqslant i, j \leqslant n\}$ where $g \in M(n, k_v)$ and g_{ij} are the entries of g. We fix once and for all a maximal k-split torus $\mathbf{T}$ in $\mathbf{G}$ and denote its centralizer in $\mathbf{G}$ by $Z(\mathbf{T})$. Then $Z(\mathbf{T})$ is a reductive k-subgroup of $\mathbf{G}$ whose commutator subgroup $\mathbf{M} = [Z(\mathbf{T}), Z(\mathbf{T})]$ is a semisimple subgroup defined over k. The group is an almost direct product $\mathbf{T}.\mathbf{C}.\mathbf{M}$ where $\mathbf{C}$ is a torus in $Z(\mathbf{T})$ defined and anisotropic over k. The following lemma will enable us to choose in $\mathbf{M}$ a maximal torus $\mathbf{D}$ defined over k such that for every $v \in S$, $\mathbf{D}$ contains a maximal k_v-split torus and $\mathbf{D}$ is anisotropic over k.

(3.10) *Lemma.* — *Let Σ be any finite set of places of k. Suppose that we are given for each $v \in \Sigma$ a maximal torus $\mathbf{D}_v$ of $\mathbf{M}$ defined over k_v. Then there is a maximal torus $\mathbf{D}$ over k in $\mathbf{M}$ such that $\mathbf{D}$ is conjugate to $\mathbf{D}_v$ by an element $\mathbf{M}(k_v)$ for all $v \in \Sigma$. Moreover $\mathbf{D}$ can be chosen to be anisotropic over k.*

Proof. — Let $D_v = \mathbf{D}_v(k_v)$ and D'_v the open set of regular elements in D_v : $g \in D'_v$ iff the centralizer of g in $\mathbf{M}$ has $\mathbf{D}_v$ as the identity connected component. Consider the map $\lambda_v : M_v \times D_v \to G_v$ (where $G_v = \mathbf{G}(k_v)$) given by $(g, t) \mapsto gtg^{-1}$. Then it is well known – and easy to see that each λ_v is of maximal rank in the open set $M_v \times D'_v$ and hence the image of this open set in M_v is an open subset Ω_v in M_v. It follows that if $g \in \Omega_v$, the identity connected component of $\mathbf{Z}(g)$, the centralizer of g in $\mathbf{G}$, is conjugate to $\mathbf{D}_v$ by an element of G_v. Now by a well known theorem due to Kneser [Kn], $\mathbf{M}(k)$ is dense in $M_\Sigma = \prod_{v \in \Sigma} M_v$. It follows that there is an element $g \in \mathbf{M}(k) \cap \prod_{v \in S} \Omega_v$. To prove the first assertion we need only take $\mathbf{D}$ to be the identity connected component of the centraliser of g in $\mathbf{G}$. The second assertion, that $\mathbf{D}$ can be chosen to be anisotropic over k, is seen as follows: Let w be a non-archimedean valuation of k not in Σ. Let $\Sigma' = \Sigma \cup \{w\}$. For each $v \in \Sigma$ choose a maximal torus $\mathbf{D}_v$ over k_v containing a maximal k_v-split torus. Let $\mathbf{D}_w$ be a maximal torus in $\mathbf{G}$ anisotropic over k_w (such a $\mathbf{D}_w$ exists – see ([PR] Theorem 6.21)). We have seen that there is a maximal torus $\mathbf{D}$ in G defined over k and such that $\mathbf{D}$ is conjugate to $\mathbf{D}_v$ by an element of $\mathbf{G}_v$ for all $v \in \Sigma'$. Since $\mathbf{D}_w$ is anisotropic over k_w, $\mathbf{D}$ is anisotropic over k.

(3.11). — Fix now a maximal torus $\mathbf{D}$ in $\mathbf{M}$ anisotropic over k which contains a maximal k_v-split torus for every $v \in S$ (such a $\mathbf{D}$ exists by Lemma 3.10). Then $\widetilde{\mathbf{T}} = \mathbf{T}.\mathbf{C}.\mathbf{D}$ is a maximal torus in $\mathbf{G}$. Let $\mathbf{T}_1 = \mathbf{T}.\mathbf{C}$. We introduce lexicographic orderings in the character groups $X(\mathbf{T})$, $X(\mathbf{T}_1)$ and $X(\widetilde{\mathbf{T}})$ compatible with the restriction maps: $\chi \in X(\widetilde{\mathbf{T}})$ is positive if its restriction to $\mathbf{T}_1$ (resp. $\mathbf{T}$) is positive. In the case when $|S| = 1$ so that $S = \{v\}$ a single valuation, we will require more of this ordering. To formulate this requirement, we fix a maximal k_v-split torus $\mathbf{C}' \subset \mathbf{C}$ and a maximal k_v-split torus $\mathbf{D}' \subset \mathbf{D}$. Let $\mathbf{T}'_1 = \mathbf{T}.\mathbf{C}'$ and $\mathbf{T}'_2 = \mathbf{T}.\mathbf{C}.\mathbf{D}'$. We demand that there are orderings on the character groups $\mathbf{T}'_1$ and $\mathbf{T}'_2$ as well such that the restriction maps induced by the inclusions

$$\mathbf{T} \subset \mathbf{T}'_1 \subset \mathbf{T}_1 \subset \mathbf{T}'_2 \subset \widetilde{\mathbf{T}}$$

are compatible with the orderings. We denote by Φ (resp. $\widetilde{\Phi}$) the k-root – (resp. absolute root) – system of $\mathbf{G}$ with respect to $\mathbf{T}$ (resp. $\widetilde{\mathbf{T}}$). Let Δ (resp. $\widetilde{\Delta}$) be the system of simple (resp. simple absolute) roots of $\mathbf{G}$ with respect to $\mathbf{T}$ (resp. $\widetilde{\mathbf{T}}$). If $\beta \in \widetilde{\Delta}$, and $\beta\,|_{\mathbf{T}} \neq 0$, then $\beta\,|_{\mathbf{T}} = \alpha \in \Delta$. For $\alpha \in \Delta$, let $\widetilde{\alpha} = \{\beta \in \widetilde{\Delta} \mid \beta\,|_{\mathbf{T}} = \alpha\}$; then $\widetilde{\alpha} \neq \phi$. For $\varphi \in \widetilde{\Phi}$, there is a unique 1-parameter unipotent subgroup $\mathbf{U}(\varphi)$ (over k_s) in $\mathbf{G}$ normalized by $\mathbf{T}$ and such that the Lie algebra $\mathscr{Lie}(\mathbf{U}(\varphi))$ of $\mathbf{U}(\varphi)$ is precisely the eigen-space corresponding to φ for the torus $\widetilde{\mathbf{T}}$. For $\varphi \in \Phi$, we denote by $\mathbf{U}(\varphi)$ the k-subgroup generated by $\{\mathbf{U}(\psi) \mid \psi\,|_{\mathbf{T}}$ is of the form φ or $2\varphi\}$ (2φ can be a k-root). $\mathbf{U}(\varphi)$ is a unipotent k-subgroup. If 2φ is not a k-root $\mathbf{U}(\varphi)$ is in a natural fashion a k-vector space. If 2φ is a root, $\mathbf{U}(2\varphi)$ is a k vector space in a natural fashion and $\mathbf{U}(\varphi)/\mathbf{U}(2\varphi)$ has a natural k-vector space structure. The $\mathbf{U}(\varphi)$, $\varphi \in \Phi$ will be referred to as the (k)-root group corresponding to φ.

(3.12). — The set Δ (resp. $\widetilde{\Delta}$) is a basis for $X(\mathbf{T})$ (resp. $X(\widetilde{\mathbf{T}})$). Thus we can write $\varphi = \sum_{\theta \in \widetilde{\Delta}} m_\theta(\varphi)\theta$ for any $\varphi \in \widetilde{\Phi}$. It is then well known that all the $m_\theta(\varphi)$, $\theta \in \widetilde{\Delta}$ are integers and that $m_\theta \geqslant 0$ or $m_\theta \leqslant 0$ for *all* $\theta \in \widetilde{\Delta}$. For $\Delta' \subset \Delta$, we set $\widetilde{\Phi}_{\Delta'} = \{\varphi \in \widetilde{\Phi} \mid m_\theta(\varphi) > 0$ for all $\theta \in \widetilde{\Delta}'\}$. Let $\mathbf{U}_{\Delta'}$ be the k-subgroup of $\mathbf{G}$ generated by $\{\mathbf{U}(\varphi) \mid \varphi \in \widetilde{\Phi}_{\Delta'}\}$ and $\mathbf{P}_{\Delta'}$ the k-subgroup generated by $\{\mathbf{U}(\varphi) \mid \varphi \in \widetilde{\Phi}, m_\theta(\varphi) \geqslant 0$ for all $\theta \in \widetilde{\Delta}'\}$. Then $\mathbf{P}_{\Delta'}$ is a k-parabolic subgroup of $\mathbf{G}$ with $\mathbf{U}_{\Delta'}$ as its unipotent radical; also $\mathbf{L}_\Delta$ the subgroup generated by $Z(\mathbf{T})$ and $\{\mathbf{U}(\varphi) \mid \in \widetilde{\Phi}, m_\theta(\varphi) = 0$ for $\theta \in \widetilde{\Delta}'\}$ is a Levi supplement to $\mathbf{U}_{\Delta'}$ in $\mathbf{P}_{\Delta'}$. Finally it is known that every k-parabolic subgroup of $\mathbf{G}$ is conjugate to $\mathbf{P}_{\Delta'}$ for a *unique* subset $\Delta' \subset \Delta$ by an element of $\mathbf{G}(k)$ (Borel-Tits [BT]).

(3.13) *Proposition.* — *When* S-*rank* $\mathbf{G} \geqslant 2$, $d_G\,|_{U(\varphi)\cap\Gamma}$ *and* $d_\Gamma\,|_{U(\varphi)\cap\Gamma}$ *are Lipschitz equivalent.*

(3.14). — We now show that (3.13) implies (3.7). The rest of the section will then be devoted to the proof of (3.13). Let $\varphi_1, \varphi_2 \ldots \varphi_N$ be the enumeration of the roots in $\Phi_{\Delta'}$ in increasing order. Let $\mu : \prod_{\varphi \in \widetilde{\Phi}_{\Delta'}} \mathbf{U}(\varphi) \to \mathbf{U}_{\Delta'}$ be the morphism

$$\mu(x_1, \ldots, x_N) = x_1 \cdot x_2 \ldots x_N$$

$x_i \in \mathbf{U}(\varphi_i)$. Then μ is an isomorphism of algebraic varieties. It follows that there are morphisms $f_i : \mathbf{U}_{\Delta'} \to \mathbf{U}(\varphi_i)$, $1 \leqslant i \leqslant N$ over k such that

$$x = f_1(x) \cdot f_2(x) \cdots f_N(x).$$

Further if Γ is a suitable congruence subgroup of $\mathbf{G}(\mathcal{O}_S)$ one sees easily that if $x \in \Gamma \cap \mathbf{U}_{\Delta'}$, $f_i(x) \in \mathbf{U}(\varphi_i)(\mathcal{O}_S)$ for all i, $1 \leqslant i \leqslant N$. Since μ is an isomorphism it follows easily from the inequality (**) of (3.5) that one has

$$d_G(1, x) \approx \sum_{1 \leqslant i \leqslant N} d_G(1, f_i(x))$$

for all $x \in \prod_{v \in S} \mathbf{U}_{\Delta'}(k_v)$. Since d_G and d_Γ are Lipschitz equivalent on $\mathbf{U}(\varphi_i) \cap \Gamma$, we see that there is a constant $C > 0$ such that $d_\Gamma(1, f_i(x)) \leqslant C d_G(1, f_i(x))$ for all $x \in \Gamma$. It follows that

$$\begin{aligned} d_\Gamma(1, x) \leqslant \sum_{1 \leqslant i \leqslant N} d_\Gamma(1, f_i(x)) &\leqslant C \textstyle\sum_{1 \leqslant i \leqslant N} d_G(1, f_i(x)) \\ &\leqslant C' d_G(1, x) \end{aligned}$$

for some constant $C' > 0$. Thus we see that d_G and d_Γ are Lipschitz equivalent on $(\mathbf{U}_{\Delta'} \cap \Gamma)$. Since Γ is commensurable with any S-arithmetic subgroup and any k-parabolic subgroup of $\mathbf{G}$ is conjugate to a $\mathbf{P}_{\Delta'}$ by an element $\mathbf{G}(k)$, we see that (3.13) implies (3.7).

(3.15). — A first step towards the proof of (3.13) is Lemma 3.16 below whose formulation requires some preliminaries. Let $v \in S$ and let $\mathbf{E}$ be unipotent algebraic group defined over k_v. Let B be any group and assume we have a homomorphism of B into the group of automorphisms of $\mathbf{E}$ defined over k_v. We assume that $\mathbf{E}$ is k-isomorphic to a vector space over k and that for this vector space structure on $\mathbf{E}$, the action of B on $\mathbf{E}$ is linear thus giving a linear representation $\sigma : B \to GL(E_v)$ where we have set $\mathbf{E}(k_v) = E_v$. For $b \in B$, let $\mathcal{S}(b)$ denote the set of eigen-values of $\sigma(b)$. Let L be the finite extension of k_v obtained by adjoining all the elements of $\mathcal{S}(b)$. We continue to denote by $| \ |_v$ the unique extension to L of the absolute value on k_v. For $\lambda \in \mathcal{S}(b)$, let $\mathbf{E}(b, \lambda)$ denote the generalized eigen-space for b corresponding to λ: it is the vector space spanned by $\{e \in E_v \mid (\sigma(b) - \lambda)^d e = 0\}$ (here $d = \dim E_v$ over k_v). Let $\mathbf{E}^{\pm}(b) = \sum_{\lambda \in \mathcal{S}(b),\, |\lambda|_v \gtrless 1} \mathbf{E}(b, \lambda)$; it is defined over k_v. Then $E^{\pm}(b)(k_v)$ can be characterized as the set of vectors $\{e \in E_v \mid \sigma(b)^{\mp n}(e)$ tends to zero as $n \to +\infty\}$. We define $\mathbf{E}_u(B)$ as the span of $\{\mathbf{E}^+(b) \mid b \in B\}$ – it is the same as the span of $\{\mathbf{E}^-(b) \mid b \in B\}$ as well. Suppose now that $\mathbf{E}' \subset \mathbf{E}$ is a B-stable k_v-subspace and let $\mathbf{F} = \mathbf{E}/\mathbf{E}'$. Then the generalized eigen-subspace in $\mathbf{E}$ for $b \in B$ corresponding to an eigen-value λ maps *onto* the generalized eigen-subspace of b in $\mathbf{F}$ corresponding to the same eigen-value. In particular $\mathbf{E}_u(B)$ maps onto $\mathbf{F}_u(B)$. With these observations we have:

(3.16) *Lemma.* — *Let* $\mathbf{L}$ *be a reductive k-group and* $\mathbf{C}$ *its central torus. Assume that one of the following conditions hold.*

(1) $\mathbf{L} = \mathbf{C}$ *is a k-split torus and* $|S| \geqslant 2$ *or*

(2) $\mathbf{C}$ *is anisotropic over k.*

Let $\Lambda \subset \mathbf{L}$ *be an* S*-arithmetic subgroup. Then for any representation* $\sigma : \mathbf{L} \to \mathrm{GL}(\mathbf{E})$ *on a* k*-vector space* E *we have for all* $v \in \mathrm{S}$, $\mathbf{E}_u(\Lambda_v) = \mathrm{E}_u(\mathbf{L}(k_v))$ *where* $\Lambda_v = \Lambda$ *regarded as a group of automorphisms of* $\mathbf{E}$ *over* k_v.

Proof. — When $\mathbf{L} = \mathbf{C}$ is split over k and $|\,\mathrm{S}\,| \geqslant 2$ this is immediate from the fact that Λ is Zariski dense in $\mathbf{C}$ so that every non-trivial character on $\mathbf{C}$ is non-trivial on Λ. To deal with the second case, let $\mathbf{L}''$ (resp. $\mathbf{C}''$) be the Zariski closure of Λ (resp. $\mathbf{C} \cap \Lambda$) in $\mathbf{L}$ and $\mathbf{L}'$ (resp. $\mathbf{C}'$) the connected component of the identity in $\mathbf{L}''$ (resp. $\mathbf{C}''$). Then $\mathbf{L}' = \mathbf{C}'.\mathbf{L}_1$ where $\mathbf{L}_1$ is the product of all the k-simple factors of $\mathbf{L}$ which are isotropic over k_v for some $v \in \mathrm{S}$. Then it is clear that $\mathrm{L}_v/\mathrm{L}'_v$ is compact where we have set L_v (resp. $\mathrm{L}'_v) = \mathbf{L}(k_v)$ (resp. $\mathbf{L}'(k_v)$). Let $\mathbf{E}' = \mathbf{E}_u(\mathrm{L}'_v)$. Then, since L'_v is normal in L_v, it is immediate that $\mathbf{E}'$ is L_v stable. Let $\mathbf{F} = \mathbf{E}/\mathbf{E}'$; then $\mathbf{F}_u(\mathrm{L}'_v) = 0$ and $\mathbf{F}_u(\mathrm{L}_v)$ = Image $\mathbf{E}_u(\mathrm{L}_v)$. We claim that $\mathbf{F}_u(\mathrm{L}_v) = 0$. To see this let $\bar{\sigma}$ be the representation of L_v on $\mathbf{F}$ induced by σ. Let $\mathbf{D}' \subset \mathbf{L}'$ be any k_v-split torus. Then for any $g \in \mathbf{D}'(k_v)$, all the eigen-values of $\bar{\sigma}(g)$ must have absolute value 1 – in view of the fact that $\mathbf{F}_u(\mathrm{L}'_v) = 0$. It follows that $\mathbf{D}'$ acts trivially on $\mathbf{F}$. We conclude thus that $\bar{\sigma}$ is trivial when restricted to L^*_v where $\mathrm{L}^*_v = \mathbf{L}^*(k_v)$ and $\mathbf{L}^*$ is the product of all the k_v-isotropic factors of $\mathbf{L}$ and the maximal k_v-split central torus in $\mathbf{L}$. It is further clear that $\mathrm{L}_v/\mathrm{L}^*_v$ is compact. Next let $\mathbf{D}$ be a k_v-split torus in $\mathbf{L}$. Then $\mathbf{D}(k_v) \subset \Omega \cdot \mathrm{L}^*_v$ with $\Omega \subset \mathrm{L}_v$ a compact set. Since L^*_v fixes every vector in $\mathbf{F}(k_v)$, we see that $\mathbf{D}(k_v)f$ is relatively compact for every f in $\mathbf{F}(k_v)$. Since $\mathbf{D}$ is k_v-split, this can happen only if $\mathbf{D}$ acts trivially on $\mathbf{F}$. We conclude that $\bar{\sigma}$ factors through $\mathbf{L}/\mathbf{L}_1$ where $\mathbf{L}_1$ is the product of the maximal central k_v-split torus in $\mathbf{L}$ and all the k_v-isotropic factors of $\mathbf{L}$. This means that $\mathbf{L}/\mathbf{L}_1$ is anisotropic over k_v and hence $\bar{\sigma}((\mathbf{L}/\mathbf{L}_1)(k_v))$ is compact. Since $\sigma(\mathrm{L}_v)$ is contained in this last compact group, $\mathbf{F}_u(\mathrm{L}_v) = 0$. We have thus shown that $\mathbf{E}_u(\mathrm{L}_v) = \mathbf{E}_u(\mathrm{L}'_v)$. Thus to show that $\mathbf{E}_u(\Lambda) = \mathbf{E}_u(\mathrm{L}_v)$, it suffices to show that $\mathbf{E}_u(\Lambda) = \mathbf{E}_u(\mathrm{L}'_v)$ in other words we may assume that Λ is Zariski dense in $\mathbf{L}$. This means that $\mathbf{E}_u(\Lambda)$ is $\mathbf{L}$-stable (since it is Λ-stable). Let $\mathbf{D}$ be a maximal k_v-split torus in $\mathbf{L}$. From Lemma 3.10 we know that there is a k-torus $\mathbf{D}$ defined over k and anisotropic over k and containing a conjugate of $\mathbf{D}$. Consider now the representation $\bar{\sigma}$ of $\mathbf{L}$ on $\mathbf{F} = \mathbf{E}/\mathbf{E}_u(\Lambda)$. The eigen-values of $\bar{\sigma}(g)$ for $g \in \mathbf{D}' \cap \Lambda$ are all of absolute value 1. Hence the $(\mathbf{D}' \cap \Lambda)$-orbit of any vector $f \in \mathbf{F}(k_v)$ is relatively compact in $\mathbf{F}(k_v)$. The same then holds for the $\mathbf{D}'(k_v)$ orbit since $\mathbf{D}'(k_v)/(\mathbf{D}' \cap \Lambda)^-$ is compact where $(\mathbf{D}' \cap \Lambda)^-$ is the closure of $(\mathbf{D}' \cap \Lambda)$. This means that the orbit of any vector in $\mathbf{F}(k_v)$ under any k_v-split torus in $\mathbf{L}$ is relatively compact. Thus $\bar{\sigma}(\mathrm{L}_v)$ is compact and hence $\mathbf{F}_u(\mathrm{L}_v) = \{0\}$. Hence $\mathbf{E}_u(\Lambda) = \mathbf{E}_u(\mathrm{L})$.

(3.17) *Corollary.* — *Let* $\mathbf{L}$, Λ *and* $\mathbf{E}$ *be as in Proposition 3.16. Then there is a finitely generated subgroup* $\mathrm{B} \subset \Lambda$ *such that* $\mathbf{E}_u(\mathrm{B}) = \mathbf{E}_u(\mathrm{L}_v)$ *for all* $v \in \mathrm{S}$.

Proof. — By Proposition 3.16, $\mathbf{E}_u(\Lambda) = \mathbf{E}_u(\mathrm{L}_v)$ so that $\mathbf{E}_u(\mathrm{L}_v)$ is spanned by vectors $\overset{v}{e}_1, \ldots, \overset{v}{e}_r$ with the property that for each i, there is an element $\overset{v}{b}_i \in \Lambda$ such that $\overset{v}{e}_i$ is in the span of generalized eigen-spaces for b_i^v corresponding to eigen-values of absolute value < 1 in the valuation v. We need only take B to be the subgroup of Λ generated by $\{b_i^v \mid v \in \mathrm{S},\ 1 \leqslant i \leqslant r\}$.

(3.18) *Lemma. — Let* $\mathbf{V}$ *a unipotent k-algebraic subgroup of* $\mathrm{GL}(n)$ *and* Γ *be an S-arithmetic subgroup of* $\mathrm{GL}(n)$. *Let* $\mathrm{B} \subset \Gamma$ *be a finitely generated subgroup of* Γ *normalizing* $\mathbf{V}$. *Suppose that there is a k-vector space structure on* $\mathbf{V}$ *compatible with the k-algebraic group structure on it such that the* B *action (by inner conjugation) on* $\mathbf{V}$ *is linear. We denote by* B_v *the group* B *and regard* $b \mapsto$ *Int* b, $b \in \mathrm{B}_v$ *as a homomorphism of* B_v *into* k_v*-automorphism of* $\mathbf{V}$. *Assume that* $\mathbf{V}_u(\mathrm{B}_v) = \mathbf{V}$ *for all* $v \in \mathrm{S}$. *Let* $\mathrm{V} = \prod_{v \in \mathrm{S}} \mathbf{V}(k_v)$ *and* H *be the subgroup of* $\mathscr{G} = \prod_{v\in \mathrm{S}} \mathrm{GL}(n, k_v)$ *generated by* B *and* V. *Then* H *is a closed subgroup of* G. *It is compactly generated. Moreover if* d_{H} *is a path space metric on* H, *for every neighbourhood* U *of 1 in* H, *there is a constant* $\mathrm{C}(=\mathrm{C}(\mathrm{U})) > 1$ *such that for all* $x \in \mathrm{V} \setminus \mathrm{U}$,

$$(*) \qquad \mathrm{C}^{-1} \log(1 + \|(x-1)\|) \leqslant d_{\mathrm{H}}(1, x) \leqslant \mathrm{C} \log(1 + \|x - 1\|)$$

where for $\mathrm{A} = \{\mathrm{A}_v\}_{v \in \mathrm{S}}$, $\|\mathrm{A}\| = \mathrm{Sup}\ \{\|\mathrm{A}_v\| \mid v \in \mathrm{S}\}$.

Proof. — Since $\mathbf{V}$ is isomorphic to a vector space over k, $\mathrm{V}/\mathrm{V} \cap \Gamma$ is compact. Suppose now that $g_n = b_n x_n$, $b_n \in \mathrm{B}$, $x_n \in \mathrm{V}$ is any sequence converging to a limit in $\mathscr{G}$. Since $\mathrm{V}/\mathrm{V}\cap\Gamma$ is compact, there is a sequence $\gamma_n \in \mathrm{V} \cap \Gamma$ such that $\{\gamma_n^{-1} x_n \mid n \in \mathbf{N}\}$ is relatively compact. Passing to a subsequence we assume that $\gamma_n^{-1} x_n$ converges to a limit y in V. This means that $b_n\gamma_n = (b_n x_n)(x_n^{-1}\gamma_n)$ converges to a limit h. On the other hand $b_n\gamma_n \in \Gamma$, a discrete subgroup of $\mathscr{G}$. It follows that $h = b_n\gamma_n$ for all large n. Thus $g = b_n\gamma_n y \in \mathrm{H}$. Hence H is closed in $\mathscr{G}$. Next for $v \in \mathrm{S}$, let $'\|\cdot\|$ be a vector space norm on $\mathbf{V}(k_v)$. Then there is a constant $\mathrm{C}_1 > 1$ such that for all $x \in \mathrm{V}(k_v)$

$$\mathrm{C}^{-1}\log(1 +' \|x\|) \leqslant \log(1 + \|(x-1)\|) \leqslant \mathrm{C}\log(1 +' \|x\|).$$

This follows from the following : let $e_1, \ldots, e_r$ be a basis of $\mathbf{V}(k_v)$ over k_v. Then the coordinates of any $x \in \mathbf{V}(k_v)$ w.r.t. this basis are polynomials in the entries of $(x-1)$ as a matrix in $\mathrm{GL}(n, k_v)$ and conversely. Thus for proving the inequality $(*)$ we may replace $\|(x-1)\|$ in that inequality by $'\|x\|$. To prove the compact generation of H, it is evidently sufficient to show that for $v \in \mathrm{S}$, $\mathbf{V}(k_v)$ is contained in the group generated by B and $\Omega = \{x \in \mathbf{V}(k_v) \mid\ '\|x\| \leqslant 1\}$. Our assumption that $\mathbf{V}_u(\mathrm{B}_v) = \mathbf{V}$ means that we can find a basis $e_1, \ldots, e_r$ of $\mathbf{V}(k_v)$ and elements $b_1, \ldots, b_r \in \mathrm{B}$ such that $'\|b_i^m e_i b_i^{-m}\|$ tends to zero as $m \to \infty$ for $1 \leqslant i \leqslant r$. One concludes in fact that there are constants $\mathrm{C}_1, \mathrm{C}_2 > 0$ and $\lambda_i > 1$ such that for all $m \in \mathbf{Z}$,

$$\mathrm{C}_1\lambda_i^{-m} \leqslant\ '\|b_i^m e_i b_i^{-m}\| \leqslant \mathrm{C}_2\lambda_i^{-m}.$$

We assume, as we may, that for $x = \sum_{1\leqslant i\leqslant r} x_i e_i$, $x_i \in k_v$,

$$'\|x\| = \max\{|x_i| \mid 1 \leqslant i \leqslant r\}.$$

One has then

$$'\|b_i^m(x_ie_i)b_i^{-m}\| \leqslant C_2\lambda_i^{-m} \mid x_i \mid^m .$$

If we now choose m so large that $\lambda_i^{-m} \mid x_i \mid < 1$ for all i, $\xi_i = b_i^m(x_ie_i)b_i^{-m} \in \Omega$ for every i, then

$$(**) \qquad x = \sum_{l \leqslant i \leqslant r} b_i^{-m}\xi_i b_i^m$$

evidently belongs to the group generated by Ω and B. Suppose now the $m = m(x)$ above is chosen as follows: $m(x) = 0$ if $x \in \Omega$; if $x \notin \Omega$, m is the smallest integer > 0 such that $c_2\lambda_i^{-m} \mid x_i \mid \leqslant 1$. If $x \notin \Omega$ one then has an integer i_0, $1 \leqslant i_o \leqslant r$ such that $C_2\lambda_{i_0}^{-m+1} \mid x_{i_o} \mid > 1$ leading to $m \leqslant A\log(1 + {}'\|x\|)$ for a suitable constant $A > 0$. The inequality evidently holds also if $x \in \Omega$. Let $M > 0$ be a constant such that $d_H(1, g) \leqslant M$ for all $g \in \Omega \cup \{b_1, ..., b_r\}$. Then from the expression (**) for x it is immediate that we have

$$d_H(1, x) \leqslant (2m + r)M \leqslant A' \log(1 +' \|x\|) + A''$$

for suitable constants A' and A''. The inequality (*) of the lemma is now clear : if we set $\mathcal{G}' = \prod_{v \in S} SL(n, k_v)$, the inequality $C^{-1}(\log 1 + \|(x - 1)\|) \leqslant d_H(1, x)$ is immediate from the discussion in 3.5 applied to the case $G = \mathcal{G}'$ since $d_H(1, x) \geqslant C' d_{\mathcal{G}'}(1, x)$ for all $x \in H$ outside a neighbourhood of 1 with a suitable constant $C' > 0$.

(3.19) *Corollary.* — *Suppose now that* **V**, Γ, B, H *are as in Lemma 3.18. Assume further that there is a semisimple k subgroup* **G** *of* GL(n) *such that* B *and* **V** *are contained in* **G**. *Then* $d_G|_V$ *is coarse Lipschitz equivalent to* $d_{H|_V}$.

(3.20) *Corollary.* — *Let* **V**, Γ, B, H *and* **G** *be as in Corollary (3.19) (with* S*-rank* **G** $\geqslant 2$*). Let* Θ *be the subgroup* B(**V** ∩ Γ). *Then* Θ *is finitely generated and* $d_G|_{V \cap \Gamma}$ *is Lipschitz equivalent to* $d_\Theta|_{V\cap\Gamma}$. *Also* $d_\Gamma|_{V\cap\Gamma}$ *is Lipschitz equivalent to* $d_\Theta|_{V\cap\Gamma}$.

Proof. — We need only observe (in the light of 3.2) that Θ is cocompact in H. For the second assertion observe that if $c > 0$ is such that $d_G(1, \gamma) \leqslant c$ for all the generators of Γ defining d_Γ then one has $d_G(1, \gamma) \leqslant c d_\Gamma(1, \gamma)$; also if we assume, as we may, that the set of generators for Γ (defining d_Γ) include a set of generators for Θ (which define d_Θ), then $d_\Theta(1, \gamma) \geqslant d_\Gamma(1, r)$ for all $\gamma \in H$.

The next lemma will be used to prove a generalization of Lemma 3.18.

(3.21) *Lemma.* — *Let* **V** *be a connected unipotent algebraic group over k and* **V**′ *be a connected k-subgroup. Assume that* **V**′ *and* **E** = **V**/**V**′ *are vector spaces on k. Further suppose that* *the commutator map* $(x, y) \mapsto xyx^{-1}y^{-1}$ *of* $\mathbf{V} \times \mathbf{V}$ *in* $\mathbf{V}'$ *induces a k-bilinear map* $c : \mathbf{E} \times \mathbf{E} \to \mathbf{V}'$ *whose image spans* $\mathbf{V}'$ *as a vector space. Then there exist k-morphisms* $L_i : \mathbf{V}' \to \mathbf{V}$ *over k and elements* $y_i \in \mathbf{V}(k)$, $1 \leqslant i \leqslant q = \dim \mathbf{V}$ *such that we have for* $z \in \mathbf{V}'$,

$$z = \prod_{1 \leqslant i \leqslant q} (L_i(z) y_i L_i(z)^{-1} y_i^{-1}).$$

Proof. — Since image c spans all of $\mathbf{V}'$ we can find vectors $\bar{x}_i, \bar{y}_i, \in \mathbf{E}(k)$ such that $e_i = c(\bar{x}_i, \bar{y}_i)$, $1 \leqslant i \leqslant q$ is a basis of $\mathbf{V}'(k)$ over k. Let $\ell_i : \mathbf{V}' \to \mathbf{E}$ be the linear map $\ell_i(\sum_{1 \leqslant j \leqslant q} z_j e_j) = z_i \bar{x}_i$ of $\mathbf{V}'$ in $\mathbf{E}$. Then clearly one has for $z = \sum_{1 \leqslant j \leqslant q} z_j e_j$, $z = \sum_{l \leqslant j \leqslant q} c(\ell_i(z), \bar{y}_i)$. Next observe that the natural map $\mathbf{V} \to \mathbf{E}$ admits a section $\sigma : \mathbf{E} \to \mathbf{V}$ defined over k. We need only take now $L_i = \sigma \circ \ell_i$.

(3.22) *Lemma.* — *Let* $\mathbf{V}$ *and* $\mathbf{V}' \subset \mathbf{V}$ *be unipotent k-subgroups of* $SL(n)$ *satisfying the conditions of Lemma 3.21. Let* $\Gamma \subset \mathbf{G}$ *be an* S*-arithmetic subgroup,* $B \subset \Gamma$ *a finitely generated subgroup normalizing* $\mathbf{V}$ *and* $\mathbf{V}'$. *We assume that the actions of* B *on* $\mathbf{E} = \mathbf{V}/\mathbf{V}'$ *and on* $\mathbf{V}'$ *are linear and that for every* $v \in S$, $\mathbf{E}_u(B_v) = \mathbf{E}$ *where* $B_v = B$ *regarded as a group acting as (linear) automorphisms of* $\mathbf{E}$ *over* k_v. *Let* $V = \prod_{v \in S} \mathbf{V}(k_v)$ *and* $H = B \cdot V$, *the subgroup generated by* B *and* V *in* $G(= \prod_{v \in S} G(k_v))$. *Then* H *is compactly generated. Moreover if* d_H *is a left translation invariant path space metric on* H, *for any neighbourhood* U *of* 1 *in* H, *there is a constant* $C = C(U) > 0$ *such that for all* $x \in H \setminus U$,

$$C^{-1} \log(1 + \|(x - 1)\|) \leqslant d_H(1, x) \leqslant C \log(1 + \|(x - 1)\|$$

If $\Theta = H \cap \Gamma$, *then* Θ *is finitely generated and* $d_\Theta |_{V \cap \Gamma}$ *Lipschitz equivalent to* $d_H |_{V \cap \Gamma}$, $d_G |_{V \cap \Gamma}$ *and also to* $d_\Gamma |_{V \cap \Gamma}$.

Proof. — Let $\mathbf{N}$ denote the Zariski closure of H. Then $\mathbf{N}$ is a k-subgroup of $GL(n)$. Let $\rho : \mathbf{N} \to GL(n')$ be a representation trivial on $\mathbf{V}'$ and inducing an isomorphism of $\mathbf{N}/\mathbf{V}'$ onto a k-subgroup of $GL(n')$. Let $\overline{B}$ = Image B and $\overline{\mathbf{V}}$ = Image $\overline{\mathbf{V}}'$ under ρ. Then the pair $(\overline{\mathbf{V}}, \overline{B})$ satisfy all the conditions imposed on $(\mathbf{V}, B)$ in Lemma 3.18. Thus if $\overline{H}$ (= Image H under ρ) is the group $\overline{B}\ \overline{V}$ where $\overline{V} = \prod_{v \in S} \mathbf{V}(k_v)$, $\overline{H}$ is compactly generated and one has for any compact neighbourhood U of 1 in $\overline{H}$, a constant $\overline{C} > 0$ such that $d_{\overline{H}}(1, \overline{V}) \leqslant \overline{C} \log(1 + \|(\overline{V} - 1)\|)$ for all $\bar{x} \in \mathbf{V} \setminus U$. Now let $x \in V$ be any element and $\bar{x}$ its image in $\mathbf{V}$. Let $\overline{\Omega}$ be a compact neighbourhood of 1 such that $\overline{\Omega}$ and $\overline{\Sigma}$ generates $\overline{H}$ where $\Sigma \subset B$ is a finite set generating B and $\overline{\Sigma}$ is its image in $\overline{B}$. Let $\Omega = \sigma(\overline{\Omega})$, where $\sigma : \mathbf{E} \to \mathbf{V}$ is a k-section for the map $\mathbf{V} \to \mathbf{E}$. We claim that $\Sigma \cup \Omega$ generates V. Since $\overline{\Sigma} \cup \overline{\Omega}$ generates $\overline{H}$, we need only show that V' is contained in the subgroup H_1 generated by Σ and Ω. Let now $z \in V'$; then by Lemma 3.21, one has

$$z = \prod_{1 \leqslant i \leqslant q} (L_i(z) y_i L_i(z)^{-1} y_i^{-1})$$

in the notation of that lemma. Clearly it suffices to show that $L_i(x)\, y_i L_i(x)^{-1} y_i$ belongs to the subgroup H_1. Now the commutator $\xi\eta\xi^{-1}\eta^{-1}$ for $\xi, \eta \in V$ depends only on the images of ξ and η in $\overline{V} = V/V'$. Suppose now that $\overline{L_i(x)}$(= image $L_i(x)$ in $\overline{V}$) is written as a product $\prod_{1 \leqslant j \leqslant \mu} \alpha_j$ with $\alpha_j \in \overline{\Omega}$ and $y_i = \prod_{1 \leqslant j \leqslant \mu} \beta_j$ with $\beta_j \in \overline{\Omega}$ then $\xi_i = \prod_{1 \leqslant j \leqslant \mu} \sigma(\alpha_j)$ and $\eta_i = \prod_{1 \leqslant j \leqslant \mu} \sigma(\beta_j)$ belong to H_1; and since ξ_i and $L_i(x)$ (resp. y_i and η_i) have the same image in $\mathbf{V}$, we see that $\xi_i \eta_i \xi_i^{-1} \eta_i^{-1} = L_i(x)\, y_i L_i(x)^{-1} y_i^{-1}$, belongs to H_1. This proves the compact generation of H. But the expression $z = \prod_{1 \leqslant i \leqslant q} L_i(z)\, y L_i(z)^{-1} y^{-1}$ contains more information. Let $x \in V$ be any element and $\overline{x}$ its image in $\overline{H}$. Now since ρ is a k-morphism one has a constant $C > 0$ such that for every $x \in \mathbf{V}(k_v)$,

$$\log(1 + \|(\overline{x} - 1)\|) \leqslant C \log(1 + \|(x - 1)\|).$$

(the norm on the left hand side is the norm in $M(n, k_v)$ while that on the right hand side is the norm on $M(n', k_v)$). Hence if $\overline{x} \in \overline{H} \setminus \overline{U}$. By Lemma 3.18, one sees that for all $\overline{x} \in \overline{H} \setminus \overline{U}$, $\overline{U}$ a compact neighbourhood of 1 in $\overline{H}$, there is a constant $\overline{C} > 0$ such that $d_{\overline{H}}(1, \overline{x}) \leqslant \overline{C} \log(1 + \|(x - 1)\|)$. Hence it follows from the definition of a path space metric, that there is a compact (generating set) neighbourhood $\overline{\Omega}$ of 1 in $\overline{V}$ such that any element $\overline{x} \in \overline{V} \setminus \overline{U}$ can be written as a product of not more than $N(\overline{x})$ elements from $\overline{\Omega} \cup \overline{\Sigma}$ where $N = N(\overline{x}) \leqslant \overline{C}' \log(1 + \|(x - 1)\|)$ for some constant $\overline{C}' > 0$. Let $\overline{x} = \overline{x}_1, \overline{x}_2, \dots, \overline{x}_N$ with $\overline{x}_i \in \overline{\Omega} \cup \overline{\Sigma}$ and let $\tilde{x} = \sigma(\overline{x}_1)\sigma(\overline{x}_2)\dots\sigma(\overline{x}_N)$: for $\overline{x} \in \overline{V}$, σ has already been defined; we extend it to all $\overline{\Omega} \cup \overline{\Sigma}$ by taking $\sigma\,|_{\overline{\Sigma}}$ to be any section to the map $\Sigma \to \overline{\Sigma}$. Then one has $x = \tilde{x}.\tilde{x}^{-1}x$ with $z = \tilde{x}^{-1}x \in V'$. Now $\|\tilde{x}^{-1}x\| \leqslant \|\tilde{x}^{-1}\|\|x\| \leqslant (\prod_{1 \leqslant i \leqslant N} \|\sigma(\overline{x}_i^{-1})\|)\|x\|$. If $A > 0$ is a constant such that $\|\sigma(g)\| \leqslant A$ for all $g \in \overline{\Omega} \cup \overline{\Sigma}$, we conclude that

$$\|z\| = \|\tilde{x}^{-1}x\| \leqslant A^N \|x\| \text{ and } N \leqslant \overline{C}' \log(1 + \|(x - 1)\|).$$

It follows that

$$\log \|z - 1\| \leqslant C' \log(1 + \|(x - 1)\|)$$

for all $x \in V$ with $\overline{x}$ outside a fixed neighbourhood $\overline{U}$ of 1 in $\overline{V}$ (C' depends on $\overline{U}$). The expression for z, $z = \prod_{1 \leqslant i \leqslant \ell} \xi_i \eta_i \xi_i^{-1} \eta_i^{-1}$, shows now that z is a product

$$z = \tau_1 \dots \tau_P$$

with $\tau_j \in \sigma(\overline{\Omega} \cup \overline{\Sigma})$ for $1 \leqslant j \leqslant P$ and $P \leqslant C'' \log(1 + \|(x - 1)\|)$ for a suitable constant $C'' > 0$. By definition $\tilde{x}$ is a product of N elements from $\sigma(\overline{\Omega} \cup \overline{\Sigma})$ with $N \leqslant \overline{C}' \log(1 + \|(x - 1)\|)$. Thus x is a product of at most $(\overline{C} + C'') \log(1 + \|(x - 1)\|)$ elements from $\sigma(\overline{\Omega} \cup \overline{\Sigma})$, it follows that $d_H(1, x) \leqslant A'(\overline{C}' + C'') \log(1 + \|(x - 1)\|)$ for all x in V with $\overline{x} \notin \overline{U}$, $\overline{U}$ a compact neighbourhood of 1 in $\mathbf{V}$ with $A' > 0$ an appropriate constant. Let $U \subset V$ be a compact neighbourhood of 1 such that image U contains

$\overline{\mathrm{U}}$. Then if $x \in \mathrm{V}$ with $\bar{x} \in \overline{\mathrm{U}}$, we can find $\xi \in \mathrm{U}$ with $\bar{\xi} = \bar{x}$ so that $z = \xi^{-1}x \in \mathrm{V}'$. Since U is compact, $\|g\| \leqslant \mathrm{M}$ for a suitable $\mathrm{M} > 0$ and all $g \in \mathrm{U}$. We conclude that $\|\xi^{-1}x\| \leqslant \mathrm{M}\|x\|$. Thus $z = \xi^{-1}x$ can be expressed as a product of not more than constant $\cdot \log(1 + \|(x-1)\|)$ elements for $\sigma(\overline{\Omega} \cup \overline{\Sigma})$ provided that $x \notin \mathrm{U}$ proving that for $x \notin \mathrm{U}$, $d_{\mathrm{H}}(1, x) \leqslant$ const $\cdot \log(1 + \|(x-1)\|)$. Thus given a neighbourhood U of 1 in V, there is a constant $\mathrm{C} > 0$ such that for all $x \in \mathrm{V} \setminus \mathrm{U}$,

$$\mathrm{C}^{-1} \log(1 + \|(x-1)\|) \leqslant d_{\mathrm{H}}(1, x) \leqslant \mathrm{C} \log(1 + \|(x-1)\|).$$

Since Θ is a cocompact subgroup of H, one concludes that $d_\Theta\,|_{\mathrm{V}\cap\Gamma}$ and $d_{\mathrm{H}}\,|_{\mathrm{V}\cap\Gamma}$ are Lipschitz equivalent. On the other hand the inequality above shows that $d_{\mathrm{H}}\,|_{\mathrm{V}\cap\Gamma}$ is Lipschitz equivalent to $d_{\mathrm{G}}\,|_{\mathrm{V}\cap\Gamma}$ (cf. (3.5); note that $\mathbf{G} \subset \mathrm{SL}(n)$). On the other hand there are constant $\mathrm{A}, \mathrm{A}' > 0$ such that

$$d_{\mathrm{G}}(1, \gamma) \leqslant \mathrm{A} d_\Gamma(1, \gamma) \quad \text{for all} \quad \gamma \in \Gamma$$
$$d_\Gamma(1, \gamma) \leqslant \mathrm{A}' d_\Theta(1, \gamma) \quad \text{for all} \quad \gamma \in \Theta.$$

The Lipschitz equivalence of $d_{\mathrm{G}}, d_\Theta, d_\Gamma$ and d_{H} all restricted to $\mathbf{V} \cap \Gamma$ now follows.

(3.23) *Lemma.* — *Let* $\mathbf{V}$, $\mathbf{V}' \subset \mathbf{V}$ *be unipotent k-subgroups of a reductive k-group* $\mathbf{G} \subset \mathrm{SL}(n)$. *Let* $\Gamma \subset \mathbf{G}$ *be a finitely generated* S*-arithmetic group and* $\mathrm{B} \subset \Gamma$ *a finitely generated subgroup normalizing* $\mathbf{V}$ *and* $\mathbf{V}'$. *Assume that* $\mathbf{E} = \mathbf{V}/\mathbf{V}'$ *carries a vector space structure such that the natural action of* B *on* $\mathbf{E}$ *is linear. Suppose further that for all* $v \in \mathrm{S}$, $\mathbf{E}_u(\mathrm{B}_v) = \mathbf{E}$ *where* B_v *is* B *regarded as* k_v*-automorphisms of* $\mathbf{E}$. *Finally assume that* $d_{\mathrm{G}}\,|_{\mathrm{V}'\cap\Gamma}$ *is Lipschitz equivalent to* $d_\Gamma\,|_{\mathrm{V}'\cap\Gamma}$. *Then* $d_{\mathrm{G}}\,|_{\mathrm{V}\cap\Gamma}$ *is Lipschitz equivalent to* $d_\Gamma\,|_{\mathrm{V}\cap\Gamma}$.

Proof. — Let $\mathbf{H}^*$ be the Zariski closure of $\mathrm{H} = \mathrm{B}.\mathrm{V}$ and ρ a faithful representation of $\mathbf{H}^*/\mathbf{V}'$ in $\mathrm{GL}(n')$ for some n'. We will treat ρ also as a representation of $\mathbf{H}^*$. Let $\rho(\mathbf{V}) = \overline{\mathbf{V}}$, $\rho(\mathrm{B}) = \overline{\mathrm{B}}$ and $\rho(\mathrm{H}) = \overline{\mathrm{H}}$ Let $\Theta = \mathrm{B}(\mathrm{V} \cap \Gamma)$ and $\overline{\Theta} = \rho(\Theta)$. Then by Lemma 3.18 and Corollary 3.20 (applied to **B, V, H**) we see that if $\overline{\Sigma} \subset \rho(\mathrm{H} \cap \Gamma)$ is a finite symmetric set of generators, then for $\gamma \in \mathrm{V} \cap \Gamma \subset \mathrm{H}$, the image $\bar{\gamma} = \rho(\gamma) \in \overline{\Theta}$ is a product $\alpha_1 \cdot \ldots \cdot \alpha_{\mathrm{N}}$ with $\alpha_i, \in \overline{\Sigma}$ and $\mathrm{N} \leqslant \mathrm{C} \log(1 + \|(\bar{\gamma} - 1)\|)$ for a suitable constant $\mathrm{C} > 0$. Let $\Sigma \subset \mathrm{H}$ be a subset that maps bijectively onto $\overline{\Sigma}$ and for $\alpha \in \overline{\Sigma}$, let $\tilde{\alpha}$ be the unique element of Σ lying over it. Let $\tilde{\gamma} = \tilde{\alpha}_1 \ldots \tilde{\alpha}_{\mathrm{N}}$, then one has $\|\tilde{\gamma}^{-1}\| \leqslant \mathrm{A}^{\mathrm{N}}$ where $\mathrm{A} = \mathrm{Sup}\,\{\|\tilde{\alpha}\| \mid \alpha \in \Sigma\}$. Now $\theta = \tilde{\gamma}^{-1}\gamma \in \mathrm{V}'$ and one has clearly $\|\theta\| = \|\tilde{\gamma}^{-1}\gamma\| \leqslant \mathrm{A}^{\mathrm{N}}\|\gamma\| \leqslant \mathrm{A}^{\mathrm{C}\log(1+\|(\gamma-1\|)}\|\gamma\|$. One concludes from this that there is a constant $\mathrm{C}' > 0$ such that

$$\log\,(1 + \|(\theta - 1)\|) \leqslant \mathrm{C}' \log(1 + \|\gamma - 1\|)$$

for all $\gamma \in \mathrm{H}$. Since $d_{\mathrm{G}}\,|_{\mathrm{V}'\cap\Gamma}$ is Lipschitz equivalent $d_{\Gamma|\mathrm{V}'\cap\Gamma}$, θ is expressible as a product of N' elements from a finite set of generators Σ_1 of Γ with $\mathrm{N}' \leqslant \mathrm{C}'' \log(1 + \|(\gamma - 1)\|)$ for some $\mathrm{C}'' > 0$. It follows that $\gamma = \tilde{\gamma}\theta$ is a product of $\mathrm{N} + \mathrm{N}' = \mathrm{N}''$ elements from $\Sigma \cup \Sigma'$

where $N'' \leqslant C'' \log(1 + \|(\gamma - 1)\|$. Thus $d_\Gamma(1, \gamma) \leqslant C_1 d_G(1, \gamma)$ for some $C_1 > 0$ and all $\gamma \in V \cap \Gamma$. This shows that $d_\Gamma|_{V\cap\Gamma}$ and $d_G|_{V\cap\Gamma}$ are Lipschitz equivalent.

(3.24) *Proof of 3.13. Case 1:* $|S| \geqslant 2$. — Let $\varphi \in \Phi$ be a root such that 2φ is *not* a root. Set $\mathbf{V} = \mathbf{U}(\varphi)$ and $B = \mathbf{T} \cap \Gamma$ where $\Gamma \subset G$ is an S-arithmetic group in $\mathbf{G}$. Then B is finitely generated. The pair $(\mathbf{V}, B)$ then satisfies all the conditions of (3.18): Note that since $\mathbf{T}$ acts on $\mathbf{V}$ linearly through the non-trivial character φ, $\mathbf{V}_u(\mathbf{T}(k_v)) = \mathbf{V} = \mathbf{V}_u(B)$ for all $v \in S$ (3.16). Proposition 3.13 for this case is now a restatement of Corollary 3.20. Next suppose that $\varphi \in \Phi$ and $2\varphi \in \Phi$. Here we appeal to Lemma 3.23 taking $\mathbf{V} = \mathbf{U}(\varphi)$, $\mathbf{V}' = \mathbf{U}(2\varphi)$ and $B = \mathbf{T} \cap \Gamma$, Γ an S-arithmetic subgroup of $\mathbf{G}$. All the assumptions made in that lemma are satisfied and we conclude that $d_G|_{V\cap\Gamma}$ is Lipschitz equivalent to $d_\Gamma|_{V\cap\Gamma}$.

(3.25) *Proof 3.13. Case 2:* $|S| = 1$, $S = \{v\}$, *k-rank* $\mathbf{G} \geqslant 2$. — Observe first that we may assume that $\varphi \in \Delta$ in proving (3.13). This is because for any $\varphi \in \Phi$ there is an element ω in the k-Weyl group of $\mathbf{G}$ such that $\omega(\varphi) \in \Delta$ or $\omega(\varphi)/2 \in \Delta$; and in the latter case $\mathbf{U}(\varphi) \subset \mathbf{U}(\varphi/2)$ ($\varphi/2 \in \Phi$). Thus we assume that φ is a simple k-root. Since $\mathbf{G}$ is k-simple there is a root $\psi \in \Delta$ with $\langle \varphi, \psi \rangle \neq 0$. We may evidently replace $\mathbf{G}$ by the group $\mathbf{G}'$ generated $\mathbf{U}(\varphi)$, $\mathbf{U}(-\varphi)$, $\mathbf{U}(\psi)$ and $U(-\psi)$ for proving (3.13). In other words, we can assume that k-rank $\mathbf{G} = 2$. Thus $\Delta = \{\alpha, \beta\}$. Let $\mathbf{V}(\alpha)$ be the group generated by $\{\mathbf{U}(\varphi) \mid \varphi \in \Phi, \varphi = m\alpha + n\beta \text{ with } m > 0\}$ and let $M(\beta) =$ group generated by $\mathbf{U}(\beta)$ and $\mathbf{U}(-\beta)$. Then $\mathbf{M}(\beta)$ normalizes $\mathbf{V}(\alpha)$ and $\mathbf{M}(\beta)$ is a k-simple group of k-rank 1. Also $\mathbf{T}(\beta)$, the identity component of $\mathbf{T} \cap \mathbf{M}(\beta)$ is a maximal split torus in $\mathbf{M}(\beta)$. For an integer $t \geqslant 0$, let $\mathbf{V}(\alpha)_t$ = group generated by $\{\mathbf{U}(\varphi) \mid \varphi \in \Phi, \varphi = m\alpha + n\beta \text{ with } m > t\}$. Then it is known that $\mathbf{E}(\alpha)_t = \mathbf{V}(\alpha)_t / \mathbf{V}(\alpha)_{t+1}$ are in a natural fashion k-vector spaces (for $t \geqslant 0$) and that the action of $\mathbf{M}(\beta)$ on $\mathbf{E}(\alpha)_t$ is linear. (Each $\mathbf{E}(\alpha)_t$ is evidently naturally isomorphic to the direct product of the $\{\mathbf{U}(\varphi) \mid \varphi \in \Phi, \ \varphi = (t+1)\alpha + s\beta\}$. Since each $\mathbf{U}(\varphi)$ carries a vector space structure we can equip $\mathbf{E}(\alpha)_t$ with the direct sum vector space structure. This vector space structure affords another description. Let $\mathbf{T}'(\beta)$ be the identity component of the kernel of β in $\mathbf{T}$. Then for $v \in \mathbf{E}(\alpha)_t(\bar{k})$ and $\lambda \in \bar{k}^*$, we define λv as the class of $\tilde{\lambda}\tilde{v}\tilde{\lambda}^{-1}$ modulo $\mathbf{V}(\alpha)_{t+1}$ where $\tilde{v} \in \mathbf{V}(\alpha)_{t+1}$ is any lift of v and $\tilde{\lambda} \in \mathbf{T}'(\beta)(\bar{k})$ is an element such that $(t+1)\alpha(\tilde{\lambda}) = \lambda$. Since $\mathbf{M}(\beta)$ and $\mathbf{T}'(\beta)$ commute, it is clear that the action of $\mathbf{M}(\beta)$ on $\mathbf{E}_\alpha(t)$ is linear for the above vector space structure.) Moreover, the eigen-characters of $\mathbf{T}(\beta)$ acting on $\mathbf{E}(\varphi)_t$ is precisely the set

$$\mathscr{E}(t) = \{\varphi \in \Phi \mid \varphi = (t+1)\alpha + m\beta \text{ for some } m\}.$$

Let $\mathscr{E}'(t) = \{\varphi \in \mathscr{E}(t) \mid \varphi \text{ is non-trivial on } \mathbf{T}(\beta)\}$ and let $\Lambda = \mathbf{M}(\beta) \cap \Gamma$. Then one has (by (3.16)) that $(\mathbf{E}(\alpha)_t)_u(\Lambda) = (\mathbf{E}(\alpha)_t)_u(\mathbf{M}(\beta)(k_v))$; on the other hand $(\mathbf{E}(\alpha)_t)_u(\mathbf{M}(\beta)(k_v)) \supset (\mathbf{E}(\alpha)_t)_u(\mathbf{T}(\beta)(k_v)) = \Sigma_{\varphi \in \mathscr{E}'(t)} \mathbf{W}(\varphi)$, where $\mathbf{W}(\varphi)$ = Image $\mathbf{U}(\varphi)$ in $\mathbf{E}(\alpha)_t$.

We observe that as the root system Φ being of rank 2, one has only the following possibilities (assuming that $\langle \alpha, \alpha \rangle \geqslant \langle \beta, \beta \rangle$):

– **Type** A_2 : $\Phi=\{\pm\alpha,\ \pm\beta,\ \pm(\alpha+\beta)\}$
– **Type** B_2 : $\Phi=\{\pm\alpha,\ \pm\beta,\ \pm(\alpha+\beta),\ \pm(\alpha+2\beta)\}$
– **Type** G_2 : $\Phi=\{\pm\alpha,\ \pm\beta,\ \pm(\alpha+\beta),\ \pm(\alpha+2\beta),\ \pm(\alpha+3\beta),\ \pm 2\alpha+3\beta\}$
– **Type** BC_2 : $\Phi=\{\pm\alpha,\ \pm\beta,\ \pm 2\beta,\ \pm(\alpha+\beta),\ \pm(\alpha+2\beta),\ \pm(2\alpha+2\beta)\}$: This is the only "non-reduced" case.

Type A_2: Here we observe that $\mathbf{V}(\alpha)_t=0$ for $t\geqslant 1$ so that $\mathbf{V}(\alpha)=\mathbf{E}(\alpha)_0=\mathbf{W}(\alpha)+\mathbf{W}(\alpha+\beta)$. It follows that $\mathbf{V}(\alpha)$ is a vector space and $\mathbf{V}(\alpha)_u(\mathbf{T}(\beta)(k_v))=\mathbf{V}(\alpha)$. From (3.17) it follows then that there is a finitely generated subgroup $B\subset\Lambda(=(M(\beta)\cap\Gamma))$ such that $(B,\mathbf{V}=\mathbf{V}(\alpha))$ satisfy all the hypotheses in (3.18). It follows now from (3.20) that Proportion 3.13 holds for $\mathbf{U}(\alpha)$. Since Φ is of type A_2 and β is a Weyl group transform of α, (3.13) holds for β as well and hence for any $\varphi\in\Phi$.

Type B_2. Here we have to show that (3.13) holds with $\varphi=\alpha$ or $\varphi=\beta$ separately as the two are not conjugates under the Weyl group. Consider first the case $\varphi=\alpha$. Then $\mathbf{V}(\alpha)$ is generated by $U(\alpha)$, $U(\alpha+\beta)$ and $U(\alpha+2\beta)$. It is a k-vector space and one has $\mathbf{V}(\alpha)_t=0$ for $t\geqslant 1$ so that $\mathbf{E}(\alpha)=\mathbf{V}(\alpha)=\mathbf{W}(\alpha)\oplus\mathbf{W}(\alpha+\beta)\oplus\mathbf{W}(\alpha+2\beta)$. The characters α and $\alpha+2\beta$ are both non-trivial on $\mathbf{T}(\beta)$. Let $\mathbf{V}=\mathbf{V}(\alpha)_u(M(\beta)(k_v))$; then $\mathbf{V}=\mathbf{V}_u(\mathbf{M}_\beta(k_v))$ and it contains $\mathbf{U}(\alpha)$. Let $B\subset\Lambda(=M\cap\Gamma)$ be a finitely generated subgroup such that $\mathbf{V}_u(B)=\mathbf{V}_u$: such a B exists by Corollary 3.17. Then the pair $(\mathbf{V},B)$ satisfy the hypotheses of Lemma 3.18. By Corollary 3.20. $d_G\,|_{V\cap\Gamma}$ and $d_\Gamma\,|_{V\cap\Gamma}$ are Lipschitz equivalent. Since $\mathbf{U}(\alpha)\subset\mathbf{V}$, $d_G\,|_{U(\alpha)\cap\Gamma}$ is Lipschitz equivalent to $d_\Gamma\,|_{U(\alpha)\cap\Gamma}$. To deal with the root β, consider the unipotent group $\mathbf{V}=\mathbf{V}(\beta)$. Let $\mathbf{V}'=\mathbf{U}(\alpha+2\beta)$. Let $\mathbf{M}(\alpha)$ be the group generated by $U(\pm\alpha)$ and $\mathbf{T}(\alpha)$ the identity connected component of $\mathbf{T}\cap\mathbf{M}(\alpha)$; then $\mathbf{T}(\alpha)$ is a maximal k-split torus in $\mathbf{M}(\alpha)$. The group $\mathbf{M}(\alpha)$ normalizes $\mathbf{V}$ as well as $\mathbf{V}'$. Since $\alpha+2\beta$ is a long root it is conjugate to α under the Weyl group. Hence by what we have shown above $d_G\,|_{V'\cap\Gamma}$ is Lipschitz equivalent to $d_\Gamma\,|_{V'\cap\Gamma}$. On the other hand in $\mathbf{E}=\mathbf{V}/\mathbf{V}'$ the eigen characters for the action of $\mathbf{T}(\alpha)$ are precisely the restrictions of β and $\beta+\alpha$ to $\mathbf{T}(\alpha)$; since β and $\beta+\alpha$ are both non-trivial on $\mathbf{T}(\alpha)$ (α is a long root) we see that $\mathbf{E}_u(T(\alpha)(k_v))=\mathbf{E}$. From (3.17) once again we can find $B\subset\Lambda=M(\alpha)\cap\Gamma$ which is finitely generated and such that $\mathbf{E}_u(\mathbf{B})=\mathbf{E}$. Thus Lemma 3.23 applies to $(\mathbf{V},\mathbf{V}',B)$ and we conclude that d_G and d_Γ are Lipschitz equivalent on $\mathbf{V}\cap\Gamma$. Since $\mathbf{V}\supset\mathbf{U}(\beta)$ we see that (3.13) holds for $\varphi=\beta$. This completes the proof in the case of Type B_2.

Type G_2. Consider here the root system generated by $(\alpha,\alpha+3\beta)$. This is of type A_2. By replacing the group $\mathbf{G}$ by the group generated by $U(\pm\alpha)$, $U(\pm(\alpha+3\beta))$ which is again k-simple with a reduced root system of type A_2, we see by the preceding that d_G and d_Γ are Lipschitz equivalent when restricted to $U(\alpha+3\beta)\cap\Gamma$. Let $\mathbf{V}=\mathbf{V}(\alpha)$ and $\mathbf{V}'=\mathbf{U}(2\alpha+3\beta)(=\mathbf{V}(\alpha)_1)$. Then $\mathbf{M}(\beta)$ normalizes $\mathbf{V}$ as well as $\mathbf{V}'$. Moreover, $\mathbf{V}'$ and $\mathbf{V}/\mathbf{V}'=\mathbf{E}$ have natural k-vector space structures for which the actions of $\mathbf{M}(\beta)$ is linear. Now the eigen-characters for $\mathbf{T}(\beta)$ acting on $\mathbf{E}$ are precisely the restrictions to

$\mathbf{T}(\beta)$ of the roots $\alpha, \alpha+\beta, \alpha+2\beta, \alpha+3\beta$ and every one of them is non-trivial. Thus $\mathbf{E}_u(\mathbf{T}(\beta)(k_v)) = \mathbf{E}$. We can now apply (3.17) and (3.22) as before to conclude that d_G and d_Γ are Lipschitz equivalent on $\mathbf{V} \mid \mathbf{V} \cap \Gamma$. Now $\mathbf{U}(\alpha+\beta) \subset \mathbf{V}$ and $\alpha+\beta$ is a short root. It is clear that d_G and d_Γ are Lipschitz equivalent on $\mathbf{U}(\alpha+\beta)$ and hence (since $\alpha+\beta$ is conjugate to β under the Weyl group) on $\mathbf{U}(\beta)$ as well. The proof in the case Φ is of type G_2 is thus complete.

Type BC_2. The root α being long, $\Delta' = \{\alpha, 2\beta\}$ is a simple root system for the k-group $\mathbf{G}'$ generated by $\mathbf{U}(\pm\alpha)$ and $\mathbf{U}(\pm 2\beta)$; and the root system of $\mathbf{G}'$ is reduced of type B_2 and $\mathbf{G}'$ is k-simple. Thus by appealing to the case of type B_2, we see that $d_{G'}$ and d_Γ are Lipschitz equivalent on $\mathbf{U}(\alpha) \cap \Gamma$ and $\mathbf{U}(2\beta) \cap \Gamma$ and hence also the metrics d_G and d_Γ are Lipschitz equivalent on $\mathbf{U}(\alpha) \cap \Gamma$ and $\mathbf{U}(2\beta) \cap \Gamma$ hence also on $\mathbf{U}(\alpha+2\beta)$ and $\mathbf{U}(2\alpha+2\beta)$ ($\alpha+2\beta$ and $2\alpha+2\beta$ are Weyl group transforms of α and 2β respectively). Consider now the group $\mathbf{V} = \mathbf{U}(\beta)\mathbf{U}(\alpha+\beta)\mathbf{U}(2\beta)\mathbf{U}(\alpha+2\beta)\mathbf{U}(2\alpha+2\beta)$. Let $\mathbf{V}' = \mathbf{U}(2\beta)\mathbf{U}(\alpha+2\beta)\mathbf{U}(2\alpha+2\beta)(= \mathbf{V}(\beta)_1)$. Then $\mathbf{M}(\alpha)$ normalizes $\mathbf{V}$ and $\mathbf{V}'$ and if $\mathbf{E} = \mathbf{V}/\mathbf{V}'$, the eigen-characters of $\mathbf{T}(\alpha)$ acting on $\mathbf{E}$ are precisely β and $\alpha+\beta$ (restricted to $\mathbf{T}(\alpha)$) and they are both nontrivial on $\mathbf{T}(\alpha)$ so that $\mathbf{E}_u(\mathbf{M}(\alpha)(k_v)) = \mathbf{E}$. We can again apply (3.23) to conclude that d_G and d_Γ are Lipschitz equivalent on $\mathbf{V} \cap \Gamma$. (d_G and d_Γ are Lipschitz equivalent on $\mathbf{V}' \cap \Gamma$ as $\mathbf{V}'$ is the direct product of $U(2\beta)$, $U(\alpha+2\beta)$ and $\mathbf{U}(2\alpha+2\beta)$). This concludes the proof of (3.13) for the case k-rank $\mathbf{G} \geqslant 2$.

(3.26). — We shall use the following corollary of Chevalley commutation relations.

Lemma. — *Let* $\mathbf{G}$ *be a connected semisimple algebraic group over an algebraically closed field* $\bar{k}$. *Let* T *be a maximal* $\bar{k}$*–split torus of* $\mathbf{G}$ *and* Φ *the root system of* $\mathbf{G}$ *with respect to* T. *For* $\alpha \in \Phi$, *let* $\mathbf{U}(\alpha)$ *denote the unipotent* 1*-parameter subgroup of* $\mathbf{G}$ *corresponding to* α. *Let* $\varphi, \theta \in \Phi$ *be a pair of distinct roots such that* $\varphi - \theta = \varphi'$ *is a root but* $\varphi - 2\theta$ *is not a root. Let* $\mathbf{V}$ *(resp* $\mathbf{V}'$, *be the subgroup of* $\mathbf{G}$ *generated by* $\mathbf{U}(m\varphi' + n\theta)$, $m > 0$, $n \geqslant 0$ *(resp.* $\mathbf{U}(m\varphi' + n\theta)$, $m > 1$, $n \geqslant 0$). *Then* $\mathbf{V}'$ *is a normal subgroup of* $\mathbf{V}$ *and* $\mathbf{V}/\mathbf{V}'$ *has a natural structure of a vector space over* $\bar{k}$ *on which the action of* $\mathbf{U}(\theta)$ *is linear. Moreover* $(\mathbf{V}/\mathbf{V}')(\bar{k}) \cong \mathbf{V}(\bar{k})/\mathbf{V}'(\bar{k})$ *is generated by* $\mathbf{U}(\varphi')$ *as a module over* $\bar{k}[\mathbf{U}(\theta)]$. *If* $\varphi + \theta$ *is not a root,* $\mathbf{U}(\varphi)(\bar{k}) = \{xyx^{-1}y^{-1} \mid x \in U(\theta), y \in U(\varphi - \theta)\}$.

Proof. — This is essentially consequence of the Chevalley commutation relations. These relations assert the following: There is a collection X_α : Add $\to$ $\mathbf{G}$ of isomorphisms of the additive group Add (over $\bar{k}$) onto the subgroup $\mathbf{U}(\alpha)$ with the following property: for α, β roots with $\alpha+\beta$ a *root*, $\alpha - \beta$ *not* a root, and length $\alpha \geqslant$ length β $X_\alpha(t)X_\beta(s)X_\alpha(t)^{-1}X_\beta(s)^{-1} = X_{\alpha+\beta}(N_{\alpha\beta} \cdot ts)\xi(t, s)$ where $\xi(t, s)$ is a product of elements belonging to the group generated by the $\{\mathbf{V}(m\alpha + n\beta) \mid n > 1\}$ with $N_{\alpha\beta} = \pm 1$. (Note that since length $\alpha \geqslant$ length β, if $m\alpha + n\beta$ is a root with $m > 1$, then $n > 1$ as well) (see [St]). We now take $\alpha = \varphi - \theta$ and $\beta = \theta$; since $\alpha + 2\beta = \varphi$ is a root, one sees that length $\alpha(\geqslant$ length $\beta)$ and the lemma is now immediate. When α, β have the same root length, then $m\alpha + n\beta$ is a root for $n \geqslant 1$, if and only if $m = 0$ or 1 and $n = 1$. This proves the second assertion.

(3.27) *Case 3.* $S = \{v\}$, *k-rank* $\mathbf{G} = 1$, $\mathbf{C}$ *contains a nontrivial* k_v*-split torus.* — Recall that $\mathbf{C}$ is the maximal anisotropic central torus in $\mathcal{Z}(\mathbf{T})$, the centralizer of $\mathbf{T}$. Thus $Z(\mathbf{T}) = \mathbf{TCM}$ with $\mathbf{M} = [Z(\mathbf{T}), Z(\mathbf{T})]$ an anisotropic semisimple group over k. The simple k-root system Δ of $\mathbf{G}$ w.r.t. $\mathbf{T}$ now consists of a single element α. In the absolute simple root system $\widetilde{\Delta}$ of $\mathbf{G}$ w.r.t. $\widetilde{\mathbf{T}}$ (cf. (3.11) for notation), the set $\widetilde{\alpha} = \{\varphi \in \widetilde{\Delta} \mid \varphi\, |_{\mathbf{T}} = \alpha\}$ consists of one or two roots (this is true for any k-rank 1, k simple group); in the case at hand i.e. when $\mathbf{C}$ is non-trivial, we assert that $|\, \widetilde{\alpha}\, | = 2$. Let $\widetilde{\Delta_{\mathbf{M}}} = \{\varphi \in \widetilde{\Delta} \mid \mathbf{U}(\varphi) \subset \mathbf{M}\}$. Since $\mathbf{C}$ is central in $Z(\mathbf{T})$ the centralizer $Z(\mathbf{T})$ of $\mathbf{T}$ is also the centralizer of $\mathbf{CT}$. Thus if $\varphi \in \widetilde{\Phi}$ is trivial on $\mathbf{T}$ (i.e. if $\mathbf{U}(\varphi)$ centralizes $\mathbf{T}$), $\mathbf{U}(\varphi)$ centralizes $\mathbf{CT}$ as well. Thus every $\varphi \in \widetilde{\Delta}$ which is trivial on $\mathbf{T}$ is trivial on $\mathbf{CT}$. On the other hand, $\bigcap_{\varphi\in\widetilde{\Delta}} \operatorname{Ker}\varphi$ is a *finite* subgroup of $\widetilde{\mathbf{T}}$. Thus we see that $\bigcap_{\{\varphi\in\widetilde{\Delta}\ |\varphi|_{\mathbf{T}}\ \text{non trivial}\}}$ (Kernel φ in $\mathbf{TC}$) has to be finite. This means, since $\dim \mathbf{C} \geqslant 1$, that $|\, \{\varphi \in \widetilde{\Delta} \mid \varphi\, |_{\mathbf{T}}$ nontrivial $\}\, | \geqslant 2$. On the other hand if $\varphi \in \widetilde{\Delta}$, and $\varphi\, |_{\mathbf{T}}$ is non-trivial, then $\varphi\, |_{\mathbf{T}} = \alpha$. Thus $|\, \widetilde{\alpha}\, | \geqslant 2$, hence $|\, \widetilde{\alpha}\, | = 2$. Since $\widetilde{\alpha} = \{\varphi \in \widetilde{\Delta} \mid \varphi\, |_{\mathbf{T}}$ is non-trivial $\}$ one sees that $\dim \mathbf{TC} = 2$ and hence $\dim \mathbf{C} = 1$.

Let $\widetilde{\alpha} = \{\beta_1, \beta_2\}$. The Galois group $\mathscr{G} = \operatorname{Gal}(k_s/k)$ of a separable closure k_s of k over k operates on the character group of $\widetilde{\mathbf{T}}$ stabilizing Φ. Moreover since $\mathbf{T}$ is split over k, it is immediate that for $\sigma \in \mathscr{G}$ and $\varphi \in \widetilde{\Phi}$, $\sigma(\varphi) = \varphi$ on $\mathbf{T}$. It follows that we have for $\sigma \in \mathscr{G}$

$$\sigma(\beta_1) = \beta_{i_\sigma} + \sum_{\varphi\in\widetilde{\Delta}_{\mathbf{M}}} m(\varphi)\varphi$$

with $i_\sigma = 1$ or 2. Since $\dim \mathbf{C} = 1$ and $\mathbf{C}$ is anisotropic, there is a $\sigma_0 \in \mathscr{G}$ such that $\sigma_0(\beta_1) = -\beta_1$ on $\mathbf{C}$. Since $\operatorname{Ker}\beta_1 \cap \operatorname{Ker}\beta_2 \cap \mathbf{C}$ is finite, at least one of β_1 or β_2 say β_1 is non-trivial on $\mathbf{C}$; then $\sigma_0(\beta_1) \neq \beta_1$ on $\mathbf{C}$ while all $\varphi \in \widetilde{\Delta}_{\mathbf{M}}$ are trivial on $\mathbf{C}$. Thus we see that $\beta_{i_{\sigma_0}} \neq \beta_1$ i.e. $\beta_{i_{\sigma_0}} = \beta_2$ and in fact $\beta_2 = \beta_1^{-1}$ on $\mathbf{C}$.

Suppose now that $\varphi \in \widetilde{\Phi}$ is any root such that $\varphi\, |_{\mathbf{T}} = \alpha$. Then one has $\varphi = \beta_i + \sum_{\varphi\in\widetilde{\Delta}_{\mathbf{M}}} m(\varphi)\varphi$ with $i = 1$ or 2. It is immediate from this that all the eigen-characters of $\mathbf{C}$ acting on $\mathbf{E} = \mathbf{U}(\alpha)/\mathbf{U}(2\alpha)$ (where we set $\mathbf{U}(2\alpha) = 0$ if 2α is not a root) are non-trivial. Set $\mathbf{V} = \mathbf{U}(\alpha)$ and $\mathbf{V}' = \mathbf{U}(2\alpha)$. Then $\mathbf{V}'$ and $\mathbf{E} = \mathbf{V}/\mathbf{V}'$ are vector spaces in a natural fashion for which the action of $Z(\mathbf{T})$ is linear. Moreover, $\mathbf{V}'$ is central in $\mathbf{V}$ and the commutation map $\mathbf{V} \times \mathbf{V} \to \mathbf{V}'$, $(x, y) \mapsto xyx^{-1}y^{-1}$ defines a k-bilinear map $c : \mathbf{E} \times \mathbf{E} \to \mathbf{V}'$. We now assert that the image of c spans $\mathbf{V}'$ as a vector space. To see this observe first that since $|\widetilde{\alpha}| = 2$, $\widetilde{\Delta}$ is simply laced (Tits classification) so that all roots lengths in $\widetilde{\Delta}$ are equal. Next let $\widetilde{\Phi}^* = \{\varphi \in \widetilde{\Phi} : \varphi\, |_{\mathbf{T}} = 2\alpha\}$. Then $\mathbf{V}'$ is spanned by the $\{\mathbf{U}(\varphi) : \varphi \in \widetilde{\Phi}^*\}$. Thus it suffices to show that $\mathbf{V}'$

contains $\mathbf{U}(\varphi)$ for every $\varphi \in \tilde{\Phi}^*$. Let k' be a Galois extension of k over which $\tilde{\mathbf{T}}$ splits and let $\mathcal{G}' = \mathrm{Gal}(k'/k)$. Then $\mathcal{G}'$ acts on the character group $\mathrm{X}(\tilde{\mathrm{T}})$ leaving $\tilde{\Phi}$ stable and hence leaves invariant the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{X}(\tilde{\mathbf{T}}) \otimes \mathbf{R}$. We identify $\mathrm{X}(\mathbf{T}) \otimes \mathbf{R}$ as the subspace orthogonal to the kernel of the natural map $\mathrm{X}(\tilde{\mathbf{T}}) \otimes \mathbf{R} \to \mathrm{X}(\mathbf{T}) \otimes \mathbf{R}$ for this inner product. Under this identification – note that $\mathrm{X}(\mathbf{T}) \otimes \mathbf{R}$ is precisely the space of $\mathcal{G}'$-invariants in $\mathrm{X}(\tilde{\mathbf{T}}) \otimes \mathbf{R}$ – one sees easily that for $\varphi \in \tilde{\Phi}$, $n\varphi \mid_{\mathrm{T}} = \sum_{\sigma \in \mathcal{G}'} \sigma(\varphi)$, where $n = |\mathcal{G}'|$. Now if $\varphi \in \tilde{\Phi}^*$, one has $0 < \langle 2\alpha, \alpha \rangle = \langle \varphi \mid_{\mathrm{T}}, \beta \mid_{\mathrm{T}} \rangle = n^{-2} \langle \sum_{\sigma \in \mathcal{G}'} \sigma(\varphi), \sum_{\sigma \in \mathcal{G}'} \sigma(\beta) \rangle = n^{-2} \langle \varphi, \sum_{\tau, \sigma \in \mathcal{G}'} \tau\sigma(\beta) \rangle$ (where $\beta \in \tilde{\alpha}$). It follows that there is a $\theta = \sigma(\beta)$ in $\tilde{\Phi}$ such that $\langle \varphi, \theta \rangle > 0$. It follows that $\varphi - \theta$ is a root. Moreover $\varphi - 2\theta$ is *not* a root since $\tilde{\Delta}$ is *simply laced*. It follows now from Lemma 3.26 that the set $\{xyx^{-1}y^{-1} : x \in \mathbf{U}(\theta)(\bar{k}),\ y \in \mathbf{U}(\varphi - \theta)(\bar{k})\}$ is all of $\mathbf{U}(\varphi)(\bar{k})$. Since $\theta \mid_{\mathrm{T}} = \beta \mid_{\mathrm{T}} = \alpha$ and $\varphi - \theta \mid_{\mathrm{T}} = 2\alpha - \alpha = \alpha$, we see that $U(\varphi)$ is contained in the image of c.

Let $\mathrm{B} = \mathbf{C} \cap \Gamma$, Γ an S-arithmetic group in $\mathbf{G}$. Then $(\mathbf{V}, \mathbf{V}', \mathrm{B})$ satisfy all the conditions in Lemma 3.22 (in view of the assumption that $\mathbf{C}$ splits over k_v, the fact proved above that all eigen characters for the action on $\mathbf{E}$ are non-trivial and 3.17). Thus d_{G} and d_Γ are Lipschitz equivalent on $\mathrm{U}(\alpha) \cap \Gamma = \mathbf{V} \cap \Gamma$.

We are now left with the last case:

(3.28). *Case 4.* $\mathrm{S} = \{v\}$, *k-rank* $\mathbf{G} = 1$, $\mathbf{C}$ *is anisotropic over* k_v *while* $\mathbf{M}$ *is isotropic over* k_v. — Suppose first that $|\ \tilde{\alpha}\ | = 2$. From the Tits classification scheme, we can then conclude from the assumption that k-rank $\mathbf{G} = 1$ the following : $\mathbf{G}$ is of type A_n, $n \geqslant 3$ or E_6. Moreover since k_v-rank $\mathbf{G} \geqslant 1$, once again from the Tits classification it is seen that $\mathrm{Z}(\mathbf{T})$ has an *absolutely simple* component $\mathbf{M}'$ which is defined over k, isotropic over k_v and its sub-diagram $\tilde{\Delta}'_{\mathrm{M}}$ in $\tilde{\Delta}$ is connected to *both* the roots in $\tilde{\alpha}$ (in the case $\mathbf{G}$ is of type $\mathbf{E}_6$, $\mathbf{M}$ itself is absolutely simple).

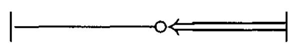
Now $\beta \in \tilde{\alpha}$ is negative dominant as a weight of $\mathbf{M}'$ for the simple system $\tilde{\Delta}'_{\mathbf{M}}$. It follows that (since $\mathbf{M}'$ is absolutely simple) that as a weight for $\mathbf{M}'$, any $\beta \in \tilde{\alpha}$ is a strictly negative linear combination of the roots in $\tilde{\Delta}'_{\mathbf{M}}$; and our choice of order on $\mathrm{X}(\tilde{\mathbf{T}})$ ensures that β is *nontrivial* on the maximal k_v-split torus of $\mathbf{M}'$ (which is contained in $\tilde{\mathbf{T}}$). Set $\mathbf{V} = \mathbf{U}(\alpha)$ and $\mathbf{V}' = \mathbf{U}(2\alpha)$. We will now examine the action of $\mathbf{M}'$ on $\mathbf{E} = \mathbf{U}(\alpha)/\mathbf{U}(2\alpha)$. $\mathbf{E}$ is a k-vector space on which $\mathbf{M}'$ acts linearly. Clearly from what we saw above $\mathbf{E}_u(\mathbf{M}'(k_v)) \supset \mathbf{U}(\beta)$ for $\beta \in \tilde{\alpha}$. Since $\mathbf{M}'$ is normal in $\mathrm{Z}(\mathbf{T})$, $\mathbf{E}_u(\mathbf{M}'(k_v))$ (denoted $\mathbf{E}'$ in the sequel) is $\mathrm{Z}(\mathbf{T})$-stable. Let $\varphi \in \tilde{\Phi}$ be such that $\mathbf{U}(\varphi) \subset \mathbf{U}(\alpha)$, $\mathbf{U}(\varphi) \not\subset \mathbf{U}(2\alpha)$. Let $\mathbf{F}(\varphi)$ be the image of $\mathbf{U}(\varphi)$ in $\mathbf{E}$. We want to show that $\mathbf{F}(\varphi) \subset \mathbf{E}'$. Since $\mathbf{E}'$ is $\mathrm{Z}(\mathbf{T})$-stable, we may by transforming $\mathbf{U}(\varphi)$ by an element of the Weyl group of $\mathrm{Z}(\mathbf{T})$ assume that φ is negative dominant with respect to $\tilde{\Delta}_{\mathbf{M}}$. This means that $\langle \varphi, \psi \rangle < 0$ for all $\psi \in \tilde{\Delta}_{\mathbf{M}}$; on the other hand since $\mathbf{U}(\varphi) \subset \mathbf{V}(\alpha)$, $\varphi > 0$. Since $\mathbf{U}(\varphi) \not\subset (2\alpha)$, $\varphi \mid_{\mathrm{T}} = \alpha$. We see thus that there is a root $\beta \in \tilde{\alpha} = \tilde{\Delta} \setminus \tilde{\Delta}_{\mathbf{M}}$ such that $\langle \varphi, \beta \rangle > 0$ so that $\varphi = \beta$ or $\varphi - \beta$ is a root. If $\varphi = \beta$, φ is non-trivial on a k_v-split torus of $\mathbf{M}'$ and hence $\mathbf{E}(\varphi) \subset \mathbf{E}'$. Suppose then that $\varphi \neq \beta$; then $\varphi - \beta$ is a root of $\mathbf{M}'$ so that $\mathbf{U}(\varphi - \beta) \subset \mathbf{M}'$. Now the root system $\tilde{\Delta}$ is simply laced and one deduces from the Chevally commutation relations (arguing as in (3.26)) that $\mathbf{E}(\varphi)$ belongs to the $\mathbf{U}(\varphi - \beta)$-submodule of $\mathbf{E}$ generated by $\mathbf{U}(\beta)$. Since $\mathbf{E}' \supset \mathbf{U}(\beta)$ and is $Z(\mathbf{T})$-stable, $\mathbf{E}' \subset \mathbf{U}(\varphi)$. We see therefore that $\mathbf{E}_u(\mathbf{M}'(k_v)) = \mathbf{E}$. The group $\mathbf{V}' = \mathbf{U}(2\alpha)$ is central in $\mathbf{V} = \mathbf{U}(\alpha)$. Also $\mathbf{V}'$ and $\mathbf{E} = \mathbf{V}/\mathbf{V}'$ are k vector spaces and the commutation map $(x, y) \mapsto xyx^{-1}y^{-1}$ of $\mathbf{V}$ yields a k-bilinear map c: $\mathbf{E} \times \mathbf{E} \to \mathbf{V}'$; image c spans $\mathbf{V}'$ as is seen from Lemma 3.26 in view of the fact that $\tilde{\Delta}$ is simply laced. One can now apply Lemma 3.22 taking for B a suitable finitely generated subgroup of $\mathbf{M}' \cap \Gamma$ (Lemma 3.17) to conclude that d_G and d_Γ are Lipschitz equivalent on $\mathbf{V} \cap \Gamma = \mathbf{U}(\alpha) \cap \Gamma$. We have thus proved (3.13) in case 4 under the additional assumption that $|\tilde{\alpha}| = 2$.

We now deal with the case $|\tilde{\alpha}| = 1$. Let $\{\beta\} = \tilde{\alpha}$. Then β is connected to every connected component of the diagram $\tilde{\Delta}_{\mathbf{M}}$ of $\mathbf{M}$. It follows that β is nontrivial on the maximal k_v-split torus of $\mathbf{M}$ contained in $\tilde{\mathbf{T}}$. As before let $\mathbf{V} = \mathbf{U}(\alpha)$ and $\mathbf{V}' = \mathbf{U}(2\alpha)$ with $\mathbf{U}(2\alpha)$ trivial if 2α is not a root. One then has vector space structures on $\mathbf{E} = \mathbf{V}/\mathbf{V}'$ and $\mathbf{V}'$ with the $Z(\mathbf{T})$-action for these structures linear. We denote by ρ the representation of $Z(\mathbf{T})$ on $\mathbf{E}$.

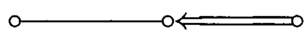
Now let $\tilde{\Psi} = \{\varphi \in \tilde{\Phi} \mid \mathbf{U}(\varphi) \subset \mathbf{U}(\alpha), \mathbf{U}(\varphi) \not\subset \mathbf{U}(2\alpha)\}$ (it is the same as the set $\{\varphi \in \tilde{\Phi} \mid \varphi|_{\mathbf{T}} = \alpha\}$. For $\psi \in \tilde{\Psi}$, let $\mathbf{E}(\psi) =$ Image $\mathbf{U}(\psi)$ in $\mathbf{E}(= \mathbf{V}/\mathbf{V}')$. We assert now that $\mathbf{E}' := \mathbf{E}_u(\mathbf{M}(k_v))$ is equal to $\mathbf{E}$. For this first observe that $\mathbf{E}'$ contains $\mathbf{E}(\beta)$ and thus it suffices to show that for $\psi \in \tilde{\Psi}$, $\mathbf{E}(\psi)$ is contained in the $Z(\mathbf{T})$ submodule $\mathbf{E}''$ of $\mathbf{E}$ generated by $\mathbf{E}(\beta)$ ($\mathbf{E}'$ is $Z(\mathbf{T})$-stable). To prove that $\mathbf{E}(\psi) \subset \mathbf{E}''$ for $\psi \in \tilde{\Psi}$ we may replace ψ by a transform of ψ under the Weyl group $Z(T)$. In particular we may assume ψ to be negative dominant for $\tilde{\Delta}_{\mathbf{M}}$. This means that $\langle \psi, \varphi \rangle \leqslant 0$ for all $\varphi \in \tilde{\Delta}_{\mathbf{M}}$. Since $\psi|_{\mathbf{T}} = \alpha$, $\psi > 0$; hence $\langle \psi, \beta \rangle > 0$. If $\psi = \beta$, $\mathbf{E}(\psi) \subset \mathbf{E}''$ and so we assume that $\psi \neq \beta$. Now $\psi = \beta + \sum_{\varphi \in \tilde{\Delta}_{\mathbf{M}}} m(\varphi)\varphi$ with $m(\varphi)$ integers $\geqslant 0$. It follows that $\psi - 2\beta$ cannot be a root. On the other hand since $\langle \psi, \beta \rangle > 0$, $\psi - \beta$ is a root. Now Lemma 3.26, taking $\psi = \varphi$ and $\beta = \theta$, shows that $\mathbf{E}(\psi)$ is contained in the $\mathbf{U}(\psi - \beta)$ submodule of $\mathbf{E}$ generated by $\mathbf{E}(\beta)$: If $x \in \mathbf{U}(\psi - \beta), y \in \mathbf{E}(\beta)$, then $\rho(x)(y) - y = z + z^1$ where $z \in \mathbf{E}(\psi)$ and z^1 belongs to sum of eigen-spaces for $\tilde{\mathbf{T}}$ corresponding to characters other then ψ (Lemma 3.26); the lemma also ensures that $z \neq 0$ if $x \neq 1$ and $y \neq 0$. Thus since $\mathbf{U}(\psi - \beta) \subset \mathbf{M}$ and $\mathbf{E}''$ are $\tilde{\mathbf{T}}$-stable, $z \in \mathbf{E}''$. As $\dim \mathbf{E}(\psi) = 1$, we have $\mathbf{E}(\psi) \subset \mathbf{E}''$. Since the $\mathbf{E}(\psi)$, $\psi \in \tilde{\Psi}$ span all of $\mathbf{E}$ we see that $\mathbf{E}_u(\mathbf{M}(k_v)) = \mathbf{E}$. By Corollary 3.17, we can find a finitely generated subgroup $B \subset \mathbf{M} \cap \Gamma$ such that $\mathbf{E}_u(B) = \mathbf{E}$. We can now appeal to Lemma 3.22 with $\mathbf{V}$, $\mathbf{V}'$ and B as above. The conditions in that lemma about the commutator map $\mathbf{V} \times \mathbf{V} \to \mathbf{V}'$ are satisfied if char $k = 0$ or if 2α is not a root or if $\tilde{\Delta}$ is simply laced. Thus we have proved (3.13) in the following situations : $S = \{v\}$, $|\tilde{\alpha}| = 1$, $\mathbf{C}$ is anisotropic over k_v and either char $k = 0$ or $\tilde{\Delta}$ is simply laced or 2α is not a root. In the case that is left out, viz when char $k > 0$, 2α is a k-root and $\tilde{\Delta}$ is not simply laced, we will appeal to Lemma 3.23. In order to do this one has to show that d_G and d_Γ are Lipschitz equivalent on $\mathbf{V}' \cap \Gamma$. Let $\mathbf{G}'$ be the k simple algebraic group generated by $\mathbf{U}(\pm 2\alpha)$. Then $\mathbf{T} \subset \mathbf{G}'$ and $\{2\alpha\}$ is a simple k-root of $\mathbf{G}'$ with respect to $\mathbf{T}$. Now if we show that S-rank $\mathbf{G}' \geqslant 2$, we can then appeal to the earlier situation (where $\mathbf{V}'$ is trivial) to conclude that $d_{G'}$ and $d_{\Gamma'}$, $\Gamma' = \mathbf{G}' \cap \Gamma$, are Lipschitz equivalent on $\mathbf{V} \cap \Gamma$. Since d_G and $d_{G'}$ are Lipschitz equivalent on $\mathbf{V}' \cap \Gamma$, this would prove the result. Thus we have to show that k_v-rank $\mathbf{G}' \geqslant 2$ under the following conditions on $\mathbf{G}$ and k

(i) Char $k > 0$ (ii) $\tilde{\Delta}$ has two root lengths (iii) 2α is a k-root.

We will appeal to the Tits classification. Since $\tilde{\Delta}$ has two root lengths, $\mathbf{G}$ is of one of the types B_n, C_n, G_2 or F_4. The fact that 2α is a root leads us to exclude (rank 1) groups of type B_n as also C_2. There are no k-rank 1 forms of type G_2 (over any field, see [Ti2]) so that G_2 is also excluded. Since k is a *global field* of positive characteristic, all anisotropic groups over k are of type of A_n (see [Ha2], cf. [Ma IX.(1.6)(viii)]). This means that $\mathbf{G}$ cannot be of type F_4 (of k-rank 1). This leaves us to consider only groups of type C_n. Here again using the fact that all anisotropic groups over k are of type A_n and examining the Tits diagrams of type C_n, we see that the Tits Diagram of $\mathbf{G}$ over k is necessarily



Over k_v, the diagram is necessarily of the form



It follows that over k_v, $\mathbf{M}$ is an almost direct product of two copies $\mathbf{H}_1, \mathbf{H}_2$ of SL(2). The representation of $\mathbf{M}$ on $\mathbf{E}$ is the tensor product $\rho_1 \otimes \rho_2$ of the natural representations ρ_1, ρ_2 of $\mathbf{H}_1, \mathbf{H}_2$ respectively. The representation of $\mathbf{M}$ on $\mathbf{V}'$ on the other hand is trivial on one of the $\mathbf{H}_i$, $i = 1, 2$, $\mathbf{H}_2$ say, and is the adjoint representation restricted to the other factor $\mathbf{H}_1$. It is easy to see now that $\mathbf{H}_1$ is the commutator subgroup of the centralizer of $\mathbf{T}$ in $\mathbf{G}'$. Thus k_v-rank $\mathbf{G}' \geqslant 2$. This completes the proof of Proportion 3.13.

4. Kazhdan conjecture

The main goal of this section is to prove:

(4.1) *Theorem.* — *Let* $\Gamma < G$ *be an irreducible lattice in* G *as in (3.6). Assume that rank* $G = \sum_{i=1}^{l} \text{rank}_{k_i} \mathbf{G}_i \geqslant 2$ *then* (Γ, d_W) *is undistorted in* (G, d_R).

(4.2) *Remarks*

(i) Clearly $d_R(\gamma, 1) \leqslant C\, d_W(\gamma, 1)$ for some fixed constant. Hence we need only show the other inequality.

(ii) Recall that by Margulis arithmeticity theorem ([Ma, chap. IX, (1.11), p. 298], [Ve]) Γ as in the theorem is an S-arithmetic group, i.e., there exists a global field k, a finite set of valuations S of k, containing all the archimedean ones, and an almost simple k-algebraic group $\mathbf{G}$, such that G is locally isomorphic to $\prod_{v \in S} \mathbf{G}(k_v)$, where k_v denotes the completion of k with respect to v, and Γ is commensurable with $\mathbf{G}(\mathcal{O}_S)$, where $\mathcal{O}_S = \{x \in k \mid |x|_v \leqslant 1 \text{ for every } v \notin S\}$. Note that we can assume that none of the factors of $\mathbf{G}$ is compact since otherwise we can project the lattice into the product of the non-compact factors. It will still be discrete and the kernel is finite. This also does not change the rank. We may, and will, assume that $\operatorname{rank}_k \mathbf{G} \geqslant 1$ since otherwise $\mathbf{G}(\mathcal{O}_S)$ is cocompact in G, in which case the theorem is easy (see also (3.2)).

Note also that Γ is indeed finitely generated (cf. [Ma, §IX.3], [Ra2]).

(iii) We will think of $\mathcal{O}_S$ as embedded discretely in $\prod_{v \in S} k_v$ via the diagonal embedding and when talking about "bounded set" etc. – it will always be with respect to this embedding. If $S_0 \subseteq S$ is a subset of valuations (e.g., $S_0 = S_\infty$ the set of the archimedean valuations) we write for $x \in k$, $|x|_{S_0} = \sum_{v \in S_0} |x|_v$. We write simply $|x|$ for $|x|_S$.

We also write for $x \in k$, $|x|^* = \prod_{v \in S} |x|_v$. Note that as S is finite (and fixed for our discussion), $|x|^*$ is bounded polynomially by $|x|$. Observe also that for $x \in \mathcal{O}_S$ we have $|x|^* = \#(\mathcal{O}_S / x\mathcal{O}_S)$.

Definition. — In what follows, we shall say that a subgroup $\Gamma_0 \leqslant \Gamma$ is (d_W, d_R)-undistorted if $(\Gamma_0, d_W|_{\Gamma_0})$ is undistorted in (G, d_R), i.e., for every element $\gamma \in \Gamma_0$ there exists a word in the generators of Γ (!!) expressing γ, whose length is $O(d_R(\gamma, 1))$. Clearly if $\Gamma_1 < \Gamma_0 < \Gamma$ and Γ_0 is (d_W, d_R)-undistorted then Γ_1 is (d_W, d_R)-undistorted. Notice also that if a finitely generated subgroup $\Gamma_0 < \Gamma$ is undistorted in G with respect to its own word metric then it is (d_W, d_R)-undistorted (with respect to d_W – the word metric of Γ).

We shall break the proof of the theorem into several lemmas.

The proof proceeds in several steps. The results of section 3 enables one to conclude first that for a k-split unipotent subgroup U of G. $U \cap \Gamma$ is (d_W, d_R)-undistorted. Since reductive k-subgroups are undistorted in G and uniform lattices are undistorted in their ambient groups, the above fact leads us to conclude that if $H \subset G$ is a k-subgroup such that $H \cap \Gamma$ is uniform in H, then $H \cap \Gamma$ is (d_W, d_R)-undistorted. We then consider a k-rank one subgroup H in G and show that $H \cap \Gamma$ is (d_W, d_R)-undistorted (note that $H \cap \Gamma$ may fail to be undistorted in H itself). This is achieved through a geometric argument involving the structure of the fundamental domain as constructed by Borel [Bo]. The next step is to show that if $P \cap \Gamma$ is (d_W, d_R)-undistorted for all maximal parabolic k-subgroups then Γ is undistorted in G. Once this is proved a simple induction on the rank of G and the result stated above for k-rank one subgroups yields the theorem. The proof that it suffices to show that $P \cap \Gamma$ is (d_W, d_R)-undistorted for all maximal parabolic k-subgroups occupies essentially all of section 4 starting (4.10). Here we exploit the Bruhat decomposition in $G(k)$ with respect to $P(k)$. One can confine oneself to elements of Γ that lie in the unique open Bruhat cell. In this cell we have a natural product decomposition for every γ as a product $\gamma = u_\gamma^- m_\gamma u_\gamma^+$ where u_γ^+ (respectively u_γ^-, m_γ) belongs to the unipotent radical of P (respectively the unipotent radical of the opposite of P, Levi component of P). The elements u_γ^-, u_γ^+ and m_γ belong to $\mathbf{U}^-(k)$, $\mathbf{U}^+(k)$ and $\mathbf{M}(k)$ respectively and are not, in general, integral. The failure of u_γ^- to be integral is measured by a function which we call $\mathrm{Den}(u_\gamma^-)$. In our case this is measured by the value of a natural representative function F on G at γ. Viz the function that describes the divisor which is the complement of the open Bruhat cell. (This function is of the form $F(g) = \langle v^*, gv \rangle$ for a suitable linear action of G on a vector space V with $v \in V$ a vector such that the line kv is stable under P.) The proof uses induction on the values of $\mathrm{Den}(u_\gamma^-)$, $F(\gamma)$ as well as $\|u_\gamma^-\|$.

(4.3) *Lemma.* — *Let* $\mathbf{U} < \mathbf{G}$ *be a k-split unipotent k-subgroup of* $\mathbf{G}$. *Then* $\Gamma_0 = \mathbf{U}(\mathcal{O}_S)$ *is* (d_W, d_R)*-undistorted.*

Proof. — Any k-split unipotent group is contained in the unipotent radical of some k-parabolic subgroup (see [BT2]). Hence the lemma follows from Theorem 3.7 and the remark in the definition following 4.2.

(4.4) *Remark.* — *When k is of characteristic zero all unipotent k-subgroups are k-split.*

(4.5) *Lemma.* — *Let* $\mathbf{H} < \mathbf{G}$ *be a reductive k-subgroup such that* $H \cap \Gamma$ *is a uniform lattice in* H. *Then* $H \cap \Gamma$ *is* (d_W, d_R)*-undistorted.*

Proof. — This follows from Proposition 3.2 and the fact that a reductive group $H < G$ is always undistorted.

(4.6) *Remark.* — *Assume k is of characteristic zero. Let* $\mathbf{H} < \mathbf{G}$ *be a k-subgroup such that* $H \cap \Gamma \backslash H$ *is compact then* $H \cap \Gamma$ *is* (d_W, d_R)*-undistorted.*

Proof. — Let $\mathbf{H} = \mathbf{RU}$ where $\mathbf{R}$ is a reductive k-subgroup and $\mathbf{U}$ is the k-unipotent radical of $\mathbf{H}$. $R \cap \Gamma$ is a uniform lattice in R and hence by Lemma 4.5 is (d_W, d_R)-undistorted. $U \cap \Gamma$ is (d_W, d_R)-undistorted by Lemma 4.3. Since $(R \cap \Gamma)(U \cap \Gamma)$ is of finite index in $H \cap \Gamma$, it follows that $H \cap \Gamma$ is (d_W, d_R)-undistorted.

(4.7) *Lemma.* — *Let* $\mathbf{H} < \mathbf{G}$ *be a k-simple k-subgroup of k-rank one. Then* $\Gamma_0 = H \cap \Gamma_{\mathcal{O}_S}$ *is* (d_W, d_R)*-undistorted where* $\Gamma_{\mathcal{O}_S} = \mathbf{G}(\mathcal{O}_S)$.

Proof. — Let $\mathbf{T}$ be a maximal k-split torus in $\mathbf{H}$, $Z(\mathbf{T})$ its centralizer and $\mathbf{N}$ and $\mathbf{N}^-$ the two (opposing) maximal unipotent k-subgroups of H normalized by $Z(\mathbf{T})$. Then $\mathbf{P} = Z(\mathbf{T})\mathbf{N}$ is the normalizer of $\mathbf{N}$ in $\mathbf{H}$. We denote by Φ^+ the positive root system of $\mathbf{H}$ with respect to $\mathbf{T}$ determined by $\mathbf{N}$ and by α the unique simple root in Φ. If ω is any non-zero invariant volume form on N one has $g(w) = \chi(g)w$, for $g \in$ P, for a character χ on $\mathbf{P}$; moreover there is an integer $r > 0$ such that $\chi = \alpha^r$ on $\mathbf{T}$. Consider now the homomorphism $|\chi| : \prod_{v \in S} \mathbf{P}(k_v) = \mathrm{P} \to \mathbf{R}^+$ given by $|\chi|(x) = |\chi(x)|$. Let ${}^0\mathrm{P}$ denote the kernel of $|\chi|$. Then there is a closed subgroup $\mathrm{A} \subset \mathrm{T}$ such that $|\chi|$ maps A isomorphically onto $|\chi|(\mathrm{P})$. The group $|\chi|(\mathrm{P})$ is all of $\mathbf{R}^+$, if char $k = 0$ while it is isomorphic to $\mathbf{Z}$ if char $k > 0$. It is well known that there is a maximal compact subgroup $\mathrm{K} \subset \mathrm{H}(= \prod_{v \in S} \mathbf{H}(k_v))$ such that $\mathrm{H} = \mathrm{K.P}(= \mathrm{K.A.}{}^0\mathrm{P})$. The quotient ${}^0\mathrm{P}/({}^0\mathrm{P} \cap \Gamma)$ is compact. We introduce the following additional notation: for a real number $c > 0$, let $\mathrm{A}[c] = \{x \in \mathrm{A} \mid |\chi(x)| \leqslant c\}$ and $\mathrm{A}(c) = \{x \in \mathrm{A} \mid |\chi(a)| < c\}$. If $\Omega \subset {}^0\mathrm{P}$ is any subset and $c \in \mathbf{R}$ is positive, define $\mathrm{F}[\Omega, c] = \mathrm{K.A}[c].\Omega$ (a "Siegel set"), and let $\mathrm{F}(\Omega, c) = \mathrm{K.A}(c).\Omega$. For $c' < c$ let $\mathrm{E}(\Omega, \mathrm{C}, \mathrm{C}') = \mathrm{F}[\Omega, c] \backslash \mathrm{F}(\Omega, c')$. Note that $\mathrm{E}(\Omega, c, c')$ is *compact.* The following result is due to Borel [Bo] when char $k = 0$ and Harder [Ha1] (see also Behr [Be]) when char $k > 0$.

There is a finite subset Σ in $\mathrm{H}(k)$ containing 1 and a real number $c_0 > 0$ such that the following holds: let $\Omega \subset {}^0\mathrm{P}$ be any compact set with $\Omega({}^0\mathrm{P} \cap \Gamma) = {}^0\mathrm{P}$ and $c > c_0$; then

$$\mathrm{F}[\Omega, c]\Sigma\Gamma_0 = \mathrm{H}.$$

Moreover the set $\{\theta \in \Gamma_0 \mid \mathrm{F}[\Omega, c]\Sigma\theta \cap \mathrm{F}[\Omega, c]\Sigma \text{ is non-empty}\}$ is finite.

We fix a metric on H which is: 1) invariant under right translations under all of H and also under left translation by elements of K, 2) compatible with the topology on H and 3) makes H into a "path space" in the sense of 3.1; as has already been observed such a metric exists. Since $\mathrm{E}(\Omega, c, c')$ $(c > c')$ is compact for compact Ω, we note that the set $\{x \mid d(x, \mathrm{E}(\Omega, c, c')) < \mathrm{M}\}$ is also relatively compact for any $\mathrm{M} > 0$; consequently the set

$$\Theta = \{\theta \in \Gamma_0 | d(\mathrm{E}(\Omega, c, c'), \mathrm{E}(\Omega, c, c')\theta) < \mathrm{M}\}$$

is finite. We need in the sequel the following assertion which is essentially known.

Assertion. — *Let Σ, c and Ω be as above. Let $\delta > 0$ be any constant. Then there exists $c' > 0$, $c' < c$ (depending on c, Ω and δ) such that the following holds: if $x, y \in \mathrm{F}[\Omega, c]$, $\xi, \eta \in \Sigma$ and $\gamma \in \Gamma_0$ are such that $d(x\xi, y\eta\gamma) < \delta$, then either $\eta\gamma\xi^{-1} \in \mathrm{N}$ or $x, y \in \mathrm{E}(\Omega, c, c')$.*

We outline a proof of the assertion. Let $q = \dim\mathbf{N}$ and $\mathbf{V}$ be the q^{th} exterior power of $\underline{h}$ (= Lie algebra of $\mathbf{H}$). Then $\mathbf{V}$ decomposes (under the natural representation of

H) into eigenspaces for $\mathbf{T}$ over k. The weights of $\mathbf{T}$ acting on $\mathbf{V}$ are necessarily of the form α^t, $t \in \mathbf{Z}$; the highest and the lowest of these weights are then $\chi(=\alpha^r)$ and $\chi^{-1}(=\alpha^{-r})$ respectively and the corresponding weight spaces are of dimension 1. Let $\| \ \|$ be a K-invariant norm on $V = \prod_{v\in S} \mathbf{V}(k_v)$. Now we assume – as we may – that $\mathbf{V}(k)$ has a k-basis $\mathscr{B}$ containing weight vectors e and f corresponding to χ and χ^{-1}, respectively, such that Γ_0 stabilizes the $\mathscr{O}_S$-span of $\mathscr{B}$. Suppose now that x, y, ξ, η and γ are as in the assertion. Then one has

$$x\xi = g.y.\eta\gamma$$

where g is in a compact subset of H determined by δ. Suppose now that $\eta\gamma\xi^{-1} \notin$ N; then $\eta\gamma\xi^{-1} = n\tau p$ where $p \in \mathbf{P}(k)$, $n \in \mathbf{N}(k)$ and τ is an element of $\mathbf{H}(k)$ normalizing $\mathbf{T}$ but not belonging to $\mathbf{T}(k)$. Such an element τ maps e into $k.f$ as is easily seen. As e belongs to q^{th} exterior power of $\underline{n}$ (= Lie subalgebra of $\underline{h}$ corresponding to $\mathbf{N}$), $pe \in k.e$ while $nf = f + w$ where w belongs to the k-span of weight vectors corresponding to weights other than χ^{-1}. Let $x = \tilde{k}.a.u = \tilde{k}.u^*.a$ where $\tilde{k} \in$ K, $a \in$ A(c) and u^* belongs to compact subset Ω^* of ^{0}P (depending on c and Ω). Similarly $y = k'u'^*a'$, $k' \in$ K, $a' \in$ A(c) and $u'^* \in \Omega^*$. Clearly then one has a constant $\beta > 0$ such that $\|xe\| \leqslant \beta|\chi(a)|\|e\|$. On the other hand, $x = g.y.\eta\gamma\xi^{-1}$. Since η, $\xi \in \Sigma$ a finite set and Γ_0 stabilizes the $\mathscr{O}_S$-span of $\mathscr{B}$, we see that there is a $\rho \in \mathscr{O}_S$ such that $\rho \neq 0$ and

$$\begin{aligned}(\eta\gamma\xi^{-1}(\mathscr{O}_S \text{ span of } \mathscr{B}) &\subset \Sigma\Gamma_0\Sigma(\mathscr{O}_S \text{ span of } \mathscr{B}) \\ &\subset \rho^{-1}(\mathscr{O}_S \text{ span of } \mathscr{B}).\end{aligned}$$

Since $\tau e \in k \cdot f$, one sees that $\eta\gamma\xi^{-1}e$ is of the form $\rho^{-1}t \cdot f + w_0$ with $t \in \mathscr{O}_S$ and ρw_0 in the $\mathscr{O}_S$-span of weight vectors other than f. Note that there exists some $\tilde{\rho} > 0$ (depending only on ρ and f) such that for any $t' \in \mathscr{O}_S$ we have $\|\rho^{-1}t f\| \geqslant \tilde{\rho}$. Now $gy = gk'u'^*a'$ and $gk'u'^*$ belongs to a fixed compact set. We see thus that there is a constant $\beta' > 0$ such that $\|gk'u'^*(w)\| \geqslant \beta'\|w\|$. It follows that

$$\begin{aligned}\|xe\| &= \|gy\eta\gamma\xi^{-1}e\| \\ &= \|gk'u'^*a'\eta\gamma\xi^{-1}e\| \\ &\geqslant \beta'\|a'\eta\gamma\xi^{-1}e\| = \beta'\|a'(\rho^{-1}tf + w_0)\| \\ &\geqslant \beta'\|a'\rho^{-1}tf\| = \beta'\chi(a')^{-1}\|\rho^{-1}tf\| \\ &\geqslant \beta'\chi(a')^{-1}\tilde{\rho} \geqslant \beta'c^{-r}\tilde{\rho}\end{aligned}$$

since $\chi(a') \leqslant c$. This leads to the inequality

$$\chi(a) \geqslant \beta^{-1}\beta'c^{-r}\tilde{\rho}\|e\|^{-1}$$

and analogously reversing the roles of x and y

$$\chi(a') \geqslant \beta^{-1}\beta'c^{-r}\tilde{\rho}\|e\|^{-1}.$$

We need only choose $c' = \beta^{-1}\beta' c^{-r}\tilde{\rho}\|e\|^{-1}$ to obtain the assertion.

With the assertion established we now go on to prove (4.7). Fix a $\delta > 0$ and choose $c' > 0$ as in the assertion. Let $F = F[\Omega, c]$ and $E = E(\Omega, c, c')$. Let

$$B = \max(\delta, \text{ diameter } E)$$

and set

$$\Theta_B = \{\gamma \in \Gamma_0 | d(E, E\gamma) \leqslant 2B\}.$$

The set Θ_B is *finite*. Suppose now $\gamma \in \Gamma_0$ with $d(1, \gamma) = |\gamma| \geqslant 2B$. Then we can find h_i, $0 \leqslant i \leqslant n$ in H with $h_0 = 1$ and $h_n = \gamma$ such that $\alpha(h_i, h_{i+1}) < \delta/2$ and

$$\begin{aligned} d(1, \gamma) &\leqslant \sum_{0 \leqslant i < n} d(h_i, h_{i+1}) \\ &\leqslant d(1, \gamma) + \delta/4. \end{aligned}$$

Passing to a subset of $h_0, \dots, h_n$ we can in fact assume that

$$\delta/4 \leqslant d(h_i, h_{i+1}) < \delta.$$

Let $J = \{0 = i_0 < i_1 \dots < i_r = n\}$ be the subset of $[0, n]$ consisting of those integers j for which $h_j \in E\Sigma\Gamma_0$. We set $g_\ell = h_{i_\ell}$. Also for each i, $0 \leqslant i \leqslant n$ pick elements $x_i \in F$, $\xi_i \in \Sigma$ and $\gamma_i \in \Gamma_0$ such that $h_i = x_i\xi_i\gamma_i$ and $x_i \in E$ whenever $i \in J$; we assume – as we may – that $x_0 = x_n = 1$, $\xi_0 = \xi_1 = 1$, $\gamma_0 = 1$ and $\gamma_n = \gamma$. We also set for $0 \leqslant \ell \leqslant r$, $y_\ell = x_{i_\ell}$, $\eta_\ell = \xi_{i_\ell}$ and $\theta_\ell = \gamma_{i_\ell}$ so that $g_\ell = y_\ell\eta_\ell\theta_\ell$.

Claim. — $d(g_\ell, g_{\ell+1}) > \delta/4$. *This is clear if* $i_{\ell+1} = i_\ell + 1$. *Suppose then that* $i_{\ell+1} > i_\ell + 1$. *Let* $\lambda = \sum_{0 \leqslant i < i_\ell} d(h_i, h_{i+1})$ *while* $\mu = \sum_{i_{\ell+1} \leqslant i < n} d(h_i, h_{i+1})$. *Then one has*

$$\begin{aligned} d(1, \gamma) + \delta/4 &\geqslant \lambda + \mu + \sum_{i_\ell \leqslant i < i_{\ell+1}} d(h_i, h_{i+1}) \\ &\geqslant \lambda + \mu + \delta/2 \end{aligned}$$

since there are at least two terms in the last summation each of which is $> \delta/4$. On the other hand

$$d(1, \gamma) \leqslant \lambda + \mu + d(g_\ell, g_{\ell+1})$$

by triangle inequality. We thus find that

$$\lambda + \mu + d(g_\ell, g_{\ell+1}) \geqslant \lambda + \mu + \delta/4$$

so that $d(g_\ell, g_{\ell+1}) \geqslant \delta/4$. Hence the claim.

Suppose now that ℓ is such that $0 \leqslant \ell < r$ and that $d(g_\ell, g_{\ell+1}) \leqslant 2B$. Then one has

$$d(E\Sigma, E\Sigma\theta_{\ell+1}\theta_\ell^{-1}) \leqslant 2B$$

so that $\theta_{\ell+1}\theta_\ell^{-1} \in \Theta_B$. Since Θ_B is finite and $d(g_\ell, g_{\ell+1}) \geqslant \delta/4$ as well, there is a constant $\mu > 0$ such that

$$d(g_\ell, g_{\ell+1}) \geqslant \mu d(1, \theta_{\ell+1}\theta_\ell^{-1}).$$

We will establish a similar inequality also in the case when $d(g_\ell, g_{\ell+1}) > 2B$. In this case one has necessarily $i_{\ell+1} > i_\ell + 1$; consequently $h_i \notin E\Sigma\Gamma_0$ for $i_\ell < i < i_{\ell+1}$. It follows now from the assertion that $\xi_i\gamma_i\gamma_{i-1}^{-1}\xi_{i-1}^{-1}$ and $\xi_{i+1}\gamma_{i+1}\gamma_i^{-1}\xi_i^{-1}$ belong to N for $i_\ell < i < i_{\ell+1}$. We conclude from this that $\eta_{\ell+1}\theta_{\ell+1}\theta_\ell^{-1}\eta_\ell^{-1} \in N$. Moreover one has:

$$\begin{aligned} d(g_\ell, g_{\ell+1}) &= d(y_\ell\eta_\ell, y_{\ell+1}\eta_{\ell+1}\theta_{\ell+1}\theta_\ell^{-1}) \\ &\geqslant -d(y_\ell\eta_\ell, y_{\ell+1}\eta_{\ell+1}) \\ &\quad + d(y_{\ell+1}\eta_{\ell+1}, y_{\ell+1}\eta_{\ell+1}\theta_{\ell+1}\theta_\ell^{-1}) \\ &\geqslant -B + d(1, y_{\ell+1}\eta_{\ell+1}\theta_{\ell+1}\theta_\ell^{-1}(y_{\ell+1}\theta_{\ell+1}^{-1})) \\ &\geqslant -B + \mu' d(1, \theta_{\ell+1}\theta_\ell^{-1}) \end{aligned}$$

for a suitable constant $\mu' > 0$; since $d(g_\ell, g_{\ell+1}) > 2B$, one sees that

$$d(g_\ell, g_{\ell+1}) \geqslant 2\mu' d(1, \theta_{\ell+1}\theta_\ell^{-1})/3.$$

We see thus that if we set $\nu = \min(\mu, 2\mu'/3)$,

$$d(g_\ell, g_{\ell+1}) \geqslant \nu d(1, \theta_{\ell+1}\theta_\ell^{-1})$$

for $0 \leqslant \ell < r$. Since for elements of $N \cap \Gamma_0$ the word metric and d are equivalent we conclude that

$$\begin{aligned} \sum_{0\leqslant\ell<r} d(g_\ell, g_{\ell+1}) &\geqslant \nu \sum_{0\leqslant\ell<r} d(1, \theta_{\ell+1}\theta_\ell^{-1}) \\ &\geqslant \nu'(\text{length } \gamma) \end{aligned}$$

(where length γ is referred to some finite set of generators of Γ_0). On the other hand we know that

$$\sum_{0\leqslant\ell<r} d(g_\ell, g_{\ell+1}) \leqslant d(1, \gamma) + \delta/2.$$

This shows that the word metric is dominated by d.

We refer the reader to [LMR] where a similar lemma is proved in a more geometric language for the special case where **H** is an $SL(2, \mathbf{R})$ in $G = SL(n, \mathbf{R})$. A similar "geometric" argument applies in the characteristic zero case to any rank one $H < G$.

(4.8) *Lemma.* — *Let* $\mathbf{G}$ *be an absolutely almost simple k-group,* $G = \prod \mathbf{G}(k_i)$ *of rank* $\geqslant 2$ *and* $\Gamma < G$ *be an* S*-arithmetic subgroup of* G. *Suppose that* $P \cap \Gamma$ *is* (d_W, d_R)*-undistorted for every maximal k-parabolic subgroup* $\mathbf{P}$ *of* $\mathbf{G}$. *Then* Γ *is* (d_W, d_R)*-undistorted.*

We postpone the proof of (4.8) to (4.10) showing first how (4.8) implies Theorem 4.1.

(4.9) *Proof of Theorem 4.1.* — We prove Theorem 4.1 by induction on k-rank of $\mathbf{G}$. When k-rank $\mathbf{G} = 0$ this follows from (3.2). When k-rank $\mathbf{G} = 1$ Theorem 4.1 is immediate from (4.7). Thus the start of the induction is secured and we assume as we may that k-rank $\mathbf{G} \geqslant 2$. Let $\mathbf{P}$ be a maximal k-parabolic subgroup of $\mathbf{G}$. We need only show that $P \cap \Gamma$ is (d_W, d_R)-undistorted. Now $\mathbf{P} = \mathbf{M}\mathbf{U}$ where $\mathbf{M}$ is a connected reductive k-subgroup and $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$. Theorem 3.7 tells us that $U \cap \Gamma$ is (d_W, d_R)-undistorted. Since $(M \cap \Gamma)(U \cap \Gamma)$ has finite index in $P \cap \Gamma$ it suffices to show that $M \cap \Gamma$ is (d_W, d_R)-undistorted. Now $\mathbf{M} = \mathbf{C}\mathbf{M}'$ where $\mathbf{M}' = [\mathbf{M}, \mathbf{M}]$ and $\mathbf{C}$ is a k-torus in the center of $\mathbf{M}$. Further $\mathbf{C} = \mathbf{C}^s\mathbf{C}^a$ where $\mathbf{C}^s$ is split over k and $\mathbf{C}^a$ is anisotropic over k. Moreover, $(C^a \cap \Gamma)(M' \cap \Gamma)$ has finite index in $M \cap \Gamma$ and $(C^a \cap \Gamma) \cap (M' \cap \Gamma)$ is finite. Now $C^a/C^a \cap \Gamma$ is compact so that $C^a \cap \Gamma$ is undistorted in C^a while C^a being a torus is undistorted in G. Thus $C^a \cap \Gamma$ is (d_W, d_R)-undistorted. We see thus that it suffices to prove that $M' \cap \Gamma$ is (d_W, d_R)-undistorted. Now k-rank $\mathbf{M}' > 0$. If k-rank $\mathbf{M}' = 1$, this follows from (4.7). If k-rank $\mathbf{M}' > 1$, we know by the induction hypothesis that $M' \cap \Gamma$ is undistorted in M'. As M' is undistorted in G, $M' \cap \Gamma$ is undistorted in G, hence (d_W, d_R)-undistorted.

(4.10) *Proof of (4.8).* — As the proof of (4.8) is rather technical it may be useful to sketch its main steps for the special case of $SL_n(\mathbf{Z}) < SL_n(\mathbf{R})$. Let $P < SL_n(\mathbf{R})$ be the maximal parabolic subgroup consisting of the stabilizer of $\mathbf{R}e_1$ in the natural action of $SL_n(\mathbf{R})$ on $\mathbf{R}^n$. Given an element $\gamma \in SL_n(\mathbf{Z})$ we would like to multiply it by elements of "controlled" length to bring it into $P \cap SL_n(\mathbf{Z})$. To this end we have $\gamma = u^- p$ where $p \in P \cap SL_n(\mathbf{Q})$ and $u^- \in U^- \cap SL_n(\mathbf{Q})$, where $U^- = \left\{ \begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & 0 & 1 & \\ \vdots & & & \\ * & 0 & & 1 \end{pmatrix} \right\}$. (This decomposition exists whenever the $(1, 1)$ entry of γ is nonzero which we may assume without loss of generality.) By multiplying by some appropriate $\delta \in U^- \cap SL_n(\mathbf{Z})$ we may assume that u^- belongs to a fixed compact set. Had u^- belonged to $U^- \cap SL_n(\mathbf{Z})$ we would have attained our goal. As this is not always the case we have to use a certain induction argument. Let $F(\gamma) \in \mathbf{Z}\setminus\{0\}$ be the $(1,1)$ entry of γ (and hence also of p). Its absolute value $|F(\gamma)|$ serves as a measure to the "failure" of u^- to be integral ($|F(\gamma)|$ is the common denominator of the entries of u^-). In order to argue by induction on $|F(\gamma)|$ we need to show how by multiplying γ by some element $\theta \in SL_n(\mathbf{Z})$ of controllable length we get a new element $\theta\gamma$ s.t. $|F(\theta\gamma)| < |F(\gamma)|$. Iterating this procedure we eventually "push" γ into $P \cap SL_n(\mathbf{Z})$. For constructing an element θ as required observe that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_n(\mathbf{Z})$ we have $\begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ hence if we have $u^- = \begin{pmatrix} 1 & & & \\ c/a & 1 & & \\ \vdots & & \ddots & \\ * & & & 1 \end{pmatrix}$ with $a > 1$ hen an element $\theta^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \subset SL_n(\mathbf{Z})$ satisfies the required properties and $|F(\theta\gamma)| = 1/a \cdot |F(\gamma)|$. Clearly similar argument works using other entries in case the $(2,1)$ entry of u^- is integral. In the general case the role of the $(2,1)$ entry of U^- is played by an element $u(-\alpha)$ belonging to the root group corresponding to the root $-\alpha$ (where α is the root associated with the maximal parabolic subgroup) s.t. $u^- = u(-\alpha)u(\varphi_2)\ldots u(\varphi_r)$. We show how one may ensure (by multiplication by an element of a fixed finite set) that $u(-\alpha)$ has a large denominator. This enables us to reduce $|F(\gamma)|$ by multiplication by some controllable element belonging to the intersection of Γ with the rank one subgroup corresponding to the root α.

We return now to the proof of (4.8).

Let $\mathbf{T}$ be a maximal k-split torus of $\mathbf{G}$. We can assume $\mathbf{T}$ is of dimension $\geqslant 1$, since otherwise $\mathbf{G}$ is k-anisotropic in which case Γ is cocompact in G, and hence undistorted in G. Let Φ be the k-root system corresponding to $\mathbf{T}$, $\Pi \subset \Phi$ a simple system of roots. We denote by $N(\mathbf{T})$ and $Z(\mathbf{T})$ the normalizer and centralizer, respectively, of $\mathbf{T}$. Let $W = N(\mathbf{T})/Z(\mathbf{T})$ be the corresponding Weyl group of $\mathbf{G}$. We may assume that there exists a set $\widetilde{W} \subset \Gamma$ of representatives of W. Here is a sketch of the ideas in showing the existence of a commensurable lattice which contains such a set of representatives: Let $\Psi' = \{\varphi \in \Phi \mid 2\varphi \notin \Phi\}$. Let $\mathbf{G}'$ be the k-subgroup of $\mathbf{G}$ generated by the root groups $\mathbf{U}_\varphi$, $\varphi \in \Psi'$. The groups $\mathbf{G}$ and $\mathbf{G}'$ share the k-split torus $\mathbf{T}$. The inclusion $\mathbf{G}' \subset \mathbf{G}$ induces an isomorphism of the corresponding Weyl groups. Let $\Gamma' < G$ be an S-arithmetic lattice and $\iota : \mathbf{G} \to GL(V)$ an embedding of $\mathbf{G}$ as a k-group (where V is a k-vector space). The induced embedding of $\mathbf{G}'$ in $GL(V)$ is also a k-embedding. The lattice Γ' leaves invariant some finitely generated $\mathcal{O}_S$-submodule L of $V(k)$. If Γ' contains a full set of representatives of W so does the lattice $\Gamma = \{\gamma \in \mathbf{G}(k) \mid \gamma L = L\}$, clearly $\Gamma' < \Gamma$. The root system of $\mathbf{G}'$ is reduced. For each $\beta \in \Pi$ choose an element $x_\beta \in \mathbf{U}_\beta(k)$ different from the identity. By a theorem of Borel and Tits [BT1] there is a unique split semisimple k-subgroup $\mathbf{G}''$ of $\mathbf{G}'$ sharing the split torus $\mathbf{T}$ and containing the elements x_β, $\beta \in \Pi$. The argument above for reducing the problem to finding an appropriate lattice in $\mathbf{G}'$ applies to reduce the problem to finding a lattice in $\mathbf{G}''$ containing representatives of W. Thus we are reduced to the case of a Chevalley group. The k-rational points of a semisimple Chevalley group contain a finite group which contains representatives for W. As this finite group is in the commensurability group of our lattice it normalizes a sublattice of finite index and we can add it to this sublattice to generate a lattice as required.

For any root $\varphi \in \Phi$, let $\mathbf{U}_\varphi$ be the corresponding root group. Let $\tilde{\mathbf{U}}_\varphi = \mathbf{U}_\varphi$ in case 2φ is not a root, and $\tilde{\mathbf{U}}_\varphi = \mathbf{U}_\varphi/\mathbf{U}_{2\varphi}$ in case 2φ is a root. In both cases $\tilde{\mathbf{U}}_\varphi(k)$ is a k-vector space. We denote by L'_φ the image of $\mathbf{U}_\varphi(\mathcal{O}_\mathrm{S})$ in $\tilde{\mathbf{U}}_\varphi(k)$. Let L_φ be the maximal $\mathcal{O}_\mathrm{S}$-submodule of $\tilde{\mathbf{U}}_\varphi(k)$ contained in L'_φ. L_φ is finitely generated projective $\mathcal{O}_\mathrm{S}$-submodule, of finite index in L'_φ and span $\tilde{\mathbf{U}}_\varphi(k)$ as a k-vector space. Let $\psi_0 \in \Phi$ be the highest root (with respect to the order determined by Π). For $\varphi \in \Phi$ and $\beta \in \Pi$ let $m(\varphi, \beta) \in \mathbf{Z}$ be defined by $\varphi = \sum_{\beta\in\Pi} m(\varphi, \beta)\beta$. Choose a simple root $\alpha \in \Pi$ so that the following conditions are satisfied:

(1) $m(\psi_0, \alpha) \leqslant m(\psi_0, \beta)$ for every $\beta \in \Pi$.

(2) If the root system Φ is not reduced then α is the unique short root in Π. (In this case, this is compatible with the first condition.)

Denote by $\Phi_-(\alpha) = \{\varphi \in \Phi \mid m(\varphi, \alpha) < 0\}$, $\Phi'_-(\alpha) = \{\varphi \in \Phi_-(\alpha) \mid \frac{1}{2}\varphi \notin \Phi\}$. Using the classification of root systems of simple Lie algebras one can check that this choice of the root α implies that for any $\varphi \in \Phi_-(\alpha)$, $m(\varphi, \alpha) \in \{-1, -2\}$. Let $\mathbf{P}$ be the maximal parabolic subgroup of $\mathbf{G}$ determined by the root α. Let $\mathbf{P} = \mathbf{M}\mathbf{U}^+$, where $\mathbf{M}$ is reductive and $\mathbf{U}^+$ is the unipotent radical of $\mathbf{P}$. Let $\mathbf{U}^-$ be the opposite of $\mathbf{U}^+$. A root $\varphi \in \Phi$ is $\mathbf{M}$-dominant if it satisfies $\langle\varphi, \beta\rangle \geqslant 0$ for all $\beta \in \Pi$ different from α. Note that $\mathbf{M}$ is generated by $\{\mathbf{U}_\varphi \mid m(\varphi, \alpha) = 0\}$ together with the centralizer of $\mathbf{T}$. Let $\mathrm{W}(\mathbf{M}) = \mathrm{N}_\mathbf{M}(\mathbf{T})/\mathrm{Z}_\mathbf{M}(\mathbf{T})$ be the k-Weyl group of $\mathbf{M}$. It is naturally embedded in W. Let $\tilde{\mathrm{W}}(\mathbf{M}) \subset \tilde{\mathrm{W}}$ be the elements representing $\mathrm{W}(\mathbf{M})$. Enumerate the roots in $\Phi'_-(\alpha)$ in decreasing order (with respect to the order determined by Π): $\Phi'_-(\alpha) = \{\varphi_1, \varphi_2, \ldots, \varphi_r\}$. Note that in completing the partial order determined by Π to a linear one we can make sure that if $m(\varphi_i, \alpha) = -1$ and $m(\varphi_j, \alpha) = -2$ then $i < j$. We have $\varphi_1 = -\alpha$. If φ_i is $\mathbf{M}$-dominant and $w \in \mathrm{W}(\mathbf{M})$ then $\varphi_j = w\varphi_i$ satisfies $i \leqslant j$.

We have $\mathbf{U}^- = \mathbf{U}_{\varphi_1}\mathbf{U}_{\varphi_2}\ldots\mathbf{U}_{\varphi_r}$ and the map $\mathbf{U}_{\varphi_1} \times \mathbf{U}_{\varphi_2} \times \ldots \times \mathbf{U}_{\varphi_r} \to \mathbf{U}^-$, $(x_1, x_2, \ldots, x_r) \to x_1 x_2 \ldots x_r$ is a k-rational isomorphism. For $u \in \mathrm{U}^-$, we define $u(\varphi_i)$, $\varphi_i \in \Phi'_-(\alpha)$, by $u = u(\varphi_1)u(\varphi_2)\ldots u(\varphi_r)$ where each $u(\varphi_i) \in \mathbf{U}_{\varphi_i}(k)$. We shall define the *denominator* $\mathrm{Den}(x)$, of an element $x \in \mathbf{U}^-(k)$, as follows: Let $x \in \mathbf{U}_\varphi(k)$ for some $\varphi \in \Phi_-(\alpha)$. If 2φ is not a root then $x \in \mathbf{U}_\varphi(k) = \tilde{\mathbf{U}}_\varphi(k)$ viewed as a k-vector space. Define the ideal $\mathrm{I}(x) = \{t \in \mathcal{O}_\mathrm{S} \mid tx \in \mathrm{L}_\varphi\}$. Let $\mathrm{Den}(x) = \#(\mathcal{O}_\mathrm{S}/\mathrm{I}(x))$. If 2φ is a root, fix a k-section $s_\varphi : \tilde{\mathbf{U}}_\varphi(k) \to \mathbf{U}_\varphi(k)$ of the natural projection map $\pi_\varphi : \mathbf{U}_\varphi(k) \to \tilde{\mathbf{U}}_\varphi(k)$. Let

$x' = s_\varphi(\pi_\varphi(x))$ and $x'' = x'^{-1}x \in \mathbf{U}_{2\varphi}(k)$. We have the ideals $\mathrm{I}(\pi_\varphi(x)) = \{t \in \mathcal{O}_S \mid t\pi_\varphi(x) \in \mathrm{L}_\varphi\}$ and $\mathrm{I}(x'') = \{t \in \mathcal{O}_S \mid tx'' \in \mathrm{L}_{2\varphi}\}$. Set $\mathrm{Den}(x) = \max\{\#(\mathcal{O}_S/\mathrm{I}(\pi_\varphi(x))), \#(\mathcal{O}_S/\mathrm{I}(x''))\}$. For $u \in \mathrm{U}^-(k)$ its denominator is defined to be $\mathrm{Den}(u) = \max\{\mathrm{Den}(u(\varphi)) \mid \varphi \in \Phi'_-(\alpha)\}$. A subset of $\mathbf{U}^-(k)$ bounded in U^-, whose elements have denominators bounded by a constant is finite. Notice that since for every $\varphi \in \Phi_-(\alpha)$, $m(\varphi, \alpha) \in \{-1, -2\}$ it follows that $\mathbf{U}^-$is at most two step nilpotent.

Let λ be a weight of $\mathbf{G}$ such that $\langle\lambda, \beta\rangle = 1$ when $\beta = \alpha$ and 0 otherwise. Let W_λ be the corresponding finite dimensional irreducible representation of $\mathbf{G}$, and r the dimension of its highest weight space. Let $\mathrm{V}_\lambda = \wedge^r \mathrm{W}_\lambda$ and $v = v_\lambda \in \mathrm{V}_\lambda$ be an integral highest weight vector. Let V^*_λ be the dual representation and $v^* = v^*_\lambda$ the lowest weight vector of V^*_λ, s.t. $\langle v^*, v\rangle = 1$. Define a function $\mathrm{F} = \mathrm{F}_\lambda : \mathbf{G}(k) \to k$ by $\mathrm{F}(g) = \langle v^*, gv\rangle$. The function F is a character on $\mathbf{P}$. Let $\Omega = \mathrm{U}^-\mathrm{M}\mathrm{U}^+$ an open dense subset of G, and let $\Omega_k = \Omega \cap \mathbf{G}(k)$, a Zariski dense subset of G. For $g \in \Omega$ we write $g = u^- m u^+$. Since $\mathbf{U}^-(\mathcal{O}_S) = \Gamma \cap \mathrm{U}^-$ is a uniform lattice in U^-, there exists some $\delta \in \Gamma \cap \mathrm{U}^-$ such that δu^- belongs to a fixed compact fundamental domain for $\mathrm{U}^- \cap \Gamma$ in U^-. Define $\rho(g)$ by $\rho(g) = \|\delta\|$. As $\mathrm{U}^- \cap \Gamma$ is discrete, ρ has values in a discrete set and it is minimal for $\delta = e$. We rescale it so that $\rho(e) = 1$.

(4.11) *Lemma.*

(i) $\Omega = \mathbf{G}\backslash\{\mathrm{F} = 0\}$,

(ii) $k[\Omega] = k[\mathbf{G}][1/\mathrm{F}]$,

(iii) *For* $\gamma \in \Gamma \cap \Omega$, *let* $\gamma = u^- m u^+$ *with* $u^- \in \mathbf{U}^-(k)$, $m \in \mathbf{M}(k)$ *and* $u^+ \in \mathbf{U}^+(k)$. *Then the denominator Den*(u^-) *is bounded by a polynomial in* $|\mathrm{F}(\gamma)|^*$.

Proof Let $\tilde{\mathbf{T}}$ be a maximal torus of $\mathbf{G}$ containing $\mathbf{T}$ and $\tilde{\Phi}$ the root system of $\mathbf{G}$ with respect to $\tilde{\mathbf{T}}$. The torus $\tilde{\mathbf{T}}$ is contained in $\mathbf{M}$ and hence in $\mathbf{P}$ as well. Fix an order on the character group of $\tilde{\mathbf{T}}$ such that for $\beta \in \tilde{\Phi}$, β is positive if β restricted to $\mathbf{T}$ is positive. Let $\tilde{\mathrm{W}}$ (resp. $\tilde{\mathrm{W}}(\mathbf{M})$) denote the Weyl group of $\mathbf{G}$ (resp. $\mathbf{M}$) with respect to $\tilde{\mathbf{T}}$. From the work of Kostant [Ko] one knows that there is a subset $\mathrm{S} \subset \tilde{\mathrm{W}}$ such that S maps bijectively onto $\tilde{\mathrm{W}}/\tilde{\mathrm{W}}(\mathbf{M})$ and for $w \in \tilde{\mathrm{W}}$, the singleton $(w\tilde{\mathrm{W}}(\mathbf{M}) \cap \mathrm{S})$ is the unique element w_0 in $w\tilde{\mathrm{W}}(\mathbf{M})$ with the property that all the root spaces corresponding to the *positive* roots $\beta \in \tilde{\Phi}$ such that $w_0(\beta) < 0$ are contained in the Lie algebra $\underline{u}^+$ of $\mathbf{U}^+$. We denote by $\underline{h}(\beta)$ the root space of $\beta \in \tilde{\Phi}$ in the sequel. Suppose now that $w \in \tilde{\mathrm{W}}$, $w \notin \tilde{\mathrm{W}}(\mathbf{M})$; then $w = w_0.w'$ with $w' \in \tilde{\mathrm{W}}(\mathbf{M})$ and for $0 < \beta \in \tilde{\Phi}$, if $w_0(\beta) < 0$, then $h(\beta) \subset \underline{u}^+$. Suppose then $g \in \mathbf{G}(\bar{k})$, $\bar{k}$ an algebraic closure of k. Then one has

$$g = u^- w p$$

where $u^- \in \mathbf{U}^-(k)$, $w \in \mathrm{S}$ and $p \in \mathbf{P}(k)$. Hence

$$\begin{aligned} \mathrm{F}(g) = \ < v^*, gv > = \ < v^*, u^- wpv > \\ = t < v^*, wv >, \quad t \neq 0. \end{aligned}$$

Now $\bar{k}v$ is an eigenspace for all of $\tilde{\mathrm{T}}$ since $\underline{u}^+$ is $\tilde{\mathrm{T}}$-stable. Consequently $w(\bar{k}.v)$ is an eigenspace for $\tilde{\mathrm{T}}$ as well. This last eigenspace coincides with $\bar{k}v$, if and only if w maps each root space $\underline{h}(\beta) \subset u^+$ into a $\underline{h}(\beta') \subset \underline{u}^+$. But then w does not change the sign of any $\beta \in \tilde{\Phi}$, $\beta > 0$ with $\underline{h}(\beta) \subset \underline{u}^+$; this means that w must be the identity element. Finally all the eigenspaces of $\tilde{\mathrm{T}}$ other than $\bar{k}v$, are orthogonal to v^*. Thus $\mathrm{F}(g) \neq 0$ if and only if $w =$ identity, i.e., $g \in \mathrm{U}^-\mathrm{P}$. The first two assertions are immediate from this. Also since the map,

$$\mathscr{B} : \mathrm{U}^- \times \mathrm{M} \times \mathrm{U}^+ \to \Omega$$

given by $(u^-, m, u^+) \mapsto u^- m u^+$ is an isomorphism (over k), its inverse is also defined over k. Since the coordinate ring of Ω is identified with $k[\mathbf{G}][1/\mathrm{F}]$, assertion (iii) follows.

(4.12) *Remark.* — To prove (4.8) we need to show that every $\gamma \in \mathbf{G}(\mathscr{O}_\mathrm{S})$ can be written as a word of length $\mathrm{O}(\log \|\gamma\|)$. It suffices to show this for $\gamma \in \Gamma \cap \Omega_k$ since a finite number of translations of Ω_k by elements of $\mathbf{G}(\mathscr{O}_\mathrm{S})$ covers $\mathbf{G}(\mathscr{O}_\mathrm{S})$.

(4.13) *Lemma.* — *There exist positive constants* A, B *and* C *so that for all* $\gamma \in \Gamma \cap \Omega_k$,

$$d_\mathrm{W}(\gamma, e) \leqslant \mathrm{A} \log \|\gamma\| + \mathrm{B} \log |\mathrm{F}(\gamma)|^* + \mathrm{C} \log \rho(\gamma).$$

(4.14) *Remark.* — Note that $|\mathrm{F}(\gamma)|^*$ and $\rho(\gamma)$ are bounded polynomially by $\|\gamma\|$, so the lemma actually says that $d_\mathrm{W}(\gamma, e) = \mathrm{O}(\log \|\gamma\|)$. For the proof, however, it is more convenient to use also $\mathrm{F}(\gamma)$ and $\rho(\gamma)$.

(4.15) *Proof of (4.13).* — Denote $l(\gamma) = d_\mathrm{W}(\gamma, e)$. Since both $|\mathscr{O}_\mathrm{S}|^*$ and $\rho(\Gamma)$ are discrete we can argue by induction on $|\mathrm{F}(\gamma)|^*$ and $\rho(\gamma)$.

Let $\gamma \in \Gamma \cap \Omega_k$ and $\gamma = u^- m u^+$. Let N_D be a fixed constant (to be determined later). Lemma 4.11 (iii) implies that there exists a constant N_F such that if $|\mathrm{F}(\gamma)|^* \leqslant \mathrm{N}_\mathrm{F}$ then $\mathrm{Den}(u^-) \leqslant \mathrm{N}_\mathrm{D}$. There exists a finite set Q of elements of U_k^- such that if $\gamma = u^- m u^+$ satisfies $\mathrm{Den}(u^-) \leqslant \mathrm{N}_\mathrm{D}$ and $\rho(\gamma) = 1$, then $u^- \in \mathrm{Q}$. Choose a fixed set $\tilde{\mathrm{Q}}$ of elements of $\Gamma \cap \Omega_k$ whose U^- parts represent all elements of Q. Thus if $\mathrm{Den}(u^-) \leqslant \mathrm{N}_\mathrm{D}$ and $\rho(\gamma) = 1$, then by multiplying γ by the inverse of a suitable element y of $\tilde{\mathrm{Q}}$, we have $y^{-1}\gamma \in \mathbf{P} = \mathbf{MU}$. As $\mathbf{P}$ is a proper k-subgroup of $\mathbf{G}$, $\mathrm{P} \cap \Gamma$ is $(d_\mathrm{W}, d_\mathrm{R})$-undistorted so $y^{-1}\gamma$ is efficiently generated and so is γ, i.e., there exists a constant A_0 such that (4.13) holds for elements $\gamma = u^- m u^+ \in \Gamma \cap \Omega_k$ which satisfy $\mathrm{Den}(u^-) \leqslant \mathrm{N_D}$ (hence in particular those satisfying $|\mathrm{F}(\gamma)|^* \leqslant \mathrm{N_F}$) and $\rho(\gamma) = 1$ provided $\mathrm{A} \geqslant \mathrm{A}_0$.

If $\rho(\gamma) > 1$, let δ be the element of $\Gamma \cap \mathrm{U}^-$ as in the definition of $\rho(\gamma)$ in (4.10). The element $\delta\gamma$ satisfies $\mathrm{F}(\delta\gamma) = \mathrm{F}(\gamma)$ and $\rho(\delta\gamma) = 1 < \rho(\gamma)$. So we can apply the induction hypothesis to deduce:

$$\begin{aligned} l(\gamma) &\leqslant l(\delta^{-1}) + l(\delta\gamma) \leqslant l(\delta) + \mathrm{A}\log\|\delta\gamma\| + \mathrm{B}\log|\mathrm{F}(\delta\gamma)|^* \\ &\quad + \mathrm{C}\log\rho(\delta\gamma) \leqslant l(\delta) + \mathrm{A}\log\|\gamma\| + \mathrm{A}\log\|\delta\| + \mathrm{B}\log|\mathrm{F}(\gamma)|^* + \mathrm{C}\cdot 0 \\ &\leqslant \mathrm{C}_{\mathrm{U}^-}\log\|\delta\| + \mathrm{A}\log\|\gamma\| + \mathrm{B}\log|\mathrm{F}(\gamma)|^* + \mathrm{A}\log\|\delta\| \\ &= \mathrm{A}\log\|\gamma\| + \mathrm{B}\log|\mathrm{F}(\gamma)|^* + (\mathrm{C}_{\mathrm{U}^-} + \mathrm{A})\log\rho(\gamma). \end{aligned}$$

This proves the claim for γ provided C was chosen to be bigger than $\mathrm{C}_{\mathrm{U}^-} + \mathrm{A}$ where $\mathrm{C}_{\mathrm{U}^-}$ is the implied constant for U^- by (4.3).

Hence we can assume that $\rho(\gamma) = 1$.

(4.16). — Recall (see 4.10) that $\mathbf{U}^-$ is a product of root groups, $\mathbf{U}^- = \prod_{\varphi\in\Phi'_{-(\alpha)}} \mathbf{U}_\varphi = \mathbf{U}_{-\alpha}\cdot\prod_{i=2}^{r}\mathbf{U}_{\varphi_i}$ $(\varphi_1 = -\alpha)$.

As in 4.10, $u^- \in \mathrm{U}^-$ may be written as $u^- = u^-(\varphi_1)u^-(\varphi_2)\dots u^-(\varphi_r)$ where $u^-(\varphi_i) \in \mathrm{U}_{\varphi_i}$. Let $u_1 = u^-(\varphi_1)$, $u_2 = u^-(\varphi_2)\dots u^-(\varphi_r)$, hence $u^- = u_1 u_2$.

(4.17) *Lemma.* — *Let* $\mathrm{N}_1 > 0$ *be a given constant. There exist* $\mathrm{N_D} > 0$, *a bounded subset* K_3 *of* $\mathbf{U}_{-\alpha}(k)$ *and a finite set* Q *of elements of* $\mathbf{G}(\mathcal{O}_\mathrm{S})$ *such that if* $\gamma \in \Gamma \cap \Omega_k$, $\gamma = u^- m u^+$, *satisfies* $\rho(\gamma) = 1$ *(or more generally* u^- *belongs to a fixed compact subset of* U^-*) and the denominator of* u^- *satisfies* $\mathrm{Den}(u^-) > \mathrm{N_D}$, *then for some* $q \in \mathrm{Q}$ *we have* $q\gamma = u_0^- m_0 u_0^+$, $u_0^-(-\alpha) \in \mathrm{K}_3$ *and* $\mathrm{Den}(u_0^-(-\alpha)) > \mathrm{N}_1$ *(notation as in 4.10). Moreover* $|\mathrm{F}(q\gamma)|^* = |\mathrm{F}(\gamma)|^*$.

(4.18) *Remark.* — We shall postpone the proof of Lemma 4.17 to (4.23) and continue with the proof of Lemma 4.13 assuming Lemma 4.17. The choice of N_1 will be made as in (4.20).

(4.19) *Lemma.* — *Let* $\mathbf{G}^\alpha$ *be the k-rank one group associated with the root* α, $\mathbf{P}^\alpha$ *be the positive parabolic subgroup of* $\mathbf{G}^\alpha$, $\mathbf{P}^\alpha = \mathbf{N}^\alpha\mathbf{M}^\alpha\mathbf{T}^\alpha$ *where* $\mathbf{N}^\alpha$ *is the unipotent radical of* $\mathbf{P}^\alpha$, $\mathbf{T}^\alpha$ *is the (one dimensional) k-split torus and* $\mathbf{M}^\alpha$ *is anisotropic and commutes with* $\mathbf{T}^\alpha$. *There exists a finite subset* Q *of* $\mathbf{U}_{-\alpha}(k)$ *such that every* $u_1 \in \mathbf{U}_{-\alpha}(k)$ *can be written as* $u_1 = \delta s p$ *where* $s \in \mathrm{Q}$, $\delta \in \mathbf{G}^\alpha(\mathcal{O})$, $p \in \mathbf{P}^\alpha(k)$. *Moreover,* p *can be written as* $p = la$ *where* l *is in a bounded subset of* $\mathbf{N}^\alpha(k)\mathbf{M}^\alpha(k)$ *and* a *is in the split torus* T^α *of* G^α, *so that* $\log\|a\| \leqslant \mathrm{K}_0\log|1/\mathrm{F}(a)|^*$, *for some fixed* K_0. *(Note that it follows that the size of* p *is controlled by* $|1/\mathrm{F}(p)|^* = |1/\mathrm{F}(a)|^*$.*)*

Proof. — It is well known that $\mathbf{G}^\alpha(\mathcal{O})\backslash\mathbf{G}^\alpha(k)/\mathbf{P}^\alpha(k)$ is finite [Bo]. Hence we can write u_1 as $u_1 = \delta' s' p'$ where $p' \in \mathbf{P}^\alpha(k)$, $\delta' \in \mathbf{G}^\alpha(\mathcal{O})$ and s' is in a finite subset Q′ of $\mathbf{G}^\alpha(k)$. If s' belongs to $\mathbf{P}^\alpha(k)$ then we can take $s = e$, $\widetilde{p} = s'p'$ and

$\tilde{\delta}=\delta'$. Otherwise we have $s' \in \mathbf{U}_\alpha(k)w\mathbf{P}^\alpha(k)$ where w is the non-trivial element of the Weyl group of $\mathbf{G}^\alpha$. Hence $s'=u''wp''$. Take $\tilde{\delta}=\delta' w$, $s=w^{-1}u''w$ and $\tilde{p}=p''p'$. Let $Q=\{e\} \cup \{w^{-1}u''w \mid s'=u''wp'' \in Q'\}$. Without loss of generality, we can assume that $w \in \mathbf{G}^\alpha(\mathcal{O})$, since we could a priori replace the S-arithmetic lattice $\Gamma=\mathbf{G}(\mathcal{O}_S)$ by a commensurable one containing a representative of w. Indeed, by [BT1] the group $\mathbf{G}^\alpha(k)$ contains a k-subgroup $SL_2(k)$ such that the usual torus of $SL_2(k)$ coincides with the split torus of $\mathbf{G}^\alpha(k)$. Hence the element $w=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a representative of the non-trivial element of the Weyl group of $\mathbf{G}^\alpha$. w is of finite order 4. Let $\tilde{\Gamma}=\Gamma \cap w\Gamma w^{-1} \cap w^2\Gamma w^{-2} \cap w^3\Gamma w^{-3}$. Then w normalizes $\tilde{\Gamma}$ and we have $\langle \tilde{\Gamma}, w\rangle$ a commensurable lattice as required. Notice that our group G satisfies the assumptions of Proposition 4.8 also with respect to this new lattice.

The above $\tilde{\delta}$, s and $\tilde{p}$ satisfy the requirement of the first part of the lemma. To justify the second part, recall that the finite set Q is in the commensurability group of $\mathbf{G}^\alpha(\mathcal{O})$, hence there exists a finite index subgroup Λ of $\mathbf{G}^\alpha(\mathcal{O})$ such that $s\Lambda s^{-1} \subseteq \mathbf{G}^\alpha(\mathcal{O})$ for every $s \in Q$. Let $\tilde{p}=\tilde{l}\tilde{a}$ with $\tilde{l} \in \mathbf{N}^\alpha(k)\mathbf{M}^\alpha(k)$ and $\tilde{a} \in T^\alpha$. There exist $a \in T^\alpha$ and $a' \in T^\alpha \cap \Lambda$ such that $\tilde{a}=a'a$ and $\log\|a\| \leqslant K_0 \log|1/F(a)|^*$ for some fixed K_0. (This follows from the fact that the elements of norm $(|\cdot|^*)$ one in $\mathcal{O}$ form a uniform lattice in the elements of norm one in k_{S_∞}.) Hence we get $u_1=\tilde{\delta}s\tilde{p}=\tilde{\delta}s\tilde{l}a'a=\tilde{\delta}'s\tilde{l}'a$ with $\tilde{\delta}' \in G^\alpha(\mathcal{O})$, $\tilde{l}' \in \mathbf{N}^\alpha(k)\mathbf{M}^\alpha(k)$. Next we use the compactness of $\Lambda \cap N^\alpha M^\alpha \backslash N^\alpha M^\alpha$ to get $u_1=\delta s l a$ as required.

(4.20) *Lemma.* — *There exist constants* $a_0<1$ *and* N_1 *such that if* $K_3 \subset \mathbf{U}_{-\alpha}(k)$ *is a bounded set as in 4.17 and* $u_1 \in K_3$ *is an element whose denominator is bigger than* N_1, *then if* $u_1=\delta s p$ *as in (4.19), then* $|F(p)|^*<a_0<1$.

Proof. — Consider the way u_1 acts on the highest weight vector $v \in V_\lambda$ (see 4.10): Since $u_1 \in U_{-\alpha}$, $u_1 v=v+v'$ where v' is a weight vector of weight $r\lambda-\alpha$ (note that the highest weight of V_λ is $r\lambda$). By our assumption the denominator of u_1 is large, hence the denominator of v' is large. (The "denominator" of a vector in V_λ is in the natural way the least common multiple of the denominators of its coordinate with respect to a fixed rational basis obtained from $\mathbf{G}(k)v_\lambda$.) At the same time $u_1=\delta s p$, so $u_1 v=\delta s p v=F(p)\delta s v$. This forces $|F(p)|^*$ to be very small. Indeed, s belongs to a finite set of $\mathbf{U}_{-\alpha}(k)$ and δ belongs to the integral points $\mathbf{G}^\alpha(\mathcal{O})$, hence $\{\delta s v\}$ is a discrete set (with respect to $\| \quad \|_{S_\infty}$). On the other hand u_1 is in a bounded subset of $\mathbf{U}_{-\alpha}(k)$.

(4.21) *Lemma.* — *There exists a constant* K *such that if* $u_1 \in \mathbf{U}_{-\alpha}(k)$ *is in a bounded set as in (4.17) and* $u_1=\delta s p$ *as in (4.19), then* $\max\{\log\|\delta^{-1}\|, \log\|\delta\|\} \leqslant K\log|1/F(p)|^*$.

Proof. — Let $p=la$ as in (4.19), so $u_1=\delta(sl)a$. As both u_1 and sl are in compact sets, the size of δ is controlled by the size of a. By Lemma 4.19 we have

$\log ||a|| \leqslant K_0 \log |1/F(a)|^*$, hence for suitable K′ and K″

$$\log ||\delta|| \leqslant K' \log |1/F(a)|^* = K' \log |1/F(p)|^*.$$

The second inequality follows from the fact that l is in a compact set. Since $\log ||\delta^{-1}||$ is Lipschitz equivalent to $\log ||\delta||$, the lemma follows.

(4.22). — We can now complete the proof of (4.13). Let $\gamma = u^- m u^+ \in \Gamma \cap \Omega_k$, we need to bound $l(\gamma) = d_W(\gamma, 1)$. Let N_1 be the constant determined by (4.20) and N_D the corresponding constant implied by Lemma 4.17. As shown in (4.15), we can assume $\rho(\gamma) = 1$. We also show there that if $\mathrm{Den}(u^-) \leqslant N_D$, then Lemma 4.13 holds for some constants. So we assume now $\mathrm{Den}(u^-) > N_D$ and that (4.13) holds by induction for smaller value of $|F(\gamma)|^*$.

Let q be the element given by Lemma 4.17 so that we have $q\gamma = u_1 u_2 m u^+$ with u_1 having a denominator $> N_1$. Let $u_1 = \delta s p$ be as in 4.19.

$$(*) \qquad \begin{aligned} |F(\delta^{-1} q\gamma)|^* &= |F(\delta^{-1} u_1 u_2 m u^+)|^* = |F(s p u_2 m u^+)|^* = \\ &= |F(s u_2' p m u^+)|^* = |F(p m u^+)|^* = |F(pm)|^* = |F(p)F(m)|^* = \\ &= |F(p)|^* |F(q\gamma)|^* = |F(p)|^* \, |F(\gamma)|^* < |F(\gamma)|^*. \end{aligned}$$

Notice that $u_2' = p u_2 p^{-1}$ is in the maximal unipotent subgroup corresponding to the negative roots.

We have the following inequalities:

$$(**) \qquad \begin{aligned} l(\delta^{-1} q\gamma) &\underset{(1)}{\leqslant} A \log ||\delta^{-1} q\gamma|| + B \log |F(\delta^{-1} q\gamma)|^* + \\ &C \log \rho(\delta^{-1} q\gamma) \underset{(2)}{\leqslant} A \log ||\gamma|| + A \log ||\delta^{-1}|| + A \log ||q|| \\ &+ B(\log(|F(\gamma)|^* \, |F(p)|^*) + C \log \rho(\delta^{-1} q\gamma) \\ &\underset{(3)}{\leqslant} A \log ||\gamma|| + B \log |F(\gamma)|^* + R \log |1/F(p)|^* - B \log |1/F(p)|^*. \end{aligned}$$

The inequality (1) follows from the induction hypotheses. In inequality (2) we have used (*) and in (3) we choose R so that

$$A \log ||q|| + A \log ||\delta^{-1}|| + C \rho(\delta^{-1} q\gamma) \leqslant R \log |1/F(p)|^*.$$

Since $\rho(\gamma) = 1$ and q belongs to a fixed finite set it follows that $u_1 u_2$ belongs to a certain compact subset of U^-. As $\delta^{-1} q\gamma = s p u_2 m u^+ = s u_2' p m u^+$ it follows that the size of $\rho(\delta^{-1} q\gamma)$ (which is determined by the size of $s u_2'$) is controlled by the size of p. Combined with (4.21) this ensures the existence of such a constant R.

We also have by (4.7), (4.20) and (4.21) that for a suitable constants S′ and S,

$$(***) \qquad l(q) + l(\delta) < l(q) + S' \log ||\delta|| < S \log |1/F(p)|^*.$$

Thus by (∗∗) and (∗ ∗ ∗):

$$l(\gamma) \leqslant l(q)+l(\delta)+l(\delta^{-1}q\gamma) \leqslant \mathrm{A}\log\|\gamma\|+\mathrm{B}\log|\mathrm{F}(\gamma)|^{*}+(\mathrm{R}-\mathrm{B}+\mathrm{S})\log|1/\mathrm{F}(p)|^{*}.$$

Thus, if we make sure to choose B bigger than R + S, (4.13) follows and hence also (4.8). As shown in (4.9), this finishes the proof of our main theorem (4.1).

(4.23) *Proof of Lemma 4.17.* — We are given an element $\gamma = u^{-}mu^{+}$ such that u^{-} belongs to a compact subset of U^{-} and $\mathrm{Den}(u^{-})$ is large, i.e., if we let $u^{-}=u^{-}(\varphi_1)u^{-}(\varphi_2)\dots u^{-}(\varphi_r)$ as in (4.10) then $\mathrm{Den}(u^{-}(\varphi_i))$ is large for some $1 \leqslant i \leqslant r$. Our goal is to multiply γ by elements belonging to a fixed finite set so that the new element will have in the corresponding decomposition a large denominator of the part belonging to $\mathbf{U}_{\varphi_1}$. We shall use the following lemmas:

(4.24) *Lemma.* — *Let* $\mathrm{K}_1 \subset \mathrm{U}^{-}$ *be a compact subset and* $\mathrm{M}_1 \in \mathbf{N}$. *There exist an integer* $\mathrm{M}_2 \in \mathbf{N}$ *and a compact subset* $\mathrm{K}_2 \subset \mathrm{U}^{-}$ *so that if* $x \in \mathrm{K}_1$, $x = x(\varphi_1)x(\varphi_2)\dots x(\varphi_r)$ *(as in (4.10)) and* $\mathrm{Den}(x(\varphi_{i_0})) \geqslant \mathrm{M}_2$ *for some* $1 \leqslant i_0 \leqslant r$, *then there exists* $w \in \widetilde{\mathrm{W}}(\mathbf{M})$ *such that* $y = wxw^{-1} \in \mathrm{K}_2$, $y = y(\varphi_1)y(\varphi_2)\dots y(\varphi_r)$, *and* $\mathrm{Den}(y(\varphi_j)) \geqslant \mathrm{M}_1$ *for some* j *such that either* $j < i_0$ *or* $j \leqslant i_0$ *and* φ_j *is* $\mathbf{M}$*-dominant.*

Proof. — Since $\widetilde{\mathrm{W}}(\mathbf{M})$ is a fixed finite set, there exists $\mathrm{M}_2 \in \mathbf{N}$ such that if $\mathrm{Den}(x(\varphi_i)) \geqslant \mathrm{M}_2$ then $\mathrm{Den}(wx(\varphi_i)w^{-1}) \geqslant \mathrm{M}_1(c\mathrm{M}_1^2)^r$ for any $w \in \widetilde{\mathrm{W}}(\mathbf{M})$, where $c > 1$ is a constant chosen so that if $z \in \mathbf{U}_{\varphi}(k)$, $z' \in \mathbf{U}_{\varphi'}(k)$ then $\mathrm{Den}([z, z']) \leqslant c\mathrm{Den}(z)\mathrm{Den}(z')$. Without loss of generality let $1 \leqslant i_0 \leqslant r$ be the first index such that $\mathrm{Den}(x(\varphi_{i_0})) \geqslant \mathrm{M}_2$. If φ_{i_0} is an $\mathbf{M}$-dominant root then the assertion holds (with $w = e$). Otherwise let $w \in \widetilde{\mathrm{W}}(\mathbf{M})$ be the (unique) element such that $w\varphi_{i_0} = \varphi_n$, $1 \leqslant n \leqslant r$, is $\mathbf{M}$-dominant. Let $y = wxw^{-1} = wx(\varphi_1)w^{-1}wx(\varphi_2)w^{-1}\dots wx(\varphi_r)w^{-1}$, each $wx(\varphi_i)w^{-1} \in \mathbf{U}_{w\varphi_i}$. Note that $\mathrm{Den}(wx(\varphi_{i_0})w^{-1}) \geqslant \mathrm{M}_1(c\mathrm{M}_1^2)^r > \mathrm{M}_1$. We have to reorder the $wx(\varphi_i)w^{-1}$ to get an expression $y = y(\varphi_1)y(\varphi_2)\dots y(\varphi_r)$. Since $\mathbf{U}^{-}$ is (at most) two step nilpotent this process produces only new elements which are commutators of the various $wx(\varphi_i)w^{-1}$'s. In case $w\varphi_{i_0}$ is not a sum of two roots from $\Phi_{-}(\alpha)$ then $y(w\varphi_{i_0}) = wx(\varphi_{i_0})w^{-1}$ and the assertion holds (note that $w\varphi_{i_0}$ being $\mathbf{M}$-dominant appears before φ_{i_0}). If $w\varphi_{i_0}$ may be expressed as a sum of two roots, say $w\varphi_{i_0} = \varphi + \varphi'$ then either for some such φ we will have $\mathrm{Den}(y(\varphi)) > \mathrm{M}_1$ and the assertion holds – note that such φ necessarily precedes $w\varphi_{i_0}$ and hence also precedes φ_{i_0}, or for all these roots φ, $\mathrm{Den}(wx(w^{-1}\varphi)w^{-1}) \leqslant \mathrm{M}_1$. This implies that their commutator has denominator at most $c\mathrm{M}_1^2$. As the number of such roots is at most r (actually much less), and $\mathrm{Den}(wx(\varphi_{i_0})w^{-1}) \geqslant \mathrm{M}_1(c\mathrm{M}_1^2)^r$ it follows that $\mathrm{Den}(y(w\varphi_{i_0})) \geqslant \mathrm{M}_1$ as required. Note that in the above we have used the fact that $m(\varphi, \alpha) \in \{-1, -2\}$ for $\varphi \in \Phi_{-}(\alpha)$, which guaranteed that a root in $\Phi'_{-}(\alpha)$ is at the sum of no more than two other roots in $\Phi'_{-}(\alpha)$. The existence of the compact set K_2 is clear.

(4.25) *Lemma.* — *Let $\varphi \in \Phi_{-}(\alpha)$ be a root such that $\langle \varphi, \alpha \rangle < 0$ and 2φ is not a root. For $z \in \mathbf{U}_\alpha(k)$ let $C_z : \tilde{\mathbf{U}}_\varphi \to \tilde{\mathbf{U}}_{\varphi+\alpha}$ be defined as follows: Given $x \in \tilde{\mathbf{U}}_\varphi = \mathbf{U}_\varphi(k)$, we have $zxz^{-1}x^{-1} \in U^*$ – the algebraic subgroup generated by $\{\mathbf{U}_{m\varphi+n\alpha} \mid m, n \in \mathbf{N},\ m\varphi + n\alpha \in \Phi\}$. Let $U^{*\prime} = < \mathbf{U}_{m\varphi+n\alpha} \mid m, n \in \mathbf{N},\ m + n \geqslant 3,\ m\varphi + n\alpha \in \Phi >$. There is a natural identification of $\tilde{\mathbf{U}}_{\varphi+\alpha}$ with $U^*/U^{*\prime}$. Let $C_z(x)$ be the image of $zxz^{-1}x^{-1}$ under this identification. If $2(\varphi + \alpha)$ is not a root and $z \neq 1$, then C_z is injective. If $2(\varphi + \alpha)$ is a root, 2α is a root; if $y, z \in \mathbf{U}_\alpha(k)$ are such that $yzy^{-1}z^{-1} \neq 1$, then $C_y \oplus C_z : \tilde{\mathbf{U}}_\varphi \to \tilde{\mathbf{U}}_{\varphi+\alpha} \oplus \tilde{\mathbf{U}}_{\varphi+\alpha}$ is injective.*

Proof. — Since $\tilde{\mathbf{U}}_\varphi$ and $\tilde{\mathbf{U}}_{\varphi+\alpha}$ are vector spaces over k, it suffices to show that the maps in question are injective at the level of k-points. Let G' denote the k-rank 2 subgroup generated by $\mathbf{U}_{\pm\varphi}$ and $\mathbf{U}_{\pm\alpha}$ and Ψ the root system of G' with respect to the torus T' (= identity component of $G' \cap T$). It is easily checked – using for instance the classification of rank 2 root systems – that $\{\varphi, \alpha\}$ constitute a *simple* system for Ψ. Consider first the case when is reduced. If $z \in E_\alpha = U_\alpha(k)$ and $x \in E_\varphi = U_\varphi(k)$ are non-trivial elements, then there is a Chevalley group over k contained in G' and containing T', z and x; and our contention is immediate from the Chevalley commutation relations. Suppose Ψ is not reduced; then, as is easily checked, 2α as well as $2(\varphi + \alpha)$ are roots. Let G'' be the k-subgroup of G' generated by $\{\mathbf{U}_{\pm\varphi}, U_{\pm 2\alpha}\}$. Then $\Psi' = \{\psi \in \Psi \mid 2\psi \notin \Psi\}$ is the root system of G'' and the preceding discussion shows that

$$C_{[y,z]} : E(\varphi) \to E(\varphi + 2\alpha)$$

where $[y, z] = yzy^{-1}z^{-1}$, is injective. Now we have the commutator identity of P. Hall:

$$[[y, z], {}^z x]\,[[z, x], {}^x y]\,[[x, y], {}^y z] = 1.$$

One sees easily from this identity that if $C_y \oplus C_z$ is not injective, $C_{[y,z]}$ is not injective either. This proves the lemma.

As a consequence we have:

(4.26) *Corollary.* — *Let $K \subset \mathbf{U}^-(k)$ be a bounded subset and an integer $M_1 > 0$ be given. Let $\varphi_{i_0} \in \Phi'_{-}(\alpha)$ be an $\mathbf{M}$-dominant root linearly independent of α. There is a finite set $J \subset \mathbf{G}(\mathcal{O}_S)$, a bounded set $K' \subset \mathbf{U}^-(k)$ and an integer $M_2 > 0$ such that if $x = x(\varphi_1)x(\varphi_2)\ldots x(\varphi_r) \in K \subset \mathbf{U}^-(k)$ satisfies $\mathrm{Den}(x(\varphi_i)) < M_1$ for all roots φ_i, $1 \leqslant i < i_0 \leqslant r$ and $\mathrm{Den}(x(\varphi_{i_0})) \geqslant M_2$, then there exists an element $g \in J$ such that $gx = uab$ with $u \in K'$, a belongs to the unipotent subgroup of $\mathbf{M}$ generated by the negative roots, $b \in \mathbf{U}_\alpha$ and $u = u(\varphi_1)u(\varphi_2)\ldots u(\varphi_r)$ with $\mathrm{Den}(u(\varphi_j)) \geqslant M_1$ for some $j < i_0$.*

Proof. — Since the root $\varphi = \varphi_{i_0}$ is a negative root which is **M**-dominant it follows that $\langle \alpha, \varphi_{i_0} \rangle < 0$. Moreover one can check by considering the rank 2 root system Σ obtained by looking at the roots in Φ lying in the linear span of α and φ_{i_0} that $\{\alpha, \varphi_{i_0}\}$ is a simple system for Σ. In the non reduced case Σ is of type BC_2, α is a short root and φ_{i_0} is a long root and in particular $2\varphi_{i_0}$ is not a root. Hence the conditions of Lemma 4.25 are satisfied. Let $\tilde{K} \subset \mathbf{U}^-(k)$ be a bounded subset such that if $x \in K$ then $x(\varphi_1) \dots x(\varphi_{i_0 - 1}) \in \tilde{K}$. Let $D = \{v \in \tilde{K} \subset \mathbf{U}^-(k) \mid \mathrm{Den}(v) \leqslant M_1\}$. D is a finite set. If $2(\varphi + \alpha)$ is not a root, for any element $v \in D$, we choose an element $f(v) \in \mathbf{G}(\mathcal{O}_S)$ such that $f(v) \in \mathbf{U}_\alpha(\mathcal{O}_S) \setminus \mathbf{U}_{2\alpha}(\mathcal{O}_S)$ and $vf(v)v^{-1} \in \mathbf{G}(\mathcal{O}_S)$. If $2(\varphi + \alpha)$ is a root choose elements $f(v), g(v) \in \mathbf{U}_\alpha(\mathcal{O}_S) \backslash \mathbf{U}_{2\alpha}(\mathcal{O}_S)$ such that $[f(v), g(v)] \neq 1$ and $vf(v)v^{-1}$ and $vg(v)v^{-1} \in \mathbf{G}(\mathcal{O}_S)$. Let $J = \{vf(v)v^{-1} \mid v \in D\}$. Let $x \in K$ be the given element. Let $v = x(\varphi_1) \dots x(\varphi_{i_0 - 1})$. By the assumptions $v \in D$. Consider the elements

$$\begin{aligned} vh(v)v^{-1}x &= vh(v)x(\varphi_{i_0}) \dots x(\varphi_r) \\ &= v(h(v)x(\varphi_{i_0})h(v)^{-1}x(\varphi_{i_0})^{-1})x(\varphi_{i_0})h(v)x(\varphi_{i_0+1}) \dots x(\varphi_r) \qquad (\#) \end{aligned}$$

where $h(v)$ denotes $f(v)$ or $g(v)$. We can write $h(v)x(\varphi_{i_0})h(v)^{-1}x(\varphi_{i_0})^{-1} = y_1 y_2$ where $y_1 \in \mathbf{U}_{\varphi_{i_0}+\alpha}(k)$ and y_2 belongs to product of other root groups (corresponding to combinations of the form $n\varphi_{i_0} + m\alpha$, where $n, m \in \mathbf{N}$). Lemma 4.25 implies that there exists a constant $M_2 \in \mathbf{N}$ so large that under the assumptions of the corollary the denominator of y_1 will be larger then M_1^2 (note that we are using the fact that $2\varphi_{i_0}$ is not a root). Taking $h(v)$ to be one of $g(v)$ or $f(v)$ as we conjugate $h(v)$ in (#) through the rest of the terms $x(\varphi_{i_0+1}) \dots x(\varphi_r)$ we will obtain

$$h(v)x(\varphi_{i_0+1}) \dots x(\varphi_r) = z_1 z_2 \dots z_s h(v).$$

Where the various z_i belong to root groups corresponding to roots of the form $m\varphi_i + n\alpha$, $i > i_0$ and $m, n \in \mathbf{N}$. We can reorder the product so that

$$vh(v)v^{-1}x = vy_1y_2x(\varphi_{i_0})z_1 \dots z_s h(v) = vy_1y_2x(\varphi_{i_0})t_1 t_2 \dots t_n a h(v).$$

Where a belongs to the unipotent subgroup of **M** corresponding to the negative roots, the t_i's belong to root groups $\mathbf{U}_\psi(k)$ where ψ is a linear combination with nonnegative integer coefficients of α and roots φ_j where $j > i_0$ and $\psi \in \Phi_-(\alpha)$. Next we can express the element $u = vy_1y_2x(\varphi_{i_0})t_1t_2 \dots t_n$ as $u = u(\varphi_1)u(\varphi_2) \dots u(\varphi_r)$. Using the above and the ordering of the roots one can check that $u(\varphi_{i_0} + \alpha) = x(\varphi_{i_0} + \alpha)y_1$. As $\mathrm{Den}(y_1) \geqslant M_1^2$ and $\mathrm{Den}(x(\varphi_{i_0} + \alpha)) \leqslant M_1$, we conclude that $\mathrm{Den}(u(\varphi_{i_0} + \alpha)) \geqslant M_1$. Clearly the root $\varphi_{i_0} + \alpha$ precede, in our ordering, the root φ_{i_0}. The existence of a bounded set K' as required is clear.

Repeated use of Lemma 4.24 and Corollary 4.26 yield the existence of a finite set $Q \subset \mathbf{H}(\mathcal{O}_S)$ as required in Lemma 4.17. The existence of the required bounded subset $K_3 \subset \mathbf{U}^-(k)$ is clear. To verify that for $q \in Q$ one has $|F(q\gamma)|^* = |F(\gamma)|^*$, note that: (i) $\widetilde{W}(\mathbf{M})$ is contained in the semisimple part of $\mathbf{M}$ and hence for $w \in \widetilde{W}(\mathbf{M})$ we have $|F(w)|^* = 1$. (ii) Applying Corollary 4.26 multiplies the "$\mathbf{M}$ part" of γ by ab where a belongs to the unipotent subgroup of $\mathbf{M}$ generated by the negative roots and $b \in \mathbf{U}_\alpha(k)$. Using the definition of $F(\cdot)$ it follows that it remains unchanged.

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