How delicate are the $f(R)$ gravity models with disappearing cosmological constant?

I. Thongkool,[1] M. Sami,[1] and S. Rai Choudhury[1]
[1] *Centre for Theoretical Physics, Jamia Millia Islamia, New Delhi-110025, India*

We consider stability of spherically symmetric solutions in $f(R)$ gravity model proposed by Starobinsky. We find that the model suffers from a severe fine tuning problem when applied to compact objects like neutron stars. The problem can be remedied by introducing a cut off on the mass of the scalar degree of freedom present in the model. A new mass scale associated with neutron stars density is then required for the stabilities of $f(R)$ gravity solutions inside relativistic stars.

PACS numbers:

I. INTRODUCTION

The cause of cosmic repulsion responsible for late time acceleration is one of the mysteries of modern cosmology. The simplest possibility to account for this effect is related to the assumption of dark energy[1]. It is, however, quite possible that late time acceleration of universe is the result of large scale modification of gravity. Amongst all the schemes of modification of gravity in the infrared regime, the f(R) theories of gravity [2]are most elegant and promising. These theories apart from the spin-2 object necessarily contain a scalar degree of freedom dubbed *scalaron*. Stability of the theory requires that the scalaron is not tachyon and graviton is not a ghost which can be ensured by demanding the positivity of the first and the second derivatives of $f(R)$ with respect to the Ricci scalar R.

The local gravity constraints impose most stringent restrictions on any scheme of large scale modification of gravity in particular the $f(R)$ theories of gravity. Most of the $f(R)$ models are either not cosmologically viable or simply reduce to ΛCDM[3]. An interesting class of models proposed by Hu-Sawicki and Starobinsky ,referred to as HSS hereafter, can reconcile with local gravity constraints and has the potential capability of being distinguished from cosmological constant[4, 5, 6](See Ref.[7] on the related theme).

The de-Sitter minimum in these models is very near to the curvature singularity which the scalaron can easily hit while evolving towards the minimum of its potential[8, 9, 10, 11, 12, 13, 14, 15]. One can try to modify HSS models by creating a large potential barrier between the de-Sitter minimum and curvature singularity[16]. However, the latter results are in clear violation of local gravity constraints[17]. Thus the presence of finite time singularity is generic in these models and should be handled carefully. The safe passage of scalaron to the minimum of its potential requires fine tuning of its initial conditions. The situation gets worse in the high curvature regime.

In this paper we examine the fine tuning problem associated with $f(R)$ theories with disappearing cosmological constant. We demonstrate that the level of fine tuning the HSS models require in case of the compact objects like neutron stars poses serious challenge to these models. The problem can be alleviated by introducing the quadratic curvature terms in the HSS scenario.

II. THE SCALAR DEGREE OF FREEDOM IN $f(R)$ THEORIES OF GRAVITY

We consider the modification of the Einstein-Hilbert action in the presence of the matter Lagrangian $\mathcal{L}_m$

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{f(R)}{2} + \mathcal{L}_m\right] \tag{1}$$

where $f(R)$ is a function of the Ricci scalar R. Variation with respect to metric leads to the following field equations

$$f_{,R}R_{\mu\nu} - \nabla_\mu\nabla_\nu f_{,R} + \left(\Box f_{,R} - \frac{1}{2}f\right)g_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

where $f_{,R} \equiv df/dR$ and $T_{\mu\nu} \equiv -2\delta\mathcal{L}_m/\delta g^{\mu\nu} + g_{\mu\nu}\mathcal{L}_m$. The $f(R)$ theories of gravity necessarily contain an additional scalar degree of freedom which becomes clear after taking the trace of (2)

$$\Box\psi = \frac{1}{3}T + \frac{dV}{d\psi}, \tag{3}$$

where $\psi \equiv f_{,R}$; $dV/d\psi = (2f(R(\psi)) - \psi R(\psi))/3$.

In what follows, it would be convenient to us to write $f(R)$ in the form of the correction term to Einstein-Hilbert action, Δ,

$$f(R) = R + \Delta, \qquad \psi = 1 + \Delta_{,R}, \tag{4}$$

where $\Delta_{,R}$ denotes the derivative of the correction term with respect to the Ricci scalar R.

In case of Starobinsky model [5], we have

$$\Delta = -\lambda R_c\left[1 - \left(1 + R^2/R_c^2\right)^{-n}\right] \tag{5}$$

$$\psi = 1 - 2n\lambda\left(\frac{R_c}{R}\right)^{2n+1} \quad \text{for} \quad R \gg R_c. \tag{6}$$

Taking $R_c = \rho_\Lambda/4$ where ρ_Λ is the cosmological density, (10^{-29} g/cm^3) and $R \sim \rho_c$ is the curvature inside the neutron star such that ($\rho_c \sim 10^{14}\mathrm{g/cm}^3$). The numerical value of ψ_0 corresponding to the de-Sitter minimum is given by

$$\psi_0 \approx \begin{cases} 1 - \mathcal{O}\left(10^{-122}\right) & n = 0.9, \\ 1 - \mathcal{O}\left(10^{-217}\right) & n = 2. \end{cases} \quad (7)$$

The Local Gravity Constraints are satisfied for $n \gtrsim 0.9$ [18] and the evolution of density perturbations during the matter-dominated epoch requires $n \gtrsim 2$ for $G_{eff} = G/f'(R)$ to be consistent with observations [5].

The scalar degree of freedom plays an important role in $f(R)$ theories of gravity, namely, its dynamics controls the space-time curvature. In generic cases, the de Sitter minimum at ψ_0 is very close to $\psi = 1$ corresponding to curvature singularity. The finite barrier between the singularity and de Sitter minimum means the curvature singularity is energetically accessible. In order to avoid it, the evolution of the field needs to be fine tuned. We shall see later that in the case of the neutron star with constant density ρ_c, the extreme fine tuning of initial conditions becomes necessary for the existence of GR-like solution ($R \sim \rho_c$) along the radius r of the star to match the correct boundary conditions at its surface, $r = r_*$.

III. THE GROWING MODE OF THE PERTURBATION AND FINE TUNING OF INITIAL CONDITIONS

The problem of fine tuning is inherent in $f(R)$ theories if they are to be consistent with local gravity constraints and can be appreciated by using the analytical arguments. An approximation scheme for the solution of $\psi(r)$ can be set up using the iterative computation for $R(r)$ as follows[4, 5],

$$R(r) = R_0(r) + \delta R_1(r) + \delta R_2(r) + \cdots , \quad (8)$$

where $R_0(r) = \rho(r) - 3P(r)$ and the first order iteration gives rise to the following expression,

$$\delta R_1 = \left[-3\nabla^2\Delta_{,R} + \Delta_{,R}R - 2\Delta\right]\big|_{R=R_0}. \quad (9)$$

In case of Starobinsky model, we have

$$|\nabla^2\Delta_{,R}|, |\Delta_{,R}R| \ll |\Delta| \qquad \text{for} \quad R \gg R_c, \quad (10)$$
$$\delta R_1(r) \approx -2\Delta \approx 2\lambda R_c = const. \quad (11)$$

In the first order iteration, ψ_1 can be expressed through the GR-like solution ψ_0 as

$$\psi_1 = 1 + \Delta_{,R}|_{R=R_0+\delta R_1}, \quad (12)$$
$$\approx 1 - 2n\lambda\left(\frac{R_c}{R_0}\right)^{2n+1}\left[1 + \delta R_1/R_0\right]^{-2n-1}, \quad (13)$$
$$\approx \psi_0 + 4n\lambda^2(2n+1)\left(\frac{R_c}{R_0}\right)^{2n+2}, \quad (14)$$

where we have used $\delta R_1 = 2\lambda R_c$.

Let us note that the scalaron mass in high curvature regime $R \approx R_0 \approx \rho \gg R_c$ is given by

$$m^2_{\psi_0} = \frac{d^2V}{d\psi^2}|_{\psi=\psi_0} \approx \frac{R_c}{6n(2n+1)\lambda}\left(\frac{R_0}{R_c}\right)^{2n+2}, \quad (15)$$

which allows us to obtain the first order iteration solution,

$$\psi_1(r) = \psi_0(r) + \frac{2}{3}\lambda\frac{R_c}{m^2_{\psi_0}}. \quad (16)$$

The GR-like solution, $\psi_1(r)$ under consideration, can deviate from $\psi_0(r)$ only near the stellar radius otherwise many known observational constraints of neutron stars will not be satisfied. Indeed, the first order iteration ψ_1 solution is approximately the Schwarzschild de Sitter solution because it is corresponding to the curvature $R - 2\lambda R_c$ which differs from R of the GR solution by a constant. In large scalaron mass limit, ψ reduces to GR solution as expected.

The configuration of the perturbative solutions of the $f(R)$ gravity and general relativity are very different in the limit of the large scalaron mass. As demonstrated Refs.[19] &[20], the sign and the size of m^2_ψ play a crucial role for the stability of solutions in time. The Dolgov-Kawasaki instability can be avoided by choosing $m^2_\psi > 0$ which makes the perturbation $\delta\psi(t)$ oscillating in time. However, a large positive m^2_ψ causes the instability of static spherically symmetric solutions [21] for a class of $f(R)$ gravity models which are carefully built to evade the local gravity constraints [4][1].

To demonstrate the catastrophic instability from a large positive value m^2_ψ, let us consider a small perturbation $\delta\psi_1(r)$ around $\psi_1(r)$. Assuming the static spherically symmetric metric, the trace equation (3) tells us that

$$\delta\psi_1'' + \frac{2}{r}\delta\psi_1' = \frac{dV}{d\psi}\Big|_{\psi=\psi_1+\delta\psi_1} - \frac{dV}{d\psi}\Big|_{\psi=\psi_1}, \quad (17)$$
$$\approx \frac{d^2V}{d\psi_1^2}\delta\psi_1 = m^2_{\psi_1}\delta\psi_1, \quad (18)$$

where primes denote derivatives with respect to r.

In case $m^2_{\psi_1} > 0$, the growth of the perturbation $\delta\psi_1(r)$ along the radius can be obtained in form

$$\delta\psi_1(r) = \delta\tilde{\psi}_1(0)\left\{C_1\frac{e^{m_0 r}}{r} + C_2\frac{e^{-m_0 r}}{r}\right\}, \quad (19)$$

where we have used the notation $m_0 \equiv m_{\psi_1} \approx m_{\psi_0}$. It follows from Eq.(19) that the exponentially growing

[1] This type of instability that we shall focus on in the subsequent discussion refers to the radial evolution.

mode of $\delta\psi_1$ is unavoidable. Thus the instability of solutions always persists for any metric $f(R)$ gravity models with the large m_ψ. This fact may be exhibited by casting the equation of perturbation $\delta\psi_1(t,r)$ in the following form,

$$\left(\partial_t^2 - \vec{\nabla}^2\right)\delta\psi = -m_0^2\delta\psi. \tag{20}$$

It should be noticed that the difference of the sign of ∂_t^2 and $\vec{\nabla}^2$ comes from the signature of the metric itself. The avoidance of Dolgov-Kawasaki time instability by choosing $m_0^2 > 0$ invokes the instability ($\delta\psi_1(r) \propto e^{\pm m_0 r}$) of the static solution. The orthogonality of the stability conditions of the time and space arising from the signature of the metric was proposed in Ref.[21] and the evidence of this instability of the static solutions was observed numerically as the problem of the existence of relativistic stars in $f(R)$ gravity theories [11],[12].

It is however difficult but necessary to maintain the small deviation from the GR-like solution $\psi_0(r)$ for the whole range of the stellar radius when the growth of perturbation is exponential. The initial value $\delta\psi_1(0)$ must be extremely fine tuned if we want to stay near the GR-like solution. The seriousness of the fine tuning is related to the size of the number $m_0 r_*$ which is typically huge. The length scale corresponding to m_0, the Compton wavelength $\lambda_c \equiv 1/m_0$, is very small in the nuclear matter density regime due to chameleon effect. Without the cut off on m_0, λ_c can shrink below *Planck length*($r_{_P} = \sqrt{\hbar G/c^3} \sim 10^{-33}$ cm). Meanwhile, the stellar radius r_* is large ($r_* \sim 10^6$ cm. for neutron stars) which means that $m_0 r_* = r_*/\lambda_c \gg 1$.

Let us estimate the number of $m_0 r_*$ for a neutron star using the following relation,

$$r_*^2 = \frac{12P_c}{\rho_c^2} = \frac{12\omega}{\rho_c} \tag{21}$$

where P_c is the pressure at the center of the neutron star and the equation of state $P_c = \omega\rho_c$ is assumed.

Using the approximation $R_0 \sim \rho_c$ and the constant equation of state parameter, $\omega \sim 0.1$, we can rewrite r_* in the term of Hubble length $r_{_{H_0}} = c/H_0 \sim R_c^{-1/2}$ as

$$r_* \sim R_0^{-1/2} = \left(\frac{R_c}{R_0}\right)^{1/2} r_{_{H_0}} \sim \left(\frac{\rho_{_\Lambda}}{\rho_c}\right)^{1/2} r_{_{H_0}}, \tag{22}$$

The faster shrinking of λ_c via Chameleon effect can be seen using Eq.(15)

$$\lambda_c = 1/m_0 \sim \left(\frac{\rho_{_\Lambda}}{\rho_c}\right)^{n+1} r_{_{H_0}}. \tag{23}$$

which means that the size of $m_0 r_*$ depends on the density contrast in the following way

$$m_0 r_* = r_*/\lambda_c \sim \left(\frac{\rho_c}{\rho_{_\Lambda}}\right)^{n+1/2} \sim 10^{43(n+1/2)}. \tag{24}$$

With the minimum requirement $n \geq 0.9$ for local gravity constraints, the growing mode at $r = r_*$ becomes

$$\delta\psi_1(r_*) = \frac{\delta\tilde{\psi_1}(0)C_1}{r_*}\exp[10^{60}]. \tag{25}$$

If $C_1 \neq 0$ or the growing mode is allowed, the initial condition $\delta\tilde{\psi_1}(0)$ must be fine tuned to a fantastic level in order to compensate the enormous factor $\exp[10^{60}]$!

As we have shown, this tuning problem arises from the same criterion as Dolgov-Kawasaki time instability. The allowance of the incredibly small set of the initial condition corresponding to the correct boundary condition of GR-like solution should be considered as a serious theoretical problem. The system which is highly sensitive to the initial value to the level of $\mathcal{O}(10^{60}\exp[10^{-60}])$ should not be considered as satisfactory.

Let us note that setting $C_1 = 0$ from the beginning is not permissible because the continuity of the gradient of the solution $\psi(r) = \psi_1(r) + \delta\psi_1(r)$ at the center of the star needs

$$\psi_1'(0) + \delta\psi_1'(0) = \psi_0'(0) + \delta\psi_1'(0) = 0, \tag{26}$$

which leads to $\delta\psi_1'(r) = 0$ or $C_1 = -C_2$ whereas the for GR-like solution to hold, $\psi_1'(0) = \psi_0'(0) = 0$ and $m_0 \approx$ *const* are assumed. Then the perturbation $\delta\psi_1(r)$ can be rewritten as

$$\delta\psi_1(r) = \delta\psi_1(0)\frac{\sinh(m_0 r)}{m_0 r}, \tag{27}$$

which gives the evolution of perturbation from centre to the surface of the star. We can estimate how the initial perturbation,$\delta\psi_1(0)$, at the center enhances as we move to the surface of the star,

$$\delta\psi_1(r_*) \approx \delta\psi_1(0)\frac{\exp[10^{60}]}{2\times 10^{60}}, \tag{28}$$

The situation becomes worse for the growth of perturbations in case $n \gtrsim 2$ with $m_0 r_* \sim 10^{107}$.

There are two ways to deviate from the first order iteration solution $\psi_1(0)$ depending on the initial sign of $\delta\psi_1(0)$. For the positive $\delta\psi_1(0)$, the perturbation drives $\psi(r)$ toward curvature singularity $\psi(r) = 1$ while in the case of $\delta\psi_1(0) < 0$, one can easily get $\psi(r)$ which is inconsistent with the observational constraints. Since $\psi(r)$ and $R(r)$ are in one to one correspondence by definition, the GR-like solution $\psi(r)$ cannot much depart from the value $\psi_0(r)$ and $\psi_1(r)$ for the entire stellar radius otherwise many known constraints of general relativity such as the one coming from double pulsar tests would not be satisfied.

The simplest way to satisfy stringent test of general relativity is provided by taking the Schwarzschild de-Sitter solution with small cosmological constant, $2\lambda R_c$. From the 1^{st} iteration, the small interval of allowed deviation is

$$\psi_1(r) - \psi_0(r) \approx 4n\lambda^2(2n+1)\left(\frac{R_c}{R_0}\right)^{2n+2}, \quad (29)$$

$$\sim \left(\frac{\rho_\Lambda}{\rho_c}\right)^{2n+2}, \quad (30)$$

where we have assumed $n, \lambda \sim \mathcal{O}(1)$.

In case $\delta\psi_1(0) < 0$, the maximum allowed deviation at r_* can be approximated by

$$\delta\psi_1(0)\frac{\exp[m_0 r_*]}{2m_0 r_*} \gtrsim -\left(\frac{\rho_\Lambda}{\rho_c}\right)^{2n+2}. \quad (31)$$

This requirement for GR-like solutions hold leads to an extreme fine tuning, for example, in case $n = 0.9$ ($m_0 r_* = 10^{60}$), we need $|\delta\psi_1(0)| \lesssim 6.3 \times 10^{-104}\exp[-10^{60}]$. With this huge size of the scalaron mass inside neutron stars, the GR-like solution is highly unstable. Any small perturbation from the exact solution, no matter how small in the physical sense, can cause a catastrophic effect such as the divergence of R or the solution cannot be compatible with observations[11].

IV. SCALARON MASS CUT OFF INDUCED BY $(\mu/R_c)R^2$ TERM

The fine tuning problem in $f(R)$ theories is closely associated with the large mass, the scalaron acquires in the high curvature regime which is essential for local gravity constraints to be evaded. It is, however, unsatisfactory that the scalaron mass can easily exceed the Planck mass[5]. Using the dimensional arguments, we can estimate the stellar radius of neutron star from the nuclear matter density, cosmological density and the Hubble radius,

$$\frac{r_*}{r_H} = \left(\frac{\rho_\Lambda}{\rho_c}\right)^{1/2} \quad \text{or} \quad r_* = 10^{-43/2} \times 1.3 \times 10^{26}\text{m}. \quad (32)$$

which gives the correct order of magnitude for the radius of neutron star, $\mathcal{O}(10^4)$ m. It is clear from Eq.(15) that mass scale (length scale) corresponding to the scalaron mass increases (decreases) faster with density for any $n > 0$,

$$m_\psi \sim \sqrt{R_c}\left(\frac{\rho}{\rho_\Lambda}\right)^{n+1}. \quad (33)$$

As pointed out in Ref.[5], the scalaron mass can exceed the Planck mass even in the regime of the density of classical relativity. Let us estimate $\sqrt{R_c}$ in the unit of M_p,

$$\frac{\sqrt{R_c}}{M_p} \sim \frac{H_0}{M_p} = \frac{2.13h \times 10^{-42}GeV/\hbar}{1.22 \times 10^{19}GeV/c^2} \sim 10^{-61}. \quad (34)$$

From Eq.(33) and Eq. (34), we find that $m_\psi \geq M_p$ when $\rho \gtrsim 10^{32}\rho_\Lambda$ for $n = 0.9$ and $\rho \gtrsim 10^{20}\rho_\Lambda$ for $n = 2$. Around the Big Bang nucleosynthesis (BBN), the density $\rho_{BBN} \sim 10^{30}\rho_\Lambda$ [22] giving rise to $m_\psi \sim 10^{-4}Mp$ for $n = 0.9$ which is heavier than the typical inflaton mass $10^{-6}Mp$. The scalaron mass exceeds the Planck mass by the factor 10^{29} in case $n = 2$.

In stars with nuclear matter density, $\rho_c \sim 10^{43}\rho_\Lambda$, m_ψ becomes $10^{20.7}M_p$ for $n = 0.9$ and $10^{68}M_p$ for $n = 2$. Hence, the length scales λ_c corresponding to this un-physical heavy masses are definitely shorter than Planck length , equivalently, the gigantic numbers, $m_0 r_* = r_*/\lambda_c$ is also un-physical due to the uncontrollable Chameleon effect.

The simplest way to remedy the fine tuning problem is provided by putting a cut off on m_ψ by carefully chosen the maximum value of m_ψ such $m_0 r_* \sim \mathcal{O}(1)$ which can be achieved by adding $(\mu/R_c)R^2$ term into the model under consideration[5, 10].

Indeed, in the limit ($R \gg R_c$) in context with the Starobinsky model, we can use the approximation

$$m_\psi^2 = \frac{1}{3}\left[\frac{1}{\Delta_{,RR}} + \frac{\Delta_{,R}}{\Delta_{,RR}} - R\right] \quad (35)$$

$$\approx \frac{1}{3}(\Delta_{,RR})^{-1} \quad (36)$$

where $\Delta_{,RR} = d\Delta_{,R}/dR$. The additional term $(\mu/R_c)R^2$ can provide a cut off on $m_\psi \sim (R_c/(6\mu))^{1/2}$ when μ is chosen to satisfy the condition[5]

$$\left(\frac{R_c}{R}\right) \gg \mu \gg \left(\frac{R_c}{R}\right)^{2(n+1)}. \quad (37)$$

In case of neutron stars, the upper limit $\mu \ll \rho_\Lambda/\rho_c \sim 10^{-43}$ is equivalent to $(\mu/R_c)R^2 \ll R$ or the correction term should be small compared to the background curvature R. It should be noted that the lower limit is always satisfied for generic values of n.

For the numerical calculation, we need not to stick to approximation (37) as m_ψ can be calculated directly. As demonstrated in Ref.[12], the carefully chosen μ corresponding to the cut off on mass about the scale of neutron star density, can give rise to GR-like solution in for neutron stars. We may explain this observation by consider the heuristic argument (32) which gives $r_* \sim 10^{-43/2}R_c^{-1/2}$ and the corresponding Compton wavelength for the mass cut off, $\lambda_c \sim 1/m_\psi \sim (6\mu)^{1/2}R_c^{-1/2}$ such that

$$r_* \sim \frac{1}{m_0} = \lambda_c \quad \text{when} \quad 6\mu \sim 10^{-43}. \quad (38)$$

which clearly allows to avoid the catastrophic fine tuning problem in compact objects like neutron stars.

V. CONCLUSIONS

We have examined the scalaron dynamics in the frame work of Starobinsky model. The curvature singularity is generic to this class of models if they are to be consistent with local gravity constraints. The finite potential barrier between the de-Sitter minimum and the curvature singularity is a serious threat to models with disappearing cosmological constant. The problem becomes grave in high curvature regime in compact objects like neutron stars. In this case, the scalaron mass becomes larger than the Planck mass due to chameleon mechanism necessary for local gravity constraints to be evaded. This in turn becomes the root cause of instability problem in the scenario under consideration. While evolving the scalaron from the centre to the surface along the radius of neutron star, we need to stay close GR. Little perturbation of initial conditions at the centre can easily destroy the desired evolution. This condition for having GR-like solutions requires extreme fine tuning of initial conditions, for instance, in case of $n = 0.9, m_0 r_* = 10^{60}$, we need $|\delta\psi_1(0)| \lesssim 6.3 \times 10^{-107} \exp[-10^{60}]$. Situation further worsens for larger values of n. This intractable level of fine tuning throws a serious challenge to $f(R)$ theories consistent with local gravity tests.

The problem can be alleviated by introducing quadratic curvature term $\mu R^2/R_0$ in the Starobinsky model which is equivalent to putting a cut off on scalaron mass corresponding to $\mu \ll \rho_\Lambda/\rho_c \sim 10^{-43}$ for nuclear matter density. This prescription, however, runs into problem if one asks for its compatibility with early universe physics *a la* inflation which leads to much larger value of the scalaron mass.

VI. ACKNOWLEDGEMENTS

We thank T. Kobayashi for discussion. IT is supported by ICCR and MS is supported by ICTP.

[1] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D**15**,1753 (2006) [arXiv:hep-th/0603057]; M. Sami, arXiv:0904.3445; V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D**9**, 373 (2000) [arXiv:astro-ph/9904398]; T. Padmanabhan, Phys. Rept. **380**, 235 (2003) [arXiv: hep-th/0212290]; E. V. Linder, Gen. Rel. Grav. **40**, 329 (2008) [arXiv:astro-ph/0704.2064]; J. Frieman, M. Turner and D. Huterer, Ann. Rev. Astron. Astrophys. **46**, 385 (2008) [arXiv:0803.0982];R. R. Caldwell and M. Kamionkowski, arXiv:0903.0866; A. Silvestri and M. Trodden, arXiv:0904.0024; V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D**15**, 2105 (2006) [arXiv:astro-ph/0610026]; T. Padmanabhan, AIP Conf. Proc. **861**, 179 (2006) [arXiv:astro-ph/0603114];

[2] S. Capozziello, Int. J. Mod. Phys. D**11**, 483 (2002) [arXiv:gr-qc/0201033]; S. Capozziello, S. Carloni, and A. Troisi Recent. Res. Dev. Astron. Astrophys. **1**, 625 (2003) [arXiv:astro-ph/0303041]; S. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D**70**, 043528 (2004) [arXiv:astro-ph/0306438]; T. P. Sotiriou and V. Faraoni [arXiv:0805.1726]; S. Nojiri and S. Odintsov, Gen. Rel. Grav. **36**, 1765 (2004) [arXiv:hep-th/0308176].

[3] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, Phys. Rev.D**75**, 083504 (2007) [arXiv:gr-qc/0612180].

[4] Wayne Hu and I. Sawicki, Phys. Rev. D**76**, 064004 (2007) [arXiv:0705.1158].

[5] A. A. Starobinsky, JETP. Lett. **86**, 157 (2007), [arXiv:0706.2041].

[6] R. Gannouji, B. Moraes and D. Polarski **JCAP 0902**, 034 (2009) [arXiv:0809.3374].

[7] S. A. Appleby and R. A. Battye, Phys. Lett. B**654**, 7 (2007) [arXiv:0705.3199].

[8] S. A. Appleby and R. A. Battye, **JCAP 0805**, 019 (2008) [arXiv:0803.1081].

[9] A. V. Frolov, Phys. Rev. Lett. **101**, 061103 (2008) [arXiv:0803.2500].

[10] Abha Dev , D. Jain , S. Jhingan , S. Nojiri , M. Sami and I. Thongkool, Phys. Rev. D**78**, 083515 (2008) [arXiv:0807.3445].

[11] T. Kobayashi and K. Maeda, Phys. Rev. D**78**, 064019 (2008) [arXiv:0807.2503].

[12] T. Kobayashi and K. Maeda, Phys. Rev. D**79**, 024009 (2009) [arXiv:0810.5664].

[13] S. Nojiri and S. D. Odintsov, arXiv:0903.5231; S. Nojiri and S. D. Odintsov, Phys. Rev. D**78**, 046006 (2008) [arXiv:0804.3519]; K. Bamba, S. Nojiri and S. D. Odintsov, **JCAP 0810**, 045 (2008) [arXiv:0807.2575]; S. Capozziello, M. De Laurentis , S. Nojiri and S. D. Odintsov, Phys. Rev. D**79**, 124007 (2009) [arXiv:0903.2753].

[14] E. Babichev and D. Langlois, arXiv:0904.1382.

[15] A. Upadhye and Wayne Hu, arXiv:0905.4055.

[16] V. Miranda, S. E. Joras, I. Waga and M. Quartin, Phys. Rev. Lett. **102**, 221101 (2009) [arXiv:0905.1941].

[17] I. Thongkool, M. Sami, R. Gannouji and S. Jhingan, arXiv:0906.2460

[18] S. Capozziello and S. Tsujikawa, Phys. Rev.D**77**, 107501 (2008) [arXiv:0712.2268].

[19] A. D. Dolgov and M. Kawasaki, Phys. Lett. B**573**, 1 (2003) [arXiv:astro-ph/0307285] .

[20] I. Sawicki and Wayne Hu, Phys. Rev. D**75**, 127502 (2007) [arXiv:astro-ph/0702278.]

[21] K. Kainulainen and D. Sunhede, Phys. Rev. D**78**, 063511 (2008) [arXiv:0803.0867].

[22] P. Zhang, Phys. Rev. D**76**, 024007 (2007) [arXiv:astro-ph/0701662].