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Factorisation of generalised theta functions. I

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Abstract. We prove a version of "factorisation", relating the space of sections of theta bundles on the moduli spaces of (parabolic, rank 2) vector bundles on curves of genus g and $g-1$.

1. Introduction

1a. Let X_1 be a smooth projective irreducible curve over $\mathbf{C}$ of genus g. Let $\mathscr{U}_{X_1} = \mathscr{U}_{X_1}(d)$ be the moduli space of semistable vector bundles of rank 2 and degree d on X_1. On $\mathscr{U}_{X_1}$ we have a natural (ample) line bundle, defined up to algebraic equivalence, which generalises the line bundle on the jacobian of X_1 defined by the Riemann theta divisor [D-N]. We call this the theta line bundle and denote it by θ_1. A section of θ_1^k over $\mathscr{U}_{X_1}$ may be called a generalised theta function of order k.

We would like to study the space $H^0(\mathscr{U}_{X_1}, \theta_1^k)$ by relating it to the space of generalised theta functions associated with a smooth curve of genus $g-1$. Such a relationship has been suggested by conformal field theory under the name of "factorisation rule" or "glueing axiom".

From the point of view of algebraic geometry it is natural to study this relationship by degenerating X_1 into an irreducible curve $X = X_0$ which is smooth except for a single node, so that the normalisation $\tilde{X}$ of X is a smooth curve of genus $g-1$. We can then consider the space of generalised theta functions on a suitable moduli space $\mathscr{U}_X$ associated to X and then seek to relate this space with a space of generalised theta function associated with the normalisation $\tilde{X}$. The space $\mathscr{U}_X$ is the moduli space of semistable torsion-free sheaves of rank 2 and degree d on X and it carries a natural theta line bundle θ. If moreover $H^1(\theta^k) = H^1(\theta_1^k) = 0$, one would have that $\dim H^0(\theta^k) = \dim H^0(\theta_1^k)$.

Let then X be an irreducible curve over $\mathbf{C}$ of genus g, smooth except for one node x_0. We denote by $\tilde{X}$ the normalisation of X, $\tilde{g} = g - 1$ the genus of $\tilde{X}$, and $\pi\colon \tilde{X} \to X$ the canonical map. Let $\{x_1, x_2\}$ be the inverse image of x_0 in $\tilde{X}$. The factorisation rule is:

$$H^0(\mathscr{U}_X, \theta^k) \sim \bigoplus_{\mu} H^0(\mathscr{U}_{\tilde{X}}^{\mu}, \theta_\mu),$$

where μ runs through a certain indexing set depending on k, $\mathscr{U}_{\tilde{X}}^{\mu}$ is the moduli space of parabolic vector bundles (of rank 2 and degree d) on $\tilde{X}$ with parabolic structures [M-S] at x_1 and x_2 (with weights depending on μ) and θ_μ is a line bundle on $\mathscr{U}_{\tilde{X}}^{\mu}$ (the generalised theta bundle).

It is clear that to carry through the induction on g one has to start with moduli spaces of parabolic torsion-free sheaves of rank 2 on a nodal curve X with parabolic structures at a finite number of smooth points and prove a factorisation rule for generalised theta functions on them, as well as a corresponding vanishing theorem for H^1. This is what is done in this paper.

1b. Statement of the main theorem

First, some preliminaries:

(1) Let X be an irreducible curve of genus g, smooth but for one node x_0. Choose a finite set $\{y_i\}_I$ of smooth points on X. Let $\tilde{X}$ be the normalisation of X, $\pi\colon \tilde{X} \to X$ the canonical map, and $\pi^{-1}(x_0) = \{x_1, x_2\}$.

(2) Fix integers $d, k > 0$, and also, for each $i \in I$ integers $0 \leqq \alpha_i < \beta_i < k$ satisfying the condition: $dk + \sum_i(\alpha_i + \beta_i)$ is even.

(3) Define "weights" $\{(a_i, b_i)\}_I$ by $a_i = \alpha_i/k$, $b_i = \beta_i/k$. We construct in the Appendix A the moduli space $\mathscr{U}_X = \mathscr{U}(X, d, \{(a_i, b_i)\}_I)$ of (s-equivalence classes of) parabolic torsion-free sheaves of rank 2 and degree d on X, with parabolic structures at the $\{y_i\}_I$, semistable with respect to the weights $\{(a_i, b_i)\}_I$. The space $\mathscr{U}_{\tilde{X}} \equiv \mathscr{U}(\tilde{X}, d, \{(a_i, b_i)\}_I)$ is constructed similarly. The definitions can be extended to the case when $a_q = b_q$ for $q \in Q \subset I$ (§2c).

(4) For $\mu = (\alpha, \beta)$, $0 \leqq \alpha \leqq \beta < k$, let $\mathscr{U}_{\tilde{X}}^{\mu}$ be the moduli space of semistable parabolic bundles on $\tilde{X}$ with parabolic structures at the $\{y_i\}_I$ and weights $\{(a_i, b_i)\}_I$, and in addition, parabolic structures at x_1 and x_2, both of weight $(a, b) = (\alpha/k, \beta/k)$.

(5) We will define (§2), up to algebraic equivalence, a natural ample line bundle $\theta = \theta(d, k, \{(a_i, b_i)\}_I)$ on $\mathscr{U}_X$. Analogous bundles θ_μ can be defined on the $\mathscr{U}_{\tilde{X}}^{\mu}$ (Definition 5.5).

We have then the

Main theorem

(A) *We have a (noncanonical) isomorphism*:

$$H^0(\mathscr{U}_X, \theta) \sim \bigoplus_{\mu} H^0(\mathscr{U}_{\tilde{X}}^{\mu}, \theta_\mu),$$

where μ runs through the integers (α, β), $0 \leqq \alpha \leqq \beta < k$.
(B) *Assume $g \geqq 4$. $H^1(\mathscr{U}_X, \theta) = 0$.*

The statement (A) is proved in §5b and (B) is a restatement of Theorem 7.

1c. We give in this sub-section a proof of factorisation in the case of rank 1 sheaves. There are few technical complications here, and the main ideas of the proof are best understood by studying this case.

If X_t is a (flat) family of curves such that $X_0 = X$, and the X_t, for $t \neq 0$ are smooth, there exists, for every integer d, a corresponding family of jacobians $J^d_{X_t}$ (of degree d line bundles) specialising to the *compactified jacobian* of X (which we denote by $\bar{J}^d_X$). The latter parametrises rank 1 torsion-free sheaves on X, and is a compactification of J^d_X, the moduli space of line bundles of degree d on X. In particular, consider J^{g-1}_X. This has a canonically defined ample line bundle on it – the theta bundle – which can be defined as Grothendieck's "determinant bundle of cohomology" [K-M] of *any* Poincaré bundle on $X_t \times J^{g-1}_{X_t}$. We shall from now on denote this bundle θ_t, and set $\theta_0 = \theta$. Given a vanishing theorem for $H^1(\bar{J}^{g-1}_X, \theta^k)$, we can compute $\dim H^0(J^{g-1}_{X_t}, \theta^k_t)$ for generic X_t by specialising to $t = 0$.

Giving a line bundle N on X is equivalent to giving one, say L, on $\tilde{X}$ together with an isomorphism between L_{x_1} and L_{x_2}. To such an isomorphism we can associate its graph, a one-dimensional subspace S of $L_{x_1} \oplus L_{x_2}$, and in turn, the quotient Q by S, thought of as a point of the projective space of $L_{x_1} \oplus L_{x_2}$. This motivates the following well-known construction. Let $J^d_{\tilde{X}}$ denote the jacobian of degree d line bundles on $\tilde{X}$. Given a Poincaré bundle $\mathscr{L}$ on $\tilde{X} \times J^{g-1}_{\tilde{X}}$, let $\mathbf{P}$ be the projective bundle on $J^{g-1}_{\tilde{X}}$ associated to the vector bundle (with an obvious notation) $\mathscr{L}_{x_1} \oplus \mathscr{L}_{x_2}$. We have on $\mathbf{P}$ the tautological exact sequence of bundles $0 \to \mathscr{S} \to \rho^*(\mathscr{L}_{x_1} \oplus \mathscr{L}_{x_2}) \to \mathscr{Q} \to 0$. Let $\pi_* \mathscr{L}$ denote the sheaf on $X \times \mathbf{P}$, got by taking the direct image of $\mathscr{L}$ by $\pi \times I_{J^{g-1}_{\tilde{X}}}$, and pulling back the resulting sheaf from $X \times J^{g-1}_{\tilde{X}}$. We can think of $\mathscr{Q}$ as a sheaf on $X \times \mathbf{P}$ supported on $\{x_0\} \times \mathbf{P}$. There is an obvious homomorphism $\pi_* \mathscr{L} \to \mathscr{Q}$ and we define $\tilde{\mathscr{N}}$ to be the kernel sheaf. Thus we have constructed a family of rank 1 torsion-free sheaves on X parametrised by $\mathbf{P}$.

There is therefore a morphism $\phi: \mathbf{P} \to \bar{J}^{g-1}_X$ such that for any Pioncaré sheaf $\mathscr{N}$ on $X \times \bar{J}^{g-1}_X$ we have $(I_X \times \phi)^* \mathscr{N} = \tilde{\mathscr{N}}$ up to tensoring by a line bundle from $\mathbf{P}$:

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\phi} & \bar{J}^{g-1}_X \\ \rho \downarrow & & \\ \bar{J}^{g-1}_{\tilde{X}} & & \end{array}$$

One can, by functoriality of the determinant bundle [L, VI, §1], compute the pull-back of θ to $\mathbf{P}$. Here it is important that we are working with line bundles of Euler characteristic 0:

$$\phi^*\theta = \rho^*(\det R\pi_{J^{g-1}_{\tilde{X}}} \mathscr{L}) \otimes \mathscr{Q}, \tag{1.1}$$

where we use the notation $\det R\pi_{Z_1} \mathscr{A}$ for the determinant bundle of cohomology of a family $\mathscr{A}$ of sheaves on $Z_1 \times Z_2$ parametrised by Z_1 (see 1f.(2)). One can check that this is independent of the choice of $\mathscr{L}$.

Let $\mathscr{D}_1, \mathscr{D}_2$ denote the two divisors in $\mathbf{P}$ given by $\mathscr{L}_{x_1}$ and $\mathscr{L}_{x_2}$, respectively. It is a fact that ϕ restricted to the complement of $\mathscr{D}_1 \cup \mathscr{D}_2$ is an isomorphism onto $J^{g-1}_X \subset \bar{J}^{g-1}_X$, and *each of the $\mathscr{D}_j$ maps isomorphically onto the singular locus $\mathscr{W}$ of* $\bar{J}^{g-1}_X$. Also, $\bar{J}^{g-1}_X$ is *seminormal* (see §4 below for the definition) and this allows us to write the exact sequence of $\mathscr{O}_{\bar{J}^{g-1}_X}$-modules:

$$0 \to \phi_* \mathscr{O}_{\mathbf{P}}(-\mathscr{D}_1 - \mathscr{D}_2) \to \mathscr{O}_{\bar{J}^{g-1}_X} \to \mathscr{O}_{\mathscr{W}} \to 0,$$

which yields

$$0 \to H^0(\phi^*\theta^k(-\mathscr{D}_1 - \mathscr{D}_2)) \to H^0(\theta^k) \to H^0(\theta^k|_{\mathscr{W}}). \tag{1.2}$$

We will argue below that the last map is a surjection. Note that $H^0(\theta^k|_{\mathscr{W}}) \sim H^0(\phi^*\theta^k|_{\mathscr{D}_1})$. Thus $H^0(\theta^k)$ is an extension:

$$0 \to H^0(\phi^*\theta^k(-\mathscr{D}_1 - \mathscr{D}_2)) \to H^0(\theta^k) \to H^0(\phi^*\theta^k|_{\mathscr{D}_1}) \to 0.$$

Now, each of the cohomology spaces on either side of the middle can be computed by taking direct images on $J^{g-1}_{\tilde{X}}$. Standard arguments, using the expression (1.1) and also $\mathcal{O}(\mathscr{D}_j) = \mathcal{O} \otimes \mathscr{L}^{-1}_{x_j}$, yield:

$$\begin{aligned}\rho_*(\phi^*\theta^k(-\mathscr{D}_1 - \mathscr{D}_2)) &= (\det R\pi_{J^{g-1}_{\tilde{X}}}\mathscr{L})^k \otimes \mathscr{L}_{x_1}\mathscr{L}_{x_2} \otimes \rho_*\mathscr{Q}^{(k-2)} \\ &= (\det R\pi_{J^{g-1}_{\tilde{X}}}\mathscr{L})^k \otimes \mathscr{L}_{x_1}\mathscr{L}_{x_2} \otimes S^{k-2}(\mathscr{L}_{x_1} \oplus \mathscr{L}_{x_2}) \\ &= (\det R\pi_{J^{g-1}_{\tilde{X}}}\mathscr{L})^k \otimes \mathscr{L}_{x_1}\mathscr{L}_{x_2} \otimes \left\{ \bigoplus_{l=0,\ldots,k-2} \mathscr{L}^l_{x_1}\mathscr{L}^{k-2-l}_{x_2} \right\},\end{aligned}$$

where S^{k-2} denotes the $(k-2)$th symmetric product. Similarly,

$$(\rho|_{\mathscr{D}_1})_*\phi^*\theta^k = (\det R\pi_{J^{g-1}_{\tilde{X}}}\mathscr{L})^k \otimes \mathscr{L}^k_{x_2}.$$

We have thus found an expression $H^0(\bar{J}^{g-1}, \theta^k)$ in terms of line bundles on $J^{g-1}_{\tilde{X}}$.

We still need to show that the sequence (1.2) is exact on the right. For this it suffices to show that $H^1(\phi^*\theta^k(-\mathscr{D}_1 - \mathscr{D}_2)) = 0$. For this observe that $R^1\rho_*(\phi^*\theta^k(-\mathscr{D}_1 - \mathscr{D}_2)) = 0$ and $\rho_*\theta^k(-\mathscr{D}_1 - \mathscr{D}_2))$ is a direct sum of ample line bundles on $J^{g-1}_{\tilde{X}}$. A similar argument shows that $H^1(\bar{J}^{g-1}_{\tilde{X}}, \theta^k) = 0$.

As a simple exercise let us compute the dimension of $H^0(\bar{J}^{g-1}, \theta^k)$. Choose Poincaré bundle $\mathscr{L}$ which is trivial on (say $\{x_1\} \times J^{g-1}_{\tilde{X}}$. Then $\det R\pi_{J^{g-1}_{\tilde{X}}}\mathscr{L}$ is in the algebraic equivalence class of theta, and the $\mathscr{L}_{x_j}$ are algebraically equivalent to the trivial bundle. Thus

$$\dim H^0(J^{g-1}_X, \theta^k) = (k-1)k^{\tilde{g}} + k^{\tilde{g}} = k^g,$$

as expected.

1d. We describe briefly the main steps in the proof of the Main Theorem.

When comparing bundles on a singular curve X and its normalisation $\tilde{X}$, we use a variant of a concept, due to Bhosle [B1], of a "generalised parabolic bundle on $\tilde{X}$ with a generalised parabolic structure over the divisor $\{x_1, x_2\}$". Such a bundle of rank 2 is given by a pair (E, Q) where E is a rank 2 vector bundle on $\tilde{X}$ and Q a two-dimensional quotient of $E_{x_1} \oplus E_{x_2}$. Given a generalised parabolic bundle (GPB from now on) one obtains a torsion-free sheaf F on X which fits into the exact sequence: $0 \to F \to \pi_*E \to {}_{x_0}Q \to 0$, where ${}_{x_0}Q$ is the skyscraper sheaf on X with support x_0 and fibre Q (it is easy to show that degree F = degree E). One can define the notion of a semistable GPB, and prove that F is a semistable torsion-free sheaf iff (E, Q) is a semistable GPB. All this goes through if there are additional parabolic structures at the $\{y_i\}_I$. There is therefore a morphism ϕ: $\mathscr{P} \to \mathscr{U}_X$, where $\mathscr{P}$ denotes a suitable moduli space of generalised parabolic bundles on $\tilde{X}$. We will study this morphism in §4c, and see that it is in particular birational – in fact $\mathscr{P}$ is the normalisation of $\mathscr{U}_X$. (One has in fact to allow for torsion at the points x_j so it is more appropriate to talk of generalised parabolic sheaves – this is done in the main body of the paper.)

We will consider a certain locally universal family (parametrised by a variety $\tilde{\mathscr{R}}_F$) of rank 2 vector bundles E on $\tilde{X}$ with degree $E = d$, and parabolic structures at the $\{y_i\}_I$: $\mathscr{U}_{\tilde{X}}$ is a geometric invariant theory quotient of the semistable points of $\tilde{\mathscr{R}}_F$ with respect to the action of a suitable reductive group and a certain linearisation by a line bundle $\hat{\theta}$. *Let* ρ: $\tilde{\mathscr{R}}'_F \to \tilde{\mathscr{R}}_F$ denote the grassmannian bundle of two-dimensional quotients of $E_{x_1} \oplus E_{x_2}$ (the reason for the notation will become clear later). Using the results of §4 (namely, "seminormality" of $\mathscr{U}_X$,) we then (§5b) characterise the subspace

$$H^0(\mathscr{U}_X, \theta_{\mathscr{U}_X}) \subset H^0(\tilde{\mathscr{R}}'_F, \rho^*\hat{\theta} \otimes \mathbf{L})^{\mathrm{inv}} = H^0(\tilde{\mathscr{R}}_F, \hat{\theta} \otimes \rho_* \mathbf{L})^{\mathrm{inv}}, \tag{1.3}$$

where $\mathbf{L}$ is essentially the line bundle $\mathrm{O}(k)$ along the fibres of the grassmannian bundle, and $\{.\}^{\mathrm{inv}}$ denotes a space of invariants for the group action. The computation of $\rho_*\mathbf{L}$ amounts to the following problem when is easily solved. Let Gr be the grassmannian of 2 dimensional subspaces of $\mathbf{C}^4$, m a positive integer: decompose the representation of $GL(4, \mathbf{C})$ on $H^0(Gr, \mathcal{O}(m))$ into irreducible representations of $GL(2) \times GL(2) \subset GL(4)$. The decomposition (A) follows from this. (Note that the (1.3) refers to invariant sections on all of $\tilde{\mathscr{R}}_F$ and not just on the open subscheme of semistable points – this is because of Lemma 4.15 below.)

We turn next to the vanishing theorem for H^1. The map ϕ: $\mathscr{P} \to \mathscr{U}_{\tilde{X}}$ is finite; and we will see that it suffices to prove the vanishing of H^1 for $\theta_{\mathscr{U}_X}$ pulled back to $\mathscr{P}$ and restricted to a "fixed determinant subvariety" $\mathscr{P}^L \subset \mathscr{P}$, $L \in J^d_{\tilde{X}}$. We will denote this pull-back bundle by $\theta_{\mathscr{P}}$. We consider a new set of data $(d, \bar{k}, \bar{\alpha}_i, \bar{\beta}_i)$ such that $\bar{k} = k + 4$, and $\bar{\beta} - \bar{\alpha} = \beta - \alpha + 2$. Let $\bar{\mathscr{P}}$ denote the corresponding moduli space of GPS's, we show that $H^1(\mathscr{P}, \theta_{\mathscr{P}}) = H^1(\bar{\mathscr{P}}, \theta_{\bar{\mathscr{P}}} \otimes \bar{\Omega})$ where $\theta_{\bar{\mathscr{P}}}$ is an ample line bundle on $\bar{\mathscr{P}}$ and $\bar{\Omega}$ is the dualising sheaf of $\bar{\mathscr{P}}$. (This would be the case, for example, if there is a common open set $\mathscr{P}_0$ in both $\mathscr{P}$ and $\bar{\mathscr{P}}$ such that the complement of $\mathscr{P}_0$ in each of them is of high codimension and such that $\theta_{\mathscr{P}}|_{\mathscr{P}_0} = \theta_{\bar{\mathscr{P}}} \otimes \bar{\Omega}|_{\mathscr{P}_0}$. Actually, we give a slightly different proof.) A Kodaira-type vanishing theorem for $\theta_{\bar{\mathscr{P}}} \otimes \bar{\Omega}$ now yields the desired vanishing theorem (§5b).

We introduce the moduli spaces of parabolic vector bundles and define the theta bundle in §2. In Appendices A and B we give a Geometric Invariant Theory construction of the moduli spaces of interest. The construction of moduli Simpson [Si]. The same method is used to construct the moduli space of generalised parabolic sheaves.

We prove in §3 that $\mathscr{U}_X$ is seminormal and in Appendix C that $\mathscr{P}$ is normal and has rational singularities. These properties are essentially used in the proof.

1e. In a subsequent work we will remove the restriction on genus in the statement of the Main Theorem (B). The results of this paper can then be used to give a proof of the "Verlinde Formula" for the dimension of generalised theta functions on the moduli space of (parabolic) bundles.

It should be mentioned that a factorisation rule for "conformal blocks", defined via representations of affine Lie algebras, has been proved in[T-U-Y].

1f. Notation

(1) We will let det $R\pi_{Z_1}$ $\mathscr{A}$ denote the determinant bundle of a flat family $\mathscr{A}$ of sheaves parameterised by Z_1. A convenient reference for the determinant bundle of a family is [L] – *our definition of the determinant bundle is, however, the inverse of the*

one used there. For example, if Z_2 is a projective curve, $\mathscr{A}$ a coherent sheaf on $Z_1 \times Z_2$ flat over Z_1, and $x \in Z_1$, we have

$$\{\det R\pi_{Z_1}\mathscr{A}\}_x = \{\det H^0(Z_2, \mathscr{A}_x)\}^{-1} \otimes \{\det H^1(Z_2, \mathscr{A}_x)\}.$$

(2) Unless otherwise mentioned, X will denote an irreducible curve of genus g, with one node x_0, $\{y_i\}_I$ a finite set of smooth points on X, and y yet another smooth point. Let $\tilde{X}$ be the normalisation of $X, \pi: \tilde{X} \to X$ the canonical map, and $\pi^{-1}(x_0) = \{x_1, x_2\}$.

(3) We shall fix an integer d, the degree, another integer $k > 0$, and also, for each $i \in I$ integers $0 \leqq \alpha_i < \beta_i < k$. We define $n = d + 2(1 - g)$ and let l denote the number determined by

$$nk = 2k|I| + 2l - \sum_i (\alpha_i + \beta_i). \tag{1.4}$$

We shall assume that the data are such that l is an integer, i.e. that $dk + \sum_i(\alpha_i + \beta_i)$ is even. Let $a_i = \alpha_i/k$, $b_i = \beta_i/k$, and set $\omega = \{(a_i, b_i)\}_I$. Finally, let $\tilde{n} = n + 2$, $\tilde{l} = l + k$.

(4) At a point $x \in X$ we let $\mathcal{O}_x$ denote the local ring and $\mathcal{M}_x$ the maximal ideal. Given a coherent sheaf F on X, we mean by F_x the vector space $F \otimes \mathcal{O}_x/\mathcal{M}_x$. The slight ambiguity of notation should not cause confusion. We let $T\,or\,F$ denote the torsion subsheaf of F. By the degree of a torsion sheaf τ on X we mean $\dim H^0(X, \tau)$. We let $h^r(F) \equiv \dim H^r(F)$.

(5) Given a vector space $\mathbf{W}$ we mean by ${}_x\mathbf{W}$ the "skyscraper sheaf" supported at the reduced point x, with fibre $\mathbf{W}$. Note $\mathbf{W} = H^0({}_x\mathbf{W})$. *We will often write simply* $\mathbf{W}$ *when we mean* ${}_x\mathbf{W}$.

(6) GIT is short for "geometric invariant theory". The GIT quotient of a G-variety V is denoted by $V//G$. By a scheme we mean a (separated) scheme of finite type over $\mathbf{C}$. By a variety we mean a reduced scheme, which will be assumed irreducible unless otherwise mentioned.

2. The theta bundles

It will be clear that the results of this section continue to be valid if the number of nodes of X is *any nonnegative integer* as long as X is irreducible.

2a. Parabolic sheaves

Let F be a torsion-free sheaf of rank 2 and degree d on X – clearly such a sheaf is a vector bundle outside the node x_0.

Definition 2.1a. By a *quasi-parabolic structure* on F at a smooth point $x \in X$ we mean a choice of a one-dimensional quotient $F_x \to Q \to 0$ of the fibre of F at the point x. If in addition real numbers ("weights") $0 \leqq a < b < 1$ are given, this is a *parabolic structure.*

We shall consider sheaves with parabolic structures at the points $\{y_i\}_I$; the weights will be $\omega = \{(a_i, b_i)\}_I$ and shall denote by Q_i the quotient at the point y_i. Such a sheaf will be called a "parabolic sheaf". The parabolic degree of a parabolic sheaf F is by definition par degree $F = d + \sum_i (a_i + b_i)$; given a rank one subsheaf

$L \subset F$ such that F/L is torsion-free, its parabolic degree is by definition par degree $L =$ degree $L + \sum_{R^c} a_i + \sum_R b_i$ where $R \equiv R(L) \subset I$ is the subset where $L_{y_i} \subset \ker(F_{y_i} \to Q_i)$ and $R^c \equiv R^c(L)$ its complement. (We shall usually write simply R when we mean $R(L)$ etc.)

Note that equation (1.4) can be rewritten:

$$\text{par degree } F = 2(|I| + l/k - 1 + g), \tag{2.1}$$

where the parabolic degree is defined with respect to the weights ω.

Definition 2.1b. A parabolic sheaf F is said to be *stable* (respectively, *semistable*) with respect to the weights $\{(a_i, b_i)\}_I$ if for every such subsheaf L we have par degree $L <_{(\text{resp } \leq)} \frac{1}{2}$ (par degree F) – in other words, if

$$2 \text{ degree } L \underset{(\text{resp } \leq)}{<} d + \sum_{R^c} (b_i - a_i) - \sum_{R} (b_i - a_i). \tag{2.2}$$

By a family of rank 2 parabolic sheaves parametrised by a variety T one means a sheaf $\mathscr{F}_T$ on $X \times T$, flat over T, and torsion-free (with rank 2 and degree d) on $X \times \{t\}$ for every point $t \in T$, together with, for each y_i, a quotient line bundle $\mathscr{Q}_{T,i}$ of $\mathscr{F}_T|_{\{y_i\} \times \mathscr{F}}$. The following theorem is proved in Appendix A.

Theorem X1. *There exists a (coarse) moduli space $\mathscr{U}^s(X, d, \omega)$ of stable parabolic sheaves F. We have an open immersion $\mathscr{U}^s(X, d, \omega) \hookrightarrow \mathscr{U}(X, d, \omega)$ where $\mathscr{U}(X, d, \omega)$ denotes the space of s-equivalence classes of semistable parabolic sheaves. The latter is a projective variety. If X is smooth, then $\mathscr{U}$ is normal, with rational singularities.*

We will set $\mathscr{U}_X = \mathscr{U}(X, d, \omega)$ and $\mathscr{U}_X^s = \mathscr{U}^s(X, d, \omega)$.

Remark 2.2. If M is a fixed line bundle on X, $F \mapsto F \otimes M$ takes (semi)stable sheaves to (semi)stable sheaves, and also preserves s-equivalence.

We begin by outlining the construction of the moduli space $\mathscr{U}(X, d, \omega)$ (see Appendix A for details). Take d to be large; let $\mathbf{Q}$ denote the Quot scheme of coherent sheaves (of degree d and rank 2) over X which are quotients of $\mathcal{O}^n$, where $n = d + 2(1 - g)$. Thus there is on $X \times \mathbf{Q}$ a sheaf $\mathscr{F}_{\mathbf{Q}}$, flat over $\mathbf{Q}$, and an exact sequence $\mathcal{O}^n \xrightarrow{p} \mathscr{F}_{\mathbf{Q}} \to 0$. Let $\mathscr{F}_{y_i}$ be the sheaf on $\mathbf{Q}$ given by restricting $\mathscr{F}_{\mathbf{Q}}$ to $\{y_i\} \times \mathbf{Q}$, and let $\text{Flag}_{(1,2)}(\mathscr{F}_{y_i})$ be the relative flag scheme of locally-free quotients of $\mathscr{F}_{y_i}$ of rank (1, 2) [EG A-I, 9.9.2]. Let $\mathscr{R}$ be the fibre product over $\mathbf{Q}$:

$$\mathscr{R} = \underset{i \in I}{\times_{\mathbf{Q}}} \text{Flag}_{(1,2)}(\mathscr{F}_{y_i})$$

Let $\mathscr{R}^s$ (respectively, $\mathscr{R}^{ss}$) denote the open subscheme of $\mathscr{R}$ corresponding to stable (respectively, semistable) parabolic sheaves such that $H^0(p)$ is an isomorphism. The variety $\mathscr{U}(X, d, \omega)$ is the "good quotient" [S1, Definitions 1.5, 1.6] of $\mathscr{R}^{ss}$ by the action of $SL(n)$ which, in fact, acts through $PSL(n)$. We will denote by ψ the projection $\mathscr{R}^{ss} \to \mathscr{U}_X$.

Choose an ample line bundle of degree 1 on X, denoted by $\mathcal{O}(1)$ from now on. For large enough m we have a $SL(n)$-equivariant embedding $\mathscr{R} \hookrightarrow \mathbf{G}$ where

$$\mathbf{G} \equiv \text{Grass}_{P(m)}(\mathbf{C}^n \otimes W) \times \underset{i}{\times} \{\text{Grass}_2(\mathbf{C}^n) \times \text{Grass}_1(\mathbf{C}^n)\},$$

$P(m) = n + 2m$, and $W \equiv H^0(X, \mathcal{O}(m))$. Each factor on the right has a canonical ample generator of the Picard group. We give $\mathbf{G}$ the polarisation (using the obvious notation):

$$\frac{l}{m} \times \underset{i}{\times} \{(k - \beta_i), (\beta_i - \alpha_i)\} \tag{2.3}$$

and take on $\mathscr{R}$ the induced polarisation. We show that the set of semistable points for the $SL(n)$ action on $\mathscr{R}$ is precisely $\mathscr{R}^{ss} \cdot \mathscr{R}^{ss}$ is reduced and irreducible and $\mathscr{U}_X$ is its GIT quotient. (The above polarisation is in general only rational since l/m need not be an integer; we will see, however, that on $\mathscr{R}^{ss}$ it is indeed given by a line bundle.)

2b. *The theta bundle*

The following Theorem characterises the theta bundle.

Theorem 1. (A) *There is a unique line bundle* $\theta_{\mathscr{U}_X} \equiv \theta(d, k, \alpha_i, \beta_i)$ *on* $\mathscr{U}_X$ *such that given any family of semistable parabolic sheaves parametrised by a variety* T, *we have* $\Phi_T^* \theta_{\mathscr{U}_X} = \theta_{\mathscr{F}_T}$ *where*

$$\theta_{\mathscr{F}_T} \equiv (\det R\pi_T \mathscr{F}_T)^k \otimes \bigotimes_i \{(\mathscr{Q}_{T,i})^{\beta_i - \alpha_i} \otimes (\det(\mathscr{F}_T)_{y_i})^{k - \beta_i}\} \otimes (\det(\mathscr{F}_T)_y)^l \tag{2.4}$$

and Φ_T *is the induced map* $T \to \mathscr{U}_X$.

(B) *The bundle* $\theta_{\mathscr{U}_X}$ *is ample.*

Proof of Theorem 1(A). We claim that $\theta_{\mathscr{F}_{\mathscr{R}^{ss}}}$ descends to $\mathscr{U}_X$. To see this we use a result of Kempf [D-N] (Lemma 2.3 below).

The bundle $\theta_{\mathscr{F}_{\mathscr{R}^{ss}}}$ is a $PGL(n)$ bundle: given $\lambda \in \mathbf{C}^*$, its action on the fibre of $\theta_{\mathscr{F}_{\mathscr{R}^{ss}}}$ at F is given by the character $\lambda \mapsto \lambda^{-kn + 2l + \sum_i (\beta_i - \alpha_i) + 2\sum_i (k - \beta_i)} = \lambda^0$ where we have used equation (1.4).

We apply Lemma 2.3 to our situation, taking $G = PGL(n)$. We first check the condition $(*)$ of Lemma 2.3 for a stable point F. By an analogue of [N, Theorem 5.3(iv)] and [S2, Proposition 9(d)], the stabiliser of the $GL(n)$-action at such a point is just the centre $\mathbf{C}^* \subset GL(n)$, and the stabiliser of the $PGL(n)$ action therefore trivial.

We turn next to a semistable point F such that the orbit through F is closed. At such a point $F = L_1 \oplus L_2$ where the L_i are rank one torsion-free sheaves, with

$$\text{par degree } L_i = \tfrac{1}{2}(\text{par degree } F) \tag{2.5}$$

Consider first the case when the (parabolic) line bundles L_1 and L_2 are not isomorphic (this is necessarily the case when $|I| > 0$). Up to $PGL(n)$ action we can write $\mathcal{O}^n = \mathcal{O}^{n_1} \oplus \mathcal{O}^{n_2}$ with $\mathcal{O}^{n_i} \sim H^0(L_i)$. The parabolic structure of F at the y_i is such that either

(1) $(L_1)_{y_i} \mapsto 0$ in Q_i, in which case the weights assigned to $(L_1)_{y_i}$ and $(L_2)_{y_i}$ are b_i and a_i respectively (we let $R_1 \subset I$ denote the set of such i), or

(2) $(L_2)_{y_i} \mapsto 0$ in Q_i, in which case the weights assigned to $(L_1)_{y_i}$ and $(L_1)_{y_i}$ are a_i and b_i respectively (we let $R_2 \subset I$ denote the set of such i).

(Note that $R_1 \cap R_2 = \emptyset$, $R_1 \cup R_2 = I$, par degree $L_1 =$ degree $L_1 + \sum_{R_1} b_i + \sum_{R_2} a_i$ and par degree $L_2 =$ degree $L_2 + \sum_{R_2} b_i + \sum_{R_1} a_i$.) Then by [S2, Proposition 25(ii)] the isotropy at F of the $GL(n)$-action is $\mathbf{C}^* \times \mathbf{C}^* \subset GL(n_1) \times GL(n_2)$. Given $(\lambda_1, \lambda_2) \in \mathbf{C}^* \times \mathbf{C}^*$ its action on the fibre of $\theta_{\mathscr{F}_{\mathscr{R}^{ss}}}$ at F is given by

$$\lambda_1^{-kn_1 + l + k|I| - \sum_i \beta_i + \sum_{R_2}(\beta_i - \alpha_i)} \times \lambda_2^{-kn_2 + l + k|I| - \sum_i \beta_i + \sum_{R_1}(\beta_i - \alpha_i)}$$

$$= \lambda_1^{-k(\text{par degree } L_1) - k(1-g) + l + k|I|} \times \lambda_2^{-k(\text{par degree } L_2) - k(1-g) + l + k|I|}$$

$$= \lambda_1^0 \lambda_2^0,$$

where we have used equations (2.1) and (2.5).

If $|I| = 0$ and the line bundles L_j are isomorphic the isotropy subgroup for the $PGL(n)$-action is $PGL(2)$ which has no nontrivial characters so again we are done.

This finishes the proof of the claim.

Arguments similar to those in [D-N, §3] show that the line bundle $\theta_{\mathscr{U}_X}$, defined as the "descendant" of $\theta_{\mathscr{F}_{\mathscr{R}^{ss}}}$ to $\mathscr{U}_X$, has the universal properties asserted in Theorem 1(A). □

Lemma 2.3. (Theorem 2.3 of [D-N]). *Let V be a variety with a G-action, where G is a reductive algebraic group. Suppose a good quotient π: $V \to V//G$ exists. Let E be a G-vector bundle on V. Then E descends to $V//G$ iff the following condition holds:*

(*) *For every point y such that the oribit Gy is closed, the stabiliser of y acts trivially on E_y.*

Remark 2.4. If there exist semistable parabolic bundles which are not parabolic stable, and $|I| > 0$, then for $i \in I$

$$\tilde{\mathscr{L}}_i \equiv \mathscr{Q}^2_{\mathscr{R}^{ss},i} \otimes (\det(\mathscr{F}_{\mathscr{R}^{ss}})_{y_i})^{-1}$$

is a $PSL(n)$ line bundle which does not satisfy the condition (*) of Lemma 2.3 at points with nontrivial isotropy. From this it follows that if $|I| > 0$, the genus g is large enough, and there exist semistable parabolic bundles which are not parabolic stable, then the moduli space of semistable bundles is *not locally factorial*. To see this note that the restriction of $\tilde{\mathscr{L}}_i$ to $\mathscr{R}^s$, which we denote by $\tilde{\mathscr{L}}_i^s$, clearly descends to a line bundle $\mathscr{L}_i^s$ on $\mathscr{U}_X^s$; if $\mathscr{U}_X$ were locally factorial $\mathscr{L}_i^s$ would extend to $\mathscr{U}_X$ as a line bundle $\mathscr{L}_i$, and its pull-back to $\mathscr{R}^{ss}$, which we denote by $\tilde{\mathscr{L}}'_i$, would be an extension of $\tilde{\mathscr{L}}_i^s$ which does indeed satisfy (*). For large enough g codimensions are high and all the above extensions would be unique, so that $\tilde{\mathscr{L}}'_i = \tilde{\mathscr{L}}_i$ (as line bundles with $PSL(n)$-action). This yields a contradiction. (cf. [D-N, §7].)

Remark 2.5. (a) Note that if $\mathscr{F}'_T = \mathscr{F}_T \otimes \mathscr{N}$ and $\mathscr{Q}'_{T,i} = \mathscr{Q}_{T,i} \otimes \mathscr{N}$, with $\mathscr{N}$ a line bundle on T, we have, by Eq. (1.4) and elementary properties of the determinant bundle of family, a canonical isomorphism $\theta_{\mathscr{F}'} \sim \theta_{\mathscr{F}}$.

(b) When a Poincaré sheaf exists, formula (2.4) can be used to define $\theta_{\mathscr{U}_X}$.

(c) Different choices of y give algebraically equivalent bundles. We sketch the proof: Let X^{reg} denote the smooth points of X, and consider the quotient $\mathscr{R}^{ss} \times X^{\text{reg}} \to \mathscr{U} \times X^{\text{reg}}$. This is a good quotient by Lemma 2.6 below. Lemma 2.3, applied to a suitable line bundle on $\mathcal{R}^{ss} \times X^{reg}$, yields, as in the proof of Theorem 1(A), a line bundle on $\mathcal{U} \times X^{reg}$ that gives the desired algebraic equivalence.

(d) Similarly, given integers v_i such that $0 \leqq \alpha_i + v_i < \beta_i + v_i < k$, $\mathcal{U}(X,d,\omega) = \mathcal{U}(X,d,a_i + v_i/k, b_i + v_i/k)$, and $\theta(d,k,\alpha_i,\beta_i)$ is algebraically equivalent to $\theta(d,k,\alpha_i + v_i, \beta_i + v_i)$.

(e) For $m \in \mathbf{Z}$, $F \mapsto F \otimes \mathcal{O}(\pm y)$ gives an isomorphism of $\mathcal{U}(X,d,\omega)$ and $\mathcal{U}(X,d \pm 2,\omega)$, such that $\theta(d \pm 2,k,\alpha_i,\beta_i)$ pulls back to $\theta(d,k,\alpha_i,\beta_i)$. Note that $l \mapsto l \pm k$.

(f) Suppose $|I| = 0$. Then Eq. (2.4) becomes: $\theta_{\mathcal{F}_T} \equiv (\det R\pi_T \mathcal{F}_T)^k \otimes \otimes (\det(\mathcal{F}_T)_y)^{\frac{1}{2}nk}$ where n is the Euler characteristic of $\mathcal{F}_t$, for $t \in T$. Note that when d is odd we have to take k even. If X is smooth the results of [D-N] show that the bundles $\theta(d,1)$ (when d is even) and $\theta(d,2)$ (when d is odd) are ample and in fact generate the Picard group of the moduli space of bundles with fixed determinant. (The first case is immediate; when d is odd one has to deform the bundle F of [D-N, p. 55] to the bundle $\mathcal{O} \oplus \mathcal{O}(-ny)$.)

Lemma 2.6. *Suppose $V \to V//G$ is a good quotient and T is any variety with trivial G-action. Then $V \times T \to V//G \times T$ is a good quotient.*

Proof. By [N, Proposition 3.10(b)] we can assume T and V are affine. The result then follows from the fact ([M-F, Theorem 1.1]) that $V \to V//G$ is a universal categorical quotient (when the base field has characteristic zero.) □

Proof of Theorem 1(B). We will show that $\theta_{\mathcal{U}_X}$ is the descendant of the ample line bundle (A.4) on $\mathcal{R}$ used to linearise the action of $SL(n)$ (cf. [D, the proof of Proposition 5.4]) if the line bundle $\mathcal{O}(1)$ on X is chosen to be $\mathcal{O}(y)$.

Note that the construction of Appendix A requires that for every semistable point the map $\mathbf{C}^n \to H^0(F)$ is an isomorphism. This implies that on $\mathcal{R}^{ss}$ we have (we will drop the suffix specifying the parameter space, which will be $\mathcal{R}^{ss}$ below)

$$\theta_{\mathcal{F}} = (\det \mathcal{O}^n)^{-k} \otimes \bigotimes_i \{\mathcal{Q}_i^{\beta_i - \alpha_i} \otimes (\det \mathcal{F}_{y_i})^{k-\beta_i}\} \otimes (\det \mathcal{F}_y)^l.$$

On the other hand one can compute the restriction of the polarisation (A.4) to $\mathcal{R}^{ss}$; this is

$$\theta_{\mathcal{F}} = (\det R\pi_{\mathcal{R}^{ss}} \mathcal{F}(my))^{l/m} \otimes \bigotimes_i \{\mathcal{Q}_i^{\beta_i - \alpha_i} \otimes (\det \mathcal{F}_{y_i})^{k-\beta_i}\}.$$

Using natural isomorphisms we see that this equals $\theta_{\mathcal{F}}$, upto tensoring by a power of the trivial line-bundle $\det \mathcal{O}^n$.

Now, some multiple of the polarisation (A.4) descends as an ample line bundle by general properties of GIT quotients. Thus some multiple of $\theta_{\mathcal{U}_X}$ is ample, and hence $\theta_{\mathcal{U}_X}$ itself. □

2c. *Parabolic weights*

We have required $0 \leqq \alpha_i < \beta_i < k$ so far, but the construction in Appendix A calls for $0 < \alpha_i < \beta_i \leqq k$. Also, in the statement of the decomposition theorems below we will need to consider the case $0 \leqq \alpha_i \leqq \beta_i < k$. We extend the range allowed in the Appendix ($0 < \alpha_i < \beta_i \leqq k$) to cover also $0 \leqq \alpha_i \leqq \beta_i < k$ as follows.

Suppose $\alpha_q = \beta_q$ for $q \in Q \subset I$. Denote by $\mathscr{U}^s(X, d, \omega)$ to moduli space of stable parabolic sheaves with parabolic structures at $\{y_i\}_{I \setminus Q}$, and parabolic weights $\{(a_i, b_i)\}_{I \setminus Q}$. A similar convention holds for $\mathscr{U}(X, d, \omega)$.

(2) Secondly if $\beta_i < k \forall i$ we define $\alpha_i^* = 1$, $\beta_i^* = \beta_i + 1$ whenever $\alpha_i = 0$. The corresponding change in weights does not alter the notion of (semi)stability, on the other hand it conforms to the convention used in the Appendix.

We need to be sure that the results above the theta bundle and its ampleness are unaffected by these redefinitions. This is true because of the following.

Remark 2.7. Suppose given smooth points z_q indexed by $q \in Q$, and integers l_q, for $q \in Q$. Let $\theta(d, k, \alpha_i, \beta_i, z_q, l_q)$ be the line bundle given by the construction of Theorem 1(A), with $(\det(\mathscr{F}_T)_y)^l$ replaced by $\otimes_{q \in Q}(\det(\mathscr{F}_T)_{z_q})^{l_q} \otimes (\det(\mathscr{F}_T)_y)^{l + l_0}$ where $\sum_{q \in Q} l_q = -l_0$. (It is easy to check that the descent conditions are satisfied with this change.) It is clear (as in 2.5(c)) that these line bundles are all algebraically equivalent to $\theta(d, k, \alpha_i, \beta_i)$. Moreover, these line bundles are also ample, because they correspond to a different choice of the line bundle $\mathcal{O}(1)$ on the curve, the new choice being such that $\mathcal{O}(l) = \mathcal{O}(\sum_{q \in Q} l_q z_q + (l + l_0)_y)$.

3. Seminormality of $\mathscr{U}_X$

3a. Torsion-free sheaves on a nodal curve

Note that a torsion-free sheaf F on X is actually free outside x_0, since $\dim X = 1$. Also, if rank $F = 2$ and if F is not locally-free at x_0, we have [S2, p. 164], either $F \otimes \mathcal{O}_{x_0} \sim \mathcal{O}_{x_0} \oplus \mathcal{M}_{x_0}$or $F \otimes \mathcal{O}_{x_0} \sim \mathcal{M}_{x_0} \oplus \mathcal{M}_{x_0}$. (We denote by $\mathcal{M}_x$ the maximal ideal at a point x.) This yields a decomposition of the space $\mathscr{R}^{ss}$: $\mathscr{R}^{ss} = \mathscr{R}_0 \cup \mathscr{R}_1 \cup \mathscr{R}_2$ where

Notation 3.1. $\mathscr{R}_a$ consists of semistable quotients $\mathcal{O}^n \to F \to 0$ satisfying

$$F \otimes \mathcal{O}_{x_0} = a\mathcal{O}_{x_0} \oplus (2 - a)\mathcal{M}_{x_0}. \tag{3.1}$$

By semicontinuity $\bigcup_{b \leq a} \mathscr{R}_b$ is closed in $\mathscr{R}^{ss}$. We will let $\hat{\mathscr{W}}$ denote the set $\bigcup_{b \leq 1} \mathscr{R}_b$, and $\hat{\mathscr{W}}'$ the set $\mathscr{R}_0$, each endowed with its reduced structure. The subschemes $\hat{\mathscr{W}}$ and $\hat{\mathscr{W}}'$ are $SL(n)$-invariant, and yield (by Lemma 4.14) closed reduced subschemes of $\mathscr{U}_X$, which we denote by $\mathscr{W}$ and $\mathscr{W}'$ respectively. Note that the $\mathscr{R}_a$ are *not* necessarily saturated sets for the quotient map, for the condition (3.1) need not be preserved by s-equivalence (see the 'Remarque' on p. 172 of [S2]).

We will prove that the spaces $\mathscr{U}_X$ and $\mathscr{W}$ are seminormal. This is a local property of a variety V, which implies in particular that any (algebraic) function on the normalisation $\tilde{V}$ that is constant on the fibres descends to an algebraic function on V. The method of the proof is to show that the variety $\mathscr{R}^{ss}$, of which $\mathscr{U}_X$ is a GIT quoteint, is seminormal. A general property of GIT quotients then yields the desired result. The seminormality of $\mathscr{R}^{ss}$ in turn is proved using Seshadri's description of its local structure. A similar proof works for $\mathscr{W}$.

We summarise Seshadri's description in the following theorem. First we make a preliminary.

Definition 3.2. Given a scheme Z and closed subschemes $Z_2 \subsetneq Z_1 \subsetneq Z$, we say that an analytical model at $p \in Z_2$ is given by schemes $Z'_2 \subsetneq Z'_1 \subsetneq Z'$ (with $(Z'_1$

and Z'_2 closed) and a point q in Z'_2 if for some r and some s, we have a diagram

$$\begin{array}{ccccc} (\hat{\mathcal{O}}_{Z_2})_p[[u_1,\ldots,u_r]] & \longleftarrow & (\hat{\mathcal{O}}_{Z_1})_p[[u_1,\ldots,u_r]] & \longleftarrow & (\hat{\mathcal{O}}_Z)_p[[u_1,\ldots,u_r]] \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ (\hat{\mathcal{O}}_{Z'_2})_q[[v_1,\ldots,v_s]] & \longleftarrow & (\hat{\mathcal{O}}_{Z'_1})_q[[v_1,\ldots,v_s]] & \longleftarrow & (\hat{\mathcal{O}}_{Z'})_q[[v_1,\ldots,v_s]] \end{array}$$

Theorem 2. (1) *$\mathcal{R}_2$ is a smooth variety.*

(2) *Let $p \in \mathcal{R}_1 \backslash \mathcal{R}_0$. The analytical local model for $\mathcal{R}_1 \hookrightarrow \mathcal{R}^{ss}$ at p is* Spec $A/(u,v) \hookrightarrow$ Spec A where $A = \mathbf{C}[u,v]/(uv)$.

(3) *Let $X = (X_{ij})$ and $Y = (Y_{lm})$ be 2×2 matrices of indeterminates. Let $A = \mathbf{C}[X,Y]/I$, $I = ((XY)_{ij}, (YX)_{lm})$. $J = (Y_{lm}, \det X) \cap (X_{ij}, \det Y)$. Let $p \in \hat{\mathcal{W}}'$. An analytical local model for $\hat{\mathcal{W}}' \hookrightarrow \hat{\mathcal{W}} \hookrightarrow \mathcal{R}^{ss}$ at p is* Spec $A/(X,Y) \hookrightarrow$ Spec $A/J \hookrightarrow$ Spec A.

Proof. This theorem follows from the results of [S2, Huitième Partie, III] and properties of smooth morphisms (see §4d). □

The following lemma is implicit in [B1].

Lemma 3.3. *Let E' be a rank 2 (semi)stable parabolic bundle on $\tilde{X}$, of degree $d-2$. Then its direct image $F = \pi_* E'$ is a (semi)stable parabolic sheaf of degree d on X, such that $F \otimes \mathcal{O}_{x_0} \sim \mathcal{M}_{x_0} \otimes \mathcal{M}_{x_0}$. We have $E' = \pi^* F/(T$ or $\pi^* F)$.*

Proof. That $E' \mapsto F \equiv \pi_* E'$, $F \mapsto E \equiv \pi^* F/(\text{Tor}\,\pi^* F)$ gives a bijection between the set of isomorphism classes of rank 2 bundles E' on $\hat{X}$ with degree $d-2$ and torsion-free sheaves F on X with degree d and $F \otimes \mathcal{O}_{x_0} \sim \mathcal{M}_{x_0} \oplus \mathcal{M}_{x_0}$ is clear from [S2, Septième Partie, Proposition 10] (see also the proof of Lemma 4.6(4).)

We check that the (semi)stability of E' implies that of F: Let L be a torsion-free rank 1 quotient of F. One checks that $L \otimes \mathcal{O}_{x_0} \sim \mathcal{M}_{x_0}$. As in the last paragraph, we have $L = \pi_* L'$, with $L' = \pi^* L/(T$ *or* $\pi^* L)$ locally-free and degree $L' =$ degree $L - 1$. One checks that L' is a quotient of E' and this gives par degree $L' >_{(\text{resp. } \geq)^t}$ par degree E' and rewriting we get par degree $L >_{(\text{resp. } \geq)^t}$ par degree F. The converse is similarly verified. □

3b. Seminormality

All rings considered in this section will be noetherian, with characteristic zero. The basic references are [T] and [Sw]. We recall from [Sw]:

Definition 3.4. An extension $A \hookrightarrow B$ of reduced rings is *subintegral* if

(1) B is integral over A

(2) Spec $B \to$ Spec A is a bijection

(3) $\forall \wp \in$ Spec B, $k_{A \cap \wp} \to k_\wp$ is an isomorphism, where $k_\wp = B_\wp/\wp B_\wp$

Definition 3.5. If $A \hookrightarrow B$, both rings reduced, we say A is *seminormal in B* if there is no extension $A \hookrightarrow C \hookrightarrow B$ with $C \neq A$ and $A \hookrightarrow C$ subintegral. We say A is seminormal if it is seminormal in its total ring of quotients.

We will use the following characterisation of seminormal rings ([Sw, Corollary 3.2]):

Proposition 3.6. *A reduced ring A is seminormal if $\forall b, c \in A$ with $b^3 = c^2$ there is a unique $a \in A$ with $b = a^2$ and $c = a^3$.*

Remark. The uniqueness of a depends only on the fact that A is reduced, and can be seen as follows. Given a_i, $i \in \{1, 2\}$ such that $b = a_i^2$ and $c = a_i^3$ we compute

$$\begin{aligned}(a_1 - a_2)^3 &= 3a_1a_2(a_1 - a_2)\\ &= 3/4\{(a_1 + a_2)^2 - (a_1 - a_2)^2\}(a_1 - a_2)\\ &= -3/4 \times (a_1 - a_2)^3,\end{aligned}$$

where we use $a_1^3 - a_2^3 = 0$, and $(a_1 + a_2)^2(a_1 - a_2) = (a_1 + a_2)(a_1^2 - a_2^2) = 0$. This shows, since A is reduced, that $a_1 = a_2$.

Recall that given a variety V, with normalisation $\sigma\colon \tilde{V} \to V$, the *conductor* $\mathscr{C}$ is the $\mathcal{O}_V$-ideal defined as the annihilator of $\mathbf{D}$, with $\mathbf{D}$ being defined by the exact sequence of sheaves on V: $0 \to \mathcal{O}_V \to \sigma_*\mathcal{O}_{\tilde{V}} \to \mathbf{D} \to 0$. In fact $\mathscr{C}$ is a $\mathcal{O}_{\tilde{V}}$-ideal as well, and the biggest such. Also, the variety defined by $\mathscr{C}$ is the non-normal locus W in V [B, Chapter 5, §1.5, Corrollary 5]. Let $\hat{W}$ be the set-theoretic inverse image of W in $\tilde{V}$.

We have then

Lemma 3.7. *If V is seminormal, then $\mathscr{C}$ is the ideal of functions vanishing on $\tilde{W}$.*

Proof. Immediate from [T, Lemma 1.3]. □

Given a local ring A, let $\hat{A}$ denote its completion w.r.t. the maximal ideal.

Lemma 3.8. *Let V be an variety. Assume that $\forall p \in V$, $\hat{\mathcal{O}}_p[[u_1, \ldots, u_n]]$ is seminormal for some n. Then V is seminormal.*

Proof. It is enough [Sw, Theorem 1] to prove $A[u_1, \ldots, u_n]$ is seminormal (where A denotes, as before, the ring of functions on V) and further, by [Sw, Proposition 4.7] that $A[u_1, \ldots, u_n]$, localised at any maximal ideal is of the form $\mathscr{I}_p + (u_1 - a_1, \ldots, u_n - a_n)$, where $\mathscr{I}_p$ is the ideal of functions vanishing at $p \in V$, and $a_i \in \mathbf{C}$. We can, without loss of generality, assume $a_i = 0$. The localisation of A at such a maximal ideal is $(\mathcal{O}_p[u_1, \ldots, u_n])_{\mathscr{I}_p + (u_1, \ldots, u_n)}$ and its completion, by [A-M, Exercise 10.5], is $\hat{\mathcal{O}}_p[[u_1, \ldots, u_n]]$. The result now follows from the next lemma. □

Lemma 3.9. *Let A be a local domain, $\hat{A}$ its completion* w.r.t. *the maximal ideal. Then if $\hat{A}$ is seminormal so is A*

Proof. Let $b, c \in A$ such that $b^3 = c^2$ (one can assume these are nonzero). Then $\exists \hat{a} \in \hat{A}$ such that $\hat{a}^3 = c$, $\hat{a}^2 = b$. Thus $\hat{a}b = c \in \hat{A}$, which implies, by faithful flatness, that $\exists\, a \in A$ such that $ab = c \in A$. One now computes: $b^2(a^2 - b) = c^2 - c^2 = 0$ which yields $b = a^2$, $c = a^3$. The uniqueness of a is clear. □

Lemma 3.10. *Let $X = (X_{ij})$ and $Y = (Y_{lm})$* be 2×2 *matrices of indeterminates. Let $A = \mathbf{C}[X, Y]/I$, $I = ((XY)_{ij}, (YX)_{lm})$. Then A is seminormal.*

Proof. We follow [S2, Theorem 30], where the proof, due to Cowsik, that A is reduced is given. One finds $I = \wp_1 \cap \wp_2 \cap \wp_3$ where $\wp_1 = (X_{ij})$, $\wp_2 = (Y_{lm})$, and $\wp_3 = (I, \det X, \det Y)$, and one checks that these are prime ideals. We claim now that

(1) $\wp_1 \cap \wp_2$ is radical, and
(2) $\wp_1 \cap \wp_2 + \wp_3$ is radical

Granting this claim, Lemma 3.11 (below) finishes the proof..

We turn now to the claim. That $\wp_1 \cap \wp_2$ is radical is clear. On the other hand we now show $\wp_1 \cap \wp_2 + \wp_3 = J_1 \cap J_2$ where $J_1 = (X_{ij}, \det Y)$ and $J_2 = (Y_{lm}, \det X)$.

That $\wp_1 \cap \wp_2 + \wp_3 \subset J_1 \cap J_2$ is clear. Consider now an element in $J_1 \cap J_2$: $\alpha = \sum a_{ij} X_{ij} + b \det Y = \sum c_{ij} Y_{ij} + d \det X$. We write $\alpha = \{\sum a_{ij} X_{ij} - d \det X\} + \{d \det X + b \det Y\}$. The second term is in $\wp_3$, and the first term, which can also be written $\sum c_{ij} Y_{ij} - b \det Y$, is in $\wp_1 \cap \wp_2$. It remains to remark that J_1 and J_2 are prime – this is because $(\det X)$ is. □

Lemma 3.11. *Let I_1 and I_2 be two radical ideals in a ring A such that $I_1 + I_2$ is radical. Then if A/I_i is seminormal for $i = 1, 2$ then so is $A/(I_1 \cap I_2)$.*

Proof. ([K-P, Lemma on p. 587]). Let $b, c \in A/(I_1 \cap I_2)$ such that $b^2 = c^3$. Then $\exists\, a_i \in A/I_i$, $i = 1, 2$ such that $b = a_i^3$, $c = a_i^2$ in A/I_i.

On the other hand, by the Remark following Proposition 3.6, we have $a_1 - a_2 = 0$ in $A/(I_1 + I_2)$ (since $A/(I_1 + I_2)$ is reduced). From the exact sequence of A-modules

$$0 \to A/I_1 \cap I_2 \to A/I_1 \oplus A/I_2 \to A/(I_1 + I_2) \to 0,$$

we see that in fact there exists an a in $A/I_1 \cap I_2$ as required. □

Lemma 3.12. *Let A be as in the statement of Lemma 3.10, $\mathcal{M}$ the maximal ideal (X_{ij}, Y_{lm}), $\hat{A}$ the completion of $A_{\mathcal{M}}$ w.r.t. $\mathcal{M} A_{\mathcal{M}}$. Then $\hat{A}[[u_1, \ldots, u_n]]$ is seminormal for any n.*

Proof. The proof of Lemma 3.10 goes through almost word for word. The only-point to note is by [Z, Theorem 2] that the ideals $\wp_3$ and $(\det X)$ remain prime under completion, since each defines a normal variety. (That $\det X$ defines a normal variety is well-known; $\wp_3$, in Cowsik's description, defines the cone over $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$, embedded in the complete linear system of $\mathcal{O}(1) \otimes \mathcal{O}(1) \otimes \mathcal{O}(1)$ where each $\mathcal{O}(1)$ comes from one of the factors. The projective normality of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ is clear, yielding normality of the cone.) □

By Theorem X1 of Appendix A, $\mathcal{U}_X$ is a variety. We can now prove (the notation of §4c is used below).

Theorem 3. *$\mathcal{U}_X$ is seminormal.*

Proof. By Lemma 3.13 below it suffices to show that $\mathcal{R}^{ss}$ is seminormal. We now use Theorem 2. The points in $\mathcal{R}_2$ are smooth and hence the local rings are seminormal. Using the Theorem 2, Lemma 3.8 and 3.12 we see that the local rings at points of $\mathcal{R}_1$ and $\mathcal{R}_0$ are seminormal as well. The Theorem now follows by [Sw, Proposition 3.7]. □

Lemma 3.13. *A GIT quotient of a seminormal variety is seminormal*

Proof. The result to be proved is: Given a seminormal domain A with a G-action, the ring of invariants (denoted A^G below) is seminormal. One needs to show that if $a \in A$, with a^2 and a^3 in A^G, then $a \in A^G$. One can assume $a \neq 0$, for if $a = 0$ the result is trivially true. For any $g \in G$, $(a - a_g)(a + a_g) = a^2 - a_g^2 = a^2 - (a^2)_g = 0$, which yields $a = \pm a_g$. On the other hand $a^3 = a_g^3$ which rules out $a = -a_g$. □

By Proposition 3.15 below $\mathscr{W}$ is a variety. We have

Proposition 3.14. *The variety $\mathscr{W}$ is seminormal.*

Proof. The analysis proceeds as above. The local result to be proved is this: Let $X = (X_{ij})$ and $Y = (Y_{lm})$ be matrices of indeterminates. Let $A = \mathbf{C}[X, Y]/I$, $I = (Y_{lm}, \det X) \cap (X_{ij}, \det Y)$. Then A is seminormal. But this is clear. □

Proposition 3.15. (1) *$\mathscr{W}$ is irreducible.*
(2) *$\mathscr{W}'$ is irreducible.*
(3) *$\mathscr{W}'$ is normal.*
(4) *$\mathscr{W}$ is the non-normal locus of $\mathscr{U}_X$.*
(5) *$\mathscr{W}'$ is the non-normal locus of $\mathscr{W}$.*
(6) *The map $E' \mapsto F = \pi_* E'$ gives a morphism $\mathscr{U}(\tilde{X}, d - 2, \omega) \to \mathscr{W}'$.*

Proof. (1–3) We will see below (Lemma 3.16) that the $\mathscr{R}_a$ $(a = 0, 1, 2)$ are irreducible. These statements are now easy consequences of Theorem 2, using general properties of GIT quotients.

(4 and 5) The proof will be given in §4, immediately following the proof of Proposition 4.11.

(6) By Lemma 3.3 there is a morphism $\mathscr{U}(\tilde{X}, d - 2, \omega) \to \mathscr{U}_X$, whose set-theoretic image is $\mathscr{W}'$. Since $\mathscr{U}(\tilde{X}, d, \omega)$ and $\mathscr{W}'$ are reduced this actually yields a morphism $\mathscr{U}(\tilde{X}, d - 2, \omega) \to \mathscr{W}'$. □

Lemma 3.16. *The $\mathscr{R}_a$ $(a = 0, 1, 2)$ are irreducible.*

Proof. In the course of the proof of Theorem X1 we show that $\mathscr{R}^{ss}$ is irreducible. Hence so is its open subset $\mathscr{R}_2$. The cases $a = 0, 1$ will be treated later, immediately following the proof of Proposition 4.11. □

4. Preliminaries

4a. Generalised parabolic sheaves

Definition 4.1a. Let E be a sheaf on $\tilde{X}$, torsion-free of rank 2 outside $\{x_1, x_2\}$. *A generalised parabolic structure on E over the divisor* $\{x_1, x_2\}$ is a two-dimensional quotient Q of $E_{x_1} \oplus E_{x_2}$.

The pair (E, Q) is said to be a "generalised parabolic sheaf" (GPS). We do not define a generalised quasiparabolic structure since a certain choice of "generalised weights" is assumed. We shall consider generalised parabolic sheaves E with, in addition, parabolic structures at the $\{y_i\}_I$ (i.e. a one-dimensional quotient $E_{y_i} \to Q_i \to 0$ of the fibre of E at each point y_i, and weights $0 \leqq a_i < b_i < 1$ as before).

Definition 4.1b. A GPS (E, Q) is said to be *stable* (respectively, *semistable*) with respect to the weights ω if for every nontrivial subsheaf E' such that E/E' is torsion-free outside the reduced points $\{x_1, x_2\}$, we have

$$\text{par degree}\, E' \underset{(\text{resp. } \leqq)}{<} \frac{\text{rank } E'}{2}(\text{par degree } E) - (\text{rank } E' - \dim Q^{E'}), \qquad (4.1)$$

where, for any subsheaf E' we denote by $Q^{E'}$ the image of $E'_{x_1} \oplus E'_{x_2}$ in Q.

Note that in the above definition the parabolic degree of E' needs to be defined. If E' is torsion this is just its degree (= length), otherwise E' is actually a subbundle of E outside $\{x_1, x_2\}$ and the earlier Definition (2.1a) extends in a clear way.

Remark 4.2. If (E, Q) is a semistable GPS, $\operatorname{Tor} E$ is supported on the reduced subscheme $\{x_1, x_2\}$ and $(\operatorname{Tor} E)_{x_1} \oplus (\operatorname{Tor} E)_{x_2} \subsetneq Q$. This follows from (4.1).

Theorem X2. *There exists a (coarse) moduli space $\mathscr{P}^s(\tilde{X}, d, \omega)$ of stable GPS's on $\tilde{X}$. We have an open immersion $\mathscr{P}^s(\tilde{X}, d, \omega) \subsetneq \mathscr{P}(\tilde{X}, d, \omega)$ where $\mathscr{P}(\tilde{X}, d, \omega)$ denotes the space of s-equivalence classes of semistable GPSs. The former is a smooth variety; the latter a normal projective variety with rational singularities.*

This theorem is proved in Appendix B. The definition of s-equivalence is given there. We shall set $\mathscr{P}^s = \mathscr{P}^s(\tilde{X}, d, \omega)$ and $\mathscr{P} = \mathscr{P}(\tilde{X}, d, \omega)$.

We make explicit the notion of a family of GPSs parametrised by a variety T. This consists of

(1) a rank 2 sheaf $\mathscr{E}_T$ (on $\tilde{X} \times T$) flat over T and locally free outside $\{x_1, x_2\} \times T$
(2) a locally-free rank 2 quotient $\mathscr{Q}_T$ (on T) of $(\mathscr{E}_T)_{x_1} \oplus (\mathscr{E}_T)_{x_2}$, and
(3) a locally-free rank 1 quotient $\mathscr{Q}_{T,i}$ (on T) of $(\mathscr{E}_T)_{y_i}$ for $i \in I$,

where we have set, for $x \in \tilde{X}$, $(\mathscr{E}_T)_x \equiv \mathscr{E}_T|_{\{x\} \times \mathscr{T}}$. (We will on occasion regard $\mathscr{Q}_T$ as a sheaf on $X \times T$ supported on $\{x_0\} \times T$.) Take now $T = \tilde{\mathscr{R}}'$, the parameter-space of the locally universal family of Appendix B:

$$\tilde{\mathscr{R}}' = \operatorname{Grass}_2(\mathscr{E}_{x_1} \oplus \mathscr{E}_{x_2}) \times_{\tilde{Q}} \left\{ \underset{i \in I}{\times_{\tilde{Q}}} \operatorname{Flag}_{(1,2)}(\mathscr{E}_{y_i}) \right\},$$

where $\tilde{Q}$ is the Quot scheme of rank 2 degree d quotients of $\mathscr{O}_{\tilde{X}}^{\tilde{n}}$. The degree d is assumed large.(We have let $\mathscr{E} = \mathscr{E}_{\tilde{\mathscr{R}}'}$; we will similarly let $\mathscr{Q} \equiv \mathscr{Q}_{\tilde{\mathscr{R}}'}$.) The polarisation on $\tilde{\mathscr{R}}'$ is defined in Appendix B (equation B-2). The moduli space $\mathscr{P}$ is the GIT quotient of $\tilde{\mathscr{R}}'^{ss}$ by $SL(\tilde{n})$. (We have $SL(\tilde{n})$ rather than $SL(n)$ because we are considering bundles of degree d on $\tilde{X}$ rather than on X.) We will denote by $\tilde{\psi}'$ the projection $\tilde{\mathscr{R}}'^{ss} \to \mathscr{P}$.

Notation 4.3a. Define $\mathscr{H}$ to be the set of (closed) points $(\mathscr{O}^{\tilde{n}} \to E \to 0, Q)$ in $\tilde{\mathscr{R}}'$, where $\mathbf{C}_{\tilde{n}} \to H^0(E)$ is an isomorphism, $H'(E(-x_1 - x_2 - x)) = 0$ for $x \in \tilde{X}$, and

(T) $\operatorname{Tor} E$ is supported on the reduced subscheme $\{x_1, x_2\}$ and $(\operatorname{Tor} E)_{x_1} \oplus (\operatorname{Tor} E)_{x_2} \subsetneq Q$.

Requiring that $H^1(E(-x_1 - x_2 - x)) = 0$ ensures that $H^1(E) = 0$, E is generated by sections, $H^0(E) \to E_{x_1} \oplus E_{x_2}$ is onto, and $E(-x_1 - x_2)$ is generated by sections.

It will be clear from Appendices B and C that $\tilde{\mathscr{R}}'^{ss} \subsetneq_{\text{open}} \mathscr{H} \subsetneq_{\text{open}} \tilde{\mathscr{R}}'$.

Notation 4.3b. Define $\tilde{\mathbf{Q}}_F$ to be the open subscheme of $\tilde{\mathbf{Q}}$ consisting of locally-free quotients $\mathcal{O}\tilde{n} \to E \to 0)$ such that

(1) $\mathbf{C}^{\tilde{n}} \to H^0(E)$ is an isomorphism, and

(2) $H^1(E(-x_1 - x_2 - x)) = 0$ for $x \in \tilde{X}$.

Notation 4.3c. Let $\tilde{\mathscr{R}}'_F$ be the inverse image of $\tilde{\mathbf{Q}}_F$ by the projection $\tilde{\mathscr{R}}' \to \tilde{\mathbf{Q}}$. This is a grassmannian bundle over $\tilde{\mathscr{R}}_F$, where

$$\tilde{\mathscr{R}}_F = \underset{i \in I}{\times_{\tilde{\mathbf{Q}}_F}} \mathrm{Flag}_{(1,2)}(\mathscr{E}_{y_i}).$$

We let ρ denote the projection $\tilde{\mathscr{R}}'_F \to \tilde{\mathscr{R}}_F$. Note that $\tilde{\mathscr{R}}'_F \subset \mathscr{H}$. On $\tilde{\mathscr{R}}'_F$, consider the morphism of vector bundles $\mathscr{E}_{x_1} \to \mathscr{Q}$ given by the generalised parabolic structure. The zero scheme of this morphism is denoted by $\hat{\mathscr{V}}_{1,F}$($\mathscr{V}$ for "vertex"). The determinant of this map defines a subscheme which we denote $\tilde{\mathscr{D}}_{1,F}$. The subschemes $\hat{\mathscr{V}}_{2,F}$ and $\tilde{\mathscr{D}}_{2,F}$ are defined similarly. Clearly $\hat{\mathscr{V}}_{j,F} \subsetneq \tilde{\mathscr{D}}_{j,F}, j = 1, 2$. As a set, $\hat{\mathscr{D}}_{1,F}$ consists of pairs (E, Q) such that the map $E_{x_1} \to Q$ is not of maximal rank and $\hat{\mathscr{V}}_{1,F}$ of pairs such that the map $E_{x_1} \to \mathscr{Q}$ is zero. Note that $\mathcal{O}(\hat{\mathscr{D}}_{j,F}) = (\det \mathscr{Q})(\det \mathscr{E}_{x_j})^{-1}$.

Notation 4.3d. The schematic closure of $\hat{\mathscr{D}}_{j,F}$ in $\mathscr{H}$ is denoted $\hat{\mathscr{D}}^f_j$. The $\hat{\mathscr{D}}_{j,F}$ are reduced and irreducible divisors and so the $\hat{\mathscr{D}}^f_j$ are also reduced prime divisors. The subscheme $\hat{\mathscr{V}}^f_j$ is defined as the schematic of $\hat{\mathscr{V}}_{j,F}$ in $\mathscr{H}$.

Notation 4.3e. We define $\hat{\mathscr{D}}^t_1$ to be component of $\mathscr{H} \backslash \mathscr{H}_F$ parametrising sheaves with *non-zero torsion at* x_2. We take $\hat{\mathscr{D}}^t_1$ to have its reduced structure. $\hat{\mathscr{D}}^t_2$ is defined similarly.

We quote from Appendix C the

Proposition C.7. (1) *The $\hat{\mathscr{D}}^f_j$ are reduced, irreducible, and normal.*

(2) *The $\hat{\mathscr{D}}^t_j$ are reduced, irreducible, and normal.*

(3) *The $\hat{\mathscr{V}}^f_j$ are smooth. We have $\hat{\mathscr{V}}^f_j \cap \{\hat{\mathscr{D}}^t_1 \cap \hat{\mathscr{D}}^t_2\} = \emptyset$.*

(4) *The closed orbits in $\hat{\mathscr{D}}^f_j$ and $\hat{\mathscr{D}}^t_j$ are contained in $\hat{\mathscr{D}}^f_j \cap \hat{\mathscr{D}}^t_j$.*

Notation 4.3f. The closed subschemes $\hat{\mathscr{D}}^f_j \cap \tilde{\mathscr{R}}'^{ss}$ and $\hat{\mathscr{V}}^f_j \cap \tilde{\mathscr{R}}'^{ss}$ are $SL(\tilde{n})$-invariant, and therefore yield (by Lemma 4.14 below) closed subschemes of $\mathscr{P}$ which we denote by $\mathscr{D}_j$ and $\mathscr{V}_j$ respectively.

Proposition C.7 has the following

Corollary 4.4. (1) *The $\mathscr{D}_j$ and the $\mathscr{V}_j$ are reduced, irreducible and normal.*

(2) $\mathscr{V}_j \cap \{\mathscr{D}_1 \cap \mathscr{D}_2\} = \emptyset$.

(3) *$\mathscr{D}_j$ is also the quotient of $(\hat{\mathscr{D}}^t_j)^{ss}$.*

4b. The map ϕ

Given a GPS on $\tilde{X}$ one obtains a sheaf F on X which fits into the exact sequence: $0 \to F \to \pi_* E \to {}_{x_0}Q \to 0$, where ${}_{x_0}Q$ is defined as in Notation 1f(5). (Note: $\pi_* E \otimes_{\mathcal{O}_x} k(x_0) = E_{x_1} \oplus E_{x_2}$). *We will often omit the subscript x_0 and simply write Q when we mean ${}_{x_0}Q$.* The sheaf F has, of course, a natural parabolic structure at the $\{y_i\}_I$.

Remark 4.5. Since π is a finite morphism, $\chi(E) = \chi(\pi_*(E))$, and $\chi(F) = \chi(\pi_*(E)) - \chi({}_{x_0}Q) = \chi(E) - 2$, which, rewritten in terms of degrees, becomes degree $F + 2(1-g) =$ degree $E + 2(1-\tilde{g}) - 2$. Thus degree $F =$ degree E. Note that the computation also gives, for any coherent sheaf E on $\tilde{X}$, degree $\pi_*E =$ degree E + rank E).

Lemma 4.6. (1) *Let* (E, Q) *be a GPS, and* F *the associated sheaf on* X. F *is torsion-free iff the condition* (T) *of Notation* 4.3*a holds.*

(2) *If* E *is a vector bundle and the maps* $E_{x_j} \to Q$ *isomorphisms, then the associated* F *is a vector bundle. Otherwise* F *is not locally free.*

(3) *If* F *is a vector bundle on* X, *there is a unique GPS* (E, Q) *which yields* F *by the above construction. Infact* $E = \pi^*F$.

(4) *If* F *is torsion-free but not locally free there is a GPS* (E, Q) *that yields* F, *with* E *a vector bundle and the map* $E_{x_2} \to Q$ *an isomorphism. The rank of the map* $E_{x_1} \to Q$ *is then*

(1) 1 *iff* $F \otimes \mathcal{O}_{x_0} \sim \mathcal{O}_{x_0} \oplus \mathcal{M}_{x_0}$, *and*

(2) 0 *iff* $F \otimes \mathcal{O}_{x_0} \sim \mathcal{M}_{x_0} \oplus \mathcal{M}_{x_0}$.

The roles of x_1 and x_2 can of course be reversed.

(5) *Every torsion-free rank* 2 *sheaf* F *on* X *comes from a pair* (E, Q), *with* E *a vector bundle.*

Proof. Many of these results are in [B1]. For completeness we sketch proofs. For any sheaf A on X define Q_A by the exact sequence $A \xrightarrow{a} \pi_*\pi^*A \to {}_{x_0}Q_A \to 0$. (The map a is generically an isomorphism and hence an injection when A is torsion-free.)

(1) It is clear that the assumption (T) is equivalent to: $\mathrm{Tor}\,\pi_*E$ $(= \pi_*(\mathrm{Tor}\,E)) \hookrightarrow {}_{x_0}Q$.

(2) If the maps $E_{x_j} \to Q$ are isomorphisms, this gives an isomorphism between E_{x_1} and E_{x_2}, which can be used to show that F is locally free. That otherwise F is not locally free follows from (3).

(3) We show next that if F is a vector bundle the GPS(E, Q) is uniquely determined. In fact E is just π^*F and $Q = Q_F$. To see this, consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{a} & \pi_*\pi^*F & \longrightarrow & {}_{x_0}(Q_F) & \longrightarrow & 0 \\ & & =\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & \pi_*E & \longrightarrow & {}_{x_0}Q & \longrightarrow & 0 \end{array}$$

If F is locally $\pi_*\pi^*F$ is torsion-free and the map b is an injection. Thus c is an injection and therefore an isomorphism because dim $Q_F = 2 =$ dim Q. The Snake Lemma now yields the isomorphism $\pi_*\pi^*F = \pi_*E$ from which it easily follows that $E = \pi^*F$.

(4) Define the vector bundle $\tilde{E}$ by the exact sequence $0 \to \mathrm{Tor}\,\pi^*F \to \pi^*F \to \tilde{E} \to 0$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \pi_*\pi^*F & \longrightarrow & {}_{x_0}(Q_F) & \longrightarrow & 0 \\ & & =\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \xrightarrow{d} & \pi_*\tilde{E} & \longrightarrow & {}_{x_0}\tilde{Q} & \longrightarrow & 0 \end{array}$$

where d is an injection (as in the above cases) because F is torsion-free, and $\tilde{Q}$ is defined to make the second sequence exact. The vertical arrows are clearly surjections, so we see that ${}_{x_0}\tilde{Q} = {}_{x_0}(Q_F)/\{\pi_*(\mathrm{Tor}\,\pi^*F)\}$. Local computation show that in case (2) $\tilde{Q} = 0$, and increase (1) dim $\tilde{Q} = 1$. In both cases it is easy to manufacture a GPS as required. We describe the case (2) which is less involved. In this case $F = \pi_* \tilde{E}$, with degree $E = d - 2$. Take $E = \tilde{E}(x_2)$, $Q = \tilde{E}_{x_2} \otimes (\Omega_{\tilde{X}})_{x_2}^{-1}$, and the maps $E_{x_j} \to Q$ as follows: the map is zero for $j = 1$ and the residue map for $j = 2$.

(5) This follows from (3) and (4) □

Proposition 4.7. (1) *If F is semistable then (E, Q) is semistable.*

(2) *If F is a stable vector bundle the GPS (E, Q) (which is unique by Lemma* 4.6(2) *is stable.*

(3) *If (E, Q) is (semi)stable then F is (semi)stable.*

Proof Given a subsheaf E' of E recall that we denote by $Q^{E'}$ the image of $E'_{x_1} \oplus E'_{x_2}$ in Q.

(1) Suppose F is semistable. Given a sub-sheaf E' of E define the subsheaf F' of F via the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \pi^* F & \longrightarrow & {}_{x_0}Q & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F' & \longrightarrow & \pi_* E' & \longrightarrow & {}_{x_0}(Q^{E'}) & \longrightarrow & 0 \end{array}$$

with the vertical arrows being inclusions. It is now easy to verify that the criterion (4.1) is satisfied and (1) is proved.

(2) It could happen in the above proof that E' is a nontrivial subsheaf of E but $F' = 0$ or $F' = F$. This is why stability of F does not guarantee stability of (E, Q), but only semistability. If F were a vector bundle a nontrivial subsheaf E' yields a nontrivial sub-sheaf F', whence the claim in part (2) of the Proposition that (E, Q) is stable if F is a stable vector bundle.

(3) Suppose now that (E, Q) is a (semi)stable GPS. Note that by Remark 4.2 F is torsion-free. Let L' be a rank 1 sub-sheaf of F such that F/L' is torsion-free. Define the sheaf K_1 to be the kernel of the composite map $\pi^* L' (\to \pi^* \pi_* E) \to E$; let E' denote the image. Consider the commutative diagram of sheaves on X:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{f} & \pi_* E & \longrightarrow & {}_{x_0}Q & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & \pi_* \pi^* L' & \longrightarrow & {}_{x_0}(Q_{L'}) & \longrightarrow & 0 \end{array}$$

The second sequence is left exact since L' is torsion-free. The first vertical arrow is an inclusion, and the quotient F/L' is torsion-free. This yields, for the subsheaf E' of E, the equality ${}_{x_0}(Q^{E'}) = {}_{x_0}(Q_{L'})/\{\pi_* K_1\}$. We have the following sequences of inequalities, each of which implies the next, and the first follows the semistability of (E, Q):

$$\begin{aligned} 2(\text{par degree } E') &\leqq \text{par degree } E - (2 - 2\dim Q^{E'}) \\ 2(\text{par degree } \pi^* L' - h^0(K_1)) &\leqq \text{par degree } E - (2 - 2\dim Q^{E'}) \\ 2(\text{par degree } \pi_* \pi^* L' - 1 - \dim K_1) &\leqq \text{par degree } E - (2 - 2\dim Q^{E'}) \\ 2(\text{par degree } L') &\leqq \text{par degree } E + (2 - 2\dim Q^{E'} \\ &\quad + h^0(K_1) - \dim Q_{L'}) \\ &= \text{par degree } E \\ &= \text{par degree } F \end{aligned}$$

(In case (E, Q) is stable all the inequalities are strict.) This proves (3). □

Remark 4.8. $\mathscr{P}^s$ is nonempty iff $\mathscr{U}_X^s$ is nonempty. (This follows from Proposition 4.7.) In this case $\dim \mathscr{P} = 4\tilde{g} + |I| + 1 = 4g + |I| - 3 = \dim \mathscr{U}_X$.

Definition 4.9a. We now define a morphism $\mathscr{P} \to \mathscr{U}_X$. For any family of GPSs as above we construct a family $\mathscr{F}_T$ of sheaves on X parameterised by T: $\mathscr{F}_T$ is defined by the exact sequence

$$0 \to \mathscr{F}_T \to (\pi \times I_T)_* \mathscr{E}_T \to \mathscr{Q}_T \to 0. \tag{4.2}$$

where $\mathscr{Q}_T$ is regarded as a sheaf on $X \times T$ supported on $\{x_0\} \times T$. Now $(\pi \times T_T)_* \mathscr{E}_T$ is flat over T since $\mathscr{E}_T$ is flat and π is finite, $\mathscr{Q}_T$ is locally-free on T and hence flat, and therefore so is $\mathscr{F}_T$. If, further, the family consists of semistable GPSs, by the above Lemma and the universal property of $\mathscr{U}_X$, we get a morphism ϕ_T: $T \to \mathscr{U}_X$. This applies in particular to $T = \tilde{\mathscr{R}}'^{ss}$, and the resulting morphism clearly induces a morphism ϕ: $\mathscr{P} \to \mathscr{U}_X$.

Definition 4.9b. Define on $\tilde{\mathscr{R}}'^{ss}$ a line bundle $\hat{\theta}'$ by

$$\hat{\theta}' \equiv (\det R\pi_{\tilde{\mathscr{R}}'} \mathscr{E})^k \otimes (\det \mathscr{Q})^k \otimes \bigotimes_i \{\mathscr{Q}_i^{\beta_i - \alpha_i} \otimes (\det \mathscr{E}_{y_i})^{k - \beta_i}\} \otimes (\det \mathscr{E}_y)^l.$$

As in §2 one can check that $\hat{\theta}'$ is the (restriction of) the ample bundle on $\tilde{\mathscr{R}}'$ used to linearise the action of $SL(\tilde{n})$, and that this descends to an (ample) line bundle $\theta_{\mathscr{P}}$ on $\mathscr{P}$.

Definition 4.9c. The variety $\tilde{\mathscr{R}}_F$ is a locally universal family of (ordinary) parabolic bundles on $\tilde{X}$. We let $\hat{\theta}$ be the line bundle on $\tilde{\mathscr{R}}_F$ defined by the data $(d, k, \alpha_i, \beta_i)$ as in §2b:

$$\hat{\theta} = (\det R\pi_{\tilde{\mathscr{R}}_F} \mathscr{E})^k \otimes \bigotimes_i \{(\mathscr{Q}_i)^{\beta_i - \alpha_i} \otimes (\det \mathscr{E}_{y_i})^{k - \beta_i}\} \otimes (\det \mathscr{E}_y)^{\tilde{l}},$$

where $\tilde{l} = l + k$.

Recall that $\tilde{\psi}'$ denotes the projection $\tilde{\mathscr{R}}'^{ss} \to \mathscr{P}$.

Lemma 4.10. (1) *Let* $\eta_x \equiv (\det \mathscr{Q})(\det \mathscr{E}_x)^{-1}$ *for a point* $x \in \tilde{X}$. *Then*

$$\hat{\theta}' = \rho^* \hat{\theta} \otimes \eta_y^k.$$

(2) $\theta_{\mathscr{P}} = \phi^* \theta_{\mathscr{U}_X}$.

Proof. The first claim is easily checked. From the exact sequence (4.2) we get

$$\begin{aligned} \det R\pi_T \mathscr{F}_T &= (\det R\pi_T(\pi_* \mathscr{E}_T)) \otimes (\det \mathscr{Q}_T) \\ &= (\det R\pi_T \mathscr{E}_T) \otimes (\det \mathscr{Q}_T). \end{aligned}$$

From this and (2.4) we see that $(\phi \circ \tilde{\psi}')^* \theta$ is equal to the restriction to $\tilde{\mathscr{R}}'^{ss}$ of $\hat{\theta}'$. This proves (2) □

Some of the notation of the next proposition is defined in §4a and §4b.

Proposition 4.11. (1) *The map* ϕ: $\mathscr{P} \to \mathscr{U}_X$ *is finite and surjective.*

(2) *Each of the* $\mathscr{D}_j$ *maps onto* $\mathscr{W}$. *This is a finite map.*

(3) $\mathscr{P} \backslash (\mathscr{D}_1 \cup \mathscr{D}_2)$ *maps isomorphically to* $\mathscr{U}_X \backslash \mathscr{W}$.

(4) *Each of the* $\mathscr{V}_j$ *maps isomorphically onto* $\mathscr{W}'$.

(5) $\mathscr{D}_1 \cap \mathscr{D}_2$ *maps to* $\mathscr{W}'$.
(6) *Let* $\mathscr{D}_j^0 = \mathscr{D}_j \backslash (\mathscr{V}_j \cup (\mathscr{D}_1 \cap \mathscr{D}_2))$. *Then* $\mathscr{D}_j^0$ *maps isomorphically onto* $\mathscr{W} \backslash \mathscr{W}'$.
(7) $\mathscr{W}$ *is irreducible.*
(8) $\mathscr{P}$ *is the normalisation of* $\mathscr{U}_X$.
(9) *Each* $\mathscr{D}_j$ *is the normalisation of* $\mathscr{W}$.

Proof. (1) Finiteness follows from Lemma 4.10(2) and ampleness of $\theta_{\mathscr{U}_X}$ and $\theta_{\mathscr{P}}$. Surjectivity follows from Lemma 4.6(5) and Proposition 4.7(1).

(2) Consider the morphism $\phi_{\tilde{\mathscr{R}}^{ss}}$. Using Lemma 4.6 and Proposition 4.7(3) we see that $\hat{\mathscr{D}}_{j,F} \cap \tilde{\mathscr{R}}'^{ss}$ maps onto $\mathscr{W}$ set-theoretically; hence so does $\hat{\mathscr{D}}_j^f \cap \tilde{\mathscr{R}}'^{ss}$. Thus $\mathscr{D}_j$ maps set-theoretically into $\mathscr{W}$. Since both schemes are reduced in fact this is a morphism. Finiteness now follows from (1).

(3) By Lemma 4.12(1) below and Corollary 4.4(3) $\phi(\mathscr{P} \backslash (\mathscr{D}_1 \cup \mathscr{D}_2)) = \mathscr{U}_X \backslash \mathscr{W}$. On the other hand $\phi|_{\mathscr{P} \backslash (\mathscr{D}_1 \cup \mathscr{D}_2)}$ has a section. To see this, note first that $\tilde{\psi}^{-1}(\mathscr{U}_X \backslash \mathscr{W}) \subset \mathscr{R}_2$. Now, given a vector bundle on X the pull-back to $\tilde{X}$ has a canonical generalised parabolic structure which is semistable iff the bundle is semistable (Proposition 4.7(b)). This gives a map from $\tilde{\psi}^{-1}(\mathscr{U}_X \backslash \mathscr{W})$ to $\mathscr{P}$ which induces a section $(\mathscr{U}_X \backslash \mathscr{W}) \to \mathscr{P} \backslash (\mathscr{D}_1 \cup \mathscr{D}_2)$. Since $\mathscr{P}$ is irreducible, so is its open subset $\mathscr{P} \backslash (\mathscr{D}_1 \cap \mathscr{D}_2)$ and we conclude that $\phi|_{\mathscr{P} \backslash (\mathscr{D}_1 \cup \mathscr{D}_2)}$ is an isomorphism.

(4) One verifies as in part (2) that $\hat{\mathscr{V}}_j^f \cap \tilde{\mathscr{R}}'^{ss}$ maps onto $\mathscr{W}'$, inducing a morphism $\mathscr{V}_j \to \mathscr{W}'$. As in the proof of (3) we can see that this map has a section. (We use Lemma 4.13 below.)

(5) One checks as above that $(\hat{\mathscr{D}}_{1,F} \cap \hat{\mathscr{D}}_{2,F}) \cap \tilde{\mathscr{R}}'^{ss}$ maps to $\mathscr{W}'$. Now as in the proof of the irreducibility of $\mathscr{H}$ (Lemma C.2) it is possible to show that the $(\hat{\mathscr{D}}_{1,F} \cap \hat{\mathscr{D}}_{2,F})$ is dense in $(\hat{\mathscr{D}}_1 \cap \hat{\mathscr{D}}_2)$. This yields the result.

The proof of (6) is similar to that of statement (4), we use Lemma 4.12(2). The claim (7) follows from (2) and Proposition 4.4(1). The statements (8) and (9) are consequences of the normality of $\mathscr{P}$ and $\mathscr{D}_j$ and statements (1–3) and (6). □

Proof of Lemma 3.16 (continued). We have in the above proofs used the following facts:

(1) One can construct a family of torsion-free (but not locally free) semi-stable sheaves on X parametrised by $\hat{\mathscr{D}}_{j,F} \cap \tilde{\mathscr{R}}'^{ss}$. This family contains every such sheaf.

(2) One can construct a family of torsion-free semi-stable sheaves F on X (with $F \otimes \mathcal{O}_{x_0} \sim \mathscr{M}_{x_0} \oplus \mathscr{M}_{x_0}$) parametrised by $\hat{\mathscr{V}}_{j,F} \cap \tilde{\mathscr{R}}'^{ss}$. This family contains every such sheaf.

The parameter spaces are in both cases reduced and irreducible. The irreducibility of $\mathscr{R}_a$ $(a = 0, 1)$ now follows by a standard argument. □

Proof of Proposition 3.15 (4 and 5). We prove (4) first. Consider the map ϕ: $\mathscr{P} \to \mathscr{U}_X$. By Proposition 4.11(8) this is the normalisation map, and by Proposition 4.11(3) the non-normal locus of $\mathscr{U}_X$ is contained in $\mathscr{W}$. Since $\mathscr{W}$ is irreducible it suffices to show that the non-normal locus is nonempty (i.e. that the map ϕ is not an isomorphism) unless $\mathscr{W}$ is empty. Suppose then that $\mathscr{W}$ is nonempty. Then so too are the divisors $\mathscr{D}_j$ in $\mathscr{P}$ (by 4.11(2)). If $\mathscr{D}_1 \cap \mathscr{D}_2$ is nonempty, $\mathscr{W}'$ is nonempty and the proof of part (5) below shows that ϕ is not an isomorphism. If $\mathscr{D}_1 \cap \mathscr{D}_2 = \emptyset$ the inverse image of a point on $\mathscr{W}$ is not connected, and we are again through.

We turn to (5) next. Consider the map $\mathscr{D}_j \to \mathscr{W}$. This is the normalisation of $\mathscr{W}$ by Proposition 4.11(9), and an isomorphism outside $\mathscr{W}'$ by Proposition 4.11(6).

On the other hand by parts (4) and (5) of the same proposition, Corollary 4.4(2), and Zariski's Main Theorem it is clear from points on $\mathscr{W}'$ are *not* normal. □

Lemma 4.12. *Let (E, Q) be a GPS, and F the associated sheaf on X.*

(1) *If F is s-equivalent to a non-locally free sheaf, then (E, Q) is s-equivalent to a GPS (E_1, Q_1) with E_1 not locally free.*

(2) *If F is s-equivalent to a non-locally free sheaf F_1 with $F_1 \otimes \mathcal{O}_{x_0} \sim \mathscr{M}_{x_0} \oplus \mathscr{M}_{x_0}$, then (E, Q) is s-equivalent to a GPS (E_1, Q_1) with E_1 having a torsion subsheaf of degree 2.*

Proof. We consider (1) first. If F is not locally free, either E is not torsion-free and we are done, or E is torsion-free and one of the maps $E_{x_j} \to Q$ is not in isomorphism. In the latter case we are again done by Proposition C.7(4). Suppose now that F is locally free. Then we have the following situation:

**There is an exact sequence $0 \to L_1 \to F \to L_2 \to 0$, with $\mathbf{L}_q$ torsion-free, 2 par degree L_q = par degree F for $q = 1, 2$, and neither L_q locally free.

(One can check, by tensoring with $\mathcal{O}_{x_0}/\mathscr{M}_{x_0}$, that if one of the L_q is not locally free then neither is.) It is clear that in case (2) also condition (**) holds so that we can now combine the two proofs.

Write $L_1 = \pi_* L_1'$ where L_1' is a line bundle on $\tilde{X}$ with degree L_1' = degree $L_1 - 1$. There is a map of sheaves $L_1' \to E$ on $\tilde{X}$ which is generically injective and hence everywhere injective since L_1' is a line bundle. Let L_1'' be the quotient. Consider the commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & L_2 & \longrightarrow & \pi_* L_1'' & \longrightarrow & {}_{x_0}Q'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & = \uparrow & & \\ 0 & \longrightarrow & F & \longrightarrow & \pi_* E & \longrightarrow & {}_{x_0}Q & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & L_1 & \xrightarrow{=} & \pi_* L_1' & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

It is easy to check (in the notation of Appendix Bb) that

$$\mu_G[(L_1', 0)] = \mu_G[(L_1'', Q'')] = \mu_G[(E, Q)].$$

Note that L_1'' is a rank one sheaf and $\dim Q = 2$. We leave it to the reader to check that such a semi-stable GPS must be s-equivalent to one with a torsion subsheaf of degree 2.

Lemma 4.13. *Let T be a variety, $\mathscr{F}$ a sheaf on $X \times T$, flat over T, such that for $t \in T$ the sheaf $\mathscr{F}_t$ on X is torsion-free of rank 2. Then $\mathscr{F}$ is torsion-free on $X \times T$. Suppose further that $\exists 0 \leqq a \leqq 2$ such that $\forall t \in T$ we have $\mathscr{F}_t \otimes \mathcal{O}_{x_0} \sim a\mathcal{O}_{x_0} \oplus (2-a)\mathscr{M}_{x_0}$. By "flat" we shall mean "flat over T". Then*

(1) *$(\pi \times I_T)^* \mathscr{F}$ is flat.*

(2) *If $a = 0$ there* exists a vector bundle $\tilde{\mathscr{E}}$ *on $\tilde{X} \times T$ such that $\mathscr{F} = (\pi \times I_{TR})_* \tilde{\mathscr{E}}$.*

(3) *If* $a = 1$ *there exists a vector bundle* $\tilde{\mathscr{E}}$ *on* $\tilde{X} \times T$ *and a line-bundle quotient* $\tilde{\mathscr{Q}}$ *of* $\tilde{\mathscr{E}}_{x_1} \oplus \tilde{\mathscr{E}}_{x_2}$ *such that the following sequence is exact*:

$$0 \to \mathscr{F} \to (\pi \times I_T)_* \tilde{\mathscr{E}} \to {}_{x_0}\tilde{\mathscr{Q}} \to 0\,.$$

Proof. It is possible to prove, as in [S2, Huitième Partie, pp. 180–182] that $\mathscr{F}$ is a subsheaf of a locally free sheaf. This implies it is torsion-free.

Consider now the sequence $\mathscr{F} \underset{i}{\to} (\pi \times I_T)^*(\pi \times I_T)^*\mathscr{F} \to \mathscr{Q}_1 \to 0$ which defines $\mathscr{Q}_1$. Since i is generically an injection and $\mathscr{F}$ is torsion-free i is an injection. Specialising, we see that $\dim h^0((\mathscr{Q}_1)_t) = 4 - a$ and hence constant. Since T is reduced $\mathscr{Q}_1$ is flat. This show $(\pi \times I_T)^*\mathscr{F}$ is flat.

Next, consider the map $(\pi \times I_T)^*\mathscr{F} \to (\pi \times I_T)^*\mathscr{F} \otimes \mathscr{Q}_{\tilde{X}}(x_1 + x_2)$. By specialising as before one sees that the cokernel is flat, and hence also the image and kernel. Let $\tilde{\mathscr{E}}$ be the image. One can now show that $\tilde{\mathscr{E}}$ is a vector bundle, and we have an exact sequence of flat sheaves $0 \to \mathscr{T}or\,(\pi \times I_T)^*\mathscr{F} \to (\pi \times I_T)^*\mathscr{F} \to \tilde{\mathscr{E}} \to 0$. We now repeat the construction of Lemma 4.6(4) "over" T to prove (2) and (3). □

It is worth pointing out that the varieties $\mathscr{W}$ or $\mathscr{W}'$ could *à priori* be empty; also it could happen that $\mathscr{U}_X = \mathscr{W}$. In fact we always have $\emptyset \neq \mathscr{W} \neq \mathscr{U}_X$ (Remark 6.19).

4c. Some general results

We collect here some general statements needed elsewhere in the paper. The following fact about GIT quotients is standard.

Lemma 4.14. *Let* V *be a projective scheme on which a reductive group* G *acts*, $\tilde{\mathscr{L}}$ *an ample line bundle linearising the* G*-action, and* V^{ss} *the open subscheme of semistable points. Let* V' *be a* G*-invariant closed subscheme of* V^{ss}, $\bar{V}'$ *its schematic closure in* V. *Then*

(1) $\bar{V}'^{ss} = V'$, *and*

(2) $V'//G$ *is a closed subscheme of* $V^{ss}//G$.

Proof. (1) See the last paragraph of the proof of [M-F, Chapter 1, §5]. (2) Clearly we can take V to be affine. Then this is a consequence of "algebraic fact number 3" on p. 29 of the same reference.

Lemma 4.15. *Suppose* V, G, *and* V^{ss} *are as in the statement of the previous lemma. Let* W *be an open* G*-invariant (irreducible) normal subscheme of* V *containing* V^{ss}. *Then* $H^0(V^{ss}, \tilde{\mathscr{L}})^{\text{inv}} = H^0(W, \tilde{\mathscr{L}})^{\text{inv}}$ *where* $(\ )^{\text{inv}}$ *denotes the invariant subspace for an action of* G.

Proof. Assume first that V is irreducible and normal. In this case we will show that any invariant section on V^{ss} in fact extends to V (cf. [S1, Theorem 4.1 (iii)). This is clear if $D = V \setminus V^{ss}$ has codimension > 1. Suppose otherwise and for simplicity assume there is only one irreducible component D_1. Consider an invariant section s on V^{ss}, and assume it has a pole along D_1. By the definition of semistability there is an invariant section s_1 on V, vanishing on D_1. For some integers l, m the section $s_1^l s^m$ will extend to D_1 and be nonvanishing there. This will contradict non-semistability of points on D_1. This shows that in fact s extends to V; it is clearly G-invariant there. In case there are more than one component, we work by induction on the number of such components. Write $D = \bigcup D_q$. As above we can find an invariant section regular along D_1 and nonzero there. If this section is everywhere regular, we have the desired contradiction. If not the polar divisor of the new section has fewer components and induction is possible.

In general, we replace V by the irreducible component V_1 containing W, and endow V_1 with its reduced structure. Using [M-F, Chapter 1, §5] (Theorem 1.19 and the remarks in the last paragraph) we see that $V_1^{ss} = V^{ss}$. The argument of the previous paragraph, applied to the normalisation of V_1 (again using the above results) finishes the proof. □

Lemma 4.16. *Let V be a normal variety with a G-action, where G is a reductive algebraic group. Suppose a good quotient π: $V \to U$ exists. Let $\tilde{\mathscr{L}}$ be a G-line bundle on V, and suppose it descends as a line bundle $\mathscr{L}$ on U. Let $V'' \subset V' \subset V$ be open G-invariant subvarieties of V, such that V' maps onto U and $V'' = \pi^{-1}(U'')$ for some nonempty open subset U'' of U. Then any invariant section of $\tilde{\mathscr{L}}$ on V' extends to V.*

Proof. (cf. the proof of [Lu, Lumme 1.8].) Clearly we can assume U and V are affine, and $\mathscr{L}$ is trivial. A nowhere vanishing section of $\mathscr{L}$ pulls back to a G-invariant trivialisation of $\tilde{\mathscr{L}}$. Thus we can assume $\tilde{\mathscr{L}}$ is the trivial line bundle with the trivial action of G. Let $k[V]$ denote the ring of regular functions on a variety V. Suppose f is an invariant regular function on V' which does not extend to V. Then $f \in k[V'']^G = k[U'']$ ([N, Theorem 3.5(iii)]) and can therefore be written as $f = g/h$, with g, h in $k[U] = k[V]^G$. Since U is normal, there exists a codimension one subset $F \subset U$ such that $h|_F = 0$, and $g|_F \neq 0$. Let $y \in F$ such that $g(y) \neq 0$ and let $x \in V'$ such that $\pi(x) = y$. Then $0 \neq g(y) = g(x) = f(x)h(y) = 0$, which is a contradiction. □

The next result is from [Kn] – *we have retained the notation of that work, and there should be no confusion with notation used elsewhere in this paper.*

Lemma 4.17. *Let X be a normal, Cohen–Macaulay variety on which a reductive group G acts, such that a good quotient π: $X \to Y$ exists. Suppose that the action is generically free and that* $\dim G = \dim X - \dim Y$, *and further suppose that*

(1) *the subset where the action is not free has codimension* $\geqq 2$, *and*

(2) *for every prime divisor D in X, $\pi(D)$ has codimension* $\leqq 1$. *Here D need not be invariant.*

Then $\omega_Y = (\pi_* \omega_X)^G$ where ω_X, ω_Y are the respective dualising sheaves and the superscript $(\ )^G$ denotes the G-invariant direct image.

Proof. This follows from Satz 5 of [Kn], noting (again in the notation of that paper) that condition (1) implies that $D_\mu = 0$, and condition (2) that $D_\pi = 0$. The result is stated in [Kn] for the case when X is an affine variety, but this is not necessary, because under our hypothesis there is a canonical morphism $(\pi_* \omega_X)^G \to \omega_Y$. □

4d. Smooth morphisms

We shall use the following device (cf. [S2, Huitième Partie]) to analyse singularities of a variety V. We shall find varieties W and V' and smooth morphisms f: $W \to V$ and f': $W \to V'$, such that the singularities of V' are easy to analyse. Recall that a *smooth morphism* of schemes f: $V \to W$ is one which is flat and has smooth scheme-theoretic fibres. Equivalently, for every $p \in V$, the completion of the local ring $\widehat{\mathcal{O}}_p$ is isomorphic, as a $\widehat{\mathcal{O}}_{f(p)}$-algebra, to $\widehat{\mathcal{O}}_{f(p)}[[x_1, \ldots, x_n]]$ for some n. There is a lifting property which characterises smooth morphisms; see [Mu, 2.1]. We have the following well-known result, see for example [Ma, Theorem 32.2 (i)]:

Lemma 4.18. *Let $f\colon W \to V$ be a smooth morphism. Then W is reduced (respectively, normal, Cohen–Macaulay, Gorenstein) if and only if V is.*

We will also need

Proposition 4.19. *Let V_1 and V_2 be varieties over $\mathbf{C}$ and, for $j = 1, 2$, let $v_j \in V_j$. Let $\mathcal{O}_j$ be the respective local rings. Suppose that the completions $\widehat{\mathcal{O}}_j$ are isomorphic. Then if V_1 has rational singularities at v_1, then so does V_2 at v_2.*

Proof. Let K_V denote the Grauert–Riemenschneider sheaf [G–R] on a variety V, obtained as the direct image of the canonical sheaf of a desingularisation of V and let Ω_V denote the dualising sheaf of V. By [K] V has rational singularities if and only if

(1) V is Cohen–Macaulay, and

(2) the canonical map $i\colon K_V \to \Omega_V$ is an isomorphism.

Now, condition (2) is equivalent to:

(3) $i^{an}\colon K_V^{an} \to \Omega_V^{an}$ is an isomorphism,

where for a coherent $\mathcal{O}_V$ sheaf F, F^{an} denotes the analytic sheaf obtained on the analytic space V^{an} associated with V. Moreover for normal V, K_V^{an} has an intrinsic characterisation in terms of V^{an}; in fact, it can be defined as the direct image of the presheaf of square-integrable holomorphic forms of top degree of the complement of the singular set [G-R, §2.2, p. 271].

Since $\widehat{\mathcal{O}}_1 = \widehat{\mathcal{O}}_2$ and $\mathcal{O}_1$ is Cohen–Macaulay and normal it follows that so is $\mathcal{O}_2$ [Z-S]. By [GAGA, §2, Proposition 3] $\widehat{\mathcal{O}}_j = \widehat{\mathcal{O}_j^{an}}$. Since $\widehat{\mathcal{O}_1^{an}} = \widehat{\mathcal{O}_2^{an}}$ there are neighbourhoods of v_1 in V_1 and v_2 in V_2 which are analytically isomorphic [A, Corollary 1.6, p. 282]. Using the intrinsic characterisation of K_V^{an} it follows that i is an isomorphism. □

5. The decomposition theorem

We assume $k > 0$. Let $\mathcal{I}_Z(Z')$ denote the ideal sheaf on Z of a subvariety Z'. (We omit the subscript Z when it is superfluous.) When Z' is of codimension one (not necessarily a Cartier divisor) we set $\mathcal{O}_Z(-Z') = \mathcal{I}_Z(Z')$.

5a. A decomposition theorem on $\mathcal{P}$

We prove first a decomposition theorem (Theorem 4) for $H^0(\mathcal{P}, \theta_{\mathcal{P}})$. This will be used in the proof of the vanishing theorem in §6; the results proved here will be of use in the next subsection as well.

For $j = 1, 2$ let E_j be two-dimensional vector spaces. Let Gr denote the grassmannian of two-dimensional quotients $E_1 \oplus E_2 \xrightarrow[P]{} Q$. We define two divisors D_1 and D_2 in Gr. Let l_j denote the line bundle $(\det E_j)^{-1} \otimes \det Q$. This has a canonical section $\det P|_{E_j}$. Its zero-scheme is the divisor D_j; thus $l_j = \mathcal{O}(D_j)$. One checks easily that the divisors D_j are reduced, irreducible and normal. As a set

$D_j = \{P \mid (\ker P) \cap E_j \neq \{0\}\}$. The action of $GL(E_1) \times GL(E_2)$ on Gr lifts to the l_j's, and (for $m \in \mathbf{Z}$) $H^0(l_j^m)$ and $H^0(l_1^m|_{D_1})$ are $GL(E_1) \times GL(E_2)$ modulus. We have then (with ζ denoting the one-dimensional representation $(\det E_1)^{-1} \otimes \det E_2$):

Lemma 5.1. *For $m \in \mathbf{Z}$ we have natural isomorphisms of $GL(E_1) \times GL(E_2)$ modulus:*
(1) $H^0(l_1^m|_{D_1 \cap D_2}) = S^m E_1^* \otimes S^m E_2$.
(2) $H^0(l_1^m|_{D_1}) = \bigoplus_{q=0,\dots,m} \zeta^{m-q} \otimes S^q E_1^* \otimes S^q E_2$.
(3) $H^0(l_1^m) = \bigoplus_{p=0,\dots,m} (\bigoplus_{q=0,\dots,p} \zeta^{p-q} \otimes S^q E_1^* \otimes S^q E_2)$
(4) *All the corresponding first cohomology groups vanish for $m \geqq 0$.*

Proof. We use the notation $H^l(l_1^q|_{D_1}) \equiv \Delta_q^l$. We will use the following easy facts:
(a) The canonical bundle of Gr is $l_1^{-4}\zeta^2$, l_1 is ample. Note that this gives
(b) $H^1(l_1^q) = \{H^3(l_1^{(-q-4)})\}^* = 0$ for $q > -4$.
(c) Also, $H^0(l_1) = \mathbf{C} \oplus \zeta \oplus E_1^* \otimes E_2$.
Consider the exact sequence:

$$0 \to l_1^{q-1} \to l_1^q \to l_1^q|_{D_1} \to 0\,. \tag{5.1}$$

This, together with (b), shows:
(d) for $q > 0$ there is an exact sequence $0 \to H^0(l_1^{q-1}) \to H^0(l_1^q) \to H^0(l_1^q|_{D_1}) \to 0$.
Let Π denote the product of two projective spaces corresponding to E_1 and E_2, and $\mathbf{q}_1, \mathbf{q}_2$ denote the respective tautological quotient bundles. Then $D_1 \cap D_2 \sim \Pi$ and $Q|_{D_1 \cap D_2} \sim \mathbf{q}_1 \oplus \mathbf{q}_2$. The assertion (1) of the Lemma follows.
Consider now, for any integer q, the exact sequence:

$$0 \to l_1^q|_{D_1}(-(D_1 \cap D_2)) \to l_1^q|_{D_1} \to l_1^q|_{D_1 \cap D_2} \to 0\,.$$

We can rewrite this:

(on D_1) $$0 \to \zeta \otimes l_1^{q-1} \to l_1^q \to (\det E_1)^{-q} \mathbf{q}_1^q \mathbf{q}_2^q \to 0\,.$$

The long exact cohomology sequence now gives:

$(n \geqq 0)$ $$0 \to \zeta \otimes \Delta_{q-1}^0 \to \Delta_q^0 \overset{P}{\to} A^q E_1^* \otimes S^q E_2 \to \zeta \otimes \Delta_{q-1}^1 \Delta_q^1 \to 0\,.$$

The map P is trivially onto for $q \leqq 0$. From (c) and (d) it follows that P is onto for $q = 1$, and therefore it is nonzero for all $q > 1$. Since $S^q E_1^* \otimes S^q E_2$ is an irreducible $GL(E_1) \times GL(E_2)$ module the map is onto (and in fact has a canonical splitting, because by induction Δ_{q-1}^0 does not contain the representation $S^q E_1^* \otimes S^1 E_2$). This yields (2). We also see that for all q, we have $\Delta_{q-1}^1 \sim \Delta_q^1$, which yields $\Delta_m^1 = 0$. Together with (b) this proves (4).
Assertion (a), together with (1), now gives (3). □

Recall that ρ denotes the projection $\tilde{\mathscr{R}}'_F \to \tilde{\mathscr{R}}_F$. The decomposition of $H^0(\mathscr{P}, \theta_{\mathscr{P}})$ is obtained by considering the projection ρ. We set, for $x \in \tilde{X}$, $\eta_x \equiv (\det \mathscr{Q})(\det \mathscr{E}_x)^{-1}$. Thus $\eta_{x_j} = (\hat{\mathscr{D}}_{j,F})$. We also set $(\det \mathscr{E}_{x_1})^{-1} \otimes (\det \mathscr{E}_{x_2}) \equiv \xi$ and $\xi_j = (\det \mathscr{E}_y)^{-1} \otimes (\det \mathscr{E}_{x_j})$.

Lemma 5.2. *Let m be an integer. Then*
(1) *If $m \geqq 0$.*

$$\rho_*(\eta_{x_1}^m|_{D_{1,F}}) = \bigoplus_{q=0,\dots,m} \xi^{m-q} \otimes S^q \mathscr{E}_{x_1}^* \otimes S^q \mathscr{E}_{x_2}\,.$$

Otherwise $\rho_*(\eta^m_{x_1}|_{\hat{\mathscr{D}}_{1,F}}) = 0$.

(2) $R^1\rho_*(\eta^m_{x_1}|_{\hat{\mathscr{D}}_{1,F}}) = 0$.

(3) *If* $m \geqq 0$.

$$\rho_*\eta^m_{x_1} = \bigoplus_{p=0,\dots,m}\left(\bigoplus_{q=0,\dots,p} \xi^{p-q}\otimes S^q\mathscr{E}^*_{x_1}\otimes S^q\mathscr{E}_{x_2}\right).$$

Otherwise $\rho_*\eta^m_{x_1} = 0$.

(4) $R^1\rho_*\eta^m_{x_1} = 0$.

Proof. Immediate corollary of Lemma 5.1. □

Lemma 5.3. *The following maps are isomorphisms:*

(1) $H^0(\tilde{\mathscr{R}}'^{ss}, \hat{\theta}')^{inv} \to H^0(\tilde{\mathscr{R}}'^{ss}\cap\tilde{\mathscr{R}}'_F, \hat{\theta}')^{inv}$, *and*

(2) $H^0((\hat{\mathscr{D}}^f_1)^{ss}, \hat{\theta}')^{inv} \to H^0((\hat{\mathscr{D}}^f_1)^{ss}\cap\hat{\mathscr{D}}_{1,F}, \hat{\theta}')^{inv}$.

Proof. (1) We use Lemma 4.16 with the identification $V = \tilde{\mathscr{R}}'^{ss}$, $U = \mathscr{P}$, $\pi = \tilde{\psi}'$, $V' = \tilde{\mathscr{R}}'^{ss}\cap\tilde{\mathscr{R}}'_F$, and $U'' = \mathscr{P}\backslash(\mathscr{D}_1\cup\mathscr{D}_2)$. To show that $\mathscr{P}\backslash(\mathscr{D}_1\cup\mathscr{D}_2)$ is nonempty it suffices, by Proposition 4.11(3), to show that $\mathscr{U}_X\backslash\mathscr{W}$ is nonempty. This is true by Remark 6.19 below. To show that $\tilde{\mathscr{R}}'^{ss}\cap\tilde{\mathscr{R}}'_F$ maps onto $\mathscr{P}$ we use Corollary B.17.

(2) We use normality of $\hat{\mathscr{D}}_1$ (Proposition C.7(1)) and Remark C.5(e). □

Proposition 5.4. *There exists a canonical isomorphism*

$$H^0(\mathscr{P}, \theta_{\mathscr{P}}) \sim \bigoplus_{(p=0,\dots,k)}\ \bigoplus_{(q=0,\dots,p)} H^0(\tilde{\mathscr{R}}_F, \hat{\theta}\otimes\xi^k_1\otimes\xi^{p-q}\otimes S^q\mathscr{E}^*_{x_1}\otimes S^q\mathscr{E}_{x_2})^{inv}. \tag{5.2}$$

Proof. We have $H^0(\mathscr{P}, \theta_{\mathscr{P}}) = H^0(\tilde{\mathscr{R}}'^{ss}, \hat{\theta}')^{inv} = H^0(\tilde{\mathscr{R}}'^{ss}\cap\tilde{\mathscr{R}}'_F, \hat{\theta}')^{inv}$ where the second equality follows from Lemma 5.3(1). On the other hand, by Lemma 4.15 and C.3 $H^0(\tilde{\mathscr{R}}'^{ss}, \hat{\theta}')^{inv} = H^0(\mathscr{H}, \hat{\theta}')^{inv}$ so that we can write $H^0(\mathscr{P}, \theta_{\mathscr{P}}) = H^0(\tilde{\mathscr{R}}'_F, \hat{\theta}')^{inv}$.

Recall Eq. (4.4): on $\tilde{\mathscr{R}}'_F$ we have $\hat{\theta}' = \hat{\theta}\otimes\eta^k_y$. Taking direct images on $\tilde{\mathscr{R}}_F$ and using Lemma 5.2.(3) we get Eq. (5.2). □

Definition 5.5. For $\tilde{\mu} = (\alpha, \beta), 0 \leqq \alpha \leqq \beta \leqq k$, let $\mathscr{U}^{\tilde{\mu}}_{\tilde{X}}$ be the moduli space of semistable parabolic bundles on $\tilde{X}$ with parabolic structures at the $\{y_i\}_I$ and weights $\{(a_i, b_i)\}_I$, and in addition, parabolic structures at x_1 and x_2, both of weight $(a, b) = (\alpha/k, \beta/k)$. Let $\theta_{\tilde{\mu}} = \theta(d, k, \alpha_i, \beta_i, x_2, l_2)$ be the line bundle on $\mathscr{U}^{\tilde{\mu}}_{\tilde{X}}$ defined as in Remark 2.7, with $Q = \{2\}$ and $l_2 = -k + \beta + \alpha$.

Theorem 4. *We have a (canonical) isomorphism:*

$$H^0(\mathscr{P}, \theta_{\mathscr{P}}) \sim \bigoplus_{\tilde{\mu}} H^0(\mathscr{U}^{\tilde{\mu}}_{\tilde{X}}, \theta_{\tilde{\mu}}).$$

where $\tilde{\mu}$ *runs through the integers* (α, β), $0 \leqq \alpha \leqq \beta \leqq k$.

Proof. We first rewrite (5.2) as follows, substituting $p = \beta$, $q = \beta - \alpha$:

$$H^0(\mathscr{P}, \theta_{\mathscr{P}}) \sim \bigoplus_{(\beta=0,\dots,k)}\ \bigoplus_{(\alpha=0,\dots,\beta)} H^0(\tilde{\mathscr{R}}_F, \hat{\theta}\otimes\xi^k_1\otimes\xi^{\alpha}\otimes S^{\beta-\alpha}\mathscr{E}^*_{x_1}\otimes S^{\beta-\alpha}\mathscr{E}_{x_2})^{inv}. \tag{5.3}$$

Note now that the bundles $S^q\mathscr{E}_{x_j}$ are direct images of line bundles on the projective bundle $\mathbf{P}(\mathscr{E}_{x_j})$; thus the cohomology groups on the right hand side of (5.3) can be written as sections of suitable line bundles $\hat{\theta}_{\tilde{\mu}}$ on $\tilde{R}_F^+$ where

$$\tilde{\mathscr{R}}^+ \equiv \underset{i\in I}{\times_{\tilde{Q}}} \mathrm{Flag}_{(1,2)}(\mathscr{E}_{y_i}) \times \underset{j=1,2}{\times_{\tilde{Q}}} \mathrm{Flag}_{(1,2)}(\mathscr{E}_{x_j}) .$$

(Recall that for a 2-dimensional vector space $\mathrm{Flag}_{(1,2)}$ is just the projective space. Thus $\tilde{\mathscr{R}}_F^+$ is the fibre product of two P^1 bundles over $\tilde{\mathscr{R}}_F$.) In fact one checks easily that

$$\hat{\theta}_{\tilde{\mu}} = \hat{\theta} \otimes \mathscr{O}_{\mathbf{P}(\mathscr{E}_{x_1})}(\beta - \alpha) \otimes \mathscr{O}_{P(\mathscr{E}_{x2})}(\beta - \alpha)$$
$$\otimes (\det \mathscr{E}_{x_1})^{k-\beta} \otimes (\det \mathscr{E}_{x_2})^{+\alpha} \otimes (\det \mathscr{E}_y)^{-k} .$$

Each $\hat{\theta}_{\tilde{\mu}}$ is the restriction to $\tilde{\mathscr{R}}_F^+$ of a line bundle linearising the $SL(\tilde{n})$-action on the projective variety $\tilde{\mathscr{R}}^+$ where $\tilde{\mathscr{R}}^+$ is the analogue of $\mathscr{R}$, for parabolic bundles on $\tilde{X}$ with parabolic structures at $\{y_i\}_I \cup \{x_1, x_2\}$ and the moduli space $\mathscr{U}_{\tilde{X}}^{\tilde{\mu}}$ is the GIT quotient of the semistable points $(\tilde{\mathscr{R}}^+)^{ss} \subset \tilde{\mathscr{R}}_F^+$. There is a small point to be checked here, namely, that the integers n, m involved in the GIT construction of $\mathscr{P}$ and $\mathscr{U}_X$ can be made to work for these additional moduli space as well. But this is clear since the index $\tilde{\mu}$ runs over a fixed finite set depending only on k.

The variety $\tilde{\mathscr{R}}_F^+$ is normal (in fact smooth) so that Lemma 4.15 applies, and we can conclude

$$HG^0(\mathscr{P}, \theta_{\mathscr{P}} \sim)\bigoplus_{\tilde{\mu}} H^0(\tilde{\mathscr{R}}_F^+, \hat{\theta}_{\tilde{\mu}})^{\mathrm{inv}} \sim \bigoplus_{\tilde{\mu}} H^0(\mathscr{U}_{\tilde{X}}^{\tilde{\mu}}, \theta_{\tilde{\mu}}) . \tag{5.4}$$

This finishes the proof. □

We close this subsection with two results which will be used in the proof of Theorem 5.

Proposition 5.6. *Let $m \geq 0$ be a integer. Consider the inclusions of sheaves:*
(1) *on $\tilde{\mathscr{R}}'_F$, $\eta_{x_1}^m(-\hat{\mathscr{D}}_{1,F}) \to \eta_{x_1}^m$,*
(2) *on $\tilde{\mathscr{R}}'_F$, $\eta_{x_1}^m(-\hat{\mathscr{D}}_{1,F} - \hat{\mathscr{D}}_{2,F}) \to \eta_{x_1}^m$, and*
(3) *on $\hat{\mathscr{D}}_{1,F}$, $\eta_{x_1}^m \otimes \mathscr{I}_{\hat{\mathscr{D}}_{1,F}}(\hat{\mathscr{V}}_{1,E} \cup (\hat{\mathscr{D}}_{1,F} \cap \hat{\mathscr{D}}_{2,F})) \to \eta_{n_1}^m|_{\hat{\mathscr{D}}_{1,F}}$*
Each of the above sheaves has $R^1\rho_*(\cdot) = 0$. The induced inclusions of the zeroth direct images by ρ (which are therefore vector bundles) *have a $SL(\tilde{n})$-invariant splitting.*

Proof. The cases (1) and (2) are an immediate consequence of Lemma 5.2.

We turn next to (3). Considser

$$(\text{on } \hat{\mathscr{D}}_{1,F}) \qquad 0 \to \eta_{x_1}^m \otimes \mathscr{I}_{\hat{\mathscr{D}}_{1,F}}(\hat{\mathscr{V}}_{1,F}) \to \eta_{x_1}^m \to \eta_{x_1}^m|_{\hat{\mathscr{V}}_{1,F}} \to 0 .$$

We have, using the fact that on $\hat{\mathscr{V}}_{1,F}$, $\det \mathscr{Q} \sim \det \mathscr{E}_{x_2}$, the equality $\eta_{x_1}|_{\hat{\mathscr{V}}_{1,F}} = \xi|_{\hat{\mathscr{V}}_{1,F}}$. By Lemma 5.2 therefore

$$\rho_*(\eta_{x_1}^m \otimes \mathscr{I}_{\hat{\mathscr{D}}_{1,F}}(\hat{\mathscr{V}}_{1,F})) = \bigoplus_{n=1,\dots,m} \xi^{m-n} \otimes S^n\mathscr{E}_{x_1}^* \otimes S^n\mathscr{E}_{x_2} ,$$

$$R^1\rho_*((\eta_{x_1}^m \otimes \mathscr{I}_{\hat{\mathscr{D}}_{1,F}}(\hat{\mathscr{V}}_{1,F}))) = 0 .$$

Consider next

(on $\hat{\mathscr{D}}_{1,F}$)

$$0 \to \eta_{x_1}^m \otimes \mathscr{I}_{\hat{\mathscr{D}}_{1,F}}(\hat{\mathscr{V}}_{1,F} \cup (\hat{\mathscr{D}}_{1,F} \cap \hat{D}_{2,F})) \to \eta_{x_1}^m \otimes \mathscr{I}_{\hat{\mathscr{D}}_{1,F}}(\hat{\mathscr{V}}_{1,F}) \to \eta_{x_1}^m|_{\hat{\mathscr{D}}_{1,F} \cap \hat{\mathscr{D}}_{2,F}} \to 0\,,$$

where we have used the fact that $(\hat{\mathscr{V}}_{1,F} \cap (\hat{\mathscr{D}}_{1,F} \cap \hat{\mathscr{D}}_{2,F})) = \emptyset$. As in the foregoing proofs we see that

$$\bigoplus_{n=1,\ldots,m-1} \xi^{m-n} \otimes S^n \mathscr{E}_{x_1}^* \otimes S^n \mathscr{E}_{x_2} = \rho_*(\eta_{x_1}^m \otimes \mathscr{I}_{\hat{\mathscr{D}}_{1,F}}(\hat{\mathscr{V}}_{1,F} \cup (\hat{\mathscr{D}}_{1,F} \cap \hat{\mathscr{D}}_{2,F})))$$

$$\to \rho_*(\eta_{x_1}^m = \bigoplus_{n=0,\ldots,m} \xi^{m-n} \otimes S^n \mathscr{E}_{x_1}^* \otimes S^n \mathscr{E}_{x_2}\,,$$

splits, and $R^1\rho_*(\eta_{x_1}^m \otimes \mathscr{I}_{\hat{\mathscr{D}}^1\,F}(\hat{\mathscr{V}}_{1,F} \cup (\hat{\mathscr{D}}_{1,F} \cap \hat{\mathscr{D}}_{2,F}))) = 0$. □

Lemma 5.7. *The following maps are surjections:*
(1) $H^0(\theta_{\mathscr{P}}) \to H^0(\theta_{\mathscr{P}}|_{\mathscr{D}_1})$,
(2) $H^0(\theta_{\mathscr{P}}) \to H^0(\theta_{\mathscr{P}}|_{\mathscr{D}_1 \cup \mathscr{D}_2})$,
(3) $H^0(\theta_{\mathscr{P}}|_{\mathscr{D}_1}) \to H^0(\theta_{\mathscr{P}}|_{\mathscr{V}_1 \cup \mathscr{D}_1 \cap \mathscr{D}_2})_2$.

Proof. Let us deal with (1) in detail. Consider the diagram:

$$\begin{array}{ccc} H^0(\tilde{\mathscr{R}}'^{\mathrm{ss}}, \hat{\theta}')^{\mathrm{inv}} & \xrightarrow{\mathrm{a}} & H^0((\hat{\mathscr{D}}_1^f)^{\mathrm{ss}}, \hat{\theta}')^{\mathrm{inv}} \\ \mathrm{e}\uparrow & & \mathrm{f}\uparrow \\ H^0(\mathscr{H}, \hat{\theta}')^{\mathrm{inv}} & \longrightarrow & H^0(\hat{\mathscr{D}}_1^f, \hat{\theta}')^{\mathrm{inv}} \\ \mathrm{b}\downarrow & & \mathrm{d}\downarrow \\ H^0(\tilde{\mathscr{R}}'_F, \hat{\theta}')^{\mathrm{inv}} & \xrightarrow{\mathrm{c}} & H^0(\hat{\mathscr{D}}_{1,F}, \hat{\theta}')^{\mathrm{inv}} \end{array}$$

We need to prove that a is surjective. The maps e, f are equalities because of Lemma 4.15. The map b is an isomorphism by Lemmas 5.3 and 4.15. The map d is similarly an isomorphism, so that the result follows by Proposition 5.6 which states that c is surjective.

The statements (2) and (3) are proved along similar lines. There is a complication in case (2) because $D_1 \cup D_2$ is not normal. In this case we have an analogous diagram, with $\hat{\mathscr{D}}_1^f$ replaced by $\hat{\mathscr{D}}_1^f \cup \hat{\mathscr{D}}_2^f$ etc. (We will continue to use the same letters to denote the maps.) We can no longer assert that f and d are equalities. But given a section σ of $H^0((\hat{\mathscr{D}}_1^f)^{\mathrm{ss}} \cup (\hat{\mathscr{D}}_2^f)^{\mathrm{ss}}. \hat{\theta}')^{\mathrm{inv}}$, it certainly extends to sections σ_j on $\hat{\mathscr{D}}_j^f$ which are equal on $(\hat{\mathscr{D}}_1)^{\mathrm{ss}} \cap (\hat{\mathscr{D}}_2)^{\mathrm{ss}}$. By seminormality of $\hat{\mathscr{D}}_{1,F} \cup \hat{\mathscr{D}}_{2,F}$, a fact easily checked, this yields a section there. The rest of the proof goes through as before. □

5b. The decomposition theorem on $\mathscr{U}_X$

We start with a general result relating sections of a line bundle on a semi-normal variety to those of the pull-back on the normalisation.

Proposition 5.8. *Suppose given a seminormal variety V, with normalisation $\sigma: V \to V$. Let the non-normal locus be W, endowed with its reduced structure. Let W be the set-theoretic inverse image of W in $\tilde{V}$, endowed with its reduced structure. Let $\mathscr{N}$ be a line bundle on V, and let $\tilde{\mathscr{N}}$ be its pull-back to $\tilde{V}$. Suppose $H^0(\tilde{V}, \tilde{\mathscr{N}}) \to H^0(\tilde{W}, \tilde{\mathscr{N}})$ is surjective. Then*

(1) *There is an exact sequence*

$$0 \to H^0(\tilde{V}, \tilde{\mathcal{N}} \otimes \mathcal{I}(\tilde{W})) \to H^0(V, \mathcal{N}) \to H^0(W, \mathcal{N}) \to 0 .$$

(2) *If* $H^1(W, \mathcal{N}) \to H^1(\tilde{W}, \tilde{\mathcal{N}})$ *is injective so is* $H^1(V, \mathcal{N}) \to H^1(\tilde{V}, \tilde{\mathcal{N}})$.

Proof. Consider the commutative diagram of sheaves on V:

$$\begin{array}{ccccccc} 0 \to & \mathcal{I}(W) & \to & \mathcal{O}_V & \to \mathcal{O}_W \to 0 \\ & = \downarrow & & \downarrow & \downarrow \\ 0 \to & \sigma_* \mathcal{I}(\tilde{W}) & \to & \sigma_* \mathcal{O}_{\tilde{V}} & \to \mathcal{O}_{\tilde{W}} \to 0 \end{array}$$

where the equality is a consequence of Lemma 3.7. Note that the vertical arrows are inclusions. Tensoring by $\mathcal{N}$ and using the projection formula we get

$$\begin{array}{ccccccc} 0 \to & \mathcal{N} \otimes \mathcal{I}(W) & \to & \mathcal{N} & \to \mathcal{N}|_W \to 0 \\ & = \downarrow & & \downarrow & \downarrow \\ 0 \to & \sigma_*(\tilde{\mathcal{N}} \otimes \mathcal{I}(\tilde{W}) & \to & \sigma_* \tilde{\mathcal{N}} & \to \tilde{\mathcal{N}}|_{\tilde{W}} \to 0 \end{array}$$

Taking cohomologies gives

$$\begin{array}{ccccccc} 0 \to & H^0(\mathcal{N} \otimes \mathcal{I}(W)) & \to & H^0(\mathcal{N}) & \to & H^0(\mathcal{N}|_W & \xrightarrow{a} H^1(\mathcal{N} \otimes \mathcal{I}(W)) \\ & = \downarrow & & \downarrow & & c \downarrow & = \downarrow \\ 0 \to & H^0(\tilde{\mathcal{N}} \otimes \mathcal{I}(\tilde{W}) & \to & H^0(\tilde{\mathcal{N}}) & \to & H^0(\tilde{\mathcal{N}}|_{\tilde{W}}) & \xrightarrow{b} H^1(\tilde{\mathcal{N}} \otimes \mathcal{I}(\tilde{W})) \end{array}$$

where we have used the fact that σ is finite to identify $H^1(\sigma_*(\tilde{\mathcal{N}} \otimes \mathcal{I}(\tilde{W})))$ with $H^1(\tilde{\mathcal{N}} \otimes \mathcal{I}(\tilde{W}))$. By assumption b is zero. Since c is an injection we see that a is zero as well. This implies the first part of the Proposition.

Continuing with the two cohomology sequences and using the above results we also get

$$\begin{array}{ccccccc} 0 \to & H^1(\mathcal{N} \otimes \mathcal{I}(W)) & \to & H^1(\mathcal{N}) & \to & H^1(\mathcal{N}|_W) \to \\ & = \downarrow & & \downarrow & & \downarrow \\ 0 \to & H^1(\tilde{\mathcal{N}} \otimes \mathcal{I}(\tilde{W})) & \to & H^1(\tilde{\mathcal{N}}) & \to & H^1(\tilde{\mathcal{N}}|_{\tilde{W}}) \to \end{array}$$

This implies the second claim. □

The subvarieties $\mathcal{W}$ and $\mathcal{W}'$ of $\mathcal{U}_X$ are defined in §3a. The seminormality of $\mathcal{U}_X$ and $\mathcal{W}$ and in particular, its main consequence, as stated in Lemma 3.7, will be used repeatedly below. Recall also (Lemma 4.10(2)) that $\theta_{\mathcal{P}} = \phi^* \theta_{\mathcal{U}_X}$.

Proposition 5.9. *There exists a (noncanonical) isomorphism:*

$$H^0(\mathcal{U}_X, \theta_{\mathcal{U}_X}) \sim H^0(\theta_{\mathcal{P}}(-\mathcal{D}_1 - \mathcal{D}_2)) \oplus H^0(\mathcal{D}_1, \theta_{\mathcal{P}}(-\mathcal{D}_2)) .$$

Proof. We use Proposition 5.8(1). By seminormality of $\mathcal{U}_X$ and Proposition 3.15(4) we have an exact sequence

$$0 \to H^0(\theta_{\mathcal{P}}(-\mathcal{D}_1 - \mathcal{D}_2)) \to H^0(\theta_{\mathcal{U}_X}) \to H^0(\theta_{\mathcal{U}_X}|_{\mathcal{W}}) \to 0 . \tag{5.5a}$$

(Proposition 5.8(1) applies because of Lemma 5.7(2).) Again, by seminormality of $\mathcal{W}$ and Proposition 3.15(5) we get

$$0 \to H^0(\theta_{\mathcal{P}} \otimes \mathcal{I}_{\mathcal{D}_1}(\mathcal{V}_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2))) \to H^0(\mathcal{W}, \theta_{\mathcal{U}_X}) \to H^0(\mathcal{W}', \theta_{\mathcal{U}_X}) \to 0 . \tag{5.5b}$$

(Proposition 5.8(1) applies because of Lem,ma 5.7(3).

On the other hand, by Corollary 4.4(2) we have on $\mathscr{P}$ an exact sequence

$$0 \to H^0(\theta_{\mathscr{P}} \otimes \mathscr{I}_{\mathscr{D}_1}(\mathscr{V}_1 \cup (\mathscr{D}_1 \cap \mathscr{D}_2))) \to H^0(\mathscr{D}_1(-\mathscr{D}_2), \theta_{\mathscr{P}}) + H^0(\mathscr{V}, \theta_{\mathscr{P}}) \to 0, \tag{5.5c}$$

where the surjectivity on the right follows from Lemma 5.7(3), again using 4.4(2).

By Proposition 4.11(3) we have $H^0(\mathscr{V}, \theta_{\mathscr{P}}) = H^0(\mathscr{W}', \theta_{\mathscr{U}_X})$, and this, together with Eqs. (5.5) yields the desired result. □

We can now prove the

Theorem 5. *Let* $\eta_x \equiv (\det \mathscr{Q})(\det \mathscr{E}_x)^{-1}$ *for a point* $x \in X$, *and* $\xi_1 \equiv \eta_y \eta_{x_1}^{-1}$. *Then there exists a noncanonical isomorphism*

$$H^0(\mathscr{U}_X, \theta_{\mathscr{U}_X}) \sim \bigoplus_{(p=0,\ldots,k-1)} \bigoplus_{(q=0,\ldots,p)} H^0(\tilde{\mathscr{R}}_F, \hat{\theta} \otimes \xi_1^k \otimes \xi^{1+p-q} \otimes S^q \mathscr{E}_{x_1}^* \otimes S^q \mathscr{E}_{x_2})^{\mathrm{inv}}. \tag{5.6}$$

Proof. By Proposition 5.9 and Lemma 4.15

$$H^0(\mathscr{U}_X, \theta_{\mathscr{U}_X}) \sim H^0(\mathscr{H}, \hat{\theta}' \otimes \mathscr{O}(-\hat{\mathscr{D}}_1 - \hat{\mathscr{D}}_2))^{\mathrm{inv}} \oplus H^0(\hat{\mathscr{D}}_1^f, \hat{\theta}' \otimes \mathscr{O}(-(\hat{\mathscr{D}}_1^f \cap \hat{\mathscr{D}}_2^f)))^{\mathrm{inv}}.$$

We have applied Lemma 4.15 with the identification $\hat{W} = \mathscr{H}$; note that the Lemma applies since, for example, sections of $\hat{\theta}' \otimes \mathscr{O}(-\hat{\mathscr{D}}_1 - \hat{\mathscr{D}}_2)$ are also sections of $\hat{\theta}'$. By Lemma 5.5 the sections on the right are determined by their restrictions to $\tilde{\mathscr{R}}'_F$ and $\hat{\mathscr{D}}_{1,F}$ respectively. Now use Lemma 5.2. □

Proof of main Theorem (A). This follows from Theorem 5 exactly as Theorem 4 follows from Proposition 5.4. □

Remark 5.10. For $j = 1, 2$, let $\mathscr{F}r_j$ denote the frame-bundle of $\mathscr{E}_{x_j}$, thought of as a principal $GL(2)$-bundle. The bundle $\tilde{\mathscr{R}}'_F \xrightarrow{\rho} \tilde{\mathscr{R}}_F$ can be regarded as associated to the principal $GL(2) \times GL(2)$ bundle $\mathscr{F}r_1 \times_{\tilde{\mathscr{R}}_F} \mathscr{F}r_2$. The various (zeroth-) direct image sheaves that we encounter can be thought of as vector bundles associated to representations of $GL(2) \times GL(2)$. In particular, equation (5.6) can be rewritten in terms of vector bundles associated to $\mathscr{F}r_1 \times_{\tilde{\mathscr{R}}_F} \mathscr{F}r_2$:

$$H^0(\mathscr{U}_X, \theta_{\mathscr{U}_X}) \sim \bigoplus_{\mu} H^0(\tilde{\mathscr{R}}_F, \hat{\theta} \otimes \xi_1^k \otimes \xi \otimes (\mathscr{E}_1^k \mathscr{O})^* \otimes \mathscr{E}_{x_2}^{\mu})^{\mathrm{inv}},$$

where μ runs over (highest weights of) irreducible representations of $GL(2)$, $\mu = (\alpha, \beta)$, $0 \leqq \alpha \leqq \beta < k$, and $\mathscr{E}_{x_j}^{\mu}$ is the bundle associated to $\mathscr{F}r_j$ through the representation with highest weight μ. (The representation corresponding to (α, β) is $(\det \varrho)^{\otimes \alpha} \operatorname{Symm}^{\beta - \alpha}(\varrho)$ where ϱ is the defining representation of $GL(2)$.

6. The vanishing theorem

Consider now a family X_t of smooth curves degenerating, as in the Introduction, to $X_0 = X$. Clearly, to be able to assert that $h^0(\mathscr{U}_{X_t}, \theta_{\mathscr{U}_{X_t}}) = h^0(\mathscr{U}_X, \theta_{\mathscr{U}_X})$ we need a vanishing theorem.

6a. A vanishing theorem on $\mathcal{U}_{\tilde{X}}$

We will first prove a vanishing Theorem for $\mathcal{U}_{\tilde{X}}$. This will (with the replacement $\tilde{X} \to X_t$) prove the constancy of $h^0(\mathcal{U}_{X_t}, \theta_{\mathcal{U}_X'})$ for $t \neq 0$. It will also be needed in the next subsection.

We begin by computing the dualising sheaf of $\mathcal{U}_{\tilde{X}}$ using Lemma 4.17. The space $\tilde{\mathcal{R}}_F$ is defined in Notation 4.3c; $\mathcal{U}_{\tilde{X}}$ is the good quotient of the open subset of semistable points $\tilde{\mathcal{R}}^{ss}$. We will denote by $\tilde{\psi}$ the projection $\tilde{\mathcal{R}}^{ss} \to \mathcal{U}_{\tilde{X}}$.

Notation 6.1. Let Det denote the morphism $\tilde{\mathcal{R}}_F \to J^d_{\tilde{X}}$ given by the determinant of the universal quotient bundle. This induces a morphism $\mathcal{U}_{\tilde{X}} \to J^d_{\tilde{X}}$, which will also be denoted det. Let $\mathcal{L}$ denote a Poincaré line bundle on $\tilde{X} \times J^d_{\tilde{X}}$ and let θ_y denote the line-bundle on $J^d_{\tilde{X}}$ defined by

$$\theta_y \equiv (\det R\pi_J \mathcal{L}) \otimes (\det \mathcal{L}_y)^{(d+1-\tilde{g})}. \tag{6.1}$$

Proposition 6.2. *Assume $\tilde{g} \geqq 1$. Let $\Omega_{\tilde{X}}$ be the canonical bundle of $\tilde{X}$, and suppose $\Omega_{\tilde{X}} = \mathcal{O}(\sum_{q \in Q} z_q)$. Let $\Omega_{\tilde{\mathcal{R}}_F}$ denote the canonical bundle of $\tilde{\mathcal{R}}_F$. We have*

$$\Omega_{\tilde{\mathcal{R}}_F}^{-1} = (\det R\pi_{\tilde{\mathcal{R}}}\mathcal{E})^4 \otimes \bigotimes_i \{\mathcal{Q}_i^2 \otimes (\det \mathcal{E}_{y_i})^{-1}\}$$
$$\otimes \bigotimes_q \{(\det \mathcal{E}_{z_q})^{-1}\} \otimes (\det \mathcal{E}_y)^{2\tilde{n}+2\tilde{g}-2} \otimes (\mathrm{Det}^* \theta_y)^{-2}. \tag{6.2}$$

Proof. $\tilde{\mathcal{R}}_F$ is a fibre-product of $\mathbf{P}^1$-bundles over $\tilde{\mathbf{Q}}_F$, and we first need an expression for $\Omega_{\tilde{\mathbf{Q}}_F}$. (The spaces $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{Q}}_F$ are defined in §4.a.) On $\tilde{X} \times \tilde{\mathbf{Q}}_F$ we have an exact sequence of vector bundles $0 \to \mathcal{K} \to \mathcal{O}^{\tilde{n}} \to \mathcal{E} \to 0$, and the tangent space at a point $0 \to K \to \mathcal{O}^{\tilde{n}} \to E \to 0$ is $H^0(\tilde{X}, K^* \otimes E)$. From the properties of $\tilde{\mathbf{Q}}_F$ (the Notation 4.3b) it follows that

$$\Omega_{\tilde{\mathcal{R}}_F}^{-1} = \det R\pi_{\tilde{\mathcal{R}}_F}(\mathcal{E} \otimes \mathcal{E}^*) \otimes \bigotimes_i \{\mathcal{Q}_i^2 \otimes (\det \mathcal{E}_{y_i})^{-1}\}.$$

We now use a variant of the method of [D–N] to evaluate $\det R\pi_{\tilde{\mathcal{R}}_F}(\mathcal{E} \otimes \mathcal{E}^*)$. Consider on $\tilde{\mathcal{R}}_F$ the projective bundle $\mathbf{P}$ associated to the vector bundle $((\pi_{\tilde{\mathcal{R}}_F})_* \mathcal{E})^*$. We have on $\tilde{X} \times \mathbf{P}$ an injection of sheaves $0 \to \mathcal{O}_{\mathbf{P}}(-1)$. Let D' denote the (reduced) subscheme where this section vanishes. Its projection to $\mathbf{P}$, which we will denote by D, is an irreducible divisor. (One sees this by intersecting with the fibre over a point of $\tilde{\mathcal{R}}_F$ – see [D-N, Lemma 7.3 and Corollaire 7.4]). Outside D' we have an exact sequence of vector bundles $0 \to \mathcal{O}_{\mathbf{P}}(-1) \to \mathcal{E} \to \det \mathcal{E} \otimes \mathcal{O}_{\mathbf{P}}(+1) \to 0$, which yields, outside D,

(1) an isomorphism $\det R\pi_{\mathbf{P}}\mathcal{E} = \det R\pi_{\mathbf{P}}\mathcal{L} \otimes (\mathcal{L}^{-1} \det \mathcal{E})^{-(d+1-\tilde{g})} \otimes \mathcal{O}_{\mathbf{P}}(-d)$.

(2) an isomorphism $\det R\pi_{\mathbf{P}}(\mathcal{E} \otimes \mathcal{E}^*) = \det R\pi_{\mathbf{P}}\mathcal{E} \otimes \det R\pi_{\mathbf{P}}\mathcal{E}^* \otimes \mathcal{O}_{\mathbf{P}}(-2d)$.

(We have written $\mathrm{Det}^* \mathcal{L} = \mathcal{L}$).

By duality $\det R\pi_{\mathbf{P}}\mathcal{E}^* = \det R\pi_{\mathbf{P}}(\mathcal{E} \otimes \Omega_{\tilde{X}})$. From this and the exact sequence $0 \to \mathcal{E} \to \mathcal{E} \otimes \Omega_{\tilde{X}} \to \bigoplus_q \mathcal{E}_{z_q} \otimes (\Omega_{\tilde{X}})_{z_q} \to 0$ we get $\det R\pi_{\mathbf{P}}\mathcal{E}^* = \det R\pi_{\mathbf{P}}\mathcal{E} \otimes \bigotimes_q \{(\det \mathcal{E}_{z_q})^{-1}\}$.

Thus we have an isomorphism outside D:

$$\det R\pi_{\mathbf{P}}(\mathcal{E} \otimes \mathcal{E}^*) = (\det R\pi_{\mathbf{P}}\mathcal{E})^4 \otimes \bigotimes_q \{(\det \mathcal{E}_{z_q})^{-1}\}$$
$$\otimes (\det R\pi_{\mathbf{P}}\mathcal{L})^{-2} \otimes (\mathcal{L}^{-1} \otimes \det \mathcal{E})^{2(d+1-\tilde{g})}.$$

If we now use the fact that $\mathscr{L}^{-1} \otimes \det \mathscr{E}$ on $\tilde{X} \times \mathbf{P}$ is a line bundle pulled back from $\mathbf{P}$ we get (6.2) outside D. (That is, the line bundles on the two sides of (6.2), when pulled back to $\mathbf{P}$, are isomorphic outside D.)

We now claim that the map $\mathrm{Pic}(\tilde{\mathscr{R}}_F) \to \mathrm{Pic}(\mathbf{P} \backslash D)$ is injective; this will clearly finish the proof. To see the truth of the claim, one uses the fact (cf. [D-N, Lemma 7.3]) that each fibre of the morphism $\mathbf{P} \backslash D \to \tilde{\mathscr{R}}_F$ is the complement of an irreducible divisor on a projective space so that any nowhere-vanishing regular function on the fibre is a constant. (This shows that if the pull-back of a line bundle is trivial then the line bundle itself is trivial, for a nowhere-vanishing section of the pull-back descends to a nowhere-vanishing section of the original bundle.) □

Lemma 6.3. *Assume $\tilde{g} \geqq 2$. Then $(\tilde{\psi}_* \Omega_{\tilde{\mathscr{R}}^{ss}})^{\mathrm{inv}} = \Omega_{\mathscr{U}_{\tilde{X}}}$ where $\Omega_{\mathscr{U}_{\tilde{X}}}$ is the dualising sheaf of $\mathscr{U}_{\tilde{X}}$.*

Proof. Consider the action of $PSL(\tilde{n})$ on $\tilde{\mathscr{R}}^{ss}$. We will see (Lemma 6.14(1)) that if $\tilde{g} \geqq 2$ the complement of the set $\tilde{\mathscr{R}}^{s}$ of stable points has codimension >1. Since $\tilde{\mathscr{R}}^{s} \to \mathscr{U}^{s}_{\tilde{X}}$ is an étale locally trivial $PSL(\tilde{n})$-bundle we see that the conditions of Lemma 4.17 are satisfied; and this implies the above result. □

We can now prove

Theorem 6. *Assume $\tilde{g} \geqq 3$. Then $H^1(\mathscr{U}_{\tilde{X}}, \theta_{\mathscr{U}_{\tilde{X}}}) = 0$.*

Proof. We use the following device: we consider a new set of data $(d, \bar{k}, \bar{\alpha}_i, \bar{\beta}_i)$ such that $\bar{k} = k + 4$, and $\bar{\beta}_i - \bar{\alpha}_i = \beta_i - \alpha_i + 2$. Let $\bar{\omega}$ denote denote the new set of parabolic weights, $\hat{\theta}_{\bar{\omega}}$ the line bundle on $\tilde{\mathscr{R}}$ defined by the new data, $\mathscr{U}_{\tilde{X},\bar{\omega}}$ the corresponding moduli space, and $\theta_{\mathscr{U}_{\tilde{X},\bar{\omega}}}$ the "descendant" of $\hat{\theta}_{\bar{\omega}}$. Recall that the parabolic data do not quite suffice to define $\hat{\theta}_{\bar{\omega}}$, but a choice of degree 1 line bundle on $\tilde{X}$ is also needed (see Remark 2.7). We make this choice so that

$$\hat{\theta}_{\bar{\omega}} = (\det R\pi_{\tilde{\mathscr{R}}_F}\mathscr{E})^{k+4} \otimes \bigotimes_i \{(\mathscr{Q}_i)^{\bar{\beta}_i - \bar{\alpha}_i} \otimes (\det \mathscr{E}_{y_i})^{k - \beta_i}\}$$
$$\otimes \bigotimes_i \{(\det \mathscr{E}_{y_i})^{\bar{\beta}_i - \beta_i - 1}\} \otimes (\det \mathscr{E}_y)^{2\tilde{n} + 2\tilde{g} - 2 + \bar{l}} .$$

We shall assume that the integers n, m in the construction of $\mathscr{U}_{\tilde{X}}$ are chosen so that they work for $\mathscr{U}_{\tilde{X},\bar{\omega}}$ as well, so that $\mathscr{U}_{\tilde{X},\bar{\omega}}$ is the quotient of the semistable points $\tilde{\mathscr{R}}^{ss}_{\bar{\omega}}$ of $\tilde{\mathscr{R}}$ with respect to the new polarisation. Using (6.2) we see that on $\tilde{\mathscr{R}}_F$ we have

$$\hat{\theta} = \hat{\theta}_{\varpi} \otimes \Omega_{\tilde{\mathscr{R}}_F} \otimes (\mathrm{Det}^* \theta_y)^{-2} . \tag{6.3}$$

Since $\tilde{\mathscr{R}}^{ss}$ is a dense open subset of $\tilde{\mathscr{R}}_F$ this continues to hold in $\tilde{\mathscr{R}}^{ss}$.

We now write

$$H^1(\mathscr{U}_{\tilde{X}}, \theta_{\mathscr{U}_{\tilde{X}}})_1 \underset{1}{=} H^1(\tilde{\mathscr{R}}^{ss}, \hat{\theta})^{\mathrm{inv}}$$
$$\underset{2}{=} H^1(\tilde{\mathscr{R}}^{ss}_{\bar{\omega}}, \hat{\theta})^{\mathrm{inv}}$$
$$\underset{3}{=} H^1(\tilde{\mathscr{R}}^{ss}_{\bar{\omega}}, \hat{\theta}_{\bar{\omega}} \otimes \Omega_{\tilde{\mathscr{R}}_F} \otimes (\mathrm{Det}^* \theta_y)^{-2})^{\mathrm{inv}}$$
$$\underset{4}{=} H^1(\mathscr{U}_{\tilde{X},\bar{\omega}}, \theta_{\mathscr{U}_{\tilde{X},\bar{\omega}}} \otimes ((\tilde{\psi}_{\bar{\omega}})_* \Omega_{\tilde{\mathscr{R}}_F})^{\mathrm{inv}} \otimes (\mathrm{Det}^* \theta_y)^{-2})$$
$$\underset{5}{=} H^1(\mathscr{U}_{\tilde{X},\bar{\omega}}, \theta_{\mathscr{U}_{\tilde{X},\bar{\omega}}} \otimes \Omega_{\mathscr{U}_{\tilde{X},\bar{\omega}}} \otimes (\mathrm{Det}^* \theta_y)^{-2}) ,$$

where $\Omega_{\mathscr{U}_{\tilde{X},\tilde{\omega}}}$ is the dualising sheaf of $\mathscr{U}_{\tilde{X},\tilde{\omega}}$ and $\tilde{\psi}_{\tilde{\omega}}$ the quotient map $\tilde{\mathscr{R}}^{ss}_{\tilde{\omega}} \to \mathscr{U}_{\tilde{X},\tilde{\omega}}$. The second equality holds because of the Lemma 6.14(2) below, using a Hartogs-type extension theorem for first cohomology. The third uses Equation 6.2, and the fourth Lemma 6.3. The first and fifth equalities follow from the fact that for good quotients the space of invariants of the cohomology of an invariant line bundle is the same as the cohomology of the invariant direct image. This fact is easily proved (as pointed out by J.M. Drezet) by taking an invariant affine cover and applying Reynold's operator to Cěch cochains.

We will prove below (Lemma 6.4) that $\theta_{\mathscr{U}_{\tilde{X},\tilde{\omega}}} \otimes (\mathrm{Det}^* \theta_y)^{-2}$ is ample. Since $\mathscr{U}_{\tilde{X},\tilde{\omega}}$ has rational singularities a Kodaira-type vanishing theorem [S-S, Theorem 7.80(f)] now applies and we can conclude that $H^1(\mathscr{U}_{\tilde{X}}, \theta_{\mathscr{U}_{\tilde{X}}}) = 0$. □

Lemma 6.4. *$\theta_{\mathscr{U}_{\tilde{X}}} \otimes (\mathrm{Det}^* \theta_y)^{-2}$ is ample if $k > 4$.*

Proof. Consider the morphism Det: $\mathscr{U}_{\tilde{X}} \to J^d_{\tilde{X}}$, and let $\mathscr{U}^L_{\tilde{X}}$, and let $\mathscr{U}^L_{\tilde{X}}$ denote the fibre above L. One has a $2^{2\tilde{g}}$-fold covering $\mathscr{U}^L_{\tilde{X}} \times J^0_{\tilde{X}} \to \mathscr{U}_{\tilde{X}}$. We will show that $\theta_{\mathscr{U}_{\tilde{X}}} \otimes (\mathrm{Det}^* \theta_y)^{-2}$ is ample when pulled back to this finite cover.

One can show by a standard method (as for example, in [S2, p. 53]) that $\mathscr{U}^L_{\tilde{X}}$ is unirational. Hence its Pic_0 is trivial, and the pull-back bundle is therefore a product of line bundles coming from the two factors. It suffices to check that the restriction to each factor is ample. The restriction to the first factor is θ, and clearly ample.

Write the restriction to $J^0_{\tilde{X}}$ as $M_1 \otimes M_2$, where M_1 the pull-back of $\theta_{\mathscr{U}_{\tilde{X}}}$ and M_2 is the pull-back of $(\mathrm{Det}^* \theta_y)^{-2}$. Now θ_y is essentially the theta bundle on $J^d_{\tilde{X}}$, and ample. We will identify $J^d_{\tilde{X}}$ and $J^0_{\tilde{X}}$, and also work up to algebraic equivalence. One checks (using well-known properties of theta bundles on abelian varieties) that M_2 is algebraically equivalent to θ_y^{-8}. Also, M_2 is algebraically equivalent to θ_y^{2k}. (Consider a family $E \otimes \mathscr{L}$ of parabolic bundles, for E a fixed parabolic bundle, and then deform E to a bundle of the form $\mathcal{O}_X \oplus \mathcal{O}_X(\sum_{h=1,\ldots,d} x_h)$.) Clearly $M_1 \otimes M_2$ is ample if $k > 4$. □

6b. *Vanishing Theorem on $\mathscr{U}_X$*

We turn now to the vanishing theorem for $\mathscr{U}_X$.

Theorem 7. *Assume $g \geqq 4$. Then $H^1(\mathscr{U}_X, \theta) = 0$.*

Proof. This is a consequence of the next lemma and Theorem 8 below. □

Lemma 6.5. *$H^1(\mathscr{U}_X, \theta_{\mathscr{U}_X})$ injects into $H^1(\mathscr{P}, \theta_{\mathscr{P}})$.*

Proof. By Proposition 5.8(2) it suffices to prove that $H^1(\mathscr{W}, \theta_{\mathscr{U}_X})$ injects into $H^1(\mathscr{D}_1 \cup \mathscr{D}_2, \theta_{\mathscr{P}})$. For this it clearly suffices to show that $H^1(\mathscr{W}, \theta_{\mathscr{U}_X})$ injects into $H^1(\mathscr{D}_1, \theta_{\mathscr{P}})$. Again using the Proposition 5.8(2) we se that it is enough to show that $H^1(\mathscr{W}', \theta_{\mathscr{U}_X})$ injects into $H^1(\mathscr{V}_1 \cup (\mathscr{D}_1 \cap \mathscr{D}_2), \theta_{\mathscr{P}})$, and as above it is enough to show that $H^1(\mathscr{W}', \theta_{\mathscr{U}_X})$ injects into $H^1(\mathscr{V}_1, \theta_{\mathscr{P}})$. This is clear because the map ϕ: $\mathscr{V}_1 \to \mathscr{W}'$ is an isomorphism. □

6c. *A vanishing theorem on $\mathscr{P}$*

We are left with the task of proving

Theorem 8. *Assume $\tilde{g} \geqq 3$. Then $H^1(\mathscr{P}, \theta_{\mathscr{P}}) = 0$.*

This in turn is proved along the lines of Theorem 6. There are complications, however. First, it takes more work to prove a formula for the dualising sheaf. Second, one cannot prove the analogue of Lemma 6.4.

Proposition 6.6. *Let $\Omega_{\tilde{X}}$ be the canonical bundle of $\tilde{X}$, and suppose $\Omega_{\tilde{X}} = \mathcal{O}(\sum_{q \in Q} z_q)$. Let $\Omega_{\tilde{\mathscr{R}}'_F}$ denote the canonical bundle of $\tilde{\mathscr{R}}'_F$. We have*

$$\Omega_{\tilde{\mathscr{R}}'_F}^{-1} = (\det R\pi_{\tilde{\mathscr{R}}'_F}\mathscr{E})^4 \otimes \bigotimes_i \{\mathscr{Q}_i^2 \otimes (\det \mathscr{E}_{yi})^{-1}\} \otimes (\det \mathscr{Q})^4 (\det \mathscr{E}_{x_1})^{-2} (\det \mathscr{E}_{x_2})^{-2}$$

$$\otimes \bigotimes_q \{(\det \mathscr{E}_{z_q})^{-1}\} \otimes (\det \mathscr{E}_y)^{2\tilde{n}+2\tilde{g}-2} \otimes (\mathrm{Det}^* \theta_y)^{-2} . \tag{6.5}$$

Proof. $\tilde{\mathscr{R}}'_F$ is a grassmannian bundle over $\tilde{\mathscr{R}}_F$. Now use Proposition 6.2. □

We need an expression for the canonical bundle of $\mathscr{H}$. (By Proposition C.3 $\mathscr{H}$ is Gorenstein and has a canonical bundle). The idea is to find an extension of the right-hand side of (6.5) to $\mathscr{H}$ as a $PSL(\tilde{n})$ line-bundle, and then to argue that this gives the canonical bundle.

Remark 6.7. (a) We have, on $\tilde{X} \times \tilde{\mathscr{R}}'$ a surjection $\mathcal{O}^{\tilde{n}} \to \mathscr{E}_{\tilde{\mathscr{R}}'} \to 0$. The kernel $\mathscr{K}$ is flat over $\tilde{\mathscr{R}}'$, and since $\tilde{X}$ is smooth, it is locally free (this needs an argument using [N, Lemma 5.4]). On $\mathscr{H}$ we have the identity (for $x \in \tilde{X} \setminus \{x_1, x_2\}$):

$$\det \mathscr{K}_x \otimes \det \mathscr{E}_x = \det \mathcal{O}^{\tilde{n}} \sim \mathcal{O} .$$

(b) In the definition of θ' (4.9b) we can replace the term $(\det \mathscr{E}_y)^l$ by $\otimes_{q \in Q} (\det \mathscr{E}_{z_q})^{l_q} \otimes (\det \mathscr{E}_y)^{l+l_0}$ (cf., Remark 2.7) as long as for every $q \in Q$ we have $z_q \notin \{x_1, x_2\}$. Using (a), we can in fact replace any (or all) of the factors $(\det \mathscr{E}_{z_q})^{l_q}$ by $(\det \mathscr{K}_{z_q})^{-l_q}$, and, after this change, allow z_q to be one of the points $\{x_1, x_2\}$. It is clear that all these choices give algebraically equivalent ample line bundles on $\mathscr{P}$.

Proposition 6.8. *Let $\Omega_{\mathscr{H}}$ denote the canonical bundle of $\mathscr{H}$. We have*

$$\Omega_{\mathscr{H}}^{-1} = (\det R\pi_{\mathscr{H}}\mathscr{E})^4 \otimes \bigotimes_i \{\mathscr{Q}_i^2 \otimes (\det \mathscr{E}_{y_i})^{-1}\} \otimes (\det \mathscr{Q})^4 (\det \mathscr{K}_{x_1})^2 (\det \mathscr{K}_{x_2})^2$$

$$\otimes \bigotimes_q \{(\det \mathscr{E}_{z_q})^{-1}\} \otimes (\det \mathscr{E}_y)^{2\tilde{n}+2\tilde{g}-2} \otimes (\mathrm{Det}^* \theta_y)^{-2} , \tag{6.6}$$

where the vector bundle $\mathscr{K}$ is defined in Remark 6.7(a) above.

Proof. Let Ω^{-1} denote the RHS of (6.6). By Proposition 6.6 the isomorphism $\Omega = \Omega_{\mathscr{H}}$ holds outside the $\hat{\mathscr{D}}^t_j$. We will check that it extends to each $\hat{\mathscr{D}}^t_j$.

For definiteness take $j = 1$ and for simplicity of notation suppose there are no ordinary parabolic points. The proof will use the methods of Appendix C (to which we refer the reader for unexplained notation) to determine $\Omega_{\mathscr{H}}$ in a neighbourhood of a suitable point of $\hat{\mathscr{D}}^t_1$. Since $\hat{\mathscr{D}}^t_1$ is irreducible, it will be enough to show that the isomorphism (6.6) extends to one such neighbourhood.

Consider then a point $(\mathcal{O}^{\tilde{n}} \to E \to 0, Q)$ in $\mathscr{H}$ where

(1) E has torsion at x_2 (i.e. the point lies on $\hat{\mathscr{D}}^t_1$),

(2) E is locally free at x_1, and

(3) the maps $E_{x_j} \to Q$ are onto for both $j = 1, 2$.

Define the vector bundle $\tilde{E}$ to be the kernel of the map sequence $E \to {}_{x_1}Q \to 0$ ($\tilde{E}$ is a vector bundle because of condition (3) in the definition of $\mathscr{H}$). The conditions (2) and (3) will continue to hold in a neighbourhood $\mathbf{U}_1$. On $\tilde{X}\times\mathbf{U}_1$ one can define a locally free sheaf $\tilde{\mathscr{E}}$ by the exact sequence $0\to\tilde{\mathscr{E}}\to\mathscr{E}\to{}_{x_1}\mathscr{Q}\to 0$ where (where $x_1\mathscr{Q}$ is the sheaf on $\tilde{X}\times\mathscr{R}'$ got by pulling back $\mathscr{Q}$ from $\mathscr{R}'$ and then restricting to $\{x_1\}\times\tilde{\mathscr{R}}'$). Suppose the vector bundle $\tilde{E}$ is stable (such points certainly exist). Then this will continue to hold in a open set U, with $(E,Q)\in\mathbf{U}\subset\mathbf{U}_1\subset\mathscr{H}$. Note that on U we have an isomorphism of vector bundles $\mathscr{E}_{x_1}\sim\mathscr{Q}$.

We construct another space E as follows. For simplicity assume that the degree d is odd so that a Poincaré bundle exists for stable bundles of degree $d-2$. (An argument with étale open sets is needed otherwise.) Denote this bundle by $\tilde{\mathscr{E}}'$: this is a vector bundle on $\tilde{X}\times\mathscr{U}_{\tilde{X}}(d-2)$. On $\tilde{X}\times\mathscr{U}(\tilde{X},d-2)$ consider the bundle of extensions **E** whose fibre over $\tilde{E}'$ is $\mathrm{Ext}^1({}_{x_2}(\tilde{E}'_{x_1}),\tilde{E}')$. On $\tilde{X}\times\mathbf{E}$ there is an universal extension $0\to\tilde{\mathscr{E}}'\to\mathscr{E}'\to{}_{x_2}(\tilde{\mathscr{E}}'_{x_1})\to 0$.

There is a morphism $H\colon\mathbf{U}\to\mathbf{E}$ such that $H^*\tilde{\mathscr{E}}'=\tilde{\mathscr{E}}\otimes\mathscr{N}$, $H^*\mathscr{E}'=\mathscr{E}\otimes\mathscr{N}$. for some line bundle $\mathscr{N}$ on U. One checks easily that

(1) H is a submersion,
(2) the fibres of H are $PSL(\tilde{n})$ orbits, and
(3) $PSL(\tilde{n})$ acts freely on U.

From this it follows that $\Omega_{\mathrm{U}}=H^*\Omega_{\mathrm{E}}$.

We now proceed to check that $H^*\Omega_{\mathrm{E}}=\Omega$. One easily computes:

$$H^*\Omega_{\mathrm{E}}^{-1}\otimes\Omega=(\det\tilde{\mathscr{E}}_y)^{-4}\otimes(\det\mathscr{K}_{x_1})^2\otimes(\det\mathscr{K}_{x_2})^2$$
$$=(\det\tilde{\mathscr{E}}_{x_2})^2\otimes(\det\mathscr{K}_{x_2})^2.$$

We will now show that $\det\mathscr{K}_{x_2}=(\det\tilde{\mathscr{E}}_{x_2})^{-1}$. Consider the commutative diagram of sheaves on $\tilde{X}\times\mathbf{U}$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & {}_{x_2}\mathscr{Q} & = & {}_{x_2}\mathscr{Q} & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & \mathscr{K} & \to & \mathscr{O}^{\tilde{n}} & \to & \mathscr{E} & \to & 0 \quad \text{(a)}\\
 & & =\uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathscr{K} & \to & \mathscr{K}' & \to & \tilde{\mathscr{E}} & \to & 0 \quad \text{(b)}\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the (b) is the pull-back of (a) by the inclusion $\tilde{\mathscr{E}}\to\mathscr{E}$ – this defines $\mathscr{K}'$. One sees easily that $\mathscr{K}'$ is a vector bundle. We have therefore the equality of line bundles on $\tilde{X}\times\mathbf{U}$: $\det\mathscr{K}\otimes\det\tilde{\mathscr{E}}=\det\mathscr{K}'$, which yields the equality of line bundles on U: $(\det\mathscr{K})_{x_2}\otimes(\det\tilde{\mathscr{E}})_{x_2}=(\det\mathscr{K}')_{x_2}$. On the other hand we get from the exact sequence $0\to\mathscr{K}'\to\mathscr{O}^{\tilde{n}}\to{}_{x_2}\mathscr{Q}\to 0$ the exact sequence of bundles on U: $0\to\mathscr{Q}\otimes(\Omega_{\tilde{X}})_{x_2}\to\mathscr{K}'_{x_2}\to\mathscr{O}^{\tilde{n}}\to\mathscr{Q}\to 0$. This shows that $(\det\mathscr{K}')_{x_2}$ is trivial. □

We next prove the analogue of Lemma 6.3:

Lemma 6.9. *Assume $\tilde{g}\geqq 2$. Then $(\tilde{\psi}'_*\Omega_{\mathscr{H}})^{\mathrm{inv}}=\Omega_{\mathscr{P}}$ where $\Omega_{\mathscr{P}}$ is the dualising sheaf of $\mathscr{P}$.*

Proof. We check that the conditions of Lemma 4.17 are satisfied. By Corollary 6.18 and Remark 6.19 there exists stable bundles on X. By Proposition 4.7(2), there exist stable generalised parabolic bundles on $\tilde{X}$. Thus there exist stable points in $\tilde{\mathscr{R}}'$, and the action of $PSL(\tilde{n})$ is therefore generically free. We now check conditions (1) and (2) of 4.17.

(1) By Lemma 6.15(1) below one sees that in $\tilde{\mathscr{R}}'^{ss}\backslash\hat{\mathscr{D}}_1 \cup \hat{\mathscr{D}}_2$ the nonstable locus has codimension $\geqq 2$. We next show that each of the $(\hat{\mathscr{D}}_j^f)^{ss}$ or $(\hat{\mathscr{D}}_j^t)^{ss}$ contains a GPS with no nontrivial automorphism. Take $j = 1$ for definiteness. Let E' be a stable (parabolic) bundle on $\tilde{X}$ of degree $d - 2$, let $E = E' \otimes {}_{x_2}\mathbf{C}$ and define the GPS structure on E as follows. We take $Q = \mathbf{C}^2$, the map $E_{x_2} \to Q$ to be the obvious projection, and the map $E_{x_1} \to Q$ any isomorphism. This yields, after an identification $H(E) \sim \mathbf{C}^{\tilde{n}}$, a point on $\hat{\mathscr{D}}_1^t$ as required. Next consider $E = E'(x_2)$, the GPS structure being given by taking $Q = E'_{x_2} \otimes (\Omega_{\tilde{X}})_{x_2}^{-1}$, the map $E_{x_1} \to Q$ being zero, and the map $E_{x_2} \to Q$ the residue. This yields a point on $\hat{\mathscr{D}}_1^f$ with no nontrivial automorphisms.

(2) If a prime divisor is not contained in the nonstable locus its projection will have codimension one. If it is contained in the nonstable locus, by (1) it will have to be one of the $(\hat{\mathscr{D}}_j^f)^{ss}$ or $(\hat{\mathscr{D}}_j^t)^{ss}$. We have already seen that the respective images of these in $\mathscr{P}$ are the $\mathscr{D}_j$. □

Consider the local universal family $\tilde{\mathscr{R}}'$ of Appendix B. The open subscheme $\mathscr{H}$ of $\tilde{\mathscr{R}}'$ is defined in §4a (Notation 4.3a).

Lemma 6.10. *There is a morphism* Det: $\mathscr{H} \to J_{\tilde{X}}^d$ *which extends the determinant morphism on the open set* $\tilde{\mathscr{R}}'_F$.

Proof. The determinant of $\mathscr{E}_{\tilde{\mathscr{R}}'}$ can be defined as the inverse of $\det \mathscr{K}$, where the vector bundle $\mathscr{K}$ is defined in Remark 6.7(a). This gives a morphism from $\tilde{\mathscr{R}}'$ to $J_{\tilde{X}}^d$. □

Restricted to $\tilde{\mathscr{R}}'^{ss}$ the map Det clearly factors through the quotient by the $SL(\tilde{n})$ action and yields a morphism $\mathscr{P} \to J_{\tilde{X}}^d$, which we again denote by Det.

Lemma 6.11. *The determinant morphism on the open set of stable torsion-free GPSs extends to a flat morphism* Det: $\mathscr{P} \to J_{\tilde{X}}^d$.

Proof. Note that $J_{\tilde{X}}^0$ does not act on $\mathscr{P}$. However, J_X does. Given a GPS (E, Q) and a line bundle M on X, the action is defined by

$$(E, Q) \mapsto M * (E, Q) \equiv (E \otimes \pi^* M, Q \otimes M_{x_0}) .$$

We have $\operatorname{Det} M * (E, Q) = \operatorname{Det}(E, Q) \otimes (\pi^* M)^2$. Now the pull-back map $J_X^0 \to J_{\tilde{X}}^0$ and the squaring map $J_{\tilde{X}}^0 \to J_{\tilde{X}}^0$ are surjective and $J_{\tilde{X}}^0$ acts transitively on $J_{\tilde{X}}^d$. By generic flatness it follows that the map Det: $\mathscr{P} \to J_{\tilde{X}}^d$ is flat. □

Let $\mathscr{H}^L$ denote the (reduced) fibre over $L \in J_{\tilde{X}}^d$. Similarly let $\mathscr{P}^L$ be the (reduced) fibre of Det above L. Clearly $\mathscr{P}^L$ is the GIT quotient of $\mathscr{H}^L$. All the properties of $\mathscr{H}$ and $\mathscr{P}$ continue to be valid for $\mathscr{H}^L$ and $\mathscr{P}^L$; the proofs require only minor modifications. We have

Proposition 6.12. *The dualising sheaf of* $\mathscr{P}^L$ *is the restriction of* $\Omega_{\mathscr{P}}$ *to* $\mathscr{P}^L$.

Proof. We first note that $\mathscr{P}^L$ is the scheme-theoretic fibre above L. For, by Bertini, the scheme-theoretic fibre is reduced for generic L, and then we can use the argument of the proof of the previous lemma to extend this to all L.

Next we use the following fact: Suppose $f: V \to U$ is a flat map of varieties, with U smooth, and V Gorenstein. Let V_p be the scheme-theoretic fibre above $p \in U$. Then the dualising sheaf of V_p is the restriction of the dualising sheaf of V. This in turn is proved by repeated use of Bertini (on U) and the adjunction formula. □

Proposition 6.13. (1) *We have a (canonical) isomorphism:*

$$H^0(\mathscr{P}^L, \theta_{\mathscr{P}}) \sim \bigoplus_{\tilde{\mu}} H^0((\mathscr{U}_{\tilde{X}}^{\tilde{\mu}})^L, \theta_{\tilde{\mu}}).$$

where $\tilde{\mu}$ runs through the integers (α, β), $0 \leqq \alpha \leqq \beta \leqq k$.

(2) *Assume $\tilde{g} \geqq 3$. Then $H^1(\mathscr{P}^L, \theta_{\mathscr{P}}) = 0$.*

(We have used the obvious notation $(\mathscr{U}_{\tilde{X}}^{\tilde{\mu}})^L$ for the fibre above L of the determinant morphism from $\mathscr{U}_{\tilde{X}}^{\tilde{\mu}}$ to $J_{\tilde{X}}^d$. The morphism itself will be denoted $\mathrm{Det}_{\tilde{\mu}}$ below.)

Proof. The first claim is proved exactly as Theorem 4. The proof of the second statement is along the lines of that of Theorem 6. Restricted to $\mathscr{H}^L$ we have the following equality (the analogue of (6.3)):

$$\hat{\theta}' = \hat{\theta}'_{\tilde{\omega}} \otimes \Omega_{\mathscr{H}},$$

for a suitable $\hat{\theta}'_{\tilde{\omega}}$, where we have to use Remark 6.7(b) to define this latter line bundle. The rest of the proof proceeds as before except that an analogue of Lemma 6.4 is not needed. Note that $\mathscr{H}$ has rational singularities, and is in particular Cohen–Macaulay, so that Hartogs-type extension theorems for cohomology are applicable. □

Proof of Theorem 8. Consider the map Det: $\mathscr{P} \to J_{\tilde{X}}^d$. Proposition 6.13 shows that $R^1(\mathrm{Det})_*(\theta_{\mathscr{P}}) = 0$. On the other hand the decomposition theorem for $\mathscr{P}^L$ shows that $R^0(\mathrm{Det})_*(\theta_{\mathscr{P}}) = \bigoplus_{\tilde{\mu}} R^0(\mathrm{Det}_{\tilde{\mu}})_*(\theta_{\tilde{\mu}})$. By Theorem 6 we have $H^1(R^0(\mathrm{Det})_*(\theta_{\mathscr{P}})) = 0$. □

6d. Codimension computations

We have to do a number of codimension computations. We do the first in some detail.

Lemma 6.14. (1) *The complement in $\tilde{\mathscr{R}}^{ss}$ of the set $\tilde{\mathscr{R}}^s$ of stable points has codimension $\geqq \tilde{g}$ if $|I| > 0$, and codimension $\geqq \tilde{g} - 1$ if $|I| = 0$*

(2) *The complement in $\tilde{\mathscr{R}}_F$ of the set $\tilde{\mathscr{R}}^{ss}$ of semistable points has codimension $\geqq \tilde{g}$.*

Proof. The dimension of $\tilde{\mathscr{R}}_F$ is easily computed to be $4\tilde{g} - 3 + |I| + \dim PLS(\tilde{n})$. (At a point $0 \to K \to \mathcal{O}^{\tilde{n}} \to E \to 0$ of $\tilde{\mathbf{Q}}_F$ the tangent space is $H^0(\tilde{X}, K^* \otimes E)$. Using the exact sequence

$$0 \to H^0(E^* \otimes E) \to \mathbf{C}^{\tilde{n}} \otimes \mathbf{C}^{\tilde{n}} \to H^0(K^* \otimes E) \to H^1(E^* \otimes E) \to 0$$

and Riemann–Roch we get $\dim H^0(K^* \otimes E) = 4\tilde{g} - 3 + (\tilde{n}^2 - 1)$.)

We first prove (1). Consider a semistable, unstable bundle E. It is an extension $0 \to L_1 \to E \to L_2 \to 0$, with par degree $L_1 = 1/2$ (par degree E). (Equivalently, $2\,\mathrm{degree}\, L_1 - d = \sum_{R^c}(b_i - a_i) - \sum_R (b_i - a_i)$.) We will now describe a (countable) number of quasi-projective varieties parametrising such bundles. (For the present we do not require a variety to be irreducible.)

For $q = 1, 2$, let d_q be integers such that $d_1 + d_2 = d$, and let $I = R_1 \cup R_2$ be a decomposition of I such that $2\,d_1 - d = \sum_{R_2}(b_i - a_i) - \sum_{R_1}(b_i - a_i)$. Let $h > 0$ be an integer, and let $v = (d_1, R_1, h)$. Choose Poincaré bundles $\mathscr{L}_q$ on $\tilde{X} \times J^{d_q}_{\tilde{X}}$. Let $\mathscr{J} \equiv J^{d_1}_{\tilde{X}} \times J^{d_2}_{\tilde{X}}$ and let $\mathscr{L}$ denote the line bundle $\mathscr{L}_2^* \mathscr{L}_1$ on $\tilde{X} \times \mathscr{J}$. Let π denote the projection $\tilde{X} \times \mathscr{J} \to \mathscr{J}$.

We define a variety $V(v) \equiv V(d_1, R_1, h)$ as follows.

(a) We first define varieties $V_2(v)$:

(1) If $h = 0$, set $V_2(v) = \mathscr{J}$. Define the bundle $\mathscr{E}_v$ on $\tilde{X} \times V_2$ to be $\mathscr{L}_1 \oplus \mathscr{L}_2$.

(2) Write $\operatorname{Supp} R^+\pi_* \mathscr{L} = \bigcup_{h>0} V_1(v)$ with $V_1(v)$ denoting the locally closed subscheme of $\mathscr{J}$ where $R^1\pi_*\mathscr{L}$ is locally free of rank h. Let $V_2(v)$ be the projective bundle $\mathbf{P}(\{R^1\pi_*\mathscr{L}\}^*)$ on $V_1(v)_{\mathrm{red}}$. On $\tilde{X} \times V_2(v)$ there is an universal extension $0 \to \mathscr{L}_1(-1) \to \mathscr{E}_v \to \mathscr{L}_2 \to 0$.

(b) Let $V_3(v)$ be the fibre product

$$\underset{i \in R_2}{\times} V_2(v\mathbf{P}((\mathscr{E}_v)_{y_i}).$$

The sub-bundle $\mathscr{L}_1(-1) \subsetneq \mathscr{E}$ yields, for each $i \in I$, a divisor in V_3.

(c) Let $V(v) = V(d_1, R_1, h)$ be the complement of the union of these divisors for $i \in R_2$.

Each $V(v)$ parametrises a family of parabolic bundles E, which occur as extensions $0 \to L_1 \to L_2 \to 0$ (the extension being split if $h = 0$), with parabolic structures at the $\{y_i\}_{R_1}$ given by the sub-bundle L_1. The dimensions of the $V(v)$ are easily bounded. These are:

(1) $\dim V(v) = 2\tilde{g} + |R_2|$ if $h = 0$,

(2) $\dim V_v \leqq 2\tilde{g} + h - 1 + |R_2|$ otherwise.

Let $V(v)^{ss}$ be the open set of semistable parabolic bundles, and let $F(v)$ be the frame-bundle of the direct image of $\mathscr{E}$ on $V(v)^{ss}$.

There is a map from each $F(v)$ to $\tilde{\mathscr{R}}^{ss} \backslash \tilde{\mathscr{R}}^s$, and the union of the images covers the latter set. We now estimate the dimension of the (closure of the) image of $F(v)$. We have ([H, Exercise 3.22]) $\dim \operatorname{Im} F(v) = \dim F(v) - e$ where e is the infimum of the dimensions of the irreducible components of the fibres. Since the E are generated by sections, any automorphism of E acts nontrivially on the frames of $H^0(E)$, and we compute

(1) $e \geqq 2 + \dim h_0$ if $h = 0$ and

(2) $e \geqq 1 + \dim h_0$ if $h > 0$,

where $h_0 = H^0(L_2^* L_1)$. In any case the codimension of the image is bounded below by $4\tilde{g} - 3 + |I| + \dim PSL(n) - \{2\tilde{g} + |R_2| + h - h_0 - 2 + \dim GL(n)\} = 2\tilde{g} - 2 + |R_1| + h_0 - h$. By Riemann–Roch this is equal to $\tilde{g} - 1 + |R_1| + 2d_1 - d = \tilde{g} - 1 + |R_1| + \sum_{R_2}(b_i - a_i) - \sum_{R_1}(b_i - a_i)$. This gives the required bound on the codimension.

We turn now to the second assertion of the lemma. The analysis is exactly as above, except that we replace the equality $2d_1 - d = \sum_{R_2}(b_i - a_i) - \sum_{R_1}(b_i - a_i)$ by $2d_1 - d > \sum_{R_2}(b_i - a_i) - \sum_{R_1}(b_i - a_i)$. □

Lemma 6.15. (1) *The complement in* $\tilde{\mathscr{R}}'^{ss} \backslash \hat{\mathscr{D}}_1 \cup \hat{\mathscr{D}}_2$ *of the set* $\tilde{\mathscr{R}}'^s$ *of stable points has codimension* $\geqq \tilde{g} + 1$ *if* $|I| > 0$, *and codimension* $\geqq \tilde{g}$ *if* $|I| = 0$.

(2) *The complement in* $\mathscr{H}$ *of the set* $\tilde{\mathscr{R}}'^{ss}$ *of semistable points has codimension* $\geqq \tilde{g} + 1$.

Proof. The dimension of $\mathscr{H}$ is easily computed to be $4\tilde{g} - 3 + |I| + 4 + \dim PSL(\tilde{n})$.

We first prove (2). Consider a point in $\mathscr{H}\backslash\tilde{\mathscr{R}}'^{ss}$. To such a point there corresponds a GPS E with a rank subsheaf L contradicting semistability. We can assume L is rank 1, and that E/L is torsion-free outside $\{x_1, x_2\}$. We have

$$d - 2 + \sum_{R^c} b_i - a_i - \sum_{R} b_i - a_i < 2\,\mathrm{degree}\, L - 2\dim Q^L . \tag{6.7}$$

In fact E/L can be assumed torsion-free. Suppose it is not, and let $L' \supset L$ be the inverse image in E of the torsion-subsheaf of E/L. Clearly the sets R and R^c are the same for L and L'. Now if (degree L' − degree L) − ($\dim Q^{L'} - \dim Q^L$) < 0 we have degree L' − degree $L = 1$ and $\dim Q^{L'} = 2$, $\dim Q^L = 0$, which is not possible. This shows that L' satisfies (6.7). Thus E is an extension $0 \to L_1 \to E \to L_2 \to 0$ with L_2 torsion-free (i.e. a line bundle) and L_1 satisfying (6.7).

Fix an integer r, with $0 \leqq r \leqq 2$. Fix two nonnegative integers s_1, s_2 with $s_1 + s_2 \equiv s \leqq r$. For $q = 1, 2$, let d_q be integers such that $d_1 + d_2 + s = d$, and let $I = R_1 \cup R_2$ be a decomposition of I such that $2\,(d_1 + s) - d - 2r > -2 + \sum_{R_2}(b_i - a_i) - \sum_{R_1}(b_i - a_i)$. Let $r' = r'(r, s)$ be defined by $r' = 0$ if $r = 2$, $r' = 1 + s$ if $r = 1$ and $r' = 4 + 2s$ if $r = 0$.

Let $h > 0$ be an integer, and let $v = (r, s_1, s_2, d_1, R_1, h)$. Choose Poincaré bundles $\mathscr{L}'_q$ on $\tilde{X} \times J^{d_q}_{\tilde{X}}$. Let $\mathscr{J} \equiv J^{d_1}_{\tilde{X}} \times J^{d_2}_{\tilde{X}}$ and let $\mathscr{L}'$ denote the line bundle $(\mathscr{L}'_2)^* \mathscr{L}'_1$ on $\tilde{X} \times \mathscr{J}$. Let π denote the projection $\tilde{X} \times \mathscr{J} \to \mathscr{J}$.

We define a variety $V(v) \equiv (r, s_1, s_2, d_1, R_1, h)$ as follows.

(a) We first define varieties $V_2(v)$:

(1) If $h = 0$, set $V_2(v) = \mathscr{J}$. Define the bundle $\mathscr{E}'_v$ on $\tilde{X} \times V_2$ to be $\mathscr{L}'_1 \oplus \mathscr{L}'_2$.

(2) Write $\mathrm{Supp}\, R^1\pi_*\mathscr{L}^1 = \bigcup_{h>0} V_1(v)$ with $V_1(v)$ denoting the locally closed subscheme of $\mathscr{J}$ where $R^1\pi_*\mathscr{L}^1$ is locally free of rank h. Let $V_2(v)$ be the projective bundle $\mathbf{P}(\{R^1\pi_*\mathscr{L}'\}^*)$ on $V_1(v)_{\mathrm{red}}$. On $\tilde{X} \times V_2(v)$ there is an universal extension $0 \to \mathscr{L}'_1(-1) \to \mathscr{E}'_v \to \mathscr{L}'_2 \to 0$.

In both cases let $\mathscr{E}_v = \mathscr{E}'_v \oplus {}_{x_1}\mathbf{C}^{s_1} \oplus {}_{x_2}\mathbf{C}^{s_2}$.

(b) Consider the bundle of two dimensional quotients $\mathscr{Q}$ of $\mathscr{E}_{x_1} \oplus \mathscr{E}_{x_2}$ such that the map ${}_{x_1}\mathbf{C}^{s_1} \oplus {}_{x_2}\mathbf{C}^{s_2} \to \mathscr{Q}$ is an injection and the map $\mathscr{L}'_{x_1} \oplus \mathscr{L}'_{x_2} \oplus {}_{x_1}\mathbf{C}^{s_1} \oplus {}_{x_2}\mathbf{C}^{s_2} \to \mathscr{Q}$ has rank r. Let $V_3(v)$ be the fibre product

$$\mathscr{Q} \times_{V_2(v)} \left\{ \times_{V_2(v)} \mathbf{P}((\mathscr{E}_v)_{y_i}) \atop i \in R_2 \right\}.$$

The sub-bundle $\mathscr{L}_1(-1) \subsetneq \mathscr{E}$ yields, for each $i \in I$, a divisor in V_3.

(c) Let $V(v) = V(r, s_1, s_2, d_1, R_1, h)$ be the complement of the union of these divisors for $i \in R_2$.

Each $V(v)$ parametrises a family of generalised parabolic sheaves $E = E' \oplus {}_{x_1}\mathbf{C}^{s_1} \oplus {}_{x_2}\mathbf{C}^{s_2}$, where E' occurs as an extension $0 \to L'_1 \to E' \to L'_2 \to 0$ (the extension being split if $h = 0$), with parabolic structures at the $\{y_i\}_{R_1}$ given by the sub-bundle L'_1. The dimensions of the $V(v)$ are easily bounded. These are:

(1) $\dim V(v) = 2\tilde{g} + |R_2| + 2s + 4 - r'(r, s)$ if $h = 0$,

(2) $\dim V(v) \leqq 2\tilde{g} + h - 1 + |R_2| + 2s + 4 - r'(r, s)$ otherwise.

Let $V(v)^{ss}$ be the open set of semistable parabolic bundles, and let $F(v)$ be the frame-bundle of the direct image of $\mathscr{E}$ on $V(v)^{ss}$.

As in the proof of the previous Lemma we take into account automorphisms, and find that the codimension is $\geqq \tilde{g} - 1 + |R_1| + 2d_1 - d + r' + 2s$, and hence strictly greater than $\tilde{g} - 1 + |R_1| + \sum_{R_2}(b_i - a_i) - \sum_{R_1}(b_i - a_i) + 2r - 2 + r'$. This proves (1). (Note that the sheaf $E' \oplus {}_{x_1}\mathbf{C}^{s_1} \oplus {}_{x_2}\mathbf{C}^{s_2}$ has an automorphism group of dimension $\geqq \dim \mathrm{aut}(E') + 2s + s$.)

The assertion (1) is proved similarly, the only change being that the inequality in (6.7) is replaced by an equality. This however does not affect the final bound. □

Remark 6.16. It is *not* true that $\tilde{\mathscr{R}}'^{ss}\backslash\tilde{\mathscr{R}}'^{s}$ has codimension $\geqq \tilde{g}$. Points on the $\mathscr{D}_j$ are never stable. The above codimension bound breaks down because one cannot assume that the sub-sheaf contradicting stability is rank 1.

We need next to consider two sets of parabolic weights ω and ω'. Write $\omega \subset \omega'$ if the indexing set I of the first set of weights is a subset of the indexing set I' of the second set $\{y_i\}_I \subset \{y_i\}_{I'}$ compatibly, and the two sets of weights agree at the points $\{y_i\}_I$. We have

Lemma 6.17. *Suppose $g > 0$ and $\omega \subset \omega'$. Then*

(1) *if $\mathscr{U}_X^s(d, \omega)$ is nonempty so is $\mathscr{U}_X^s(d, \omega')$.*

(2) *if X is irreducible with a node and there exist ω-stable non-locally-free sheaves then there also exist ω'-stable non-locally-free sheaves.*

Proof. We prove (1). The other statement has a similar proof.

Clearly it is enough to consider the case $I' = I \cup \{0\}$. For simplicity we assume that a Poincaré family $\mathscr{F}$ exists on $X \times \mathscr{U}_X^s(d, \omega)$. (By working with an étale open set in $\mathscr{U}_X$ one can avoid this assumption.) Consider the projective bundle $\mathbf{P} \equiv \mathbf{P}(\mathscr{F}_{y_0})$. This parametrises a $(4g - 3 + |I'|$-dimensional family of parabolic bundles with weights ω'. We will show that there exist ω'-stable bundles in this family.

Let (F, Q_i, Q_0) be a bundle in the family which is *not* ω'-stable. Then it has a line sub-bundle L such that $L_{y_0} = \ker(E_{y_0} \to Q_0)$ and

$$\sum_{R^c}(b_i - a_i) - \sum_{R'}(b_i - a_i) - (b_0 - a_0) \leqq 2\,\text{degree}\, L - d < \sum_{R^c}(b_i - a_i) - \sum_{R}(b_i - a_i)$$

where $R \equiv R(L) \subset I$ is the subset where $L_{y_i} \subset \ker(F_{y_i} \to Q_i)$ and $R^c \equiv R^c(L)$ its complement. As in the proof of Lemma 6.14 we find that such bundles (F, Q_i, Q_0) are parametrised by a (finite) number of subvarieties of $\mathbf{P}$ (labelled by (R_1, d_1, h)), of dimension $\leqq 2g + |I| - |R_1| + h - 1 - h_0$. The codimension is therefore greater than

$$2g - 1 + |R_1| + h_0 - h_1 = g + |R_1| + 2d_1 - d \geqq g + |R_1| + \sum_{R_2}(b_i - a_i) - \sum_{R_1}(b_i - a_i) - (b_0 - a_0).$$

Grouping the terms on the right as $\{|R_1| - \sum_{R_1}(b_i - a_i)\} + \sum_{R_2}(b_i - a_i) + \{g - (b_0 - a_0)\}$ we get a positive lower bound on the codimension. □

Corollary 6.18. *Suppose X irreducible with one node. Then there exist stable (non-locally-free) sheaves on X except when $g = 1$, d even, $|I| = 0$.*

Proof. It is well-known that $\mathscr{U}_X(d, \omega)$ is nonempty when X is smooth, $|I| = 0$, $g \geqq 2$. Now suppose X irreducible with one node. Using Lemma 3.3 we get stable non-locally-free sheaves when

(1) $|I| = 0$, $g \geqq 3$.

If $g = 1$, $|I| = 0$ and d is odd, we get stable sheaves by taking a nontrivial extension $0 \to \mathcal{O} \to E \to L \to 0$ where L is a rank one torsion-free sheaf of degree 1. This covers the case

(2) $|I| = 0$, $g = 1$, d odd.

Further, if $g = 1$ and d is even, one constructs a stable parabolic bundle with parabolic structure at one point y_1 as follows. Take two different rank one torsion-free sheaves L_1 and L_2 (one is then necessarily locally free), let $E = L_1 \oplus L_2$, and take a quasi-parabolic structure $E_{y_1} \to Q_1 \to 0$ such that $(L_i)_{y_1} \neq \ker(E_{y_1} \to Q_1)$, $i = 1, 2$, and arbitrary weights $a_1 < b_1$. This yields a stable parabolic sheaf with

(3) $|I| = 1$, $g = 1$, d even.

The above constructions of course work for nosingular X as well, and again using Lemma 3.3, we can add the cases

(4) $|I| = 0$, $g = 2$, d odd.

(5) $|I| = 1$, $g = 2$, d even.

where again we get non-locally-free sheaves.

The case

(6) $|I| = 0$, $g = 2$, d even,

can be covered by taking a suitable extension $0 \to L_1 \to E \to L_2 \to 0$, with degree $L_1 = -1$, degree $L_2 = +1$. We omit the details.

We now use Lemma 6.17 to finish the proof. □

Remark 6.19. Note that since stability is an open condition, if stable non-locally-free sheaves exist, stable locally free sheaves also must exist. Thus Corollary 6.18 implies that if X is a nodal curve,

$$\emptyset \neq \mathcal{W} \neq \mathcal{U}_X, \tag{6.8}$$

except possibly when $g = 1$, d even, $|I| = 0$. In fact in this case it is easy to see (normalising $d = -2$) that $\mathcal{U}_X = (X \times X)/\sim$ where $\sim$ is the involution exchanging the two factors, and that (6.8) holds in this case as well.

Appendix A. The moduli space of parabolic sheaves

There exist two constructions of parabolic moduli spaces on curves – that of [M-S] and that of [B2]. Neither works in the case of a singular curve. We present in this Appendix a construction of the moduli space, which generalises the work of C. Simpson, and is applicable when the underlying curve has a nodal singularity (and presumably more generally). This approach to the construction of parabolic moduli spaces arose out of conversations with A. Ramanathan.

For ease of reference we have tried to make this Appendix self-contained, at the risk of some repetition.

Unless otherwise mentioned, X will denote an irreducible projective curve of genus g over $\mathbf{C}$, smooth but for one node x_0. Let $\mathcal{O}_X(1)$ be an ample line bundle on X of degree 1, $\{y_i\}_I$ a finite set of smooth points on X. Let d denote an integer, the degree (to be chosen below). Fix another integer $k > 0$, and also, for each $i \in I$ integers $0 < \alpha_i < \beta_i \leqq k$. We set $n = d + 2(1 - g)$ and let l denote the number determined by

$$nk = 2k|I| + 2l - \sum_i (\alpha_i + \beta_i). \tag{A.1}$$

We assume that the data are such that l is integer, i.e. that $dk + \sum_i(\alpha_i + \beta_i)$ is even Let $a_i = \alpha_i/k$, $b_i = \beta_i/k$. Set $\omega \equiv \{(a_i, b_i)\}_I$. Note that $0 < a_i < b_i \leqq 1$. The usual range assumed is $0 \leqq a_i < b_i < 1$. (This is not a significant difference since the definition of stability only involves the difference $b_i - a_i$. However, the construction below certainly requires $a_i > 0$.)

We wish to construct the moduli space $\mathscr{U}_X$ of s-equivalence classes of semistable rank 2 torsion-free sheaves on X with parabolic structures at the $\{y_i\}_I$ (with weights ω). It will be clear from the construction that it works for an irreducible curve with an arbitrary of nodes. In particular X could be smooth.

Definition A.1a. Let F be rank 2 torsion-free sheaf on X. By a *quasi-parabolic structure* on F at a smooth point $x \in X$ we mean a choice of a one-dimensional quotient $F_x \to Q \to 0$ of the fibre of F at the point x. If in addition real numbers ("weights") $a < b$ are given, this is a *parabolic structure.*

We shall refer to a torsion-free sheaf with parabolic structures at the $\{y_i\}_I$ (with weights ω) as a "parabolic sheaf".

Definition A.1b. A parabolic sheaf F is said to be *stable* (respectively, *semistable*) with if for every rank one subsheaf L of F such that F/L is torsion-free we have

$$\text{par degree } L \underset{(\text{resp.} \leqq 2)}{<} \tfrac{1}{2}(\text{par degree } F')\,.$$

The parabolic degree of F is by definition par degree $F = d + \sum_i(a_i + b_i)$; given a rank one subsheaf $L \subset F$ such that F/L is torsion-free, its parabolic degree is by definition par degree $L =$ degree $L + \sum_{R^c} a_i + \sum_R b_i$ where $R \subset I$ is the subset of $i \in I$ such that $L_{y_i} \subset \ker(F_{y_i} \to Q_i)$ and R^c its complement.

Remark A.2. The condition for (semistability can be written

$$2\,\text{degree } L \underset{(\text{resp.} \leqq)}{<} d + \sum_{R^c}(b_i - a_i) - \sum_{R}(b_i - a_i)\,. \tag{A.2}$$

In particular this implies

$$2\,\text{degree } L < d + |I|\,. \tag{A.3}$$

Theorem X1. *There exists a (coarse) moduli space $\mathscr{U}^s(X, d, \omega)$ of stable parabolic sheaves F. We have an open immersion $\mathscr{U}^s(X, d, \omega) \subsetneq \mathscr{U}(X, d, \omega)$ where $\mathscr{U}(X, d, \omega)$ denotes the space of s-equivalence classes of semistable parabolic sheaves. The latter is a projective variety. If X is smooth, then $\mathscr{U}$ is normal, with rational singularities.*

(The notion of s-equivalence of parabolic sheaves is defined as in the case of vector bundles, using [S2, Troisième Partie, Theorem 12]. In the notation of that theorem we say that two parabolic sheaves F_1 and F_2 are s-equivalent if $Gr(F_1) = Gr(F_2)$.)

The rest of this Appendix will be devoted to a proof of Theorem X1. By Remark 2.2 we are free to choose d as large as we wish.

Lemma A.3. *There exists an integer $N_1 > 0$ such that for any semistable parabolic sheaf F of rank 2 and euler characteristic $> N_1$*

(1) *F is generated by its sections, and*

(2) $H^1(F) = 0$.

Proof. One imitates the proof of [N, Lemma (5.2)′] and uses equation (A.3). Note that the constant δ' in the statement of the quoted lemma can be majorised by q [N, page 165]. □

Remark A.4. The method of proof shows the following: Suppose F is a rank 2 parabolic sheaf (not necessarily semistable) such that for every torsion-free quotient $F \to L \to 0$ we have $h^0(L) \geqq N_1/2 - |I|$. Then $H^1(F) = 0$.

Choose d large enough that for any parabolic semistable F of degree d, $H^0(F)$ generates F, $H^1(F) = 0$. (One can do this without loss of generality because of Remark 2.2). Let $\mathbf{Q}$ denote the Quot scheme [G] of coherent sheaves over X which are quotients of $\mathcal{O}^n$, where $n = d + 2(1 - g)$, with Hilbert Polynomial P equal to that of any such F, i.e. $P(m) = 2m + n$. Thus there is on $X \times \mathbf{Q}$ a sheaf $\mathcal{F}_{\mathrm{Q}}$, flat over $\mathbf{Q}$, and a surjection $\mathcal{O}^n \to \mathcal{F}_{\mathrm{Q}} \to 0$. The Quot scheme is a projective scheme [G]: there exists an integer $M_1(n)$ such that for $m \geqq M_1(n)$ we have (denoting the vector space $H^0(\mathcal{O}_X(m))$ by W):

(1) for every point $\mathcal{O}^n \to F \to 0$ in the Quot scheme, if we let K be the kernel, we have $H^1(K(m)) = 0$, so that the map $\mathbf{C}^n \otimes W \to H^0(F(m))$ is onto, and

(2) the map $\mathbf{Q} \to Grass_{P(m)}(\mathbf{C}^n \otimes W)$ given by (1) is an closed embedding.

We define another (complete) scheme $\mathcal{R}$ as follows. For $i \in I$, consider the sheaf $\mathcal{F}_{y_i}$ on $\mathbf{Q}$ given by restricting $\mathcal{F}_{\mathrm{Q}}$ to $\{y_i\} \times \mathbf{Q}$, and let $\mathrm{Flag}_{(1,2)}(\mathcal{F}_{y_i})$ be the relative Flag variety of locally-free quotients of $\mathcal{F}_{y_i}$ of rank (1, 2) [EGA-I, 9.9.2]. The scheme $\mathcal{R}$ is then a fibre product over $\mathbf{Q}$:

$$\mathcal{R} = \underset{i \in I}{\times}{}_{\mathrm{Q}}\mathrm{Flag}_{(1,2)}(\mathcal{F}_{y_i}).$$

Notation A.5. A (closed) point of $\mathcal{R}$ will be given by a point $\mathcal{O}^n \xrightarrow{p} F \to 0$ in the Quot scheme, together with quotients $F \xrightarrow{Pr,i} Q_{r,i} \to 0$, where $Q_{r,i}$ is a skyscraper sheaf supported at the (reduced) point y_i, with $h^0(Q_{r,i}) = r$, $r = 1, 2$, the $p_{r,i}$ satisfying $\ker p_{2,i} \subset \ker p_{1,i}$. We let p_m denote the map $\mathcal{O}^n(m) \to F(m)$.

We have a $SL(n)$-equivariant embedding $\mathcal{R} \hookrightarrow \mathbf{G}$ where

$$\mathbf{G} \equiv Grass_{P(m)}(\mathbf{C}^n \otimes W) \times \underset{i}{\times} \{Grass_2(\mathbf{C}^n) \times Grass_1(\mathbf{C}^n)\}.$$

Each factor on the right has a canonical ample generator of the Picard group. We give $\mathbf{G}$ the polarisation (using the obvious notation):

$$\frac{l}{m} \times \underset{i}{\times} \{(k - \beta_i), (\beta_i - \alpha_i)\}. \tag{A.4}$$

This gives a linearisation of the $SL(n)$ action.

Let $\mathcal{R}^{ss}$ denote the subset of closed points of $\mathcal{R}$ such that the corresponding parabolic sheaves are semistable (in particular torsion-free), and the map $H^0(p)\colon \mathbf{C}^n \to H^0(F)$ is an isomorphism. We will prove below that for large enough choices of n and m these are precisely the semistable points for the action of $SL(n)$ on $\mathcal{R}$ (in the sense of Geometric Invariant Theory) w.r.t. this polarisation. This will yield the existence of $\mathcal{U}$ and also show, incidentally, that semistability is an open condition for parabolic sheaves and that $\mathcal{R}^{ss}$ is (the set of closed points of) an open subscheme.

At a point $(P, \{(P_{2,i}, P_{1,i})\} \in \mathbf{G}$ we shall denote by $(U, \{(U_{2,i}, U_{1,i})\}_I)$ the respective quotients. Note that if the point $(P, \{(P_{2,i}, P_{1,i})\}_I)$ is the image of $(p, \{(p_{2,i}, p_{1,i})\}_I) \in \mathscr{R}$ then $P = H^0(p_m)$, $P_{r,i} = H^0(p_{r,i})$ and $H^0(Q_{r,i}) = U_{r,i}$.

We have a straightforward generalisation of [N-T, Proposition 5.1.1] (see also [Si, Proposition 4.3]) whose proof we omit:

Proposition A.6. *A point $(P, \{(P_{2,i}, P_{1,i})\}_I) \in \mathbf{G}$ is stable (respectively, semistable) for the action of $SL(n)$, with respect to the polarisation (A.4) (we refer to this from now on as GIT-stability), iff for all nontrivial subspaces $H \subset \mathbf{C}^n$ we have (with $h \equiv \dim H$)*

$$\frac{l}{m}(hP(m) - n\dim P(H \otimes W)) + \sum_i (k - \beta_i)(2h - n\dim P_{2,i}(H))$$

$$+ \sum_i (\beta_i - \alpha_i)(h - n\dim P_{1,i}(H)) \underset{(\text{resp. } \leq)}{<} 0. \tag{A.5}$$

Notation A.7. Given a point $(p, \{(p_{2,i}, p_{1,i})\}_I) \in \mathscr{R}$ (as in A.5), and a subsheaf F' of F we set $Q^{F'}_{r,i} \equiv p_{r,i}(F')$. Similarly, given a quotient $F \xrightarrow{T} G \to 0$, set $G/T(\ker p_{r,i}) = Q/p_{r,i}(\ker T) = Q^G_{r,i}$.

Lemma A.8. *Suppose $(p, \{(p_{2,i}, p_{1,i})\}_I) \in \mathscr{R}$ is a point such that F is torsionfree and let m be a positive integer. Then F is stable (respectively, semistable) iff for every subshear $0 \neq F' \neq F$ we have:*

$$\frac{l}{m}(\chi(F')P(m) - n\chi(F'(m))) + \sum_i (k - \beta_i)(2\chi(F') - nh^0(Q^{F'}_{2,i}))$$

$$+ \sum_i (\beta_i - \alpha_i)(\chi(F') - nh^0(Q^{F'}_{1,i})) \underset{(\text{resp. } \leq)}{<} 0. \tag{A.6}$$

Proof. For any subsheaf F' of F let LHS(F') denote the left-hand side of (A.6). Assume first that the inequality holds for every proper subsheaf. Let F' be a proper nonzero subsheaf such that F/F' is torsion-free. For any such F' (which is necessarily of rank 1) we have by Riemann–Roch,

$$\begin{aligned}
\text{LHS}(F') &= \frac{l}{m}(\chi(F')(2m+n) - n(m + \chi(F'))) + \chi(F')\left(2\sum_i (k - \beta_i) + \sum_i (\beta_i - \alpha_i)\right) \\
&\quad - n\left(\sum_i (k - \beta_i) + \tfrac{1}{2}\sum_i (\beta_i - \alpha_i)\right) - \frac{n}{2}\left(\sum_{R^c} (\beta_i - \alpha_i) - \sum_R (\beta_i - \alpha_i)\right) \\
&= l(2\chi(F') - n) + (2\chi(F') - n)\left(\sum_i (k - \beta_i) + \tfrac{1}{2}\sum_i (\beta_i - \alpha_i)\right) \\
&\quad - \frac{n}{2}\left(\sum_{R^c} (\beta_i - \alpha_i) - \sum_R (\beta_i - \alpha_i)\right) \\
&= \frac{1}{2}(2\,\text{degree}\, F' - d)\left(2l + 2k|I| - \sum_i (\beta_i + \alpha_i)\right) \\
&\quad - \frac{n}{2}\left(\sum_{R^c} (\beta_i - \alpha_i) - \sum_R (\beta_i - \alpha_i)\right) \\
&= \frac{nk}{2}\left\{(2\,\text{degree}\, F' - d) + \sum_R (b_i - a_i) - \sum_{R^c} (b_i - a_i)\right\},
\end{aligned}$$

where in the last step we use (A.1). Comparison with equation (A.2) shows that F is (semi)stable if (A.6) is satisfied.

Suppose now that F is (semi)stable and F' is a proper subsheaf. The above computations yield the desired inequality when F/F' is torsion-free. Suppose now that F/F' is not torsion-free. We will show that $\mathrm{LHS}(F') < 0$. Write $0 \to F' \to \tilde{F}' \to \mathscr{T} \to 0$ where $\mathscr{T}$ is torsion, $\tilde{F}' \subsetneq F$ and $F/\tilde{F}'$ is torsion-free. Let $\mathscr{T} = \tilde{\mathscr{T}} + \sum_i \mathscr{T}_i$ where $\mathscr{T}_i$ is the subsheaf of $\mathscr{T}$ determined by the requirement that its stalk at y_i is the same as that of $\mathscr{T}$. Clearly $\mathrm{LHS}(\tilde{F}') \leqq 0$. We will now show that $\mathrm{LHS}(F') - \mathrm{LHS}(\tilde{F}') < 0$:

$$\begin{aligned}
\mathrm{LHS}(F') - \mathrm{LHS}(\tilde{F}') = & -\frac{l}{m}(h^0(\mathscr{T})(2m+n) - nh^0(\mathscr{T})) \\
& - h^0(\mathscr{T})\left(2\sum_i (k-\beta_i) + \sum_i (\beta_i - \alpha_i)\right) \\
& - n\left\{\sum_i (k-\beta_i)(h^0(Q^{F'}_{2,i}) - h^0(Q^{\tilde{F}'}_{2,i}))\right. \\
& \left. + \sum_i (\beta_i - \alpha_i))(h^0(Q^{F'}_{1,i}) - h^0(Q^{\tilde{F}'}_{1,i}))\right\} \\
\leqq & - nkh^0(\tilde{\mathscr{T}}) - n\sum_i h^0(\mathscr{T}_i)\{k - (k-\beta_i) - (\beta_i - \alpha_i)\} \\
= & - nkh^0(\tilde{\mathscr{T}}) - n\sum_i h^0(\mathscr{T}_i)\alpha_i,
\end{aligned}$$

where we have used $h^0(Q^{\tilde{F}'}_{r,i}) - h^0(Q^{F'}_{r,i}) \leqq h^0(\mathscr{T}_i)$. Since by assumption $\alpha_i > 0$ we have the required inequality. □

The next two lemmas are generalisations of [Si, Lemmas 2.8 and 2.9] respectively.

Lemma A.9. *There exists $M_2(n) \geqq M_1(n)$ such that for $M \geqq M_2(n)$ the following holds. Suppose $(p, \{(p_{2,i}, p_{1,i})\}_I) \in \mathscr{R}$ is a point such that $H^0(p)$: $\mathbf{C}^n \to H^0(F)$ is an isomorphism and, for every subsheaf F' of F generated by sections we have*

$$\frac{l}{m}(h^0(F')P(m) - n\chi(F'(m))) + \sum_i (k-\beta_i)(2h^0(F') - nh^0(Q^{F'}_{2,i}))$$

$$+ \sum_i (\beta_i - \alpha_i)(h^0(F') - nh^0(Q^{F'}_{1,i})) \underset{(\text{resp. } \leqq)}{<} 0. \tag{A.7}$$

Then the point is GIT-(semi)stable.

Proof. For $H \subset \mathbf{C}^n$ let F'_H denote the subsheaf of F generated by H, define K_H by the exact sequence: $0 \to K_H \to H \otimes \mathcal{O}_X \to F'_H \to 0$. Now, for all points of $\mathbf{Q}$ and all subspaces H the sheaves F'_H run over a bounded family, as do the sheaves K_H. Therefore we can find $M_2(n)$ such that for $m \geqq M_2(n)$ we have $h^1(F'_H(m)) = 0$ and $h'(K_H(m)) = 0$ for all such F'_H and K_H.

Note now that

(1) $\dim H \leqq h^0(F'_H)$,

(2) $P_{r,i}(H)) = H^0(Q^{F'_H}_{2,i})$ and

(3) $\dim P(H \otimes W) = \chi(F'_H(m))$ for $m \geqq M_2(n)$ (by the previous paragraph).
The Lemma now follows from Proposition (A.6) □

Lemma A.10. *There exists $M_3(n) \geqq M_2(n)$ such that for $m \geqq M_3(n)$ the following holds. Suppose $(p, \{(p_{2,i}, p_{1,i})\}_I) \in \mathscr{R}$ is a point which is GIT-semistable then $\mathbf{C}^n \to H^0(F)$ is an isomorphism, and for all quotients $F \xrightarrow{T} G \to 0$ we have*

$$l(-2h^0(G) + nr(G)) + \sum_i (k - \beta_i)(-2h^0(G) + n h^0(Q^G_{2,i}))$$

$$+ \sum_i (\beta_i - \alpha_i)(-h^0(G) + n h^0(Q^G_{1,i})) \leqq 0. \tag{A.8}$$

Proof. Denote by H_1 the kernel of the map $\mathbf{C}^n \to H^0(F)$. Note that $P_{r,i}(H_1) = 0$, and $P(H_1 \otimes W) = 0$. But this implies, by (A.5), that $H_1 = 0$. This proves that $\mathbf{C}^n \to H^0(F)$ is an injection.

Suppose now that G is a quotient contradicting (A.8), i.e.

$$l(2h^0(G) + nr(G)) + \sum_i (k - \beta_i)(2h^0(G) + n h^0(Q^G_{2,i}))$$

$$+ \sum_i (\beta_i - \alpha_i)(-h^0(G) - n h^0(Q^G_{1,i})) < 0. \tag{A.9}$$

Let H be the kernel of the map $\mathbf{C}^n \to H^0(G)$ and let F' be the subsheaf of F generated by H. From (A.9) we conclude that $h^0(G) < n$, and from this and the definition of H and F' that

(1) $\dim H \geqq n - h^0(G) > 0$,
(2) $r(F') + r(G) \leqq 2$, and
(3) $h^0(Q^G_{r,i}) + h^0(Q^{F'}_{2,i}) \leqq r$,
(4) $h^0(Q^{F'}_{2,i}) = \dim P_{r,i}(H)$.

Combining these inequalities with (A.9) we get (with $h = \dim H$ as before)

$$-l(2h - nr(F')) - \sum_i (k - \beta_i)(2h - n \dim P_{2,i}(H))$$

$$- \sum_i (\beta_i - \alpha_i)(h - n \dim P_{1,i}(H)) < 0.$$

For large $m \geqq M(F')$ we can replace the first term by $l/m(h\,P(m) - n\chi(F'(m)))$, which equals $l/m(h\,P(m) - n \dim P(H \otimes W))$ provided $m \geqq M_2$. Since the F''s range over a bounded family we can fine $M_3(n) \geqq M_2(n)$ so that $M_3(n) \geqq M(F')$ for all F''s. Now, if $m \geqq M_3(n)$ we have

$$-\frac{l}{m}(hP(m) - n \dim P(H \otimes W)) - \sum_i (k - \beta_i)(2h - n \dim P_{2,i}(H))$$

$$- \sum_i (\beta_i - \alpha_i)(h - n \dim P_{1,i}(H)) < 0.$$

But this contradicts (A.5) which holds by GIT semistability. Thus (A.8) is now established.

Let now L be a rank 1 torsion-free quotient of F. Then we have, by (A.8), $h^0(L) \geqq n/2 - |I|$. This implies, since $n > N_1$, that $H^1(F) = 0$ and therefore $h^0(F) = n$. (See Remark A.4). This proves that the map $\mathbf{C}^n \to H^0(F)$ is an isomorphism. □

We now state the analogue of [Si, Theorem 2]:

Proposition A.11. *There exists an integer $N > 0$, and given $n \geqq N$ and, an integer $M(n) > 0$ such that for $m \geqq M(n)$ the following is true. A point $(p, \{(p_{2,i}, p_{1,i})\}_I) \in \mathscr{R}$ is GIT-stable (respectively, GIT-semistable) iff the quotient F is torsion-free and a stable (respectively, semistable) sheaf, and the map $\mathbf{C}^n \to H^0(F)$ is an isomorphism.*

We will need the following

Lemma A.12. *There exists $N_2 \geqq N_1$ such that the following holds. If F is a semistable parabolic sheaf with Euler characterstic $n \geqq N_2$:*

(1) *$\forall F' \subset F$ we have*

$$l(2h^0(F') - r(F')n) + \sum_i (k - \beta_i)(2h^0(F') - n\,h^0(Q_{2,i}^{F'}))$$

$$+ \sum_i (\beta_i - \alpha_i)(h^0(F') - n\,h^0(Q_{2,i}^{F'})) \leqq 0. \tag{A.10}$$

(2) *If, for some $F' \subset F$, equality holds in* (A.10) *then, for any $m \geqq 1$,*

$$\frac{l}{m}(\chi(F')P(m) - n\chi(F'(m))) + \sum_i (k - \beta_i)(2\chi(F') - n\,h^0(Q_{2,i}^{F'}))$$

$$+ \sum_i (\beta_i - \alpha_i)(\chi(F') - n\,h^0(Q_{1,i}^{F'})) = 0. \tag{A.11}$$

Proof. Let F'_i denote the terms in the canonical filtration [H-N] of F' (the filtration being defined ignoring parabolic structures), let $Q_i = F'_i/F'_{i-1}$. Let $\mu(F)$ denote the slope (degree F)/(rank F). Then $h^0(F') \leqq \sum_i h^0(Q_i)$, $\mu(Q_i) \leqq \mu(F) + c|I|$ for some constant c. Also, by [Si, Corollary 2.5] we have, when $h^0(Q_i) > 0$ the inequality $h^0(Q_i) \leqq r(Q_i)(\mu(Q_i) + B_1)$ for some constant B_1. Let $v = \inf\{\mu(Q_i) \| h^0(Q_i) > 0\}$. Then $h^0(F') \leqq (r(F') - 1)(\mu(F) + c|I| + B_1) + (v + B_1)$. If $v \leqq \mu(F) - C$ (C to be fixed below) this yields $h^0(F') \leqq r(F')2n + B_2 - C$ for some constant B_2; thus for such F' the left hand side of (A.10) is less than or equal to

$$h^0(F')\left(2l + 2k|I| - \sum_i(\beta_i + \alpha_i)\right) - nlr(F')$$

$$\leqq \left(\frac{r(F')}{2}n + B_2 - C\right)\left(2l + 2k|I| - \sum_i(\beta_i + \alpha_i)\right) - nlr(F')$$

$$\leqq 2l(B_2 - C) + \left(2k|I| - \sum_i(\beta_i + \alpha_i)\right)(n + B_2 - C)$$

$$= nk(B_2 - C) + n\left(2k|I| - \sum_i(\beta_i + \alpha_i)\right) \quad \text{using (A.1)}$$

$$= nk(B_3 - C)$$

where the last equality defines B_3. Choosing $C > B_3$ we get the desired inequality (which is in fact strict – this will be relevant for the proof of part (2) of the lemma) for subsheaves F' satisfying $v \leqq \mu(F) - C$. On the other hand we can arrange (by taking $n \geqq N_2$, N_2 large enough) that all stable bundles Q with rank $\leqq 2$ and $\mu(Q) \geqq \mu(F) - C$ have $H'(Q) = 0$, yielding, for F' contradicting $v \leqq \mu(F) - C$, the equality $\chi(F'(m)) = h^0(F'(m))$ for $m \geqq 0$. Then (A.6) implies (A.10) for such F'. Part (2) of the lemma now follows easily. □

Proof of Proposition A.11. Choose $N = N_2$ where N_2 is determined by the above lemma. The proof of the "if" part is now similar to the proof of [Si, Theorem 2], where the first step of the proof has been isolated in Lemma A.12.

We sketch the proof of the "only if" part. Suppose $(p, \{(p_{2,i}, p_{1,i})\}_I) \in \mathscr{R}$ is a point which is GIT-(semi)stable. Note that by Lemma A.10, $\mathbf{C}^n \to H^0(F)$ is an isomorphism. Let $\tau = \operatorname{Tor} F$, $G = F/\tau$ and apply Lemma A.10, noting that $h^0(G) = n - h^0(\tau)$, $h^0(Q^G_{2,i}) = 2 - h^0(Q^\tau_{2,i})$ and $h^0(Q^G_{1,i}) = 1 - h^0(Q^\tau_{1,i})$. We get

$$kh^0(\tau) \leqq \sum_i (k - \beta_i)h^0(Q^\tau_{2,i} + (\beta_i - \alpha_i)h^0(Q^\tau_{1,i}),$$

from which one can easily conclude (since $\alpha_i > 0$) that $\tau = 0$.

Proof of Theorem X1. The proof of the first part of the theorem (existence of $\mathscr{U}$) is now similar to the proof of [Si, Theorem 2]. That $\mathscr{U}$ is projective follows from the GIT construction. The other properties of $\mathscr{U}$ follow from the corresponding facts about $\mathscr{R}^{ss}$, again by GIT. Consider for example the case when X is smooth. Define $\mathbf{Q}_F$ to be the open subscheme of $\mathbf{Q}$ consisting of locally-free quotients $\mathscr{O}^n \to F \to 0)$ such that

(1) $\mathbf{C}^n \to H^0(F)$ is an isomorphism, and
(2) $H^1(F) = 0$.

Let $\mathscr{R}_F$ be the inverse image of $\mathbf{Q}_F$ by the projection $\mathscr{R} \to \mathbf{Q}$. This is a bundle over $\mathbf{Q}_F$,

$$\mathscr{R}_F = \times_{\substack{\mathbf{Q}_F \\ i \in I}} \operatorname{Flag}_{(1,2)}(\mathscr{F}_{y_i}).$$

The projection $\mathscr{R}_F \to \mathbf{Q}_F$ is smooth, and $\mathbf{Q}_F$ itself can be proved to be smooth (in particular irreducible) as in [N, Remark 5.5]. Thus $\mathscr{R}_F$ is smooth, and hence so is its open subscheme $\mathscr{R}^{ss}$. This yields irreducibility and normality of $\mathscr{U}$; it also follows that $\mathscr{U}$ has rational singularities [Bo].

For X a nodal curve, $\mathscr{R}^{ss}$ can be similarly proved to be reduced and irreducible defining $\mathbf{Q}_F$ as above, but replacing "locally-free quotients" by "torsion-free quotients". In this case $\mathbf{Q}_F$ is not smooth. That it is irreducible can be seen as before. That it is reduced is the main result of [S2, Huitième Partie, III], where in fact it is proved that given $q \in \mathbf{Q}_F$ the completion $\hat{\mathscr{O}}_q$ is reduced. □

Appendix B. Generalised parabolic sheaves

Ba. The moduli space of generalised parabolic sheaves

The notation of the previous Appendix holds. In addition let $\tilde{X}$ be the normalisation of X, $\tilde{g} = g - 1$ the genus of $\tilde{X}$, and $\pi: \tilde{X} \to X$ the canonical map. Let $\{x_1, x_2\}$

be the inverse image of x_0 in $\tilde{X}$. Set $\tilde{n} \equiv d + 2(1 - \tilde{g})$, and define $\tilde{l}$ by

$$\tilde{n}k = 2k|I| - 2\tilde{l} - \sum_i (\alpha_i + \beta_i).$$

(Note $\tilde{n} = n + 2$, and $\tilde{l} = l + k$.)

We wish to construct the moduli space $\mathscr{P}$ of s-equivalence classes of semistable rank 2 sheaves on $\tilde{X}$ with parabolic structures at the $\{y_i\}_I$ (with weights ω) and a generalised parabolic structure over $\{x_1, x_2\}$.

Definition B.1. Let E be a rank 2 sheaf, torsion-free outside $\{x_1, x_2\}$, with parabolic structures over $\{y_i\}_I$. A *generalised parabolic structure* on E over the divisor $\{x_1, x_2\}$ is a choice of a two-dimensional quotient Q of $E_{x_1} \oplus E_{x_2}$. We do not define a generalised quasiparabolic structure since a certain choice of "generalised weights" is assumed. A parabolic sheaf with, in addition, a generalised parabolic structure over $\{x_1, x_2\}$, is a *generalised parabolic sheaf* (GPS). A GPS E is said to be *stable* (respectively, *semistable*) with respect to the weights ω if for every proper subsheaf E' such that E/E' is torsion-free outside $\{x_1, x_2\}$, we have

$$\text{par degree } E' \underset{(\text{resp. } \leqq)}{<} \frac{\text{rank } E'}{2} (\text{par degree } E) - (\text{rank } E' - \dim Q^{E'}) \qquad (\text{B.1})$$

where, for any subsheaf E' we denote by $Q^{E'}$ the image of $E'_{x_1} \oplus E'_{x_2}$ in Q.

Theorem X2. *There exists a (coarse) moduli space $\mathscr{P}^s(\tilde{X}, d, \omega)$ of stable GPSs on X. We have an open immersion $\mathscr{P}^s(\tilde{X}, d, \omega) \hookrightarrow \mathscr{P}(\tilde{X}, d, \omega)$ where $\mathscr{P}(\tilde{X}, d, \omega)$ denotes the space of s-equivalence classes of semistable GPS's. The former is a smooth variety; the latter a normal projective variety with rational singularities.*

B.2. Outline of Proof of Theorem X2

(1) Lemma A.3 is replaced by the following result: *There exists an integer $N'_1 > 0$ such that for any semistable generalised parabolic sheaf E of rank 2 and euler characteristic $> N'_1$ we have $H'(E(-x_1 - x_2 - x)) = 0$, $x \in \tilde{X}$.* This ensures that $H^1(E) = 0$, E is generated by sections, $H^0(E) \to E_{x_1} \oplus E_{x_2}$ is onto, and $E(-x_1 - x_2)$ is generated by sections.

(2) Let $\tilde{P}(m) = 2m + \tilde{n}$. Define

$$\mathscr{R}' = \text{Grass}_2(\mathscr{E}_{x_1} \oplus \mathscr{E}_{x_2}) \times_{\tilde{Q}} \{ \underset{i \in I}{\times_{\tilde{Q}}} \text{Flag}_{(1,2)}(\mathscr{E}_{y_i}) \}$$

(3) Define

$$\mathbf{G}' \equiv \text{Grass}_{P(m)}(\mathbf{C}^{\tilde{n}} \otimes W) \times \text{Grass}_2(\mathbf{C}^{\tilde{n}} \otimes \mathbf{C}^2) \times \underset{i}{\times} \{ \text{Grass}_2(\mathbf{C}^{\tilde{n}}) \times \text{Grass}_1(\mathbf{C}^{\tilde{n}}) \}.$$

(4) Define the polaristion on $\mathbf{G}'$:

$$\frac{(\tilde{l} - k)}{m} \times k \times \underset{i}{\times} \{ (k - \beta_i), (\beta_i - \alpha_i) \}. \qquad (\text{B.2})$$

(5) Replace (A.5) by

$$\frac{(\tilde{l}-k)}{m}(hP(m) - \tilde{n}\dim P(H\otimes W)) + k(2h - \tilde{n}\dim P_G(H\otimes \mathbf{C}^2))$$

$$+\sum_i (k-\beta_i)(2h - \tilde{n}\dim P_{2,i}(H)) + \sum_i (\beta_i - \alpha_i)(h - \tilde{n}\dim P_{1,i}(H)) \underset{(\text{resp. } \leqq)}{<} 0,$$

where P_G is the projection in the second factor of $\mathbf{G}'$.

(6) Replace (A.6) by

$$\frac{(\tilde{l}-k)}{m}(\chi(E')P(m) - \tilde{n}\chi(E'(m))) + k(2\chi(E') - \tilde{n}h^0(Q^{E'}))$$

$$+\sum_i (k-\beta_i)(2\chi(E') - \tilde{n}h^0(Q^{E'}_{2,i})) + \sum_i (\beta_i - \alpha_i)(\chi(E') - \tilde{n}h^0(Q^{E'}_{1,i})) \underset{(\text{resp. } \leqq)}{<} 0.$$

The rest of the proof of Theorem X1 goes through with obvious modifications *except that we cannot assume that the sheaves involved are torsion-free at* x_1 *and* x_2. The fact that $\tilde{\mathscr{R}}'^{ss}$ is reduced, irreducible and normal is proved in Appendix C (Lemma C.2 and Proposition C.3).

For example, the analogue of Proposition A.11 is the following result. (We denote a point of $\mathrm{Grass}_2(\mathbf{C}^n\otimes\mathbf{C}^2)$ by p_2.)

Proposition B.3. *There exists* N' *and* M' *such that for* $\tilde{n}\geqq N'$ *and* $m\geqq M'$ *the following is true. A point* $(p, p_2, \{(p_{2,i}, p_{1,i})\}_I)\in\tilde{\mathscr{R}}'$ *is GIT-stable (respectively, GIT-semistable) iff the quotient E is torsion-free outside* $\{x_1, x_2\}$ *and a stable (respectively, semistable) generalised parabolic sheaf, and the map* $\mathbf{C}^{\tilde{n}}\to H^0(E)$ *is an isomorphism.*

Remark B.4. Note that if (E, Q) is a semistable GPS, $\mathrm{Tor}\, E$ is supported on the reduced subscheme $\{x_1, x_2\}$ and

$$(\mathrm{Tor}\, E)_{x_1}\oplus(\mathrm{Tor}\, E)_{x_2}\subsetneq Q. \tag{B.3}$$

This follows from (B.1).

Remark B.5. The above construction of the moduli space also shows that $\tilde{\mathscr{R}}^{ss}$ is open in $\tilde{\mathscr{R}}'$ and hence, by a standard argument, semistability is an open property for GPS's.

Bb. S-equivalence of generalised parabolic sheaves

We enlarge the category of GPS's by adopting the following more general definition. For simplicity we assume that no "ordinary" parabolic points are present. It should be noted that the detailed description of s-equivalence given below (Proposition B.15) is not really needed. Corollary B.17 and Proposition C.7(4) are the 1only places where it is used; and one can give direct proofs of these without using Proposition B.15.

Definition B.6. A *generalised m-parabolic structure* on a sheaf E over the divisor $\{x_1, x_2\}$ is a choice of an m-dimensional quotient Q of $E_{x_1}\oplus E_{x_2}$. A sheaf with a generalised m-parabolic structure will be called a m-GPS, or GPS for short.

A GPS E is said to be *stable* (respectively, *semistable*) if E is torsion-free outside $\{x_1, x_2\}$, and

(1) if rank $E > 0$ then for every proper subsheaf E' such that E/E' is torsion-free outside $\{x_1, x_2\}$ we have

$$\text{rank } E(\text{degree } E' - \dim Q^{E'}) \underset{(\text{resp. } \leq)}{<} \text{rank } E'(\text{degree } E - m). \qquad \text{(B.4)}$$

(2) If rank $E = 0$, then we have $E_{x_1} \oplus E_{x_2} = Q$ and $\dim Q = 1$ (respectively $E_{x_1} \oplus E_{x_2} = Q$).

(For any subsheaf E', we denote by $Q^{E'}$ the image of $E'_{x_1} \oplus E'_{x_2}$ in Q).

Definition B.7. If (E, Q) is a GPS, and rank $E > 0$ set

$$\mu_G[(E, Q)] = \frac{(\text{degree } E - \dim Q)}{\text{rank } E}$$

Examples B.8. (1) Any torsion-sheaf τ supported on $\{x_1, x_2\}$ is in a canonical way a semistable GPS: one takes $Q = \tau_{x_1} \oplus \tau_{x_2}$. Such a GPS is stable iff degree $\tau = 1$.

(2) A line bundle L with a one-dimensional quotient Q of $L_{x_1} \oplus L_{x_2}$ is a semistable GPS. It is stable iff each map $L_{x_j} \to Q$ is nonzero.

(3) A line bundle L with a two-dimensional quotient Q of $L_{x_1} \oplus L_{x_2}$ is a semistable 2-GPS. It is never stable.

It is useful to think of a m-GPS as a sheaf E on $\tilde{X}$ together with a map $\pi_* E \to {}_{x_0}Q \to 0$, with Q being thought of as a sheaf on X supported on the reduced point x_0, with $h^0(Q) = m$. In this subsection we will omit the (pre-)subscript x_0. Let K_E denote the kernel of the sheaf map $\pi_* E \to Q$.

Definition B.9. *A morphism of GPS's* $(E, Q) \to (E'', Q'')$ is a sheaf map $E \to E''$ which maps K_E to $K_{E''}$ (and therefore induces a map $Q \to Q''$).

Definition B.10. Given an exact sequence $0 \to E' \to E \to E'' \to 0$ of sheaves on $\tilde{X}$, and $\pi_* E \to Q \to 0$ a GP structure on E we define GP structures on E' and E'' via the diagram:

$$\begin{array}{ccccccccc} 0 & \to & \pi_* E' & \to & \pi_* E & \to & \pi_* E'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Q' & \to & Q & \to & Q'' & \to & 0 \end{array}$$

(The first horizontal sequence is exact because π is finite, Q' is defined as the image in Q of $\pi_* E'$ so that the first vertical arrow is onto, Q'' is defined by demanding that the second horizontal sequence is exact, and finally the third vertical arrow is onto by the snake lemma.) We will sometimes write $0 \to (E', Q') \to (E, Q) \to (E'', Q'') \to 0$; the meaning of such a sequence is clear.

A morphism $(E, Q) \to (E'', Q'')$ of GPS's factors:

$$\begin{array}{ccccc} (E, Q) & \to & (E'', Q'') & & \\ \downarrow & & \uparrow & & \\ (W, Q_1) & \to & (W, Q''_1) & \to & 0 \\ \downarrow & & \uparrow & & \\ 0 & & 0 & & \end{array}$$

We have the following Lemmas, whose proofs we omit.

Lemma B.11. *Let (E, Q) be a GPS with* rank $E > 0$, *and suppose E is torsion-free outside* $\{x_1, x_2\}$. *Then the following are equivalent:*

(1) (E, Q) *is (semi)stable.*

(2) *For every proper sub-GPS* (E', Q') *we have*

$$\operatorname{rank} E(\operatorname{degree} E' - \dim Q') \underset{(\text{resp. } \leq)}{<} \operatorname{rank} E'(\operatorname{degree} E - \dim Q).$$

(3) *For every proper quotient GPS*(E'', Q'') *we have*

$$\operatorname{rank} E(\operatorname{degree} E'' - \dim Q'') \underset{(\text{resp. } \geq)}{<} \operatorname{rank} E''(\operatorname{degree} E - \dim Q).$$

Lemma B.12. *Let* $(E, Q) \to (E'', Q'')$ *be a morphism of semistable GPS's. Assume that if* rank $E \neq 0$ *and* rank $E'' \neq 0$, *then* $\mu_G[(E, Q)] = \mu_G[(E'', Q'')]$. *Then the kernel and cokernel are semistable GPS's. If both* (E, Q) and (E'', Q'') *are stable GPS's the morphism must be an isomorphism or zero.*

Proposition B.13. *Fix* μ *a rational number. Then the category of semistable GPS's* (E, Q) *such that* rank $E = 0$ *or,* rank $E > 0$, *with* $\mu_G[(E, Q)] = \mu$, *is an abelian, artinian, noetherian category whose simple objects are the stable GPS's in the category.*

One can conclude as usual that given a semi-stable GPS it has a Jordan-Holder filtration.

Definition B.14. Two semistable GPS's are said to be *s-equivalent* if they have the same "associated graded" GPS.

Proposition B.15. *The s-equivalence classes of* rank 2 *2-GPS's are the following:*

(1) *If* (E, Q) *is a stable GPS then E is necessarily a vector bundle, and both maps* $E_{x_j} \to Q$ *are isomorphisms. Two such GPS's are s-equivalent iff they are isomorphic.*

(2) *If d is even, consider GPS's* (E, Q) *such that E is an extension* $0 \to L_1 \to E \to L_2 \to 0$ *with* degree $L_p = d/2$, $p = 1, 2$, *and such that the induced parabolic structure on* L_1 *is stable (i.e. the maps* $(L_1)_{x_j} \to Q$ *have the same one-dimensional image* Q_1 – *denote by* Q_2 *the quotient* Q/Q_1.) *All such GPS's with* (L_1, Q_1) *and* (L_2, Q_2) *fixed form an s-equivalence class.*

(3) *Consider extensions* $0 \to \tilde{E} \to E \to \tau \to 0$, *or extensions* $0 \to \tau \to E \to \tilde{E} \to 0$, *with* τ *a torsion-sheaf of degree* 1 *supported at* x_1, *with the induced structure on* $\tilde{E}$ *that of a stable* 1-*GPS–denote by* $(\tilde{E}, \tilde{Q})$ *this structure. All such GPS's with* $(\tilde{E}, \tilde{Q})$ *fixed form an s-equivalence class. (Included in this equivalence class is the case when E is locally-free, the map* $E_{x_2} \to Q$ *has one-dimensional image* $\tilde{Q}$, *the map* $E_{x_1} \to Q$ *is an isomorphism, and* $\tilde{E}$ *is the kernel of the sheaf map* $E \to Q/\tilde{Q} \to 0$, $Q/\tilde{Q}$ *being thought of as a sheaf supported at* x_1.)

(4) *If d is even, consider extensions as in the previous case, with* $\tilde{E}$ *an extension* $0 \to L_1 \to \tilde{E} \to L_2 \to 0$ *or* $0 \to L_2 \to E \to L_1 \to 0$ degree $L_1 = d/2$, *degree* $L_1 = d/2 - 1$, *the induced generalised parabolic structure on* L_1 *is stable, and that on* L_2 *trivial. Such GPS's with fixed* $(L_1, \tilde{Q})$ *and* L_2 *form an s-equivalence class.*

(5) *Same as* (3) *with* x_2 *in place of* x_1.

(6) *Same as* (4) *with* x_2 *in place of* x_1.

(7) (i) *Extensions* $0 \to \tilde{E} \to E \to \tau_1 \oplus \tau_2 \to 0$, *or extensions* $0 \to \tau_1 \oplus \tau_2 \to E \to \tilde{E} \to 0$, *with* τ_j *a torsion-sheaf of* degree 1 *supported at* x_j, *with the induced generalised parabolic structure on* $\tilde{E}$ *trivial,* $\tilde{E}$ *a stable bundle.* (ii) *Extensions*

$0 \to \tilde{E}_1 \to E \to \tau \to 0$, *or extensions* $0 \to \tau_1 \to E \to \tilde{E}_1 \to 0$, *with the induced structure on* $\tilde{E}_1$ *that of a unstable 1-GPS, with* $\tilde{E}_1$ *in turn an extension of* τ_2 *by* $\tilde{E}$, *with the induced parabolic structure on* $\tilde{E}$ *trivial.* (iii) *The same as* (ii) *with the roles of* x_1 *and* x_2 *reversed. All such GPS's with a fixed* $\tilde{E}$ *form an s-equivalence class.* (*Included in this equivalence class are the cases when E is locally-free, the maps* $E_{x_j} \to Q$ *have one-dimensional images* Q_j, *and* $\tilde{E}$ *is the kernel of the sheaf map* $E \to Q_1 \oplus Q_2$, *the* Q_j *being thought of as sheaves supported on the* $\{x_j\}$.)

(8) *If d is even, the same as above, with* $\tilde{E}$ *an extension* $0 \to L_1 \to E \to L_2 \to 0$, degree $L_p = d/2 - 1$.

(9) *Extensions* $0 \to \tilde{E} \to E \to \tau \to 0$, *or extensions* $0 \to \tau \to E \to \tilde{E} \to 0$, *with* τ *a torsion-sheaf of* degree 2 *supported at* x_1, *with the induced generalised parabolic structure on* $\tilde{E}$ *trivial,* $\tilde{E}$ *a stable bundle. All such extensions, with* $\tilde{E}$ *fixed, form an s-equivalence class.* (*Included in this equivalence class is the case when E is locally-free, the map* $E_{x_2} \to Q$ *is zero.*)

(10) *The same as above, with* $\tilde{E}$ *an extension* $0 \to L_1 \to E \to L_2 \to 0$, degree $L_p = d/2 - 1$, $p = 1, 2$.

(11) *Same as* (9) *with* x_2 *in place of* x_1.

(12) *Same as* (10) *with* x_2 *in place of* x_1.

Remark B.16. In case (3) above the Jordan–Holder filtration has two terms, with one of the factors a torsion sheaf of length one and the other a stable 1-GPS. In case (7) and (9) the filtration has three terms, with one term a stable rank two bundle and the other two torsion sheaves of length one each.

Corollary B.17. *Every semistable GPS*(E', Q') *is sequivalent to a semistable GPS* (E, Q) *with E locally free.*

Appendix C. The singularities of moduli space of generalised parabolic sheaves

The notation of the previous Appendix holds. For simplicity we assume $|I| = 0$. Including ordinary parabolic points makes no difference to the following considerations.

Notation C.1. Define $\mathscr{H}$ to be the set of (closed) points $(\mathcal{O}^{\bar{n}} \to E \to 0, Q)$ in $\tilde{\mathscr{R}}'$, where $\mathbf{C}^{\bar{n}} \to H^0(E)$ is an isomorphism, $H^1(E(-x_1 - x_2 - x)) = 0$ for $x \in \tilde{X}$, and

(T) $\operatorname{Tor} E$ is supported on the reduced subscheme $\{x_1, x_2\}$ and $(\operatorname{Tor} E)_{x_1} \oplus (\operatorname{Tor} E)_{x_2} \subsetneq Q$.

Requiring that $H^1(E(-x_1 - x_2 - x)) = 0$ ensures that $H^1(E) = 0$, E is generated by sections, $H^0(E) \to E_{x_1} \oplus E_{x_2}$ is onto, and $E(-x_1 - x_2)$ is generated by sections.

We will see below that $\mathscr{H}$ is the set of closed points of an open subscheme of $\tilde{\mathscr{R}}'$. We will continue to denote this subscheme by $\mathscr{H}$. Clearly then

$$\tilde{\mathscr{R}}'^{ss} \underset{\text{open}}{\subsetneq} \mathscr{H} \underset{\text{open}}{\subsetneq} \tilde{\mathscr{R}}'.$$

Lemma C.2. *The set of points where the conditions of Notation C.1 hold is open.* $\mathscr{H}$ *is irreducible, as in* $\tilde{\mathscr{R}}'^{ss}$.

Proof. We first check that $\mathscr{H}$ is open. Consider the flat family of sheaves F on X, parametrised by $\tilde{R}'$, constructed as in §4b via the sequence:

$$0 \to \mathscr{F} \to (\pi_X I_{\tilde{\mathscr{R}}'})^* \mathscr{E} \to {}_{x_0}\mathscr{Q} \to 0.$$

Consideration (T) precisely determines the points (E, Q) where F is torsion-free on X (Lemma 4.6(1)). This can be seen to be an open condition using [EGA-IV, (12.2.1)]. The other conditions in the definition of $\mathscr{H}$ are clearly open.

Next we prove the irreducibility of $\mathscr{H}$ (which, clearly, implies that of $\tilde{\mathscr{R}}'^{ss}$.) Let $\tilde{\mathscr{R}}'_F$ be the open subscheme of $\mathscr{H}$ consisting of locally-free sheaves. This is a grassmannian bundle over $\tilde{\mathbf{Q}}_F$ (4.5b). That $\tilde{\mathbf{Q}}_F$ is irreducible is easy to see by a standard argument [N, Remark 5.5]; hence so is $\tilde{\mathscr{R}}'_F$. We will show, in the course of the proof of the next proposition, that $\tilde{\mathscr{R}}'_F$ is dense in $\mathscr{H}$. □

Proposition C.3. *$\mathscr{H}$ is reduced, normal, Gorenstein and has rational singularities. Hence the same holds for $\tilde{\mathscr{R}}'^{ss}$.*

Proof. The claim is obvious at a point (E, Q) corresponding to a torsion-free sheaf, where in fact the space is smooth.

We divide the rest of the proof into steps. Let $(\mathcal{O}^{\tilde{n}} \to E \to 0, Q)$ be a point of $\mathscr{H}$, with P denoting the projection $E_{x_1} \oplus E_{x_2} \to Q$, and assume E is not locally free. We shall use Lemma 4.18 and Proposition 4.19 without comment.

Step 1. The simplest nontrivial case is when $\tau_{x_2} = 0$ and the map $\mathscr{E}_{x_1} \to \mathscr{Q}$ is surjective at (E, Q) and hence in an open neighbourhood $\mathbf{U} \subset \tilde{\mathscr{R}}'$. Define the sheaf $\tilde{\mathscr{E}}$ in this neighbourhood by the exact sequence $0 \to \tilde{\mathscr{E}} \to \mathscr{E} \to {}_{x1}\mathscr{Q} \to 0$ (where ${}_{x1}\mathscr{Q}$ is the sheaf on $\tilde{X} \times \tilde{\mathscr{R}}'$ got by pulling back $\mathscr{Q}$ from $\tilde{\mathscr{R}}'$ and then restricting to $\{x_1\} \times \tilde{\mathscr{R}}'$); at the point (E, Q) we have, with obvious notation $0 \to \tilde{E} \to E \to {}_{x_1}Q \to 0$. It follows from the definition of $\mathscr{H}$ that $\tilde{E}$ is locally free, $\dim H^0(\tilde{E}) = \tilde{n} - 2$, $\tilde{E}$ is generated by global sections, and $H^1(\tilde{E}) = 0$ – all this will continue to be true in a possibly smaller neighbourhood, say $\mathbf{U}'$. (To see why $\tilde{E}$ is generated by sections use the following fact: there are exact sequences $0 \to \tilde{E}' \to \tilde{E} \to \tau_1 \to 0$ and $0 \to \tau_2 \to E(-x_1 - x_2) \to \tilde{E}' \to 0$ where $\tilde{E}'$ is the image of the map $E(-x_1 - x_2) \to E$, and τ_1 and τ_2 are torsion sheaves.)

Consider the fibre-product $\mathbf{B}$ of the frame bundle of the zeroth direct image of $\tilde{\mathscr{E}}$ onto $\mathbf{U}'$ with the frame bundle of $\mathscr{Q}$. One has a smooth morphism $\mathbf{B} \to \mathbf{U}'$.

Let now $\tilde{\mathbf{Q}}^1$ be the Quot scheme of rank 2 degree $d-2$ quotients $\mathcal{O}^{\tilde{n}-2} \to \tilde{E} \to 0$, and $\tilde{\mathbf{Q}}^1_F$ the open subset of locally-free quotients with vanishing first cohomology such that the map $\mathbf{C}^{\tilde{n}-2} \to H^0(\tilde{E})$ is an isomorphism. The space $\tilde{\mathbf{Q}}^1_F$ is smooth. Consider on $\tilde{\mathbf{Q}}^1_F$ the bundle $\mathbf{E} \equiv \mathrm{Ext}^1({}_{x_1}\mathbf{C}^2, \tilde{\mathscr{E}})$ of extensions [La] where ${}_{x_1}\mathbf{C}^2$ is the skyscraper sheaf on the (reduced) point x_1 with $\mathbf{C}^2$ as fibre. On $\tilde{X} \times \mathbf{E}$ there is an exact sequence of sheaves flat over $\mathbf{E}$:$0 \to \tilde{\mathscr{E}} \to \mathscr{E} \to {}_{x_1}\mathbf{C}^2 \to 0$. Let $\mathbf{W}$ denote the total space of the vector bunle $\mathrm{Hom}(\mathscr{E}_{x_2}, \mathcal{O}^2)$ on $\mathbf{E}$.

There is a smooth morphism $\mathbf{B} \to \mathbf{W}$. On the other hand $\mathbf{W}$ is smooth which shows the same for the original point (E, Q).

Step 2. We next turn to the case when $\tau_{x_2} = 0$, $\tau_{x_1} \neq 0$, and the map $(\mathscr{E})_{x_1} \to \mathscr{Q}$ is not surjective. Let $\mathbf{F}$ denote the frame-bundle of $\mathscr{Q}$, and consider a point $(Fr{:}Q \to \mathbf{C}^2) \in \mathbf{F}$ above (E, Q) where $p_1 \circ Fr \circ P$: $\tau_{x_1} \to \mathbf{C}$ is an isomorphism (p_1 denoting the projection to the first co-ordinate $\mathbf{C}^2 \to \mathbf{C}$.). The map $P_1 \equiv p_1 \circ Fr \circ P$: $\mathscr{E}_{x_1} \to \mathbf{C}$ is nonzero in some neighbourhood, say $\mathbf{F}_1$. On $\tilde{X} \times \partial\mathbf{F}_1$ define $\tilde{\mathscr{E}}$ by the sequence $0 \to \tilde{\mathscr{E}} \to \mathscr{E} \xrightarrow[P_1]{} {}_{x1}\mathbf{C} \to 0$. As in Step 1 one sees that in a possibly smaller neighbourhood $\mathbf{F}_2$, $\tilde{\mathscr{E}}$ is locally-free, $H^0(\tilde{E})$ generates $\tilde{E}$, $h^0(\tilde{E}) = \tilde{n} - 1$ and $H^1(\tilde{E}) = 0$. Let $\mathbf{B} \to \mathbf{F}_2$ now be the bundle of frames of the direct image of $\tilde{\mathscr{E}}$ with respect to $\pi_{\mathbf{E}_2}$.

On the other hand let $\tilde{\mathbf{Q}}^1$ be the Quot scheme of rank 2 degree $d-1$ quotients $\mathcal{O}^{\tilde{n}-1}\tilde{\mathcal{E}} \to 0$, and let $\tilde{\mathbf{Q}}^1_F$ denote the open subset of locally-free quotients with vanishing first cohomology such that $\mathbf{C}^{\tilde{n}-1} \to H^0(\tilde{E})$ is an isomorphism. Let $\mathbf{E} \equiv \mathrm{Ext}^1({}_{x_1}\mathbf{C}, \tilde{\mathcal{E}})$ be the bundle of extensions [La] where ${}_{x_1}\mathbf{C}$ is the skyscraper sheaf on the (reduced) point x_1 with $\mathbf{C}$ as fibre. On $\tilde{X} \times \mathbf{E}$ there is an exact sequence of sheaves flat over $\mathbf{E}$: $0 \to \tilde{\mathcal{E}} \to \mathcal{E} \to {}_{x_1}\mathbf{C} \to 0$. Let $\mathbf{W}$ denote the total space of the vector bundle $\mathrm{Hom}(\tilde{\mathcal{E}}_{x_2}, \mathcal{O}^2)$ on $\mathbf{E}$. Finally let $\mathbf{V} \equiv \mathbf{V}(\mathcal{E}_{x_1}) = \mathrm{Spec}(\mathbf{S}(\mathcal{E}_{x_1}))$ be defined as in [EGA-I, (9.4.8)].

There is a smooth morphism $\mathbf{B} \to \mathbf{V} \times_{\mathbf{E}} \mathbf{W}$. We need, therefore, to analyse the singularities of $\mathbf{V}$. The map $\mathbf{V} \times_{\mathbf{E}} \mathbf{W} \to \tilde{\mathbf{Q}}^1_F$ is locally trivial, so clearly we can hold $\tilde{E}$ fixed for this purpose. Lemma C.4 concludes the proof in this case.

Step 3. We next consider the case when both τ_{x_1} and τ_{x_2} are one-dimensional. The nontrivial case is when $(\mathcal{E})_{x_j} \to \mathcal{Q}$ is not surjective at either point. (The other cases can be reduced to at most a combination of the two earlier ones.) We now imitate Step 2 and reduce the proof to Lemma C.6 below. □

Lemma C.4. *Let $\tilde{E}$ be a* rank 2 *locally-free sheaf on $\tilde{X}$, let $x \in \tilde{X}$ be a smooth point. Let* $\mathbf{E} \equiv \mathrm{Ext}^1({}_x\mathbf{C}, \tilde{E})$ *and consider the universal extension* $0 \to \tilde{E} \to E \to {}_x\mathbf{C} \to 0$ *on* $\tilde{X} \times \mathbf{E}$. *Then the space* $\mathbf{V}(E_x)$ (cf., [EGA-I, (9.4.8)]) *is reduced, normal, Gorenstein, with rational singularities.*

Proof. Clearly we can replace $\tilde{X}$ by an affine neighbourhood of x where $\tilde{E}$ is trivial, and then by using Noether normalisation, by the affine line $\mathbf{A}^1$. We let w denote the affine co-ordinate, and identify $\tilde{E} \sim \mathcal{O}^2$. We have then natural co-ordinates (u_1, u_2) on $\mathbf{E}$.

Let E be the sheaf on $\mathbf{A}^1 \times \mathbf{E}$ defined by the exact sequence: $0 \to \mathcal{O} \to \mathcal{O}^3 \to E \to 0$, the map $\mathcal{O} \to \mathcal{O}^3$ being $\tilde{h} \mapsto (u_1\tilde{h}, u_2\tilde{h}, -\omega\tilde{h})$. The inclusion $\mathcal{O}^2 \to \mathcal{O}^3$ given by $(f, g) \mapsto (f, g, 0)$ induces an inclusion $\mathcal{O}^2 \to E$. We have thus the diagram on $U \times \mathbf{E}$, the middle horizontal sequence eing split:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O} & = & \mathcal{O} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}^2 & \to & \mathcal{O}^3 & \to & \mathcal{O} & \to & 0 \\
 & & =\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}^2 & \to & E & \to & {}_x\mathbf{C} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It is clear that E is the universal extension that we seek. (The map $\mathcal{O} \to \mathcal{O}$ is given by $\tilde{h} \mapsto wh$.)

Restricitng to $\{x\} \times \mathbf{E}$ we get

$$0 \to \mathcal{O} \xrightarrow{a} \mathcal{O}^3 \to E_x \to 0. \tag{C.3}$$

The map a is given by $\tilde{h} \mapsto \tilde{h}(u_1, u_2)$ (and is therefore an injection.) This shows that $\mathbf{V}(E_x)$ is the subscheme of $\mathbf{V}(\mathcal{O}^3)$ defined as follows. The scheme $\mathbf{V}(\mathcal{O}^3)$ is the total space of the dual bundle of $\mathcal{O}^3$; with respect to the natural co-ordinates (u_1, u_2, v_1, v_2, s) on $\mathbf{V}(\mathcal{O}^3)$ the subscheme defined by the ideal $(u_1v_1 + u_2v_2)$ is $\mathbf{V}(E_x)$. This is the product of the affine line with the cone over the nonsingular quadric surface in P^3, and is easily seen to be reduced, normal and Gorenstein; also, it has a rational singularity at the vertex. □

Remark C.5. (a) $\mathscr{H}$ *is not* locally factorial. It is well-known that the cone over the nonsingular quadric in $\mathbf{P}^3$ is not factorial at the vertex, with class group equal to $\mathbf{Z}$.

(b) The canonical map c: $E_x \to \mathcal{O}$ on $\mathbf{V}(E_x)$ is induced by the map $\mathcal{O}^3 \to \mathcal{O}$, $(f, g, h) \mapsto fv_1 + gv_2 + hs$.

(c) The locus of non-locally-free extensions is given by the (non-Cartier) divisor defined by the ideal (u_1, u_2).

(d) Let c be the map $E_x \to \mathcal{O}$ on $\mathbf{V}(E_x)$ defined in (b) above, and let b be the map obtained by restricting the map $E \to {}_x\mathbf{C}$. Consider the map $E_x \to \mathcal{O}^2 \sim Q$, given by $t \mapsto (c(t), b(t))$. In the complement of the non-free locus this map is of rank one precisely when $\ker c = \ker b = \mathcal{O}^2$. This yields the equation $v_1 = 0$, $v_2 = 0$.

(e) $\hat{\mathscr{D}}_1^f \backslash \hat{\mathscr{D}}_{1,F}$ has codimension $\geqq 3$ in $\hat{\mathscr{D}}_1^f$. This follows from (c–d).

Lemma C.6. *Let* $\tilde{E}$ *be a* rank 2 *locally-free sheaf on* $\tilde{X}$, *let (for* $j = 1, 2$) $x_j \in \tilde{X}$ *be smooth points. Let* $\mathbf{E}' \equiv \operatorname{Ext}^1({}_{x_1}\mathbf{C} \oplus {}_{x_2}\mathbf{C}, \tilde{E})$ *and consider the universal extension* $0 \to \tilde{E} \to E \to {}_{x_1}\mathbf{C} \oplus {}_{x_2}\mathbf{C} \to 0$ *on* $\tilde{X} \times \mathbf{E}'$. *Then the space* $\mathbf{V}(E_{x_1} \oplus E_{x_2})$ *is normal, Gorenstein, with rational singularities.*

Proof. Clear extension of the proof of Lemma C.4. □

We are now in a position to state

Theorem X3. $\mathscr{P}$ *is reduced, irreducible and normal, with rational singularities.*

Proof. We use Lemma 4.18 and Proposition 4.19. These are all then immediate consequences of well-known properties of GIT quotients. The relevant result about rational singularities is that of [Bo]. □

The codimension one subschemes $\hat{\mathscr{D}}_j^f$ and $\hat{\mathscr{D}}_j^t$ in $\mathscr{H}$ are defined in §4a, and also the subscheme $\hat{\mathscr{V}}_j^f$ of each $\hat{\mathscr{D}}_j^f$. The following description of the varieties $\mathscr{D}_j$ should be kept in mind; it follows easily from Proposition B.14: $\mathscr{D}_1$ consists of s-equivalence classes of GPS's such that the "associated graded" GPS has torsion at x_2.

Proposition C.7. (1) *The* $\hat{\mathscr{D}}_j^f$ *are reduced, irreducible, and normal.*

(2) *The* $\hat{\mathscr{D}}_j^t$ *are reduced, irreducible, and normal.*

(3) *The* $\hat{\mathscr{V}}_j^f$ *are smooth. We have* $\hat{\mathscr{V}}_j^f \cap \{\hat{\mathscr{D}}_1^f \cap \hat{\mathscr{D}}_2^f\} = \emptyset$.

(4) *The closed orbits in* $\hat{\mathscr{D}}_j^f$ *and* $\hat{\mathscr{D}}_j^t$ *are contained in* $\hat{\mathscr{D}}_j^f \cap \hat{\mathscr{D}}_j^t$.

Proof. We will prove these claims for $j = 1$. The proofs depend heavily on the local description of $\mathscr{H}$ obtained during the proof of Proposition C.3.

(1) We will give the proof of (1) in some detail. By definition $\hat{\mathscr{D}}_1^f$ is reduced. The divisor $\hat{\mathscr{D}}_{1,F}$ is irreducible, hence so is its closure. Normality of $\hat{\mathscr{D}}_{1,F}$ is also clear (because, for example, it is a complete intersection and the singular set, $\hat{\mathscr{V}}_{1,F}$ has codimension 2). It remains to prove normality of $\hat{\mathscr{D}}_1^f$ at points of $\hat{\mathscr{D}}_1^f \backslash \hat{\mathscr{D}}_{1,F}$. By semicontinuity, at such a point (E, Q), the map $E_{x_1} \to Q$ must be either zero or have one-dimensional image.

(i) Suppose first that E is locally free at x_1. Then it is not so at x_2 and the local model of $\mathscr{H}$ at such a point is either as in Step 1 (if $E_{x_2} \to Q$ is surjective) or as in Step 2 (if $E_{x_2} \to Q$ has one-dimensional image.) of the proof of C.3. Note, however, that the roles of x_1 and x_2 are reversed vis-à-vis that proof. In either case the inverse image of $\hat{\mathscr{D}}_1^f$ by the smooth map $\mathbf{B} \to \mathbf{U}'$ (respectively, $\mathbf{B} \to \mathbf{F}_2$) is the pull-back, in turn, of $\tilde{\mathscr{D}}$ via the map $\mathbf{B} \to \mathbf{W}$ (respectively, $\mathbf{B} \to \mathbf{V} \times_E \mathbf{W}$) where $\mathbf{W}$ is the total space of the vector bundle $\mathrm{Hom}(\tilde{\mathscr{E}}_{x_1}, \mathcal{O}^2)$ and $\tilde{D} \subset \mathbf{W}$ is defined by the determinantal ideal. In either case we have normality.

(ii) If E is not locally free at x_1, $E_{x_1} \to Q$ must have one-dimensional image, and there are again two cases to consider: (1) If E is locally free at x_2 the local model is the divisor given by the ideal (x, y) in $\mathbf{C}[u, v, x, y]/(ux + vy)$ (C.5(d)). (2) If E is not locally free at x_2 the local model is the product of the above with another normal variety.

(2) We prove irreducibility. Consider the open subset of $\hat{\mathscr{D}}_1^t$ where the torsion subsheaf has degree 1; this set is easily seen to be dense. Such a sheaf E is necessarily of the form $\tilde{E} \oplus \mathbf{C}_{x_1}$, with $\tilde{E}$ generated by global sections. It is now straightforward to imitate the proof of [N, Remark 5.5]. The other facts are proved as in (ii) above. The relevant result is C.5 (4).

(3) It is easily seen that $\hat{\mathscr{V}}_1^f$ is the set of (E, Q) such that the map $E_{x_1} \to \mathscr{Q}$ is zero. E is therefore locally free at x_1, and the map $E_{x_2} \to \mathscr{Q}$ surjective. The local model is as in Step 1 if E is not locally free at x_2. In any case it is clear that $\hat{\mathscr{V}}_1^f$ is smooth. The other statements have similar lproofs.

(4) This follows from Proposition B.15. □

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