ON ARITHMETIC FUNCTIONS

BY P. KESAVA MENON
(*Annamalai University*)

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Introduction

IN a paper on the theory of multiplicative arithmetic functions R. Vaidyanathaswamy has investigated two operations on arithmetic functions—'Composition' and 'Compounding'. In the same paper he has introduced the notion of a 'principal function' and has applied it to prove an interesting theorem on multiplicative functions which are functions of the g.c.d. of the arguments, the proof being by the method of 'generating series'. The objects of the present paper are (1) to study an operation which generalizes composition and compounding, (2) to study in detail principal functions, to give a purely arithmetic proof of Vaidyanathaswamy's result mentioned above and to obtain other similar results, (3) to evaluate certain multiple Dirichlet series, and (4) to evaluate certain multiple Lambert series.

The notions and notations in Vaidyanathaswamy's paper are employed here.

Section 1. *A Generalised Composition.*

1. Let (D) denote the matrix

$$\begin{pmatrix} d_{11} & \cdots & d_{1r} \\ \vdots & & \vdots \\ d_{s1} & \cdots & d_{sr} \end{pmatrix}$$

and let f and ψ be two arithmetic functions of a $(r \times s)$ matrix set of arguments. Consider the function

$$\Sigma f(\mathrm{D}),$$

the summation being for all divisions d_{ij} such that $\prod_{i=1}^{s} d_{ij} = \mathrm{M}_j \; (j = 1, 2, \ldots r)$ and $\psi(\mathrm{D}) = 1$. We shall show that this operation on f includes composition and compounding as particular cases.

Let the number of rows in the matrix be two and suppose that

$$f\begin{pmatrix} \mathrm{M}_1, \ldots, \mathrm{M}_r \\ \mathrm{N}_1, \ldots, \mathrm{N}_r \end{pmatrix} = f_1(\mathrm{M}_1, \ldots, \mathrm{M}_r) \times f_2(\mathrm{N}_1, \ldots, \mathrm{N}_r).$$

If we take ψ to be E (the function whose value is unity for all values of the arguments) it is clear that the above operation reduces to the composition of f_1 and f_2 $(f_1 \cdot f_2)$.

Let g (M, N) denote the greatest common divisor (g.c.d.) of M and N. If we take $\psi\begin{pmatrix} M_1, & \ldots, & M_r \\ N_1, & \ldots, & N_r \end{pmatrix}$ to be $\prod_{i=1}^{r} g(M_i, N_i)$, then it can easily be seen that the generalized operation is simply the compounding of f_1 and f_2 $(f_1 \oplus f_2)$.

THEOREM 1: If f and ψ are multiplicative functions the latter being positive integral valued, then

$$\sum_{\substack{\psi(D)=1 \\ \prod_{i=1}^{s} d_{ig} = M_g \ (g=1,\cdots,r)}} f(D) = F(M_1, \ldots, M_r)$$

is a multiplicative function of r arguments.

Proof.—Let $\prod_{i=1}^{r} M_i$ be prime to $\prod_{i=1}^{r} N_i$ and let (G), (DG) denote the matrices

$$\begin{pmatrix} \delta_{11}, & \ldots\ldots, & \delta_{1r} \\ \vdots & & \vdots \\ \delta_{s1}, & \ldots\ldots, & \delta_{sr} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d_{11}\,\delta_{11}, & \ldots\ldots, & d_{1r}\,\delta_{1r} \\ \vdots & & \vdots \\ d_{s1}\,\delta_{s1}, & \ldots\ldots, & d_{sr}\,\delta_{sr} \end{pmatrix}$$

respectively. Then it follows that

$$F(M_1N_1, \ldots, M_rN_r) = \sum_{\substack{\psi(DG)=1 \\ \prod_{i=1}^{s} d_{ig}=M_g,\ \prod_{i=1}^{s} \delta_{ig}=N_g,\ (g=1,\cdots,r)}} f(DG)$$

$$= \sum_{\psi(D)\psi(G)=1} f(D) f(G)$$

$$= \Big\{ \sum_{\substack{\psi(D)=1 \\ \prod_{i=1}^{s} d_{ig}=M_g}} f(D) \Big\} \times \Big\{ \sum_{\substack{\psi(G)=1 \\ \prod_{i=1}^{s} \delta_{ig}=N_g}} f(G) \Big\} \quad (g=1, 2, \cdots, r)$$

$$= F(M_1, \ldots, M_r) \times F(N_1, \ldots, N_r)$$

so that F is multiplicative.

2. The following are some other particular cases of the generalized composition.

(1) Let $g(M_1, \ldots, M_r)$ denote the g.c.d. of $M_1, \ldots, M_r$, and define $\psi\begin{pmatrix} M_1, \ldots, M_r \\ N_1, \ldots, N_r \end{pmatrix}$ to be $g(M_1, \ldots, M_r) \times E(N_1, \ldots, N_r)$. Then ψ is clearly multiplicative. Therefore

$$\sum_{g(d_1, \cdots, d_r)=1 \,;\; d_i \mid M_i} f\begin{pmatrix} d_1, \ldots, d_r \\ M_1/d_1, \ldots, M_r/d_r \end{pmatrix}$$

is multiplicative if f is multiplicative.

(2) Take ψ to be $\prod_{i \neq g} g(M_i, M_g)\; E(N_1, \ldots, N_r)$. Then the multiplicativity of F follows that of f.

(3) Take $\psi\begin{pmatrix} M_1, \ldots, M_r \\ N_1, \ldots, N_r \end{pmatrix}$ to be

$$\prod_{i=1}^{r} g(M_i, m_i)\, E(N_1, \ldots N_r)$$

where m_i $(i = 1, \ldots, r)$ are given numbers. Then ψ is multiplicative. Therefore if f is multiplicative so also is the function

$$\sum_{\substack{g(d_i, m_i) = 1 \\ (d_i \mid M_i \,;\; i = 1, \cdots, r)}} f\begin{pmatrix} d_1, \ldots, d_r \\ M_1/d_1 \ldots, M_r/d_r \end{pmatrix}.$$

(4) Let $\psi\begin{pmatrix} M_1, \ldots, M_r \\ N_1, \ldots, N_r \end{pmatrix}$ be $g(M_1, \ldots, M_r, N_1, \ldots, N_r)$; then we see that if f be multiplicative

$$\sum_{g(d_1, \cdots, d_r, M_1/d_1, \cdots, M_r/d_r)=1,\; d_i \mid M_i} f\begin{pmatrix} d_1, \ldots, d_r \\ M_1/d_1, \ldots, M_r/d_r \end{pmatrix}$$

is also multiplicative.

3. *Principal functions.*—f is called a principal function of r arguments equivalent to θ if

$$f(M_1, \ldots, M_r) = 0, \text{ unless } M_1 = \ldots = M_r,$$

$$\text{and } f(M, \ldots, M) = \theta(M).$$

We shall write $f = \text{princ}\ \theta$.

THEOREM 2: If f_1, f_2 be two principal functions of r arguments equivalent to θ_1 and θ_2 respectively, then

$$f_1 \cdot f_2 = \text{princ}\ \theta_1 \cdot \text{princ}\ \theta_2 = \text{princ}\ (\theta_1 \cdot \theta_2),$$

and

$$f_1 \oplus f_2 = \text{princ}\ \theta_1 \oplus \text{princ}\ \theta_2 = \text{princ}\ (\theta_1 \oplus \theta_2).$$

l terms in the sum

$$\Sigma\, f_1 (d_1, \ldots, d_r)\, f_2 \left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$$

ere the summation is for any set of divisors d_i of M_i $(i = 1, \ldots, r)$ l vanish unless

$$d_1 = d_2 \ldots = d_r, \; \frac{M_1}{d_1} = \ldots = \frac{M_r}{d_r}$$

ıultaneously, and so it is a principal function. In particular it follows .t the composite or the compound of two principal functions is a principal ıction.

;o

$$(\text{princ } \theta_1 \cdot \text{princ } \theta_2)(M, \ldots, M) = \sum_{d \mid M} f_1 (d_1 \ldots, d)\, f_2 \left(\frac{M}{d}, \ldots, \frac{M}{d}\right)$$

$$= \sum_{d \mid M} \theta_1 (d)\, \theta_2 \left(\frac{M}{d}\right) = (\theta_1 \cdot \theta_2)(M);$$

l

$$(\text{princ } \theta_1 \oplus \text{princ } \theta_2)(M, \ldots, M) = \sum_{d \mid M;\; g(d, M/d)=1} f_1 (d, \ldots, d)\, f_2 \left(\frac{M}{d}, \ldots, \frac{M}{d}\right)$$

$$= \sum_{d \mid M;\; g(d, M/d)=1} \theta_1 (d)\, \theta_2 (M/d) = (\theta_1 \oplus \theta_2)(M);$$

ence the required results follow.

THEOREM 3: The function

$$\Sigma\, f \left(\frac{M_1}{d}, \ldots, \frac{M_r}{d}\right) \theta (d)$$

ere the summation is for all common divisors d of $M_1, \ldots, M_r$ is equal to

$$(f \cdot \text{princ } \theta)(M_1, \ldots, M_r).$$

: $(f \cdot \text{princ } \theta)(M_1, \ldots, M_r) = \sum_{d_i \mid M_i} f \left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right) (\text{princ } \theta)(d_1, \ldots, d_r)$

: $(\text{princ } \theta)(d_1, \ldots, d_r) = 0$ unless $d_1 = \ldots = d_r$,

$$= \theta (d) \text{ if } d_1 = \ldots = d_r = d,$$

that $\sum_{d_i \mid M_i} f \left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right) (\text{princ } \theta)(d_1, \ldots, d_r)$

$$= \Sigma\, f \left(\frac{M_1}{d}, \ldots, \frac{M_r}{d}\right) \theta (d)$$

summation being for all common divisors d of $M_1, \ldots, M_r$.

In a similar manner we prove

THEOREM 4: The function

$$\sum_{d|M_1, \cdots, M_r;\, g(d, M_i/d)=1} f\left(\frac{M_1}{d}, \ldots, \frac{M_r}{d_r}\right) \theta(d)$$

is equal to $(f \oplus \text{princ } \theta)$ $(M_1, \ldots, M_r)$.

4. Let us write $f_1(M_1, \ldots, M_r)$ for

$$\sum_{g(d_1, \cdots, d_r)=1;\, d_i|M_i} f\left(\frac{M_1}{d}, \ldots, \frac{M_r}{d}\right),$$

and g for $g(M_1, \ldots, M_r)$. Then

$$\sum_{d_i|M_i} f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right) = \sum_{d|g}\ \sum_{g(d_1, \cdots, d_r)=d;\, d_i|M_i} f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$$

$$= \sum_{d|g}\ \sum_{g(\delta_1, \cdots, d_r)=1;\, \delta_i \left| \frac{M_i}{d}\right.} f\left(\frac{M_1}{d\delta_1}, \ldots, \frac{M_r}{d\delta_r}\right)$$

$$= \sum_{d|g} f_1\left(\frac{M_1}{d}, \ldots, \frac{M_r}{d}\right)$$

$$= (f_1 \cdot \text{princ E})(M_1, \ldots, M_r), \text{ by Theorem 3.}$$

Thus we get

THEOREM 5: If $f_1(M_1, \ldots, M_r) = \sum_{g(d_1, \cdots, d_r)=1;\, d_i|M_i} f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$, then

$f \cdot E = f_1 \cdot \text{princ } E$;

or, equivalently (denoting the inverse of E by E^{-1})

$$f_1 = f \cdot E \cdot \text{princ } E^{-1}$$

Corollary.—By definition* $f_1(M_1, \ldots, M_r)$

$$= \sum_{d_i|M_i} E_0\left(g(d_1, \ldots, d_r)\right) f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$$

$$= \{E_0(g) \cdot f\}(M_1, \ldots, M_r), \text{ say.}$$

Therefore, taking f to be E_0 we get

$$E_0\{g(M_1, \ldots, M_r)\} = (E \cdot \text{princ } E^{-1})(M_1, \ldots, M_r).$$

* E_0 is the function which vanishes unless all the arguments are unity and then is equal to unity.

Example.—Take f to be the function $\phi_k(M_1, \ldots, M_r)$

$$= \prod_i \phi_k(M_i)$$

where $\phi_k(M)$ is the Jordan function representing the number of sets of k numbers not greater than M whose g.c.d. is prime to M. Then

$$\phi_k(M_1, \ldots, M_r) = (I_k \cdot E^{-1})(M_1, \ldots, M_r)$$

where $I_k(M_1, \ldots, M_r) = \prod_i I_k(M_i)$, $I_k(M) = M^k$,

and $E^{-1}(M_1, \ldots, M_r) = \prod_i E^{-1}(M_i)$.

Applying Theorem 5 we get

$$\sum_{g(d_1, \cdots, d_r)=1} \phi_k\left(\frac{M_1}{d_1}\right) \cdots \phi_k\left(\frac{M_r}{d_r}\right) = (I_k \cdot E^{-1} \cdot E \cdot \text{princ } E^{-1})(M_1, \ldots, M_r)$$

$$= (I_k \cdot \text{princ } E^{-1})(M_1, \ldots, M_r)$$

$$= \sum \left(\frac{M_1, \ldots, M_r}{d^r}\right)^k \mu(d)$$

$$= (M_1, \ldots, M_r)^k \sum \mu(d)/d^{rk},$$

where μ is the Mobius function and the summations on the right-side are for all common divisors d of $M_1, \ldots, M_r$.

5. Let $f(M_1, \ldots, M_r)$ be the principal function equivalent to $\psi(M)$ and consider the sum

$$\sum_{g(d_1, \cdots, d_r)=1\,;\, d_i | M_i} f_1(d_1, \ldots, d_r) f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right)$$

f_1 being an arbitrary function. Obviously we need consider only such divisors which make $\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}$ equal, say, to t. Also we are to have $g(d_1, \ldots, d_r) = 1$, so that it follows that t is the g.c.d. of $M_1, \ldots, M_r$. Hence there is only one set of divisors $\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right)$ for which the terms of the sum can be non-vanishing. Therefore we have

$$\sum_{g(d_1, \cdots, d_r)=1\,;\, d_i | M_i} f_1(d_1, \ldots, d_r) f\left(\frac{M_1}{d_1}, \ldots, \frac{M_r}{d_r}\right) = f_1\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right) \psi(t)$$

Taking f_1 and ψ to be E we have

$$E_0(g) \cdot \text{princ } E = E, \text{ or } E_0(g) = E \cdot \text{princ } E^{-1}$$

which is the corollary of Th. 5.

Hence we have

THEOREM 6: If t denotes the g.c.d. of $M_1, \ldots, M_r$, then

$$f_1\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right)\psi(t) = \left\{ ((f_1 \times E_0(g)) \cdot \text{princ}\,\psi) \right\}(M_1, \ldots, M_r)$$

$$= \left\{(f_1 \times (E \cdot \text{princ}\ E^{-1})) \cdot \text{princ}\,\psi \right\}(M_1, \ldots, M_r).$$

Taking $f_1 = E$ we get

$$\psi\{g(M_1, \ldots, M_r)\} = (E \cdot \text{princ}\ E^{-1}\,\psi)(M_1, \ldots, M_r)$$

$$= [E \cdot \text{princ}\,(\psi \cdot E^{-1})](M_1, \ldots, M_r).$$

Thus we have

THEOREM 7*: If $f(M_1, \ldots, M_r)$ be a function $\psi(t)$ of the g.c.d. t of $M_1, \ldots, M_r$, then it is the integral of the principal function equivalent to the function whose integral is ψ.

Taking ψ to be E in Theorem 6 we get

THEOREM 8: If t is the g.c.d. of $M_1, \ldots, M_r$, then

$$f_1\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right) = \left\{(f \times (E \cdot \text{princ}\ E^{-1})) \cdot \text{princ}\ E\right\}(M_1, \ldots, M_r).$$

Analogous to Theorem 6 we can easily obtain

THEOREM 9: If t is the g.c.d. of $M_1, \ldots, M_r$, then

$$\left\{(f_1 \times E_0(g)) \oplus \text{princ}\,\psi\right\}(M_1, \ldots, M_r)$$

$$= f_1\left(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\right)\psi(t), \text{ if } g\left(t, \frac{M_i}{t}\right) = 1$$

$$= 0, \text{ otherwise.}$$

6. We shall require the following considerations in Section 3.

Let $P(a_1, \ldots, a_r; M)$ denote the number of solutions in non-zero positive integers of the equation

$$a_1 x_1 + \ldots + a_r x_r = M,$$

the a's and M being positive integers, the g.c.d. of $x_1, \ldots, x_r$ being unity. Consider the function

$$\theta(M) = \sum_{a_1 M_1 + \cdots + a_r M_r = M} f\{g(M_1, \ldots, M_r)\}.$$

Let $g(M_1, \ldots, M_r) = d$. Then d is also a divisor of M.

* This result, when f is multiplicative, is the theorem of R. Vaidyanathaswamy mentioned in the Introduction.

It is therefore clear that we may write

$$\theta(\mathrm{M}) = \sum_{d|\mathrm{M}} \sum_{\substack{a_1\mathrm{N}_1+\cdots+a_r\mathrm{N}_r=\mathrm{M}/d \\ g(\mathrm{N}_1,\cdots,\mathrm{N}_r)=1}} f(d)$$

$$= \sum_{d|\mathrm{M}} f(d)\,\mathrm{P}(a_1, \ldots, a_r;\ \mathrm{M}/d).$$

This relation may also be written in the form

$$(\theta \cdot f^{-1})(\mathrm{M}) = \mathrm{P}(a_1, \ldots, a_r;\ \mathrm{M}),$$

provided $f(1) \neq 0$.

Section 2. *Multiple Dirichlet Series.*

1. A series of the form

$$\sum_{\mathrm{M}_i=1;\ i=1,\cdots,r}^{\infty} f(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})$$

shall be called a multiple Dirichlet series. In all cases under consideration we shall suppose that $s_1, \ldots, s_r$ are so chosen as to ensure the absolute convergence of the series.

Let $\Sigma\ f_1(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})$,

$\Sigma\ f_2(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})$

be two multiple Dirichlet series. Then it can readily be shown that their product is the series

$$\Sigma\ (f_1 \cdot f_2)(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r}).$$

Let f_2 be the principal function equivalent to ψ. Then

$$\Sigma\ f_2(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r}) = \Sigma\ \psi(\mathrm{M})/\mathrm{M}^{s_1+\cdots+s_r}.$$

It follows that

$$\{\Sigma f_1(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})\}\{\Sigma \psi(\mathrm{M})/\mathrm{M}^{s_1+\cdots+s_r}\}$$

$$= \Sigma \left\{ \sum_d f_1\left(\frac{\mathrm{M}_1}{d}, \ldots, \frac{\mathrm{M}_r}{d}\right)\psi(d)\right\} \Big/ (\mathrm{M}_1^{s_1}\ldots\mathrm{M}_r^{s_r})$$

where $\sum_d$ denotes summation for all common divisors d of $\mathrm{M}_1, \ldots, \mathrm{M}_r$.

2. Now $\sum_{g(\mathrm{M}_1,\cdots,\mathrm{M}_r)=1} 1/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})$

$$= \Sigma\,\mathrm{E}_0\,[g(\mathrm{M}_1, \ldots, \mathrm{M}_r)]/(\mathrm{M}_1^{s_1}\ldots\mathrm{M}_r^{s_r})$$

$$= \Sigma\,(\mathrm{E} \cdot \mathrm{princ}\ \mathrm{E}^{-1})(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots \mathrm{M}_r^{s_r})$$

$$= \{\Sigma\,\mathrm{E}(\mathrm{M}_1, \ldots, \mathrm{M}_r)/(\mathrm{M}_1^{s_1}, \ldots, \mathrm{M}_r^{s_r})\}\Sigma\,\mu(\mathrm{M})/\mathrm{M}^{s_1+\cdots+s_r})$$

$$= \{\prod_{i=1}^{r} \zeta(s_i)\}/\zeta(\Sigma s_i),$$

where $\zeta(s)$ is the Riemann Zeta function $\Sigma \frac{1}{n^s}$.

Since $\psi\,[g\,(M_1, \ldots, M_r)] = [E_0\,(g) \cdot \text{princ}\,\psi]\,(M_1, \ldots, M_r)$ we have also

$$[\{\Pi\,\zeta\,(s_i)\}/\zeta\,(\Sigma s_i)] \times \Sigma\,\psi\,(M)/M^{s_1+\cdots+s_r}$$
$$= \Sigma\,\psi\,[g\,(M_1, \ldots, M_r)]/(M_1^{s_1}, \ldots,$$

Examples.—(1) Put $\psi = \mu\,(M)$. Then we get

$$\Sigma\,\mu\big(g\,(M_1, \ldots, M_r)\big)/(M_1^{s_1}, \ldots M_r^{s_r}) = \{\Pi\,\zeta\,(s_i)\}/\{\zeta\,(\Sigma s_i)\}^2,$$

(2) Put $\psi =$ the Jordan function ϕ_k. Then

$$\Sigma\,\phi_k\big(g\,(M_1, \ldots, M_r)\big)/(M_1^{s_1}, \ldots, M_r^{s_r}) = \frac{\Pi\,\zeta\,(s_i)}{\zeta\,(\Sigma s_i)} \times \frac{\zeta\,(\Sigma s_i - k)}{\Pi\,\zeta\,(s_i)}$$
$$= \zeta\,(\Sigma s_i - k)/\zeta\,(\Sigma s_i).$$

(3) Put $\psi\,(M) = \sigma_a\,(M)$ representing the sum of the ath powers of the divisors of M. Then

$$\Sigma\,\sigma_a\big(g\,(M_1, \ldots, M_r)\big)/(M_1^{s_1} \ldots M_r^{s_r}) = \frac{\Pi\,\zeta\,(s_i)}{\zeta\,(\Sigma s_i)} \times \zeta\,(\Sigma s_i)\,\zeta\,(\Sigma s_i - a)$$
$$= \{\Pi\,\zeta\,(s_i)\}\,\zeta\,(\Sigma s_i - a).$$

(4) Put $\psi\,(M) = \sigma_a\,(M)\,\sigma_b\,(M)$. Then

$$\Sigma\,\sigma_a\big(g\,(M_1, \ldots, M_r)\big)\,\sigma_b\big(g\,(M_1, \ldots, M_r)\big)/\,(M_1^{s_1} \ldots M_r^{s_r})$$
$$= \frac{\Pi\,\zeta\,(s_i)}{\zeta\,(\Sigma s_i)} \times \frac{\zeta\,(\Sigma s_i)\,\zeta\,(\Sigma s_i - a)\,\zeta\,(\Sigma s_i - b)\,\zeta\,(\Sigma s_i - a - b)}{\zeta\,(2\,\Sigma s_i - a - b)}$$
$$= \{\Pi\,\zeta\,(s_i)\}\,\zeta\,(\Sigma s_i - a)\,\zeta\,(\Sigma s_i - b)\,\zeta\,(\Sigma s_i - a - b)/\zeta\,(2\,\Sigma s_i - a - b).$$

3. Theorem 6 of Section 1 gives

$$\psi\,(t)\,f\Big(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\Big) = \{\big(f \times E_0\,(g)\big) \cdot \text{princ}\,\psi\}\,(M_1, \ldots, M_r)$$

where $t = g\,(M_1, \ldots, M_r)$.

Therefore, $\Sigma\,\psi\,(t)\,f\Big(\frac{M_1}{t}, \ldots, \frac{M_r}{t}\Big)\Big/(M_1^{s_1}, \ldots, M_r^{s_r})$

$$= \{\Sigma f\,(M_1, \ldots, M_r)\,E_0\,\{g\,(M_1, \ldots, M_r)\}/(M_1^{s_1} \ldots M_r^{s_r})\} \times \Sigma\,\psi\,(M)/M^{s_1+\cdots+s_r}.$$

But $\sum\limits_{g\,(M_1, \cdots, M_r)=1} f\,(M_1, \ldots, M_r)/(M_1^{s_1}, \ldots, M_r^{s_r})$

$$= \Sigma f\,(M_1, \ldots, M_r)\,E_0\,\{g\,(M_1, \ldots, M_r)\}/(M_1^{s_1} \ldots M_r^{s_r})$$

Hence we have $\Sigma\,\psi\,(t)\,f\Big(\frac{M_1}{t}, \frac{M_2}{t}, \ldots, \frac{M_r}{t}\Big)\Big/(M_1^{s_1} \ldots M_r^{s_r})$

$$= \{\Sigma\,\psi\,(M)/M^{s_1+\cdots+s_r}\} \times \sum_{g,\,(M_1, \cdots, M_r)=1} f\,(M_1, \ldots, M_r)/(M_1^{s_1} \ldots M_r^{s_r}),$$

where t is the g.c.d. of $M_1, \ldots, M_r$.

Section 3. Multiple Lambert Series.

The series $\sum_{\substack{m_i=1\\ i=1,\cdots,r}}^{\infty} f(m_1, \ldots, m_r) \dfrac{x_1^{m_1}, \ldots, x_r^{m_r}}{(1-x_1^{m_1})\ldots(1-x_r^{m_r})}$

shall be called a multiple Lambert series. Here again, as in the previous section we shall suppose that the series considered are absolutely convergent.

Expanding the series as a power series in $x_1, \ldots, x_r$, we get

$$\Sigma f(m_1, \ldots, m_r) \frac{x_1^{m_1}\ldots x_r^{m_r}}{(1-x_1^{m_1})\ldots(1-x_r^{m_r})} = \Sigma (f\cdot \mathrm{E})(m_1, \ldots, m_r)\, x_1^{m_1}\ldots x_r^{m_r}.$$

Replacing f by $f\cdot \mathrm{E}^{-1}$ we see that

$$\Sigma (f\cdot \mathrm{E}^{-1})(m_1, \ldots, m_r) \frac{x_1^{m_1}\ldots x^{m_r}}{(1-x_1^{m_1})\ldots(1-x_r^{m_r})} = \Sigma f(m_1, \ldots, m_r)\, x_1^{m_1}\ldots x_r^{m_r}.$$

Let $f(m_1, \ldots, m_r) = \psi\{g(m_1, \ldots, m_r)\}$,

$$= \{\mathrm{E}\cdot \mathrm{princ}(\mathrm{E}^{-1}\cdot \psi)\}(m_1, .., m_r).$$

Then we get

$$\Sigma\ \mathrm{princ}(\mathrm{E}^{-1}\cdot\psi)(m_1, \ldots, m_r)\frac{x_1^{m_1}\ldots x_r^{m_r}}{(1-x_1^{m_1})\ldots(1-x_r^{m_r})} = \Sigma\psi\{g(m_1, \ldots, m_r)\}\, x_1^{m_1}, \ldots, x_r^{m_r}.$$

The left-side is equal to

$$\sum_{m=1}^{\infty}(\psi\cdot\mathrm{E}^{-1})(m)\frac{(x_1, \ldots, x_r)^m}{(1-x_1^m)\ldots(1-x_r^m)}$$

so that we have

$$\sum_{m=1}^{\infty}(\psi\cdot\mathrm{E}^{-1})(m)\frac{(x_1..x_r)^m}{(1-x_1^m)..(1-x_r^m)} = \Sigma\psi\{g(m_1, .., m_r)\}\, x_1^{m_1}..x_r^{m_r}.$$

Examples.—(1) Take $\psi(m) = m^k$ so that $\psi\cdot\mathrm{E}^{-1}$ is the Jordan function ϕ_k. Then we get

$$\sum_{m=1}^{\infty}\phi_k(m)\frac{(x_1\ldots x_r)^m}{(1-x_1^m)..(1-x_r^m)} = \Sigma\{g(m_1, \ldots, m_r)\}^k\, x_1^{m_1}\ldots x_r^{m_r}.$$

(2) Put $\psi = \sigma_a$, the sum of the ath powers of the divisors of the argument. Since $(\sigma_a\cdot\mathrm{E}^{-1})(m) = m^a$ we have

$$\sum_{m=1}^{\infty}\frac{m^a(x_1\ldots x_r)^m}{(1-x_1^m)\ldots(1-x_r^m)} = \Sigma\sigma_a\{g(m_1, \ldots, m_r)\}\, x_1^{m_1}\ldots x_r^{m_r}.$$

Let us put $x_1 = x^{a_1}, \ldots, x_r = x^{a_r}$. Then we have

$$\Sigma\, \psi\{g(m_1, \ldots, m_r)\}\, x^{a_1 m_1 + \ldots + a_r m_r}$$

$$= \sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \frac{x^{m(a_1 + \ldots + a_r)}}{(1 - x^{m a_1}) \ldots (1 - x^{m a_r})}$$

But the left-side is equal to

$$\sum_{m=1}^{\infty} \left\{ \sum_{a_1 m_1 + \cdots + a_r m_r = m} \psi\{g(m_1, \ldots, m_r)\} \right\} x^m$$

$$= \sum_{m=1}^{\infty} \left\{ \sum_{d \mid m}^{\infty} \psi(d)\, \mathrm{P}\left(a_1, \ldots, a_r;\ \frac{m}{d}\right) \right\} x^m$$

where $\mathrm{P}(a_1, \ldots, a_r; m)$ is the number of solutions in non-zero positive integers of the equation

$$a_1 x_1 + \ldots + a_r x_r = m$$

the g.c.d. of $x_1, \ldots, x_r$ being unity (see Section 1 § 6).

Thus we get

$$\sum_{m=1}^{\infty} (\psi \cdot \mathrm{P})(m)\, x^m = \sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \frac{x^{m(a_1 + \ldots + a_r)}}{(1 - x^{a_1 m}) \ldots (1 - x^{a_r m})}.$$

Examples.—(1) Put $\psi = \mathrm{E}$. Then we find

$$\sum_{m=1}^{\infty} \left\{ \sum_{d \mid m} \mathrm{P}(a_1, \ldots, a_r;\ d) \right\} x^m = \frac{x^{+a_1 \ldots + a_r}}{(1 - x^{a_1}) \ldots (1 - x^{a_r})}.$$

(2) Put $\psi = \mathrm{E}_0$. Then

$$\sum_{m=1}^{\infty} \mathrm{P}(a_1, \ldots, a_r;\ m)\, x^m = \sum_{m=1}^{\infty} \mu(m) \frac{x^{m(a_1 + \ldots + a_r)}}{(1 - x^{m a_1}) \ldots (1 - x^{m a_r})}.$$

(3) Put $\psi(m) = m^k$. Then

$$\sum_{m=1}^{\infty} \left\{ m^k \sum_{d/m} \mathrm{P}(a_1, \ldots, a_r;\ d/d^k) \right\} x^m = \Sigma \frac{\phi_k(m)\, x^{m(a_1 + \ldots + a_r)}}{(1 - x^{m a_1}) \ldots (1 - x^{m a_r})}.$$

In particular if we put $a_1 = \ldots = a_r = 1$, and write

$$\mathrm{P}(1, \ldots, 1;\ m) = \mathrm{P}_r(m),$$

we have

$$\sum_{m=1}^{\infty} \sum_{m_1 + \cdots + m_r = m} \psi\{g(m_1, \ldots, m_r)\}\, x^m = \sum_{m=1}^{\infty} (\psi \cdot \mathrm{P}_r)(m)\, x^m$$

$$= \sum_{m=1}^{\infty} (\psi \cdot \mathrm{E}^{-1})(m) \left(\frac{x^m}{1 - x^m}\right)^r.$$

Equating coefficient of x^m we get

$$\sum_{\substack{m_1+\dots+m_r=m \\ m_i \geqslant 1}} \psi\{g(m_1, \dots, m_r)\} = \sum_{d|m} \psi\left(\frac{m}{d}\right) P_r(d)$$

$$= \sum_{d|m} \binom{d-1}{r-1} (\psi \cdot E^{-1}) \left(\frac{m}{d}\right),$$

where $\binom{n}{r}$ denotes $n(n-1) \dots (n-r+1)/r\,!$

In particular $\sum_{d|m} P_r(d) = \binom{m-1}{r-1}$,

i.e., $$P_r(m) = \sum_{d|m} \binom{d-1}{r-1} \mu\left(\frac{m}{d}\right);$$

and $m^k \sum_{d|m} P_r(d)/d^k = \sum_{d|m} \binom{d-1}{r-1} \phi_k\left(\frac{m}{d}\right)$.

Let the series $\sum_{m=1}^{\infty} (\psi \cdot E^{-1})(m)/m^r$ be absolutely convergent. Then as x tends to 1,

$$\sum_{m=1}^{\infty} (\psi \cdot E^{-1})(m) \left(\frac{x^m}{1-x^m}\right)^r \sim \left\{\sum_{m=1}^{\infty} (\psi \cdot E^{-1})(m)/m^r\right\} (1-x)^{-r}.$$

Therefore, if we denote

$\sum_{m_1+\cdots+m_r=m} \psi\{g(m_1, \dots, m_r)\}$ by $\Psi(m)$, we have

$$\sum_{m \leqslant n} \Psi(m) \sim \frac{n^r}{\Gamma(r+1)} \times \Sigma\, (\psi \cdot E^{-1})(m)/m^r *$$

$$= \frac{n^r}{\zeta(r)\, \Gamma(r+1)} \sum_{m=1}^{\infty} \psi(m)/m^r.$$

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2. Titchmarsh .. *Theory of Functions*.

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* See Titchmarsh, *Theory of Functions*, p. 242.