3-D KINEMATICAL CONSERVATION LAWS (KCL): EVOLUTION OF A SURFACE IN $\mathbb{R}^3$ - IN PARTICULAR PROPAGATION OF A NONLINEAR WAVEFRONT

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Abstract. 3-D KCL are equations of evolution of a **propagating** surface (or a wavefront) Ω_t in 3-space dimensions and were first derived by Giles, Prasad and Ravindran in 1995 assuming the motion of the surface to be isotropic. Here we discuss various properties of these 3-D KCL. These are the most general equations in conservation form, governing the evolution of Ω_t with singularities which we call **kinks** and which are curves across which the normal $\mathbf{n}$ to Ω_t and amplitude w on Ω_t are discontinuous. From KCL we derive a system of six differential equations and show that the KCL system is equivalent to the ray equations of Ω_t. The six independent equations and an energy transport equation (for small amplitude waves in a polytropic gas) involving an amplitude w (which is related to the normal velocity m of Ω_t) form a completely determined system of seven equations. We have determined eigenvalues of the system by a very novel method and find that the system has two distinct nonzero eigenvalues and five zero eigenvalues and the dimension of the eigenspace associated with the multiple eigenvalue 0 is only 4. For an appropriately defined m, the two nonzero eigenvalues are real when $m > 1$ and pure imaginary when $m < 1$. Finally we give some examples evolution of weakly nonlinear wavefronts.

The symbols used in this paper are listed in the Appendix B

1. Introduction

Propagation of a nonlinear wavefront and a shock front in three dimensional space $\mathbb{R}^3$ are very complex physical phenomena and both fronts share a common property of possessing curves of discontinuities across which the normal direction to the fronts and the amplitude distribution on them suffer discontinuities. These are discontinuities of the first kind, i.e., the limiting values of the discontinuous functions and their derivatives on a front as we approach a curve of discontinuity from either side are finite. Such a discontinuity was first analysed by Whitham in 1957 (see [19]), who called it shock-shock, meaning shock on a shock front. However, a discontinuity of this type is geometric in nature and can arise on any propagating surface Ω_t, and we give it a general name **kink**. In order to explain the existence of a kink and study its formation and propagation, we need the governing equations in the form a system of physically realistic conservation laws. In this paper we derive and analyse such conservation laws in a specially defined *ray coordinate system* and since they are derived purely on geometrical consideration and we call them *kinematical conservation laws (KCL)*. When a discontinuous solution of the KCL system in the ray coordinates has a shock satisfying Rankine - Hugoniot conditions, the image of the shock in $\mathbb{R}^3$ is a kink.

Before we start any discussion, we assume that all variables, both dependent and independent, used in this paper are non-dimensional. There is one exception, the dependent variables in the first paragraph in section 4 are dimensional.

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KCL governing the evolution of a moving curve Ω_t in two space dimensions (x_1, x_2) were first derived by Morton, Prasad and Ravindran in 1992 [13], and the kink (in this case, a point on Ω_t) phenomenon is well understood (see [14]-section 3.3). We call this system of KCL as 2-D KCL which we describe in the next paragraph.

Consider a one parameter family of curves Ω_t in (x_1, x_2)-plane, where the subscript t is the parameter whose different values give different positions of a moving curve (which may represent a wavefront). We assume that this family of curves has been obtained with the help of a ray velocity $\boldsymbol{\chi} = (\chi_1, \chi_2)$, which is a function of x_1, x_2, t and $\mathbf{n}$, where $\mathbf{n}$ is the unit normal to Ω_t. We assume that motion of this curve Ω_t is isotropic so that we take the ray velocity $\boldsymbol{\chi}$ in the direction of $\mathbf{n}$ and write it as

$$\boldsymbol{\chi} = m\mathbf{n}, \tag{1.1}$$

where we assume throughout this paper that the scalar function m depends on $\mathbf{x}$ and t but is independent of $\mathbf{n}$. The ray equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = m\mathbf{n}, \ \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\left(-n_2\frac{\partial}{\partial x_1} + n_1\frac{\partial}{\partial x_2}\right)m, \tag{1.2}$$

where $\mathbf{n} = (n_1, n_2) = (\cos\theta, \sin\theta)$ are derived from the Charpit's equations (or Hamilton's canonical equations) of the eikonal equation (see section 2). The normal velocity m of Ω_t is a non-dimensionalized velocity with respect to a characteristic velocity (say the sound velocity a_0 in a uniform ambient medium in the case Ω_t is a wavefront in such a medium). Given a representation of the curve Ω_0 at the time $t = 0$ in the form $\mathbf{x} = \mathbf{x}_0(\xi)$, we determine the unit normal $\mathbf{n}_0(\xi)$ and then we solve the system (1.2) with these as initial values (this is a simplified view - the system (1.2) is usually under-determined as explained below). Thus we get a representation of the curve Ω_t at time t in the form $\mathbf{x} = \mathbf{x}(\xi, t)$. We assume (for development of the theory) that this gives a mapping: $(\xi, t) \to (x_1, x_2)$ which is one to one. In this way we have introduced a ray coordinate system (ξ, t) such that t = constant represents the curve Ω_t and ξ = constant represents a ray. Then $m\mathrm{d}t$ is an element of distance along a ray, i.e., m is the metric associated with the variable t. Let g be the metric associated with the variable ξ, then

$$\frac{1}{g}\frac{\partial}{\partial\xi} = -n_2\frac{\partial}{\partial x_1} + n_1\frac{\partial}{\partial x_2}. \tag{1.3}$$

Simple, geometrical consideration gives (see [14]-section 3.3 and also the section 3 of this paper)

$$\mathrm{d}\mathbf{x} = (g\mathbf{u})\mathrm{d}\xi + (m\mathbf{n})\mathrm{d}t, \tag{1.4}$$

where $\mathbf{u}$ is the tangent vector to Ω_t, i.e., $\mathbf{u} = (-n_2, n_1)$. Equating $(x_1)_{\xi t} = (x_1)_{t\xi}$ and $(x_2)_{\xi t} = (x_2)_{t\xi}$, we get the 2-D KCL

$$(gn_2)_t + (mn_1)_\xi = 0, \quad (gn_1)_t - (mn_2)_\xi = 0. \tag{1.5}$$

Using these KCL we can derive the Rankine-Hugonoit conditions (i.e., the jump relations) relating the quantities on the two sides of a shock path in (ξ, t)-plane or a kink path in (x_1, x_2)-plane. The system (1.5) is under-determined since it contains only two equations in three variables θ, m and g. It is possible to close it in many ways. One possible way is to close it by a single conservation law

$$(gG^{-1}(m))_t = 0, \tag{1.6}$$

where G is a given function of m. Baskar and Prasad [3] have studied the Riemann problem for the system (1.5) or (1.6) assuming some physically realistic conditions on $G(m)$. For a weakly nonlinear

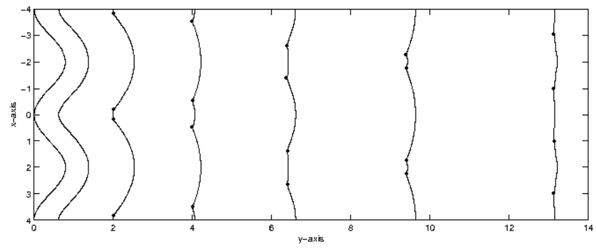


FIGURE 1. successive positions of a nonlinear wavefront at $t = 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ starting from a periodic pulse $y = 0.4(1 - \cos(\pi x/2))$. The wavefront develops four kinks and ultimately becomes plane.

wavefront ([14]-chapter 6) in a polytropic gas, conservation of energy along a ray tube gives (with a suitable choice of ξ)

$$G(m) = (m - 1)^{-2} e^{-2(m-1)}, \tag{1.7}$$

(see also the equation (6.6) in this article). Prasad and his collaborators have used this closure relation to solve many interesting problems and obtained many new results [4, 6, 12, 17]. KCL with (1.6) and (1.7) is a very interesting system. It is hyperbolic for $m > 1$ and has elliptic nature for $m < 1$.

Fig. 1 shows successive positions of a nonlinear wavefront with initially periodic shape. As the front propagates, the concave part bulges out and the convex part becomes concave. Four well defined kinks (shown by dots) are seen on Ω_t from $t = 1$ onwards. The upper two kinks (as well as the lower ones) interact and separate away. The nonlinear wavefront ultimately tends to become planar (corrugational stability). for further details, see [12, 17].

In this article, we shall discuss an extension of 2-D KCL to 3-D KCL. We start with a review of the ray theory in section 2. A derivation of 3-D KCL of Giles, Prasad and Ravindran (GPR) [9] is given in section 3. In section 4 we give an explicit differential form of the KCL and in section 5 we show its equivalence to the ray equations. In section 6, we derive a conservation form of the energy transport equation along rays for a small amplitude waves in a polytropic gas and then we close the 3-D KCL by this energy transport equation. We call the system of 7 conservation laws, six KCL and the energy transport equation, the equations of weakly nonlinear ray theory (WNLRT). We have two systems of equations in differential form: system-I consists of two of the ray equations, which are equations for first two components n_1 and n_2 of $\mathbf{n}$ and the energy transport equation; and system-II consists of seven differential forms of the equations of WNLRT (i.e., the KCL and the energy transport equation). In section 7, we discuss the eigenvalues and eigenvectors of the system-I and in section 8 we do that for the system-II. In section 8.4, we derive the nonzero eigenvalues of the system-I from those of the system-II and vice versa. This article, therefore, puts the theory of 3-D KCL on a strong foundation and the theory can be used to discuss evolution of a surface Ω_t in 3-space dimensions and formation and propagation of curves of singularities on Ω_t. In section 9 we give some results showing successive positions of a nonlinear wavefront in 3-D.

2. A brief discussion of the ray equations of an isotropically evolving front Ω_t

Though it is possible to derive KCL for a more general motion of a moving surface Ω_t (following [14] for 2-D KCL), we consider here only to the case when the motion of Ω_t is isotropic in the sense that the associated ray velocity $\boldsymbol{\chi}$ depends on the unit normal $\mathbf{n}$ by the relation (1.1). An example of this is the wave equation

$$u_{tt} - m^2\left(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}\right) = 0, \tag{2.1}$$

where m need not be constant. For this equation, we shall take only a forward facing wavefront Ω_t, so that the associated characteristic surface Ω in $(\mathbf{x}, t)$-space, given by $\varphi(\mathbf{x}, t) = 0$, satisfies the eikonal equation

$$\varphi_t + m\left\{\varphi_{x_1}^2 + \varphi_{x_2}^2 + \varphi_{x_3}^2\right\}^{1/2} = 0. \tag{2.2}$$

Note that Ω is a surface in space-time (i.e., $\mathbb{R}^4$) and Ω_t given by $\varphi(\mathbf{x}, t) = 0$, for $t =$ constant is a surface in 3-D $\mathbf{x}$-space. For m as a given function of $\mathbf{x}$ and t, bicharacteristic equations or the ray equations, ([14]-sections 2.4, 6.1, [15]) are

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = m\mathbf{n}, \quad |\mathbf{n}| = 1, \tag{2.3}$$

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t} = -\mathbf{L}m := -\left(\nabla - \mathbf{n}\langle\mathbf{n}, \nabla\rangle\right)m. \tag{2.4}$$

The bicharacteristics in $(\mathbf{x}, t)$-space form a 5 parameter family of curves. Now, we take a characteristic surface Ω and note that its level set at $t = 0$, i.e., the surface $\Omega_0\colon \varphi(\mathbf{x}, 0) = 0$ in the $\mathbf{x}$-space is a two dimensional manifold. Thus Ω_0 is represented parametrically as $\mathbf{x} = \mathbf{x}_0(\xi_1, \xi_2)$, from which the unit normal $\mathbf{n}_0(\xi_1, \xi_2)$ of Ω_0 can be calculated. Now the bicharacteristics, which generate Ω can be obtained by solving the equations (2.3) and (2.4) with initial data

$$\mathbf{x}|_{t=0} = \mathbf{x}_0(\xi_1, \xi_2) \text{ and } \mathbf{n}|_{t=0} = \mathbf{n}_0(\xi_1, \xi_2). \tag{2.5}$$

Thus the bicharacteristic curves which generate a given characteristic surface Ω in space-time form a two parameter family

$$\mathbf{x} = \mathbf{x}(\mathbf{x}_0(\xi_1, \xi_2), t). \tag{2.6}$$

The rays, starting from the various points of Ω_0 are projections on $\mathbf{x}$-space of the above bicharacteristic curves. Fermat's method of construction of the wavefront Ω_t at any time t consists of generating the surface Ω_t from the solution (2.6) by keeping t constant and varying ξ_1 and ξ_2. A front Ω_t having an isotropic motion need not come from the wave equation. An example of this is the crest line of a curved solitary wave on the surface of a shallow water [2]. However, every isotropically evolving wavefront would satisfy an eikonal equation (2.2) with a suitable front velocity m. Evolution of such a front Ω_t is given by the ray equations (2.3)-(2.4).

3. 3-D KCL of Giles, Prasad and Ravindran (1995)

Following the discussion in the last section consider a surface Ω in $\mathbb{R}^4$, $\Omega\colon \varphi(\mathbf{x}, t) = 0$ and let us assume that Ω is generated by a two parameter family of curves in $\mathbb{R}^4$, such that projection of these curves on $\mathbf{x}$-space are rays which are orthogonal to the successive position of the front $\Omega_t\colon \varphi(\mathbf{x}, t) = 0, t =$ constant.

We introduce a ray coordinate system (ξ_1, ξ_2, t) in $\mathbf{x}$-space such that $t =$ constant represents the surface Ω_t, see [11]. The surface Ω_t in $\mathbf{x}$-space is now generated by a one parameter family of curves such that along each of these curves ξ_1 varies and the parameter ξ_2 is constant. Similarly Ω_t is generated by another one parameter family of curves along each of these ξ_2 varies and ξ_1 is constant. Through each point (ξ_1, ξ_2) of Ω_t there passes a ray orthogonal (in $\mathbf{x}$-space) to the successive positions of Ω_t, thus rays form a two parameter family as mentioned above. Given ξ_1, ξ_2 and t, we uniquely identify a point P in $\mathbf{x}$-space. For the development of theory, we assume that the mapping from (ξ_1, ξ_2, t)-space to (x_1, x_2, x_3)-space is one to one. On Ω_t let $\mathbf{u}$ and $\mathbf{v}$ be unit tangent vectors of the curves ξ_2 = constant and ξ_1 = constant respectively and $\mathbf{n}$ be unit normal to Ω_t. Then

$$\mathbf{n} = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}. \tag{3.1}$$

Let an element of length along a curve (ξ_2 = constant, t = constant) be $g_1 \mathrm{d}\xi_1$ and that along a curve (ξ_1 = constant, t = constant) be $g_2 \mathrm{d}\xi_2$. The element of length along a ray (ξ_1 = constant, ξ_2 = constant) is $m\mathrm{d}t$. The displacement $\mathrm{d}\mathbf{x}$ in $\mathbf{x}$-space due to increments $\mathrm{d}\xi_1$, $\mathrm{d}\xi_2$ and $\mathrm{d}t$ is given by (this is an extension of the result (1.4))

$$\mathrm{d}\mathbf{x} = (g_1\mathbf{u})\mathrm{d}\xi_1 + (g_2\mathbf{v})\mathrm{d}\xi_2 + (m\mathbf{n})\mathrm{d}t. \tag{3.2}$$

This gives

$$J := \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, t)} = g_1 g_2 m \sin\chi, \quad 0 < \chi < \pi, \tag{3.3}$$

where $\chi(\xi_1, \xi_2, t)$ is the angle between the $\mathbf{u}$ and $\mathbf{v}$, i.e.,

$$\cos\chi = \langle \mathbf{u}, \mathbf{v} \rangle. \tag{3.4}$$

As explained after (4.5) and (6.6) in the next section, we shall like to choose $\sin\chi = |\mathbf{u} \times \mathbf{v}|$ which requires the restriction $0 < \chi < \pi$ on χ. For a smooth moving surface Ω_t, we equate $\mathbf{x}_{\xi_1 t} = \mathbf{x}_{t\xi_1}$ and $\mathbf{x}_{\xi_2 t} = \mathbf{x}_{t\xi_2}$, and get the 3-D KCL of Giles, Prasad and Ravindran [9],

$$(g_1\mathbf{u})_t - (m\mathbf{n})_{\xi_1} = 0, \tag{3.5}$$

$$(g_2\mathbf{v})_t - (m\mathbf{n})_{\xi_2} = 0. \tag{3.6}$$

We also equate $\mathbf{x}_{\xi_1\xi_2} = \mathbf{x}_{\xi_2\xi_1}$ and derive 3 more scalar equations contained in

$$(g_2\mathbf{v})_{\xi_1} - (g_1\mathbf{u})_{\xi_2} = 0. \tag{3.7}$$

Equations (3.5)-(3.7) are necessary and sufficient conditions for the integrability of the equation (3.2) (see [8], section 1.9).

From the equations (3.5) and (3.6) we can show that $(g_2\mathbf{v})_{\xi_1} - (g_1\mathbf{u})_{\xi_2}$ does not depend on t. If any choice of coordinates ξ_1 and ξ_2 on Ω_0 implies that the condition (3.7) is satisfied at $t = 0$ then it follows that (3.7) is automatically satisfied. Thus, the 3-D KCL is a system of six scalar evolution equations (3.5) and (3.6). However, since $|\mathbf{u}| = 1$, $|\mathbf{v}| = 1$, there are 7 dependent variables in (3.5) and (3.6): two independent components of each of $\mathbf{u}$ and $\mathbf{v}$, the front velocity m of Ω_t, g_1 and g_2. Thus KCL is an under-determined system and can be closed only with the help of additional relations or equations, which would follow from the nature of the surface Ω_t and the dynamics of the medium in which it propagates.

We derive a few results from (3.5) and (3.6) without considering the closure equation (or equations) for m. The system (3.5) and (3.6) consists of equations which are conservation laws, so its weak solution may contain shocks which are surfaces in (ξ_1, ξ_2, t)-space. Across these **shock surfaces** m, g_1, g_2 and vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{n}$ will be discontinuous. Image of a shock surface into $\mathbf{x}$-space will be another surface, let us call it a **kink surface**, which will intersect Ω_t in a curve, say **kink curve** $\mathscr{K}_t$. Across this kink curve or simply the kink, the normal direction $\mathbf{n}$ of Ω_t will be discontinuous as shown in Figure 2. As time t evolves, $\mathscr{K}_t$ will generate the kink surface. A **shock front** (a phrase very commonly used in literature) is a curve in (ξ_1, ξ_2)-plane and its motion as t changes generates the shock surface in (ξ_1, ξ_2, t)-space. We assume that the mapping between (ξ_1, ξ_2, t)-space and (x_1, x_2, x_3)-space continues to be one to one even when a kink appears.

The distance $\mathrm{d}\mathbf{x}$ between two points $P(\mathbf{x})$ and $Q'(\mathbf{x} + \mathrm{d}\mathbf{x})$ on Ω_t and $\Omega_{t+\mathrm{d}t}$ respectively satisfies the relation (3.2), where (ξ_1, ξ_2, t) and $(\xi_1 + \mathrm{d}\xi_1, \xi_2 + \mathrm{d}\xi_2, t + \mathrm{d}t)$ are corresponding coordinates in (ξ_1, ξ_2, t)-space. If the points P and Q' are chosen to be points on the kink surface (see [14] for

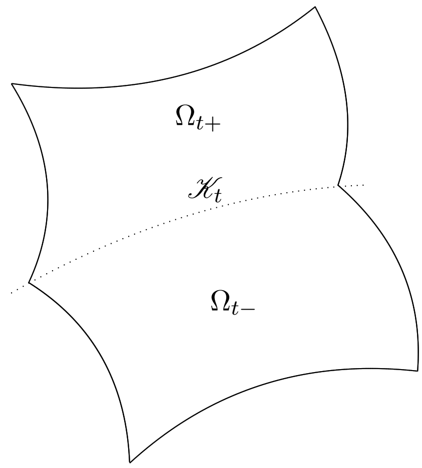


FIGURE 2. Kink curve $\mathscr{K}_t$ (shown with dotted lines) on $\Omega_t = \Omega_{t+} \cup \Omega_{t-}$

a two dimensional analog), then the conservation of d$\mathbf{x}$ implies that the expression for $(\mathrm{d}\mathbf{x})_+$ on one side of the kink surface must be equal to the expression for $(\mathrm{d}\mathbf{x})_-$ on the other side. Denoting quantities on the two sides of the kink by subscripts + and -, we get

$$\begin{aligned} & g_{1+}\mathrm{d}\xi_1\mathbf{u}_+ + g_{2+}\mathrm{d}\xi_2\mathbf{v}_+ + m_+\mathrm{d}t\mathbf{n}_+ \\ & \qquad = g_{1-}\mathrm{d}\xi_1\mathbf{u}_- + g_{2-}\mathrm{d}\xi_2\mathbf{v}_- + m_-\mathrm{d}t\mathbf{n}_-. \end{aligned} \tag{3.8}$$

We take the direction of the line element PQ' such that its projection on (ξ_1, ξ_2)-plane is in the direction of the normal to the shock curve in (ξ_1, ξ_2)-plane, then the differentials are further restricted. Let the unit normal of this shock curve be (E_1, E_2) and let K be its velocity of propagation in this plane, then the differentials in (3.8) satisfy $\frac{\mathrm{d}\xi_1}{\mathrm{d}t} = E_1 K$ and $\frac{\mathrm{d}\xi_2}{\mathrm{d}t} = E_2 K$, and (3.8) now becomes

$$\begin{aligned} & (g_{1+}E_1\mathbf{u}_+ + g_{2+}E_2\mathbf{v}_+)K + m_+\mathbf{n}_+ \\ & \qquad = (g_{1-}E_1\mathbf{u}_- + g_{2-}E_2\mathbf{v}_-)K + m_-\mathbf{n}_-. \end{aligned} \tag{3.9}$$

Thus (3.9) is a condition for the conservation of distance (in three independent directions in $\mathbf{x}$*-space) across a kink surface when a point moves along the normal to the shock curve in* (ξ_1, ξ_2)*-plane.*

Using the usual method for the derivation of jump conditions across a shock, we deduce the from conservation laws (3.5) and (3.6)

$$K[g_1\mathbf{u}] + E_1[m\mathbf{n}] = 0, \quad K[g_2\mathbf{v}] + E_2[m\mathbf{n}] = 0, \tag{3.10}$$

where a jump $[f]$ of a quantity f is defined by

$$[f] = f_+ - f_-. \tag{3.11}$$

Multiply the first relation in (3.10) by E_1 and the second relation by E_2, adding and using $E_1^2 + E_2^2 = 1$, we get

$$E_1 K[g_1\mathbf{u}] + E_2 K[g_2\mathbf{v}] + [m\mathbf{n}] = 0. \tag{3.12}$$

which is the same as (3.9). Thus we have proved a theorem of GPR, [9].

Theorem 3.1. *The six jump relations (3.10) imply conservation of distance in* x_1, x_2 *and* x_3 *directions (and hence in any arbitrary direction in* $\mathbf{x}$*-space) in the sense that the expressions for a vector* *displacement* $(d\mathbf{x})_{\mathscr{K}_t}$ *of a point of the kink line* $\mathscr{K}_t$ *in an infinitesimal time interval dt, when computed in terms of variables on the two sides of a kink surface, have the same value. This displacement of the point is assumed to take place on the kink surface and that of its image in* (ξ_1, ξ_2, t)*-space takes place on the shock surface such that the corresponding displacement in* (ξ_1, ξ_2)*-plane is with the shock front (i.e., it is in direction* $\frac{d}{dt}(\xi_1, \xi_2) = (E_1, E_2)K$*).*

This theorem assures that the 3-D KCL are physically realistic.

Consider a point P on a kink line $\mathscr{K}_t$ on Ω_t and two straight lines T_- and T_+ orthogonal to the kink line at P and lying in the tangent planes at P to Ω_{t-} and Ω_{t+} on the two sides of $\mathscr{K}_t$. Let N_- and N_+ be normals to the two tangent planes at P. Then the four lines T_+, N_+, N_- and T_-, being orthogonal to the kink line at P, are coplanar. A kink phenomenon is basically two dimensional. Locally, the two sides Ω_{t-} and Ω_{t+} of Ω_t can be regarded to be planes separated by a straight kink line. Hence the evolution of the kink phenomena can be viewed locally in a plane which intersects the planes Ω_{t-}, Ω_{t+} and $\mathscr{K}_t$ orthogonally as shown in the Figure 3.3.4 of [14].

We state an important result which will be very useful in proving many properties of the KCL. *Let* $P_0(\mathbf{x}_0)$ *be a given point on* Ω_t*. Then there exist two one parameter families of smooth curves on* Ω_t *such that the unit vectors* $\mathbf{u}_0$ *and* $\mathbf{v}_0$ *along the members of the curves through the chosen point* P_0 *can have any two arbitrary directions and the metrics* g_{10} *and* g_{20} *at this point can have any two positive values.*

4. An explicit differential form of KCL

Writing the differential form of (3.5) and taking inner product with $\mathbf{u}$ and using $\langle \mathbf{u}, \mathbf{n}_{\xi_1} \rangle = -\langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle$ we get

$$g_{1t} = -m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle. \tag{4.1}$$

Similarly,

$$g_{2t} = -m \langle \mathbf{n}, \mathbf{v}_{\xi_2} \rangle. \tag{4.2}$$

In the differential form of (3.5), we use the expression (4.1) for g_{1t} and get

$$g_1 \mathbf{u}_t = m_{\xi_1} \mathbf{n} + m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle \mathbf{u} + m \mathbf{n}_{\xi_1}. \tag{4.3}$$

Similarly

$$g_2 \mathbf{v}_t = m_{\xi_2} \mathbf{n} + m \langle \mathbf{n}, \mathbf{v}_{\xi_2} \rangle \mathbf{v} + m \mathbf{n}_{\xi_2}. \tag{4.4}$$

In order that

$$\cos\chi = \langle \mathbf{u}, \mathbf{v} \rangle \text{ and } \sin\chi = |\mathbf{u} \times \mathbf{v}| \tag{4.5}$$

is valid, we choose χ the angle between $\mathbf{u}$ and $\mathbf{v}$ to satisfy $0 < \chi < \pi$. Then

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}|_{\xi_1} &= (\sin\chi)_{\xi_1} = -\frac{\cos\chi}{\sin\chi}(\cos\chi)_{\xi_1} \\ &= -\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u} \times \mathbf{v}|} \langle \mathbf{u}, \mathbf{v} \rangle_{\xi_1}. \end{aligned} \tag{4.6}$$

Hence, from (3.1)

$$\mathbf{n}_{\xi_1} = \frac{1}{|\mathbf{u} \times \mathbf{v}|} \left\{ (\mathbf{u} \times \mathbf{v})_{\xi_1} + \frac{\mathbf{n} \langle \mathbf{u} \times \mathbf{v} \rangle}{|\mathbf{u} \times \mathbf{v}|} \langle \mathbf{u}, \mathbf{v} \rangle_{\xi_1} \right\}. \tag{4.7}$$

Substituting the expressions (3.1) for $\mathbf{n}$ and (4.7) for $\mathbf{n}_{\xi_1}$ in (4.3) we get a form of an equation for $\mathbf{u}$ in which $\mathbf{u}_t$ expressed purely in terms of $m, g_1, \mathbf{u}$ and $\mathbf{v}$. Similarly, we can get a form of an equation for $\mathbf{v}$. It is simple to show that the third scalar equation in (4.3), i.e., the equation for u_3 (or in

(4.4), i.e., the equation for v_3) can be derived from the first two equations in (4.3) (or in (4.4)). Thus, (4.1)-(4.4) contain a set of 6 independent differential forms of equations of the KCL (3.5) and (3.6) purely in terms of $m, g_1, \mathbf{u}$ and $\mathbf{v}$.

We now proceed to derive an explicit form of the equations (4.1)-(4.2) and the first two in each of the two equations (4.3) and (4.4), i.e., we write these equations in terms of variables g_1, g_2, u_1, u_2, v_1 and v_2 only (i.e., free from u_3 and v_3). This involves long calculations. We first express derivatives of u_3 in terms of those of u_1 and u_2 using the relation $u_3^2 = 1 - u_1^2 - u_2^2$. This immediately leads to equations for g_1 and g_2 in the forms

$$g_{1t} - m\frac{n_3u_1 - n_1u_3}{u_3}u_{1\xi_1} + m\frac{n_2u_3 - n_3u_2}{u_3}u_{2\xi_1} = 0, \tag{4.8}$$

$$g_{2t} + m\frac{n_1v_3 - n_3v_1}{v_3}v_{1\xi_2} - m\frac{n_3v_2 - n_2v_3}{v_3}v_{2\xi_2} = 0. \tag{4.9}$$

We take $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ to be a right handed set of vectors, then

$$(n_1, n_2, n_3) = \frac{1}{\sin\chi}\left(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\right). \tag{4.10}$$

Using these expressions for components of $\mathbf{n}$ and using $u_1^2+u_2^2+u_3^2 = 1$ and $u_1v_1+u_2v_2+u_3v_3 = \cos\chi$ we get the following expressions for some terms in the coefficients in (4.8)

$$n_3u_1 - n_1u_3 = \frac{v_2}{\sin\chi} - u_2\cot\chi, \quad n_2u_3 - n_3u_2 = \frac{v_1}{\sin\chi} - u\cot\chi. \tag{4.11}$$

We can do similar calculation for the coefficients in equation (4.9). Thus, we get the required equations for g_1 and g_2

$$g_{1t} - m\frac{v_2 - u_2\cos\chi}{u_3\sin\chi}u_{1\xi_1} + m\frac{v_1 - u_1\cos\chi}{u_3\sin\chi}u_{2\xi_1} = 0, \tag{4.12}$$

$$g_{2t} + m\frac{u_2 - v_2\cos\chi}{v_3\sin\chi}v_{1\xi_2} - m\frac{u_1 - v_1\cos\chi}{u_3\sin\chi}v_{2\xi_2} = 0. \tag{4.13}$$

When we use (4.6) in the equation (4.3) for u_1, we note that we need to find expressions for $u_{3\xi_1}$, and the first components of $(\mathbf{u}\times\mathbf{v})_{\xi_1}$ and $\mathbf{n}\langle\mathbf{u}, \mathbf{v}\rangle\langle\mathbf{u}, \mathbf{v}\rangle_{\xi_1}$ expressed purely in terms of u_1, u_2, v_1 and v_2. This requires long calculations and finally yields

$$\begin{aligned} g_1u_{1t} = n_1m_{\xi_1} + m\Bigg\{ & \frac{u_1u_2 + n_1n_2}{u_3\sin\chi}\cos\chi u_{1\xi_1} - \frac{u_1^2 + n_1^2 - 1}{u_3\sin\chi}\cos\chi u_{2\xi_1} \\ & - \frac{u_2v_1 + n_1n_2 cos\chi}{v_3\sin\chi}v_{1\xi_1} + \frac{u_1v_1 + (n_1^2 - 1)\cos\chi}{v_3\sin\chi}v_{2\xi_1}\Bigg\} \end{aligned} \tag{4.14}$$

which is of the desired form since u_3, v_3 and components of $\mathbf{n}$ can be expressed in terms of u_1, u_2, v_1 and v_2. Similarly, we can find equations of evolution for u_2, v_1 and v_2. Collecting all these results, we get the following explicit differential forms of the equations for u_1, u_2, v_1 and v_2. In these equations derivatives of $\mathbf{n}, u_3$ and v_3 do not appear.

$$g_1u_{1t} - n_1m_{\xi_1} + b_{11}^{(1)}u_{1\xi_1} + b_{12}^{(1)}u_{2\xi_1} + b_{13}^{(1)}v_{1\xi_1} + b_{14}^{(1)}v_{2\xi_1} = 0, \tag{4.15}$$

$$g_1u_{2t} - n_2m_{\xi_1} + b_{21}^{(1)}u_{1\xi_1} + b_{22}^{(1)}u_{2\xi_1} + b_{23}^{(1)}v_{1\xi_1} + b_{24}^{(1)}v_{2\xi_1} = 0, \tag{4.16}$$

$$g_2v_{1t} - n_1m_{\xi_2} + b_{31}^{(2)}u_{1\xi_2} + b_{32}^{(2)}u_{2\xi_2} + b_{33}^{(2)}v_{1\xi_2} + b_{34}^{(2)}v_{2\xi_2} = 0, \tag{4.17}$$

$$g_2v_{2t} - n_2m_{\xi_2} + b_{41}^{(2)}u_{1\xi_2} + b_{42}^{(2)}u_{2\xi_2} + b_{43}^{(2)}v_{1\xi_2} + b_{44}^{(2)}v_{2\xi_2} = 0, \tag{4.18}$$

where the coefficients $b_{ij}^{(1)}$ and $b_{ij}^{(2)}$ are given in the Appendix A.

We find a very interesting between the coefficients $b_{ij}^{(1)}$ and $b_{ij}^{(2)}$. This is obtained as a consistency condition between two different expressions for m_{ξ_1} and m_{ξ_2}. We proceed to derive the expressions for m_{ξ_1}. Differentiate $m^2 = m^2(n_1^2 + n_2^2 + n_3^2)$ to derive

$$m_{\xi_1} = n_1(mn_1)_{\xi_1} + n_2(mn_2)_{\xi_1} + n_3(mn_3)_{\xi_1},$$

use (2.3) for the expressions in the brackets and interchange the order of derivatives to get

$$\begin{aligned} m_{\xi_1} = & \ \langle \mathbf{n}, (\mathbf{x}_{\xi_1})_t \rangle = \langle \mathbf{n}, (g_1 \mathbf{u})_t \rangle \\ & = g_{1t} \langle \mathbf{n}, \mathbf{u} \rangle + g_1 \langle \mathbf{n}, \mathbf{u}_t \rangle = g_1 \langle \mathbf{n}, \mathbf{u}_t \rangle \\ & = g_1 \alpha_1 u_{1t} + g_1 \alpha_2 u_{2t}, \quad \text{after using} \quad u_3^2 = 1 - u_1^2 - u_2^2, \end{aligned} \tag{4.19}$$

where

$$\alpha_1 = -\frac{n_3 u_1 - n_1 u_3}{u_3} =: \frac{1}{m} b_{61}^{(1)}, \quad \alpha_2 = \frac{n_2 u_3 - n_3 u_2}{u_3} =: \frac{1}{m} b_{62}^{(1)}. \tag{4.20}$$

Using the expressions for u_{1t} and u_{2t} in (4.19) from (4.15) and (4.16), we get an identity. Equating coefficients $m_{\xi_1}, u_{1\xi_1}, u_{2\xi_1}, v_{1\xi_1}, v_{2\xi_1}$ we get 5 consistency conditions

$$\begin{aligned} n_1\alpha_1 + n_2\alpha_2 = 1, \quad \alpha_1 b_{11}^{(1)} + \alpha_2 b_{21}^{(1)} = 0, \quad \alpha_1 b_{21}^{(1)} + \alpha_2 b_{22}^{(1)} = 0, \\ \alpha_1 b_{13}^{(1)} + \alpha_2 b_{23}^{(1)} = 0, \quad \alpha_1 b_{14}^{(1)} + \alpha_2 b_{24}^{(1)} = 0. \end{aligned} \tag{4.21}$$

Similarly, starting from m_{ξ_2}, we get another set of consistency conditions in terms of β_3 and β_4, where

$$\beta_3 = \frac{n_1 v_3 - n_3 v_1}{v_3} =: \frac{1}{m} b_{73}^{(2)}, \ \beta_4 = -\frac{n_3 v_2 - n_2 v_3}{v_3} =: \frac{1}{m} b_{74}^{(2)}. \tag{4.22}$$

$$\begin{aligned} n_1\beta_3 + n_2\beta_4 = 1, \quad \beta_3 b_{31}^{(2)} + \beta_4 b_{41}^{(2)} = 0, \quad \beta_3 b_{32}^{(2)} + \beta_4 b_{42}^{(2)} = 0, \\ \beta_3 b_{33}^{(2)} + \beta_4 b_{43}^{(2)} = 0, \quad \beta_3 b_{34}^{(2)} + \beta_4 b_{44}^{(2)} = 0. \end{aligned} \tag{4.23}$$

(4.21) and (4.23) show nice relations in the coefficients of the equations (4.15)-(4.18). We have used these relations to simplify numerical computation of eigenvalues in the section 8.2.

3D-KCL being only 6 equations in seven quantities $u_1, u_2, v_1, v_2, m, g_1$ and g_2, it is an under-determined system. This is expected as KCL is purely a mathematical result and the dynamics of a particular moving surface Ω_t has played no role in the derivation of KCL. In our previous investigations, we have closed the 2-D KCL for three different types of Ω_t ([2], [14]-chapters 6 and 10), one of them being the case when Ω_t is a weakly nonlinear wavefront in a polytropic gas, which we shall consider again in the section 6 for 3D-KCL. We shall use the energy transport equation along rays of the weakly nonlinear ray theory in a polytropic gas to derive the closure equation in the form of an additional conservation law.

5. Equivalence of KCL and ray equations

Let us start with a given smooth function m of $\mathbf{x}$ and t and let $\mathbf{x}, \mathbf{n}$ (with $|\mathbf{n}| = 1$) satisfy the ray equations (2.3) and (2.4), which give successive positions of a moving surface Ω_t. Choose a coordinate system (ξ_1, ξ_2) on Ω_t with metrics g_1 and g_2 associated with ξ_1 and ξ_2 respectively. Let $\mathbf{u}$ and $\mathbf{v}$ be unit tangent vectors along the curves $\xi_2 =$ constant and $\xi_1 =$ constant respectively. Then the derivation of the section 3 leads to the equations (3.5)-(3.7) and hence the 3-D KCL. Thus the ray equations imply 3-D KCL.

In addition to the above proof, let us give a direct derivation of the KCL equations (4.1) and (4.2) from the first ray equations, i.e, equation (2.3). Definition of the metric g_1 gives $g_1^2 = x_{1\xi_1}^2 +$

${x_2}^2_{\xi_1} + {x_3}^2_{\xi_1} = |\mathbf{x}_{\xi_1}|^2$. Differentiating it with respect to t, using $\mathbf{x}_{\xi_1 t} = \mathbf{x}_{t\xi_1}$ and $\mathbf{x}_{\xi_1} = g_1\mathbf{u}$ (from (3.2)) we get

$$g_{1t} = \langle \mathbf{u}, (\mathbf{x}_t)_{\xi_1} \rangle. \tag{5.1}$$

Using (2.3) in this

$$\begin{aligned} g_{1t} &= \langle \mathbf{u}, m_{\xi_1}\mathbf{n} + m\mathbf{n}_{\xi_1} \rangle \\ &= \langle \mathbf{u}, m\mathbf{n}_{\xi_1} \rangle = -m\langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle. \end{aligned} \tag{5.2}$$

which is the equation (4.1). Similarly the equation (4.2) can be derived.

Now we take up the proof of the converse, i.e., the derivation of the ray equations from the KCL (3.5)-(3.6). We are given three smooth unit vector fields $\mathbf{u}, \mathbf{v}, \mathbf{n}$ and three smooth scalar functions m, g_1 and g_2 in (ξ_1, ξ_2, t)-space such that $\mathbf{n}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, i.e.,

$$\langle \mathbf{n}, \mathbf{u} \rangle = 0 \quad \text{and} \quad \langle \mathbf{n}, \mathbf{v} \rangle = 0 \tag{5.3}$$

and they satisfy the KCL (3.5)-(3.7).

According to the fundamental integability theorem ([8]-page 104), the conditions (3.5)-(3.7) imply the existence of a vector $\mathbf{x}$ satisfying (3.2), i.e.,

$$(\mathbf{x}_t, \mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2}) = (m\mathbf{n}, g_1\mathbf{u}, g_2\mathbf{v}). \tag{5.4}$$

This gives a one to one mapping between $\mathbf{x}$-space and (ξ_1, ξ_2, t)-space as long as the Jacobian (3.3) is neither zero nor infinity. Let t = constant in (ξ_1, ξ_2, t)-space is mapped on to a surface Ω_t in $\mathbf{x}$-space on which ξ_1 and ξ_2 are surface coordinates. Then $\mathbf{u}$ and $\mathbf{v}$ are tangent to Ω_t and (5.3) show that $\mathbf{n}$ is orthogonal to Ω_t. Let $\varphi(\mathbf{x}, t) = 0$ be the equation of Ω_t, then $\mathbf{n} = \nabla\varphi/|\nabla\varphi|$. The relation $\mathbf{x}_t = m\mathbf{n}$ in (5.4) is nothing but the first part of the ray equation and shows that m is the normal velocity of Ω_t. The function φ now satisfies the eikonal equation (2.2) which implies (2.4), see also [15]. Thus, we have derived the ray equations from KCL.

Now we have completed the proof of the theorem

Theorem 5.1. *For a given smooth function m of $\mathbf{x}$ and t, the ray equations (2.3) and (2.4) are equivalent to the KCL as long as their solutions are smooth.*

Though we have established equivalence of two systems, it is instructive to derive the second part of the ray equations, i.e., the equations (2.4) from KCL (3.5)-(3.6) by direct calculation, which we do below.

Transformation (5.4) between $\mathbf{x}$-space and (ξ_1, ξ_2, t)-space, implies relations

$$\frac{\partial}{\partial t} = \langle m\mathbf{n}, \nabla \rangle, \quad \frac{\partial}{g_1\partial\xi_1} = \langle \mathbf{u}, \nabla \rangle, \quad \frac{\partial}{g_2\partial\xi_2} = \langle \mathbf{v}, \nabla \rangle \tag{5.5}$$

between the partial derivatives in (x, y, z) and (ξ_1, ξ_2, t) coordinates. Solving $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ in terms of $\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial t}$, we get

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \frac{\mathbf{n}}{m}\frac{\partial}{\partial t} + \frac{\mathbf{u} - \mathbf{v}\cos\chi}{\sin\chi}\frac{1}{g_1}\frac{\partial}{\partial\xi_1} + \frac{\mathbf{v} - \mathbf{u}\cos\chi}{\sin\chi}\frac{1}{g_2}\frac{\partial}{\partial\xi_2}. \tag{5.6}$$

Differentiating the relations $|\mathbf{n}| = 1, \langle \mathbf{n}, \mathbf{u} \rangle = 0$ and $\langle \mathbf{n}, \mathbf{v} \rangle = 0$ with respect to t, and solving for n_{1t} and n_{2t}, we get

$$\begin{aligned} n_{1t} = &\frac{(n_3v_2 - n_2v_3)}{u_3}\{(n_3u_1 - n_1u_3)u_{1t} - (n_2u_3 - n_3u_2)u_{2t}\} \\ &+ \frac{(n_2u_3 - n_3u_2)}{v_3}\{-(n_1v_3 - n_3v_1)v_{1t} + (n_3v_2 - n_2v_3)v_{2t}\}. \end{aligned} \tag{5.7}$$

and

$$n_{2t} = \frac{(n_1v_3 - n_3v_1)}{u_3}\{(n_3u_1 - n_1u_3)u_{1t} - (n_2u_3 - n_3u_2)u_{2t}\} + \frac{(n_3u_1 - n_1u_3)}{v_3}\{-(n_1v_3 - n_3v_1)v_{1t} + (n_3v_2 - n_2v_3)v_{2t}\}. \tag{5.8}$$

Now we substitute the expressions for u_{1t}, u_{2t}, v_{1t} and v_{2t} from (4.15)-(4.18) in the terms on the right hand side of above equations and after long calculations we find that all terms $u_{i\xi_i}, v_{i\xi_i}, i = 1, 2$ drop out and only derivatives of m appear. The final equations are

$$\frac{\partial n_1}{\partial t} = -\frac{1}{\sin\chi}\left\{(u_1 - v_1\cos\chi)\frac{\partial m}{g_1\partial\xi_1} + (v_1 - u_1\cos\chi)\frac{\partial m}{g_2\partial\xi_2}\right\}, \tag{5.9}$$

$$\frac{\partial n_2}{\partial t} = -\frac{1}{\sin\chi}\left\{(u_2 - v_2\cos\chi)\frac{\partial m}{g_1\partial\xi_1} + (v_2 - u_2\cos\chi)\frac{\partial m}{g_2\partial\xi_2}\right\}. \tag{5.10}$$

These are precisely the same equations for n_1 and n_2 if we substitute the expression (5.6) for $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ in the second part of the ray equations, i.e., the equations (2.4) for n_1 and n_2. Thus, we have derived (2.4) from the differential form of KCL.

6. Energy transport equation from a WNLRT for a polytropic gas and the complete set of equations

In this section we shall derive a closure relation in a conservation form for the 3D-KCL so that we get a completely determined system of conservation laws. Let the mass density, fluid velocity and gas pressure in a polytropic gas [7] be denoted by $\varrho, \mathbf{q}$ and p. Consider now a high frequency small amplitude curved wavefront Ω_t running into a polytropic gas in a uniform state and at rest ($\varrho_0 =$ constant, $\mathbf{q} = 0$ and $p_0 =$ constant, [14]-section 6.1). Then a perturbation in the state of the gas on Ω_t can be expressed in terms of an amplitude w and is given by

$$\varrho - \varrho_0 = \left(\frac{\varrho_0}{a_0}\right)w, \;\; \mathbf{q} = \mathbf{n}w, \;\; p - p_0 = \varrho_0 a_0 w. \tag{6.1}$$

where a_0 is the sound velocity in the undisturbed medium $= \sqrt{\gamma p_0/\varrho_0}$ and w is a quantity of small order, say $\mathcal{O}(\epsilon)$. Let us remind, what we stated in the section 1, all dependent variables are dimensional in this (and only in this) paragraph. Note that w here has the dimension of velocity.

The amplitude w is related to the non-dimensional normal velocity m of Ω_t by

$$m = 1 + \frac{\gamma+1}{2}\frac{w}{a_0}. \tag{6.2}$$

The operator $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + m\langle\mathbf{n}, \nabla\rangle$ in space-time becomes simply the partial derivative $\frac{\partial}{\partial t}$ in the ray coordinate system (ξ_1, ξ_2, t). Hence the energy transport equation of the WNLRT ([14]-equation (6.1.3)) in non-dimensional coordinates becomes

$$m_t = (m-1)\mathit{\Omega} = -\frac{1}{2}(m-1)\langle\nabla, \mathbf{n}\rangle, \tag{6.3}$$

where the italic symbol $\mathit{\Omega}$ is the mean curvature of the wavefront Ω_t. Ray tube area A for any ray system ([19]-pages 244, 280, [14]-relation (2.2.23)) is related to the mean curvature $\mathit{\Omega}$ (we write here in non-dimensional variables) by

$$\frac{1}{A}\frac{\partial A}{\partial l} = -2\mathit{\Omega}, \;\; \frac{\partial}{\partial l} \text{ in ray coordinates,} \tag{6.4}$$

where l is the arc length along a ray. In non-dimensional variables $\mathrm{d}l = m\mathrm{d}t$. From (6.3) and (6.4) we get

$$\frac{2m_t}{m-1} = -\frac{1}{mA}A_t. \tag{6.5}$$

This leads to a conservation law, which we accept to be the required one,

$$\left\{(m-1)^2 e^{2(m-1)} A\right\}_t = 0. \tag{6.6}$$

Integration gives $(m-1)^2 e^{2(m-1)} A = F(\xi_1, \xi_2)$, where F is an arbitrary function of ξ_1 and ξ_2. The ray tube area A is given by $A = g_1 g_2 \sin\chi$, where χ is defined by (3.4). In order that A is positive, we need to choose $0 < \chi < \pi$. Now the energy conservation equation becomes

$$\left\{(m-1)^2 e^{2(m-1)} g_1 g_2 \sin\chi\right\}_t = 0. \tag{6.7}$$

After a few steps of calculation the differential form of this conservation law becomes

$$g_2 g_{1t} + g_1 g_{2t} + g_1 g_2 \cot\chi \left\{-\frac{n_2}{u_3}u_{1t} + \frac{n_1}{u_3}u_{2t} + \frac{n_2}{v_3}v_{1t} - \frac{n_1}{v_3}v_{2t}\right\} + \frac{2g_1 g_2 m}{m-1} m_t = 0 \tag{6.8}$$

or

$$a_{51}u_{1t} + a_{52}u_{2t} + a_{53}v_{1t} + a_{54}v_{2t} + a_{55}m_t + a_{56}g_{1t} + a_{57}g_{2t} = 0, \text{ say.} \tag{6.9}$$

The complete set of conservation laws for the weakly nonlinear ray theory (WNLRT) for a polytropic gas are: the six equations in (3.5)-(3.6) and the equation (6.7). The equations (3.7) need to be satisfied at any fixed t, say at $t = 0$. A complete set of equations of WNLRT in differential form are: the equation (4.15)-(4.18), (6.8) and the two equations (4.12) and (4.13), i.e.,

$$g_{1t} + b^{(1)}_{61} u_{1\xi_1} + b^{(1)}_{62} u_{2\xi_1} = 0, \tag{6.10}$$

$$g_{2t} + b^{(2)}_{73} v_{1\xi_2} + b^{(2)}_{74} v_{2\xi_2} = 0, \tag{6.11}$$

where the coefficients are given in Appendix B.

A matrix form of these equations for the vector $\mathbf{U} = (u_1, u_2, v_1, v_2, m, g_1, g_2)^T$ is

$$A\mathbf{U}_t + B^{(1)}\mathbf{U}_{\xi_1} + B^{(2)}\mathbf{U}_{\xi_2} = 0, \tag{6.12}$$

where

$$A = \begin{bmatrix} g_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_2 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{6.13}$$

$$B^{(1)} = \begin{bmatrix} b^{(1)}_{11} & b^{(1)}_{12} & b^{(1)}_{13} & b^{(1)}_{14} & -n_1 & 0 & 0 \\ b^{(1)}_{21} & b^{(1)}_{22} & b^{(1)}_{23} & b^{(1)}_{24} & -n_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b^{(1)}_{61} & b^{(1)}_{62} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.14}$$

$$B^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{31}^{(2)} & b_{32}^{(2)} & b_{33}^{(2)} & b_{34}^{(2)} & -n_1 & 0 & 0 \\ b_{41}^{(2)} & b_{42}^{(2)} & b_{43}^{(2)} & b_{44}^{(2)} & -n_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{73}^{(2)} & b_{74}^{(2)} & 0 & 0 & 0 \end{bmatrix}. \tag{6.15}$$

7. Eigenvalues and eigenvectors of the equations of WNLRT in terms the unknowns (n_1, n_2, m, g_1, g_2)

Let us define two operators

$$\frac{\partial}{\partial \lambda_1} = n_1 \frac{\partial}{\partial x_3} - n_3 \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial \lambda_2} = n_3 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_3}. \tag{7.1}$$

They represent derivatives in two independent tangential directions on Ω_t and hence the operator $\mathbf{L}$ in (2.4) can be expressed in terms of $\frac{\partial}{\partial \lambda_1}$ and $\frac{\partial}{\partial \lambda_2}$. Two independent equations in (2.4), say for n_1 and n_2, can be written as

$$\frac{\partial n_1}{\partial t} - \frac{n_2^2 + n_3^2}{n_3} \frac{\partial m}{\partial \lambda_1} - \frac{n_1 n_2}{n_3} \frac{\partial m}{\partial \lambda_2} = 0, \tag{7.2}$$

$$\frac{\partial n_2}{\partial t} + \frac{n_1 n_2}{n_3} \frac{\partial m}{\partial \lambda_1} + \frac{n_1^2 + n_3^2}{n_3} \frac{\partial m}{\partial \lambda_2} = 0. \tag{7.3}$$

The expression $\langle \nabla, \mathbf{n} \rangle$, when we use $n_3^2 = 1 - n_1^2 - n_2^2$, can also be written in terms of the operators $\frac{\partial}{\partial \lambda_1}$ and $\frac{\partial}{\partial \lambda_2}$. The transport equation (6.3) now takes the form

$$\frac{\partial m}{\partial t} - \frac{(m-1)}{2n_3} \frac{\partial n_1}{\partial \lambda_1} + \frac{(m-1)}{2n_3} \frac{\partial n_2}{\partial \lambda_2} = 0. \tag{7.4}$$

The equations of WNLRT in terms of unknowns n_1, n_2 and m are just the three equations (7.2)-(7.4). However, we have shown in section 5 that (7.2) and (7.3) along with (2.3) form a system equivalent to the KCL. Thus the system of equations (7.2)-(7.4) is equivalent to a bigger system of seven equations (6.12) in terms of another set of variables $\{u_1, u_2, v_1, v_2, m, g_1, g_2\}$.

For some analysis later on, it is worth adding to the equations (7.2)-(7.4) equations for g_1 and g_2 also in terms of $\frac{\partial}{\partial \lambda_1}$ and $\frac{\partial}{\partial \lambda_2}$. But this can done only by freezing the operators and the derivatives at a given point.

Freezing of coefficients at a given point $P_0(\mathbf{x_0})$ of Ω_t. In the definition (7.1), λ_1 and λ_2 are not variables, but $\frac{\partial}{\partial \lambda_1}$ and $\frac{\partial}{\partial \lambda_2}$ are simply symbols for the two operators. It is quite unlikely that we can globally choose $\mathbf{u}$ to be $\frac{1}{\sqrt{n_1^2+n_3^2}}(-n_3, 0, n_1)$ and $\mathbf{v}$ to be $\frac{1}{\sqrt{n_2^2+n_3^2}}(0, n_3, -n_2)$. However, the result mentioned at the end of the section 3 tells us that a given point $P_0(\mathbf{x}_0)$, we can choose

$$\mathbf{u}_0 = \frac{1}{\sqrt{n_1^2 + n_3^2}}(-n_3, 0, n_1), \quad \mathbf{v}_0 = \frac{1}{\sqrt{n_3^2 + n_2^2}}(0, n_3, -n_2). \tag{7.5}$$

Similarly at this point we can choose

$$g_1 = g_{1_0} \text{ and } g_2 = g_{2_0}. \tag{7.6}$$

where g_{10} and g_{20} are arbitrary values. Now at P_0

$$\begin{aligned}\frac{1}{g_{10}}\frac{\partial}{\partial \xi_1} &= \text{space rate of change in the direction of } \mathbf{u}_0 \\ &= \langle \mathbf{u}_0, \nabla \rangle = \frac{1}{\sqrt{n_1^2+n_3^2}}\frac{\partial}{\partial \lambda_1}.\end{aligned} \tag{7.7}$$

Similarly at P_0

$$\frac{1}{g_{20}}\,\frac{\partial}{\partial \xi_2} = \frac{1}{\sqrt{n_2^2+n_3^2}}\frac{\partial}{\partial \lambda_2}. \tag{7.8}$$

Freezing the equations (7.2), (7.3) and (7.4) at P_0 and using (7.7) and (7.8) we get at P_0

$$\frac{\partial n_1}{\partial t} - \frac{(n_2^2+n_3^2)\sqrt{n_1^2+n_3^2}}{n_3 g_{10}}\frac{\partial m}{\partial \xi_1} - \frac{n_1 n_2\sqrt{n_2^2+n_3^2}}{n_3 g_{20}}\frac{\partial m}{\partial \xi_2} = 0, \tag{7.9}$$

$$\frac{\partial n_2}{\partial t} + \frac{n_1 n_2\sqrt{n_1^2+n_3^2}}{n_3 g_{10}}\frac{\partial m}{\partial \xi_1} + \frac{(n_1^2+n_3^2)\sqrt{n_2^2+n_3^2}}{n_3 g_{20}}\frac{\partial m}{\partial \xi_2} = 0, \tag{7.10}$$

$$\frac{\partial m}{\partial t} - \frac{(m-1)\sqrt{n_1^2+n_3^2}}{2n_3 g_{10}}\frac{\partial n_1}{\partial \xi_1} + \frac{(m-1)\sqrt{n_2^2+n_3^2}}{2n_3 g_{20}}\frac{\partial n_2}{\partial \xi_2} = 0. \tag{7.11}$$

The eigenvalues of these three frozen equations are

$$\mu_{1,2} = \pm\left[\frac{m-1}{2n_3^2}\left\{\left(n_2^2+n_3^2\right)\overline{e}_1^2 + 2n_1n_2\overline{e}_1\overline{e}_2 + \left(n_1^2+n_3^2\right)e_2^{-2}\right\}\right]^{1/2},\ \mu_3 = 0, \tag{7.12}$$

where

$$\overline{e}_1 = \frac{\sqrt{n_1^2+n_3^2}}{g_{10}}e_1, \quad \overline{e}_2 = \frac{\sqrt{n_2^2+n_3^2}}{g_{20}}e_2 \tag{7.13}$$

and (e_1, e_2) is an arbitrary nonzero 2-D vector. Since these are distinct, the eigenspace is complete. It is easy to see that

$$(n_2^2+n_3^2)\overline{e}_1^2 + 2n_1n_2\overline{e}_1\overline{e}_2 + (n_1^2+n_3^2)\overline{e}_2^2 = (n_2\overline{e}_1 + n_1\overline{e})^2 + n_3^2(\overline{e}_1^2+\overline{e}_2^2) > 0. \tag{7.14}$$

Thus the frozen system of 3 equations (7.9)-(7.11) is hyperbolic if $m > 1$ and has elliptic nature if $m < 1$ (not strictly elliptic because one eigenvalue is real).

To the equations (7.9)-(7.11), we can add the equations for g_1 and g_2 in terms of the variables n_1 and n_2. Writing differential form of (3.5) and taking inner product with $\mathbf{u}$, we get $g_{1t} = m\langle \mathbf{u}, \mathbf{n}_{\xi_1}\rangle$. We eliminate the derivative of n_3 by using $n_3^2 = 1 - n_1^2 - n_2^2$ and get

$$g_{1t} = \frac{m}{n_3}\left\{(n_3u_1 - n_1u_3)n_{1\xi_1} - (n_2u_3 - n_3u_2)n_{2\xi_1}\right\}. \tag{7.15}$$

We freeze this equation at P_0 and use (7.5), then the equation becomes

$$g_{1t} + \frac{m\sqrt{n_1^2+n_3^2}}{n_3}n_{1\xi_1} + \frac{mn_1n_2}{n_3\sqrt{n_1^2+n_3^2}}n_{2\xi_1} = 0. \tag{7.16}$$

Similarly, we can get the frozen equation for g_2 at P_0 as

$$g_{2t} - \frac{mn_1n_2}{n_3\sqrt{n_2^2+n_3^2}}n_{1\xi_2} - \frac{m\sqrt{n_2^2+n_3^2}}{n_3}n_{2\xi_2} = 0. \tag{7.17}$$

It is important to note that these two equations (like the equations (7.9)-(7.11)) are frozen equations at P_0 with special directions $\mathbf{u}_0$ and $\mathbf{v}_0$.

Since g_{10} and g_{20} appear in (7.9)-(7.11), we should actually consider the system of five frozen equation (7.9)-(7.11), (7.16) and (7.17). The eigenvalues of this system are

$$\mu_1,\ \mu_2,\ \mu_3 = 0,\ \mu_4 = 0,\ \mu_5 = 0, \tag{7.18}$$

where μ_1 and μ_2 are given by (7.12). It is now simple to check that the number of linearly independent eigenvectors corresponding to the triple eigenvalue 0 is only 2. The system now becomes degenerate. This is an important result which will be noticed to be true for the system of 7 equations for the vector $(u_1, u_2, v_1, v_2, m, g_1, g_2)$.

8. Eigenvalues and eigenvectors of WNLRT in terms of $(u_1, u_2, v_1, v_2, m, g_1, g_2)$

We have not been able to find out expressions for the eigenvalues of the system of equations (6.12) directly by solving the 7th degree equation for the eigenvalues. We can find them in a special case by choosing the vectors $\mathbf{u} = \mathbf{u}'$ and $\mathbf{v} = \mathbf{v}'$, where $\mathbf{u}'$ and $\mathbf{v}'$ are orthogonal at P_0 and then freezing the coefficients at this point.

8.1. **Freezing of the coefficients at P_0 where $\mathbf{u}'$ and $\mathbf{v}'$ are orthogonal.** When $\mathbf{u}'$ is orthogonal to $\mathbf{v}'$, $\cos\chi' = 0$ and $\sin\chi' = 1$. The entries in the coefficient matrices $A', B'^{(1)}$ and $B'^{(2)}$ of the system (6.12) simplify considerably. Then the equation for the eigenvalues, i.e., $\det\left(-\nu A' + e_1' B'^{(1)} + e_2' B'^{(2)}\right) = 0$, where $(e_1', e_2') \in \mathbb{R}^2 \backslash (0,0)$, becomes

$$\det\begin{bmatrix} -\nu g_1' & 0 & \frac{m u_2' v_1'}{v_3'} e_1' & \frac{m u_1' v_1'}{v_3'} e_1' & -n_1 e_1' & 0 & 0 \\ 0 & -\nu g_1' & \frac{m u_2' v_2'}{v_3'} e_1' & \frac{m u_1' v_2'}{v_3'} & -n_2 e_1' & 0 & 0 \\ -\frac{m u_1' v_2'}{u_3'} e_2' & \frac{m u_1' v_1'}{u_3'} e_2' & -\nu g_2' & 0 & -n_1 e_2' & 0 & 0 \\ -\frac{m u_2' v_2'}{u_3'} e_2' & \frac{m u_2' v_1'}{u_3'} e_2' & 0 & -\nu g_2' & -n_2 e_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu \frac{2m}{m-1} g_1' g_2' & -\nu g_2' & -\nu g_1' \\ -\frac{m v_2'}{u_3'} e_1' & \frac{m v_1'}{u_3'} e_1' & 0 & 0 & 0 & -\nu & 0 \\ 0 & 0 & \frac{m u_2'}{v_3'} e_2' & -\frac{m u_1'}{v_3'} e_2' & 0 & 0 & -\nu \end{bmatrix} = 0. \tag{8.1}$$

A long calculation leads to the following eigenvalues

$$\nu_{1,2} = \pm \left\{ \frac{(m-1)(e_1'^2 g_2' + e_2'^2 g_1')}{2 g_1'^2 g_2'^2} \right\}^{1/2}, \quad \nu_3 = \nu_4 = \nu_5 = \nu_6 = \nu_7 = 0. \tag{8.2}$$

It is also found that the number of independent eigenvectors corresponding to the multiple eigenvalue 0 is 4 resulting in the loss of hyperbolicity of the system for $m > 1$.

8.2. **Numerical computation of the eigenvalues and eigenvectors for the general case.** In this case we take any point P on Ω_t and choose $g_1 = 1,\ g_2 = 1$. Now we choose different values of $m > 1$ and < 1 and of vectors $\mathbf{u}$ and $\mathbf{v}$ such that $u_3 \neq 0$ and $v_3 \neq 0$. Choice of $u_3 \neq 0$ and $v_3 \neq 0$ is required because we discounted the equations for u_3 and v_3 in the section 4 and this lead to appearance of u_3 and v_3 in the denominators of many coefficients in the equations (6.12). We solved equation $\det\left(-\lambda A + e_1 B^{(1)} + e_2 B^{(2)}\right) = 0$ numerically for computing the eigenvalues $\lambda_i, i = 1, 2, \ldots, 7$, for a number of values of scalars e_1 and e_2. To simplify the numerical computation we used the relations (4.21) and (4.23).

All results for different choices of $\mathbf{u}$ and $\mathbf{v}$ gave values of $\lambda_3 = \ldots = \lambda_7 = 0$ and $\lambda_1(= -\lambda_2)$ real for $m > 1$ and purely imaginary for $m < 1$. From numerical experiments, we postulate that *the* 7×7 *system (6.12) has two distinct eigenvalues* λ_1 *and* $\lambda_2 = -\lambda_1$ *for* $m \neq 1$ *and five coincident eigenvalues* $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$*. The eigenvalues* λ_1 *and* λ_2 *are real for* $m > 1$*, imaginary for* $m < 1$ *and corresponding to the multiple eigenvalue* 0 *of multiplicity* 5*, there exist only* 4 *linearly independent eigenvectors.* We shall prove this postulate in the next subsection.

8.3. **Transformation of frozen coordinates at a given point to get the eigenvalues in the most general case.** We have not been able to get the expressions for the eigenvalues of the system (6.11) in its general form by solving the algebraic equation $\det\left(-\lambda A + e_1 B^{(1)} + e_2 B^{(2)}\right) = 0$. We could find these expressions in a particular case when $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')$, with $\langle \mathbf{u}', \mathbf{v}' \rangle = 0$. In this subsection, we shall obtain the expressions of the nonzero eigenvalues in the general case from the expressions of ν_1 and ν_2 in (8.2) by transforming the set $\{\mathbf{u}', \mathbf{v}'\}$ to $\{\mathbf{u}, \mathbf{v}\}$ in the tangent plane at a given point P_0 on Ω_t (all vectors frozen at the point P_0).

Let (η_1, η_2) be a coordinate system on Ω_t, which is orthogonal at P_0 with unit tangent vectors $\mathbf{u}'$ and $\mathbf{v}'$ to the curves $\eta_2 =$ constant and $\eta_1 =$ constant respectively at this point. Then

$$\mathbf{u}' = \frac{1}{g_1'} \frac{\partial \mathbf{x}}{\partial \eta_1}, \quad \mathbf{v}' = \frac{1}{g_2'} \frac{\partial \mathbf{x}}{\partial \eta_2}. \tag{8.3}$$

Let the set $\{\mathbf{u}', \mathbf{v}'\}$ and $\{\mathbf{u}, \mathbf{v}\}$ at P_0 be related by

$$\mathbf{u}' = \gamma_1 \mathbf{u} + \delta_1 \mathbf{v}, \quad \mathbf{v}' = \gamma_2 \mathbf{u} + \delta_2 \mathbf{v}. \tag{8.4}$$

¿From expressions (8.3) for $\mathbf{u}', \mathbf{v}'$, similar expressions for $\mathbf{u}, \mathbf{v}$ in (5.4) and relations (8.4) we get the following transformation of derivations

$$\frac{1}{g_1'} \frac{\partial}{\partial \eta_1} = \gamma_1 \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \delta_1 \frac{1}{g_2} \frac{\partial}{\partial \xi_2}, \tag{8.5}$$

$$\frac{1}{g_2'} \frac{\partial}{\partial \eta_2} = \gamma_2 \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \delta_2 \frac{1}{g_2} \frac{\partial}{\partial \xi_2}. \tag{8.6}$$

The frozen system of equations for $\mathbf{U}' = (u_1', u_2', v_1', v_2', m, g_1', g_2')$ in terms of operators $\frac{\partial}{\partial \eta_1}$ and $\frac{\partial}{\partial \eta_2}$, i.e., the equation

$$A \frac{\partial \mathbf{U}'}{\partial t} + B'^{(1)} \frac{\partial \mathbf{U}'}{\partial \eta_1} + B'^{(2)} \frac{\partial \mathbf{U}'}{\partial \eta_2} = 0. \tag{8.7}$$

gets transformed into the system

$$\begin{aligned} A \frac{\partial \mathbf{U}}{\partial t} &+ g_1' B'^{(1)} \left(\gamma_1 \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \delta_1 \frac{1}{g_2} \frac{\partial}{\partial \xi_2} \right) \mathbf{U} \\ &+ g_2' B'^{(2)} \left(\gamma_2 \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \delta_2 \frac{1}{g_2} \frac{\partial}{\partial \xi_2} \right) \mathbf{U} = 0. \end{aligned} \tag{8.8}$$

Comparing this equation with (6.12) we get

$$g_1 B^{(1)} = \gamma_1 g_1' B'^{(1)} + \gamma_2 g_2' B'^{(2)}, \tag{8.9}$$

$$g_2 B^{(2)} = \delta_1 g_1' B'^{(1)} + \delta_2 g_2' B'^{(2)}. \tag{8.10}$$

The characteristic equation of (6.12) is

$$\det\left(-\lambda A + e_1 B^{(1)} + e_2 B^{(2)}\right) = 0 \tag{8.11}$$

which with the help of (8.9) and (8.10) becomes

$$\det\left[-\lambda A + g_1'\left(\frac{e_1}{g_1}\gamma_1 + \frac{e_2}{g_2}\delta_1\right)B'^{(1)} \right.$$

$$\left. + g_2'\left(\frac{e_1}{g_1}\gamma_2 + \frac{e_2}{g_2}\delta_2\right)B'^{(2)}\right] = 0. \tag{8.12}$$

This is same as the characteristic equation of (8.7) with an eigenvalue λ' if

$$\lambda' = \lambda,\ \ \frac{e_1'}{g_1'} = \gamma_1\frac{e_1}{g_1} + \delta_1\frac{e_2}{g_2},\ \ \frac{e_2'}{g_2'} = \gamma_2\frac{e_1}{g_1} + \delta_2\frac{e_2}{g_2} \tag{8.13}$$

Thus, we have proved an important result

Theorem 8.1. *Let λ' be an expression of an eigenvalue of (8.7) in terms of e_1'/g_1' and e_2'/g_2'. Then the expression for the same eigenvalue of (6.12) in terms of e_1/g_1 and e_2/g_2 can be obtained from it by replacing e_1'/g_1' and e_2'/g_2' by (8.13).*

Using this theorem, we derive the expressions for the eigenvalues of (6.12) from those of (8.2). The first result that we conclude is that these eigenvalues are

$$\lambda_1 \neq 0,\ \lambda_2(= -\lambda_1),\ \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0 \tag{8.14}$$

and then from (8.2) we get the expression for λ_1 as

$$\lambda_1 = \left[\frac{m-1}{2}\left\{(\gamma_1^2+\gamma_2^2)\frac{e_1^2}{g_1^2} + 2(\gamma_1\delta_1+\gamma_2\delta_2)\frac{e_1}{g_1}\frac{e_2}{g_2} + (\delta_1^2+\delta_2^2)\frac{e_2^2}{g_2^2}\right\}\right]^{1/2}. \tag{8.15}$$

The rank of the pencil matrix for the eigenvalue $\lambda = 0$, i.e., the rank of $e_1B^{(1)} + e_2B^{(2)}$ will be the same as the rank of $e_1'B'^{(1)} + e_2'B'^{(2)}$ when the relations (8.9) and (8.10) are valid and hence it would be 3. Thus, the number of linearly independent eigenvectors corresponding to $\lambda = 0$ is only 4. We have now proved the main theorem.

Theorem 8.2. *The system (6.12) has* 7 *eigenvalues $\lambda_1, \lambda_2(= -\lambda_1), \lambda_3 = \lambda_4 = \ldots = \lambda_7 = 0$, where λ_1 and λ_2 are real for $m > 1$ and purely imaginary for $m < 1$. Further, the dimension of the eigenspace corresponding to the multiple eigenvalue* 0 *is* 4.

8.4. **Application of the theory in the section 8.3.** We have shown the equivalence of the ray equations to the differential form of KCL in section 5. Hence, we expect to derive the expressions of the eigenvalues μ_1 and μ_2 of the section 7 from the eigenvalues ν_1 and ν_2 simply by using the general formula (8.15). It is really remarkable to do this derivation and show the power of the transformation of coordinates introduced in the last subsection.

Note that the frozen coordinates (ξ_1, ξ_2, t) at P_0 are associated with the unit vectors $\mathbf{u}_0$ and $\mathbf{v}_0$ given in (7.5). We now choose orthogonal coordinates (η_1, η_2) at P_0 such that these coordinates are associated with unit tangent vectors

$$\mathbf{u}'_0 = \mathbf{u}_0 = \frac{1}{\sqrt{n_1^2+n_3^2}}(-n_3, 0, n_1) \tag{8.16}$$

and

$$\mathbf{v}'_0 = \frac{1}{\sqrt{n_1^2+n_3^2}}(-n_1n_2, n_1^2+n_3^2, -n_2n_3). \tag{8.17}$$

Note that $\mathbf{v}'_0$ is not only orthogonal to $\mathbf{u}'_0$ but also to $\mathbf{n}$. Now, the relation (8.4) becomes

$$\mathbf{u}'_0 = \gamma_1\mathbf{u}_0 + \delta_1\mathbf{v}_0,\ \ \mathbf{v}'_0 = \gamma_2\mathbf{u}_0 + \delta_2\mathbf{v}_0, \tag{8.18}$$

where

$$\gamma_1 = 1,\ \delta_1 = 0;\ \gamma_2 = \frac{n_1 n_2}{n_3},\ \delta_2 = \frac{1}{n_3}\sqrt{n_1^2+n_3^2}\sqrt{n_2^2+n_3^2}. \tag{8.19}$$

Substituting these values in the general formula (8.15) we find that the eigenvalue ν_1 (in (8.2)) of the orthogonal system becomes μ_1, where

$$\mu_1^2 = \frac{m-1}{2n_3^2}\left[(n_3^2+n_1^2n_2^2)\frac{e_1^2}{g_{10}^2} + (n_1^2+n_3^2)(n_2^2+n_3^2)\frac{e_2^2}{g_{20}^2} \right.$$
$$\left. +2n_1n_2\sqrt{n_1^2+n_3^2}\sqrt{n_2^2+n_3^2}\frac{e_1}{g_{10}}\frac{e_2}{g_{20}}\right]. \tag{8.20}$$

We note that

$$n_3^2 + n_1^2n_2^2 = n_3^2(n_1^2+n_2^2+n_3^2) + n_1^2n_2^2 = (n_1^2+n_3^2)(n_2^2+n_3^2). \tag{8.21}$$

Using this result in (8.21), we find that μ_1 and $\mu_2(= -\mu_1)$ here are exactly the same as μ_1 and μ_2 in (7.12).

Thus we have obtained a beautiful result-*derivation of the eigenvalues of the a system (say the system-I) consisting of ray equations and the energy transport equation from the differential form of another system (say, system II) consisting of the KCL along with the energy transport equation.* The process can be reversed. We can obtain the eigenvalues of the system-II from those of the smaller and simpler system-I. The invariance of eigenvalues and eigenvectors of two equivalence systems of same number equations is well known (see [16]-Theorem 6.1, page 220) but the invariance which we see here is in two system of different number of equations.

The problem of evolution of a moving surface Ω_t in $\mathbb{R}^3$ is quite complex. Ray theory represented by the system I is a complete system of equations except that it would not describe formation and propagation of kink lines. This system in ray-coordinates has 3 eigenvalues μ_1, μ_2 and μ_3 given in (7.12). The first two non zero eigenvalues for $m > 1$ carry with them changes in the geometry of Ω_t and the amplitude w (or m) on Ω_t. The third zero eigenvalue carries with it the total energy and represents conservation of the total energy in a ray tube. It will be interesting to study in (ξ_1, ξ_2, t)-space geometry of bicharateristics associated with these characteristic fields. The system II coming from KCL is essential to study the evolution of Ω_t with singularities. However, this leads to 4 more eigenvalues, which are all equal to $\lambda_3 = 0$. These additional eigenvalues are results of an increase in the dependent variables $u_1, u_2, v_1, v_2, m, g_1, g_2$ which are necessary in the formulation of KCL (in reality only three variables n_1, n_2, m along with the equations (2.3) for $\mathbf{x}$ suffice to describe the evolution of Ω_t). We need to study some exact and numerical solutions of the KCL with energy transport equation to see the effect of these additional eigenvalues, which cause a loss in the hyperbolicity of the system of $m > 1$ (the number of independent eigenvectors corresponding to $\lambda_i = 0, i = 3, 4, \ldots, 7$, is only four).

9. Some Examples of Propagation of Nonlinear Wavefronts

The KCL (3.5)-(3.6) and energy transport equation (6.7) of WNLRT can be written as a system of conservation laws

$$W_t + F_1(W)_{\xi_1} + F_2(W)_{\xi_2} = 0, \tag{9.1}$$

with the conserved variables W and the fluxes $F_i(W)$ given as

$$W = \Big(g_1u_1, g_1u_2, g_1u_3, g_2v_1, g_2v_2, g_2v_3, (m-1)^2e^{2(m-1)}g_1g_2\sin\chi\Big)^t,$$
$$F_1(W) = (mn_1, mn_2, mn_3, 0, 0, 0, 0)^t, \quad F_2(W) = (0, 0, 0, mn_1, mn_2, mn_3, 0)^t. \tag{9.2}$$

We first formulate the initial data for the system of conservation laws (9.1). Let the initial position of a weakly nonlinear wavefront Ω_t be given as

$$\Omega_0 \colon x_3 = f(x_1, x_2). \tag{9.3}$$

On Ω_0, we choose $\xi_1 = x_1,\ \xi_2 = x_2$, then

$$\Omega_0 \colon x_{10} = \xi_1,\ x_{20} = \xi_2,\ x_{30} = f(\xi_1, \xi_2) \tag{9.4}$$

and

$$g_{10} = \sqrt{1 + f_{\xi_1}^2}, \quad g_{20} = \sqrt{1 + f_{\xi_2}^2}, \quad \mathbf{u}_0 = \frac{(1, 0, f_{\xi_1})}{\sqrt{1 + f_{\xi_1}^2}}, \quad \mathbf{v}_0 = \frac{(0, 1, f_{\xi_2})}{\sqrt{1 + f_{\xi_2}^2}}. \tag{9.5}$$

We can easily check that (3.7) is satisfied on Ω_0. The unit normal $\mathbf{n}_0$ on Ω_0 is

$$\mathbf{n}_0 = -\frac{(f_{\xi_1}, f_{\xi_2}, -1)}{\sqrt{1 + f_{\xi_1}^2 + f_{\xi_2}^2}} \tag{9.6}$$

in which the sign is so chosen that $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ form a right handed system. Let the distribution of the front velocity be given by

$$m = m_0(\xi_1, \xi_2). \tag{9.7}$$

We have now completed formulation of the initial data for the KCL system (9.1).

The problem is to find solution of the system (9.1) satisfying the initial data given by (9.5) and (9.7). Having solved these equations, we can get Ω_t by solving the first part of the ray equations, namely (2.3) at least numerically for a number of values of ξ_1 and ξ_2.

Since the system of conservation laws (9.1) is only weakly hyperbolic, it is a nontrivial task to get a stable numerical approximation of the solution. Weakly hyperbolic systems are known to be very sensitive to numerical schemes than regular hyperbolic systems. Since central finite volume schemes are less dependent on hyperbolic nature of the conservation laws we employ a staggered Lax-Friedrichs scheme [10] to obtain a numerical solution of (9.1). We refer the reader to [1] for a comprehensive study of several numerical experiments with the 3-D KCL system (9.1).

9.1. **Propagation of a wavefront with initially periodic shape.** We choose the initial front to be of a periodic shape

$$\Omega_0 \colon x_3 = \kappa \left(2 - \cos\left(\frac{\pi x_1}{a}\right) - \cos\left(\frac{\pi x_2}{b}\right) \right), \tag{9.8}$$

with the constants $\kappa = 0.4, a = b = 2$. The initial velocity has a constant value $m_0 = 1.2$. The computational domain $[-4, 4] \times [-4, 4]$ is divided into 801×801 mesh points. The simulations are done up to $t = 10.0$ with a CFL number 0.45. In Fig. 3 we give a surface plots of the initial wavefront Ω_0 and the wavefronts Ω_t at times $t = 0.0, 2.0, 4.0, 6.0, 8.0, 10.0$. The front Ω_t has moved up in the x_3-direction and has developed several kink lines. In the Fig. 4 (a) and (b) we plot respectively the shapes of Ω_t with respect to x_1 along the cross section $x_2 = 0$ and with respect to the distance along the section $x = y$. In these plots the kinks are marked with dots. During time evolution the front tends to become planar. A pair of kinks develop in each period of the initial front Ω_0. In Fig. 5 we have plotted the variation of the normal velocity m with respect to ξ_1 along $\xi_2 = 0$. It is evident that four shock have developed in m which corresponds to the above mentioned kinks appearing on the front. A plot of m along $\xi_2 = \text{constant} > 0$ will show a quite complex pattern with kinks.

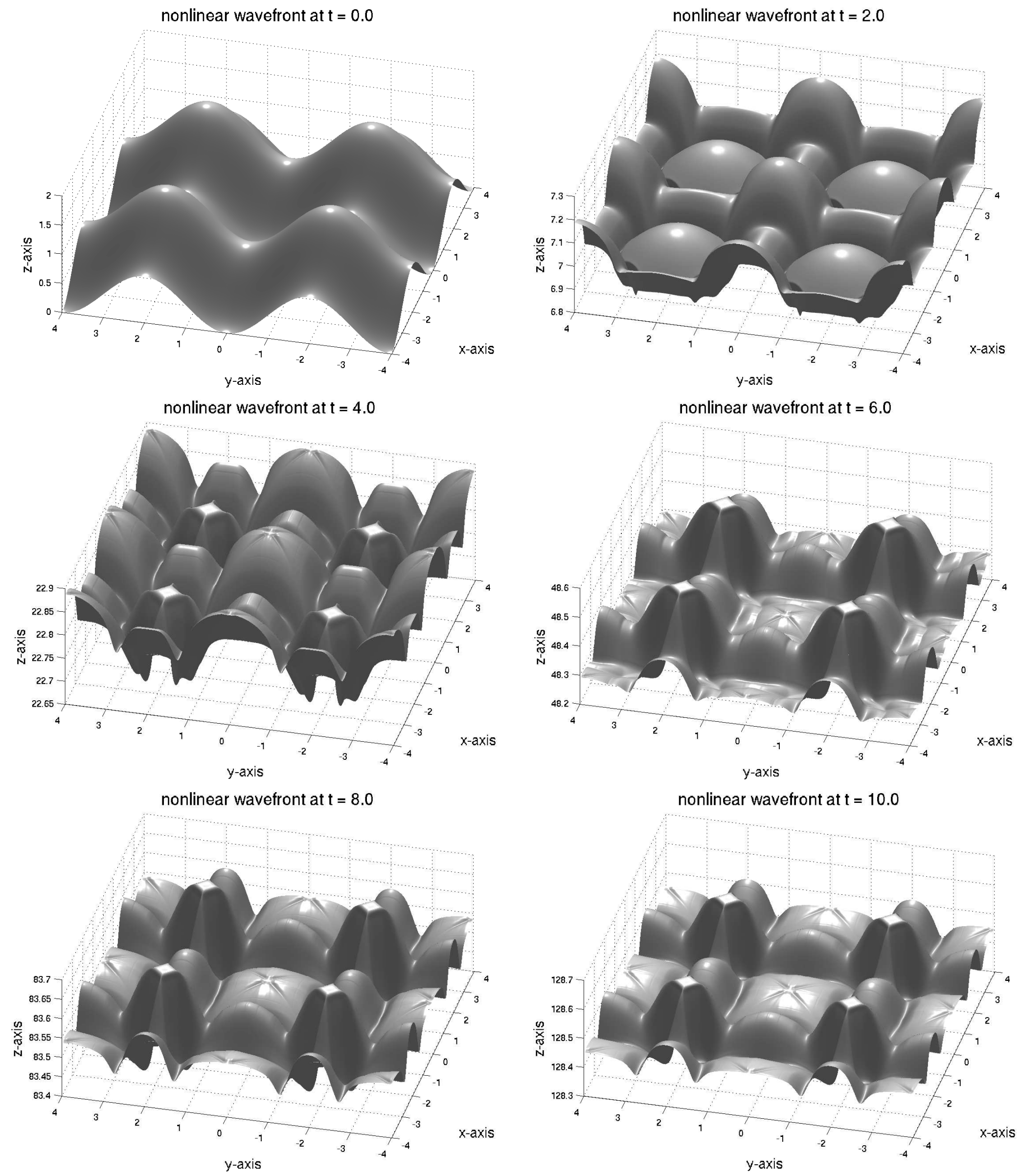
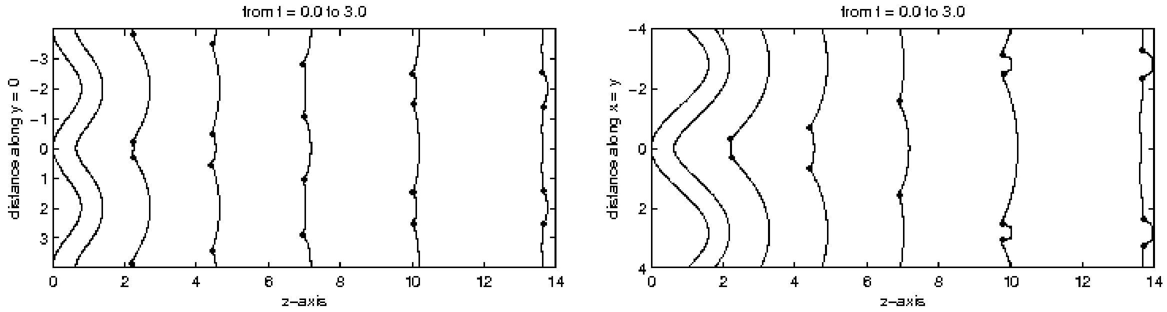


FIGURE 3. The nonlinear wavefront Ω_t starting initially in a periodic shape. Figures at time $t \geq 4$ show a complex pattern on kink lines on Ω_t, some horizontal and some slanted.

9.2. **Propagation of a nonlinear wavefront non-symmetric to the coordinate axes.** We choose initial wavefront Ω_0 in a such a way that it is non-symmetric with respect to the coordinate directions. The front Ω_0 has a single smooth dip. The data reads,

$$\Omega_0 \colon x_3 = \kappa \left(1 - e^{-\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}\right)} \right), \tag{9.9}$$

where the parameter values are set to be $\kappa = 3, a = 4, b = 8$.



(a) (b)

FIGURE 4. Planar slices of the nonlinear wavefront Ω_t starting initially in a periodic shape. (a) - section along $y = 0$. (b) - section along $x = y$.

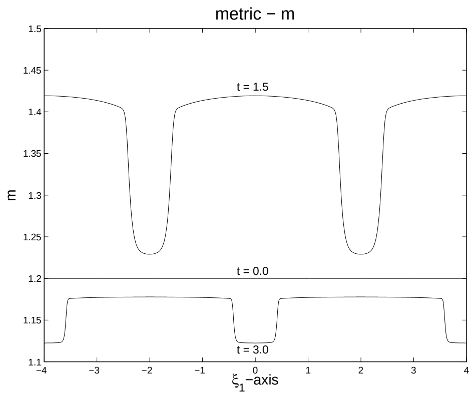
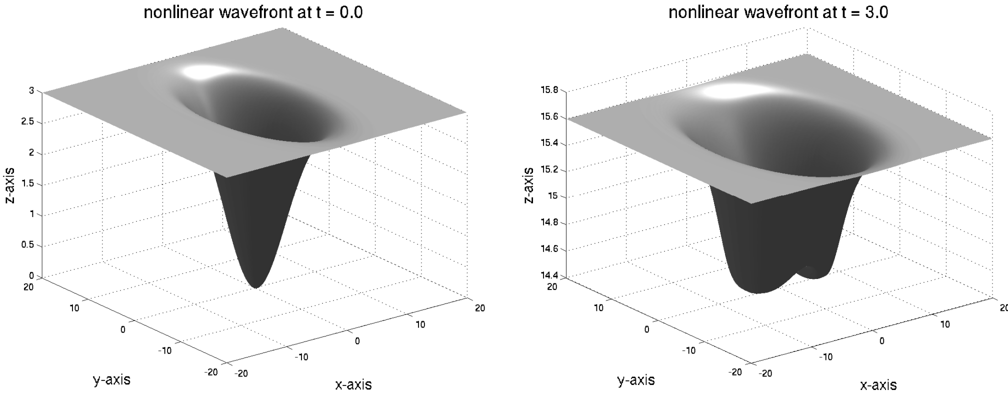


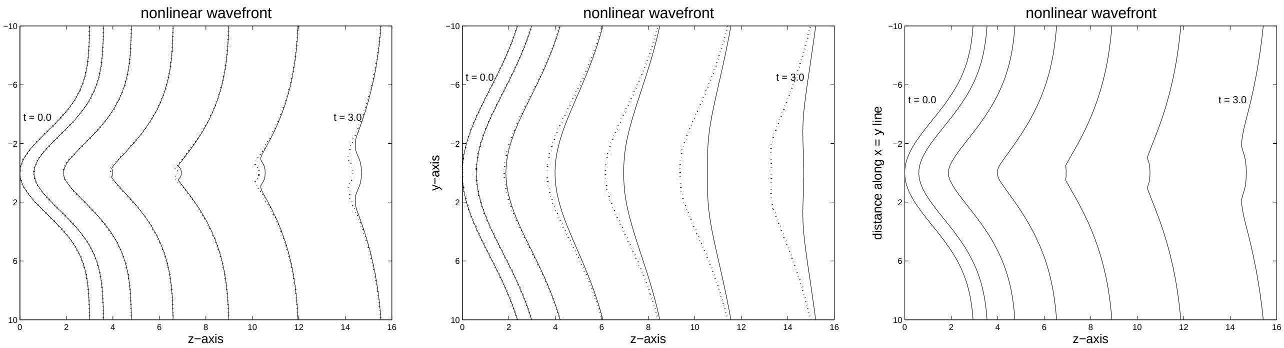
FIGURE 5. Time evolution of the wavefront velocity - m along ξ_1 direction in the section $\xi_2 = 0$.

In Fig. 6 we plot the initial wavefront Ω_0 and the wavefront Ω_t at time $t = 3.0$. The wavefront has moved up in the x_3-direction and the dip has spread over more area. The lower part of the front moves up fast leading to a change in shape of the initial front Ω_0. To explain the results of convergence of the wavefront we give in Fig. 7 (a), (b) and (c) the slices of the wavefront along x-section, y-section and $x = y$-section respectively. Due to the particular choice of the parameters a and b in the initial data, the initial front Ω_0 has larger principal radius of curvature in the $x-$direction. This causes a stronger convergence of the rays in the $x-$direction compared to the $y-$direction as evident from the Fig. 7 (a) and (b). The slice (c) along the diagonal line $x = y$ shows an intermediate effect. Also to illustrate the truly 3-dimensional effects of convergence, we have plotted the corresponding results obtained from the 2-D KCL system in Fig. 7 (a) and (b) with dotted lines. It is observed that both the results agree qualitatively. Note that both the 2-D and 3-d results match for small times, but the 3-dimensional wavefront moves faster than 2-dimensional one. We close this section by giving the plots of the normal velocity m along ξ_1- and ξ_2- directions. It is observed that m has two shocks in the ξ_1 direction which corresponds to the two kinks in the x-direction.



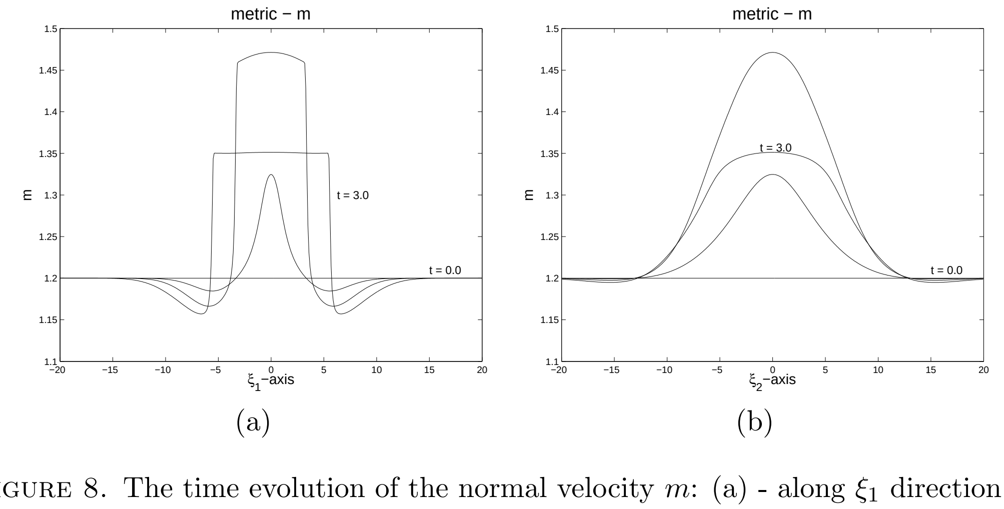
(a) (b)

FIGURE 6. Nonlinear wavefront Ω_t with an initial smooth dip.



(a) (b) (c)

FIGURE 7. The sections of the nonlinear wavefront. (a) - along $y = 0$. (b) - along $x = 0$. (c) - along $x = y$. The dotted lines in (a) and (b) are results obtained with 2-D KCL.



(a) (b)

FIGURE 8. The time evolution of the normal velocity m: (a) - along ξ_1 direction in the section $\xi_2 = 0$, (b) - along ξ_2 direction in the section $\xi_1 = 0$.

10. Concluding Remarks

The numerical computation of KCL along with the energy transport equation reveals fascinating shapes of a nonlinear wavefront. There appears to be no other method to give such intricate shapes.

Appendix A

Non-zero elements of the matrices A, $B^{(1)}$ and $B^{(2)}$.

$$a_{11} = g_1,\ a_{22} = g_1,\ a_{33} = g_2,\ a_{44} = g_2,\ a_{66} = 1,\ a_{77} = 1,$$
$$a_{51} = -\frac{1}{u_3} g_1 g_2 n_2 \cot\chi,\ a_{52} = \frac{1}{u_3} g_1 g_2 n_1 \cot\chi,$$
$$a_{53} = \frac{1}{v_3} g_1 g_2 n_2 \cot\chi,\ a_{54} = -\frac{1}{v_3} g_1 g_2 n_1 \cot\chi,$$
$$a_{55} = \frac{2m}{m-1} g_1 g_2,\ a_{56} = g_2,\ a_{57} = g_1.$$

$$b^{(1)}_{11} = -\frac{m}{u_3}(u_1 u_2 + n_1 n_2)\cot\chi,\ b^{(1)}_{12} = \frac{m}{u_3}(u_1^2 + n_1^2 - 1)\cot\chi,$$
$$b^{(1)}_{13} = \frac{m}{v_3 \sin\chi}(u_2 v_1 + n_1 n_2 \cos\chi),\ b^{(1)}_{14} = -\frac{m}{v_3 \sin\chi}(u_1 v_1 + (n_1^2 - 1)\cos\chi),\ b^{(1)}_{15} = -n_1.$$

$$b^{(1)}_{21} = -\frac{m}{u_3}(u_2^2 + n_2^2 - 1)\cot\chi,\ b^{(1)}_{22} = \frac{m}{u_3}(u_1 u_2 + n_1 n_2)\cot\chi,$$
$$b^{(1)}_{23} = \frac{m}{v_3 \sin\chi}(u_2 v_2 + (n_2^2 - 1)\cos\chi),\ b^{(1)}_{24} = -\frac{m}{v_3 \sin\chi}(u_1 v_2 + n_1 n_2 \cos\chi),\ b^{(1)}_{25} = -n_2.$$

$$b^{(1)}_{61} = -\frac{m}{u_3 \sin\chi}(v_2 - u_2\cos\chi),\ b^{(1)}_{62} = \frac{m}{u_3 \sin\chi}(v_1 - u_1\cos\chi).$$

$$b^{(2)}_{31} = -\frac{m}{u_3 \sin\chi}(u_1 v_2 + n_1 n_2 \cos\chi),\ b^{(2)}_{32} = \frac{m}{u_3 \sin\chi}(u_1 v_1 + (n_1^2 - 1)\cos\chi),$$
$$b^{(2)}_{33} = \frac{m}{v_3}(v_1 v_2 + n_1 n_2)\cot\chi,\ b^{(2)}_{34} = -\frac{m}{v_3}(v_1^2 + n_1^2 - 1)\cot\chi,\ b^{(2)}_{35} = -n_1.$$

$$b^{(2)}_{41} = -\frac{m}{u_3 \sin\chi}(u_2 v_2 + (n_2^2 - 1)\cos\chi),\ b^{(2)}_{42} = \frac{m}{u_3 \sin\chi}(u_2 v_1 + n_1 n_2 \cos\chi),$$
$$b^{(2)}_{43} = \frac{m}{v_3}(v_2^2 + n_2^2 - 1)\cot\chi,\ b^{(2)}_{44} = -\frac{m}{v_3}(v_1 v_2 + n_1 n_2)\cot\chi,\ b^{2}_{45} = -n_2.$$

$$b^{(2)}_{73} = \frac{m}{v_3 \sin\chi}(u_2 - v_2\cos\chi),\ b^{(2)}_{74} = -\frac{m}{v_3 \sin\chi}(u_1 - v_1\cos\chi).$$

Appendix B. (Notations)

All variables are suitably non-dimensionalized.

$(\mathbf{x}, t) = (x_1, x_2, t)$	for 2-D KCL.
$(\mathbf{x}, t) = (x_1, x_2, x_3, t)$	for 3-D KCL.
$\Omega\colon \varphi(\mathbf{x}, t) = 0$	a surface in $(\mathbf{x}, t)$ space.
$\Omega_t\colon \varphi(\mathbf{x}, t) = 0,\ t = \text{constant},$	a moving surface in $\mathbf{x}$-space at a fixed time t.
$\mathit{\Omega}$	mean curvature of Ω_t.
m	normal velocity of Ω_t and is the metric associated with t in ray coordinates.
$\mathbf{n}$	unit normal of Ω_t.
θ	the angle $\mathbf{n}$ makes with x-axis for 2-D KCL.
$\boldsymbol{\chi}$	ray velocity $= m\mathbf{n}$ for isotropic evolution of Ω_t.
(ξ, t)	ray coordinates for 2-D KCL.
g	metric associated with ξ.
$\mathbf{u}$	unit tangent vector to Ω_t for 2-D KCL.
(ξ_1, ξ_2, t)	ray coordinates for 3-D KCL.
g_i	metric associated with $\xi_i, i = 1, 2$.
$\mathbf{u}, \mathbf{v}$	unit tangent vectors on Ω_t in direction of the coordinates ξ_1 and ξ_2 for 3-D KCL.
$\mathbf{L}$	$\nabla - \mathbf{n}\langle \mathbf{n}, \nabla \rangle$.
χ	$\cos\chi = \langle \mathbf{u}, \mathbf{v} \rangle,\ 0 < \chi < \pi$.
$\mathscr{K}_t$	kink curve on Ω_t.
(E_1, E_2)	unit normal to the shock curve in (ξ_1, ξ_2)-plane.
K	velocity of propagation of a shock curve in (ξ_1, ξ_2)-plane.
$[f] := f_+ - f_-$	jump of a quantity across a shock curve in (ξ_1, ξ_2)-plane.
$a_{ij}, b_{ij}^{(1)}, b_{ij}^{(2)}$	components of 7×7 matrices $B^{(1)}$ and $B^{(2)}$ in equation (6.12)
w	amplitude of a nonlinear wavefront in a polytropic gas (see relation (6.2))
A	ray tube area.
$\frac{\partial}{\partial \lambda_1},\ \frac{\partial}{\partial \lambda_2}$	$n_1 \frac{\partial}{\partial x_3} - n_3 \frac{\partial}{\partial x_1},\ n_3 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_3}$.
$\mu_1, \mu_2 (= -\mu_1), \mu_3 = 0$	eigenvalues of the frozen equations (7.9)-(7.11).
$\nu_1, \nu_2 (= -\nu_1), \nu_3 = \ldots = \nu_7 = 0$	eigenvalues of the frozen system (6.12) with $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')$, where $\langle \mathbf{u}', \mathbf{v}' \rangle = 0$.
$\lambda_1, \lambda_2 (= -\lambda_1), \lambda_3 = \ldots = \lambda_7 = 0$	eigenvalues of the system (6.12).
(η_1, η_2)	orthogonal coordinates on Ω_t with unit tangent vectors $\mathbf{u}', \mathbf{v}'$ frozen at a point P_0.
(g_1', g_2')	metrics associated with η_1 and η_2 frozen at a point P_0.
$\gamma_1, \gamma_2, \delta_1, \delta_2, e_1, e_2, e_1', e_2'$	coefficients occurring in section 8.3.

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