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NONPARAMETRIC ESTIMATION OF THE DERIVATIVES OF A DENSITY BY THE METHOD OF WAVELETS

By

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Abstract

A method of estimation of the derivatives of a probability density using wavelet systems is proposed. Precise order for the integrated mean square of the proposed estimator is obtained.

Key words and Phrases : Nonparametric estimation of derivative of density; Wavelets.

AMS (1991) Subject classification : Primary 62 G 05 .

1. Introduction

Methods of nonparametric estimation of density function and regression function are widely discussed in the literature (cf. Prakasa Rao (1993)). Rosenblatt (1991) gives a short review of stochastic curve estimation. It is known that the estimation of the derivative of a density as well as that of the regression function are of importance and interest to detect possible bumps in the case of the density and to detect concavity or convexity properties of the regression function if any. Asymptotic properties of the kernel type estimators for the derivatives of density have been investigated earlier (cf. Prakasa Rao (1983), p. 237).

Here we discuss the estimation of the derivatives of a density using the method of wavelets. Antoniadis and Carmona (1991), Antoniadis et al. (1994) and Masry (1994) discuss the estimation of density and regression function by using the method of wavelets. Masry (1994) obtained the *exact* orders for the integrated mean square error (IMSE) of density estimator using a wavelet basis. All the earlier results on IMSE give a bound only on the IMSE for estimators of kernel type or other estimators derived by the method of orthogonal series. We generalize the result of Masry (1994) to estimators of the derivatives of a density. For an overview of recent advances in nonparametric functional estimation, see Prakasa Rao (1996).

2. Introduction to Wavelets

A wavelet system is an infinite collection of translated and scaled versions of functions ϕ and ψ called the *scaling function* and the *primary wavelet function* respectively.

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The function $\phi(x)$ is a solution of the equation

$$\phi(x) = \sum_{k=-\infty}^{\infty} C_k \, \phi(2x-k) \tag{2.1}$$

with

$$\int_{-\infty}^{\infty} \phi(x)dx = 1 \tag{2.2}$$

and the function $\psi(x)$ is defined by

$$\psi(x) = \sum_{k=-\infty}^{\infty} (-1)^k C_{-k+1} \, \phi(2x-k). \tag{2.3}$$

Note that the choice of the sequence $\{C_k\}$ determines the wavelet system. It is easy to see that

$$\sum_{k=-\infty}^{\infty} C_k = 2. \tag{2.4}$$

Define

$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k), \; -\infty < j,k < \infty \tag{2.5}$$

and

$$\psi_{jk}(x) = 2^{j/2} \, \psi(2^j x - k), \; -\infty < j,k < \infty. \tag{2.6}$$

Suppose the coefficients $\{C_k\}$ satisfy the condition

$$\begin{aligned} \sum_{k=-\infty}^{\infty} C_k C_{k+2\ell} &= 2 \text{ if } \ell = 0 \\ &= 0 \text{ if } \ell \neq 0. \end{aligned} \tag{2.7}$$

It is known that, under some additional condition on ϕ, the collection $\{\psi_{jk}, -\infty < j,k < \infty\}$ is an orthonormal basis for $L^2(R)$ and $\{\phi_{jk}, -\infty < k < \infty\}$ is an orthonormal system in $L^2(R)$ for each $-\infty < j < \infty$ (cf. Daubechies (1990)).

DEFINITION 2.1. A scaling function $\phi \in C^{(r)}$ is said to be *r-regular* for an integer $r \geq 1$ if for every non-negative integer $\ell \leq r$ and for any integer k,

$$|\phi^{(\ell)}(x)| \leq c_k(1+|x|)^{-k}, \; -\infty < x < \infty \tag{2.8}$$

for some $c_k \geq 0$ depending only on k where $\phi^{(\ell)}(\cdot)$ denotes the ℓ-th derivative of ϕ.

DEFINITION 2.2. A *multiresolution analysis* of $L^2(R)$ consists of an increasing sequences of closed subspaces $\{V_j\}$ of $L^2(R)$ such that

(i) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$;

(ii) $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(R)$;

(iii) there is a scaling function $\phi \in V_0$ such that

$$\{\phi(x-k), \ \ -\infty < k < \infty\}$$

is an orthonormal basis for V_0; and for all $h \in L^2(R)$,

(iv) for all $-\infty < k < \infty, h(x) \in V_0 \Rightarrow h(x-k) \in V_0$;

(v) $h(x) \in V_j \Rightarrow h(2x) \in V_{j+1}$.

Mallat (1989) has shown that given any multiresolution analysis, it is possible to derive a function ψ (primary wavelet function) such that for any fixed $j, -\infty < j < \infty$, the family $\{\psi_{jk}, -\infty < k < \infty\}$ is an orthonormal basis of the orthogonal complement W_j of V_j in V_{j+1} so that $\{\psi_{jk}, -\infty < j, k < \infty\}$ is an orthonormal basis of $L^2(R)$. Conversely, given any compactly supported wavelet system, it gives rise to a multiresolution analysis of $L^2(R)$ (cf. Daubechies (1990)). When the scaling function ϕ is r-regular, the corresponding multiresolution analysis is said to be r-regular.

Let H_2^s denote the space of all functions $g(\cdot)$ in $L^2(R)$ whose first $(s-1)$ derivatives are absolutely continuous and define the norm

$$\|g\|_{H_2^s} = \sum_{j=0}^{s} \left[\int_{-\infty}^{\infty} |g^{(j)}(t)|^2 dt \right]^{1/2}.$$

LEMMA 2.3. *(Mallat (1989)). Let a multiresolution analysis be r-regular. Then, for every $0 < s < r$, any function $g \in L^2(R)$ belongs to H_2^s iff*

$$\sum_{\ell=-\infty}^{\infty} e_\ell^2 e^{2s\ell} < \infty \tag{2.9}$$

where $e_\ell^2 = \|g - g_\ell\|_2^2$ and g_ℓ is the orthogonal projection of g on V_ℓ.

REMARK. The above introduction is based on Antoniades et al. (1994). For a detailed introduction to wavelets, see Chui (1992) or Daubechies (1992). For a brief survey, see Strang (1989).

3. Estimation of the d-th derivative $f^{(d)}$ of the density f

Suppose $X_1, X_2, ..., X_n$ are i.i.d. (independent and identically distributed) random variables with density f and f is d-times differentiable, $d \geq 0$. We interpret $f^{(0)}$ as f. The problem of interest is the estimation of $f^{(d)}$.

Assume that $f^{(d)} \in L^2(R)$ and there exist $D_j \geq 0$, $\beta_j \geq 0$ such that

$$|f^{(j)}(x)| \leq D_j |x|^{-\beta_j} \text{ for } \ |x| \geq 1, \ 0 \leq j \leq d \tag{3.1}$$

where $\beta_0 > 4d + 1$.

Consider a multiresolution as discussed in Section 2. Let ϕ be the corresponding scaling function. Suppose that the multiresolution is r-regular for some $r \geq d$. Then, by definition, $\phi \in C^{(r)}, \phi$ and its derivative $\phi^{(j)}$ up to order r are rapidly decreasing i.e., for every integer $m \geq 1$, there exists a constant $A_m > 0$ such that

$$|\phi^{(j)}(x)| \leq \frac{A_m}{(1+|x|)^m}, \quad 0 \leq j \leq r. \tag{3.2}$$

Let

$$\phi_{\ell k}(x) = 2^{\ell/2}\phi(2^\ell x - k), \; -\infty < k, \ell < \infty. \tag{3.3}$$

Then

$$\phi_{\ell,k}^{(j)}(x) = 2^{(\ell/2)+\ell j}\phi^{(j)}(2^\ell x - k), \; 0 \leq j \leq r \tag{3.4}$$

and

$$|\phi_{\ell k}^{(j)}(x)| \leq \frac{2^{(\ell/2)+\ell j}A_m}{(1+|x|)^m}, \; 0 \leq j \leq r. \tag{3.5}$$

If $d \geq 1$, then it is clear that

$$\lim_{|x|\to\infty} \phi_{\ell,k}^{(j)}(x)\, f^{(d-j-1)}(x) = 0, \; 0 \leq j \leq d-1 \tag{3.6}$$

for any fixed ℓ and k. Let $f_{\ell d}$ be the orthogonal projection of $f^{(d)}$ on V_ℓ. Note that

$$f_{\ell d}(x) = \sum_{j=-\infty}^{\infty} a_{\ell j}\, \phi_{\ell j}(x) \tag{3.7}$$

where

$$\begin{aligned} a_{\ell j} &= \int_{-\infty}^{\infty} f^{(d)}(u)\, \phi_{\ell j}(u) du \\ &= (-1)^d \int_{-\infty}^{\infty} f(u)\, \phi_{\ell,j}^{(d)}(u) du \end{aligned} \tag{3.8}$$

by (3.6) for $d \geq 1$. Clearly the equation (3.8) holds for $d = 0$. Hence, for all $d \geq 0$,

$$a_{\ell j} = (-1)^d E\left[\phi_{\ell j}^{(d)}(X)\right]. \tag{3.9}$$

Let

$$\hat{a}_{\ell_j} = \frac{(-1)^d}{n}\sum_{i=1}^{n} \phi_{\ell_j}^{(d)}(X_i) \tag{3.10}$$

be an estimator of $a_{\ell j}$ based on the i.i.d. sample $X_1, ..., X_n$. Estimate $f^{(d)}$ by

$$\hat{f}_{nd}(x) = \sum_{j=-k_n}^{k_n} \hat{a}_{\ell_n,j}\, \phi_{\ell_n,j}(x). \tag{3.11}$$

Let

$$f_{\ell,k,d}(x) = \sum_{j=-k}^{k} a_{\ell,j}\ \phi_{\ell,j}(x).$$

The problem is to estimate

$$E\|f^{(d)} - \hat{f}_{nd}\|_2^2. \tag{3.12}$$

Note that

$$\|f^{(d)} - \hat{f}_{nd}\|_2^2 = \|f^{(d)} - f_{\ell,k_n,d}\|_2^2 + \|f_{\ell,k_n,d} - \hat{f}_{nd}\|_2^2 \tag{3.13}$$

and hence

$$\begin{aligned} \varepsilon_n^2 \equiv E\|f^{(d)} - \hat{f}_{nd}\|_2^2 &= \|f^{(d)} - f_{\ell_n,k_n,d}\|_2^2 + E\|f_{\ell_n,k_n,d} - \hat{f}_{nd}\|_2^2 \\ &= \|f^{(d)} - f_{\ell_n d}\|_2^2 + \|f_{\ell_n,d} - f_{\ell_n,k_n,d}\|_2^2 \\ &\quad + E\|f_{\ell_n,k_n,d} - \hat{f}_{nd}\|_2^2 \\ &= e_{\ell_n}^2 + Q_n^2 + J_n^2, \text{ (say)}. \end{aligned} \tag{3.14}$$

Suppose that

$$f^{(d)} \in H_2^s. \tag{3.15}$$

It follows that

$$e_\ell^2 = 0(e^{-2s\ell}) \tag{3.16}$$

by Lemma 2.1. Note that

$$\begin{aligned} Q_n^2 &= \|f_{\ell_n,d} - f_{\ell_n,k_n,d}\|_2^2 \\ &= \sum_{|j|>k_n} |a_{\ell_n,j}|^2. \end{aligned} \tag{3.17}$$

But

$$\begin{aligned} a_{\ell j} &= (-1)^d \int_{-\infty}^{\infty} f(u)\phi_{\ell j}^{(d)}(u)du \\ &= (-1)^d 2^{\ell(\frac{1}{2}+d)} \int_{-\infty}^{\infty} \phi^{(d)}(2^\ell u - j) f(u)du \\ &= (-1)^d 2^{\ell d-(\ell/2)} \int_{-\infty}^{\infty} \phi^{(d)}(v) f\left(\frac{v+j}{2^\ell}\right) dv. \end{aligned} \tag{3.18}$$

Hence

$$|a_{\ell j}| \le 2^{\ell d-(\ell/2)} \left\{ \int_{|v|\le\frac{|j|}{2}} \phi^{(d)}(v) f\left(\frac{v+j}{2^\ell}\right) dv + \int_{|v|>\frac{|j|}{2}} \phi^{(d)}(v) f\left(\frac{v+j}{2^\ell}\right) dv \right\}$$

$$
\begin{aligned}
&\leq 2^{\ell d-(\ell/2)}\left\{\left[\sup_{|v|\leq|j|/2} f\left(\frac{v+j}{2^\ell}\right)\right]\int_{-\infty}^{\infty}|\phi^{(d)}(v)|dv\right.\\
&\quad\left.+\left[\sup_{|v|>|j|/2}|\phi^{(d)}(v)|\right]\int_{-\infty}^{\infty} f\left(\frac{v+j}{2^\ell}\right)dv\right\}\\
&\leq 2^{\ell d-(\ell/2)}\left\{\frac{D_0}{(|j|/2^{\ell+1})^{\beta_0}}\|\phi^{(d)}\|_1+\frac{A_m}{(1+|j|/2)^m}2^\ell\right\}\\
&\leq 2^{\ell d-(\ell/2)}\left\{\frac{D_0\|\phi^{(d)}\|_1 2^{(\ell+1)\beta_0}}{|j|^{\beta_0}}+\frac{2^{\ell+m}}{|j|^m}A_m\right\}. \qquad (3.19)
\end{aligned}
$$

Hence

$$
\begin{aligned}
Q_n^2 &\leq 2^{2\ell_n d-\ell_n+1}\left\{D_0^2\|\phi^{(d)}\|_1^2\, 2^{2\beta_0(\ell_n+1)}\sum_{|j|>K_n}\frac{1}{|j|^{2\beta_0}}\right.\\
&\quad\left.+\ 2^{2(\ell_n+m)}A_m^2\sum_{|j|>K_n}\frac{1}{|j|^{2m}}\right\}\\
&\leq 2^{2\ell_n d-\ell_n+1}\left\{\frac{D_0^2\|\phi^{(d)}\|_1^2\, 2^{2\beta_0(\ell_n+1)}}{(2\beta_0-1)K_n^{2\beta_0-1}}+\frac{A_m^2}{(2m-1)}\frac{2^{2(\ell_n+m)}}{K_n^{2m-1}}\right\} \qquad (3.20)
\end{aligned}
$$

from (3.17) and (3.19) for any integer $m \geq 1$. Let $m > \beta_0$. Then

$$
\begin{aligned}
Q_n^2 &\leq 2\left\{\frac{D_0^2\|\phi^{(d)}\|_1^2}{(2\beta_0-1)}\frac{2^{\ell_n(2d-1)+2\beta_0(\ell_n+1)}}{K_n^{2\beta_0-1}}+\frac{A_m^2 2^{2(\ell_n+m)+\ell_n(2d-1)}}{(2\beta_0-1)K_n^{2\beta_0-1}}\right\}\\
&\leq \frac{2^{\ell_n\{(2d-1)+2\beta_0\}}}{K_n^{2\beta_0-1}}\frac{2^{2\beta_0+1}}{(2\beta_0-1)}D_0^2\|\phi^{(d)}\|_1^2(1+O(2^{2\ell_n(1-\beta_0)}))\\
&\leq \frac{2^{\ell_n\{(2d-1)+2\beta_0\}}}{K_n^{2\beta_0-1}}\frac{2^{2\beta_0+1}}{(2\beta_0-1)}D_0^2\|\phi^{(d)}\|_1^2(1+0(1)) \qquad (3.21)
\end{aligned}
$$

since $\beta_0 > 1$ and $\ell_n \to \infty$. If

$$K_n = 2^{\{(2d-1)+2\beta_0+2s\}(\ell_n/(2\beta_0-1))}\log\, n, \qquad (3.22)$$

then

$$\frac{2^{\ell_n\{(2d-1)+2\beta_0\}}}{K_n^{2\beta_0-1}} = \frac{1}{(\log\, n)^{2\beta_0-1}\, 2^{2s\ell_n}} \to 0 \text{ as } n\to\infty \qquad (3.23)$$

since $\beta_0 > 1$ and $\ell_n \to \infty$ and in fact

$$Q_n^2 = 0(2^{-2s\ell_n}). \qquad (3.24)$$

Note that

$$
\begin{aligned}
B_n^2 &= \|f^{(d)}-f_{\ell n,k_n,d}\|^2 = Q_n^2+\|f^{(d)}-f_{\ell_n,d}\|_2^2\\
&= 0(2^{-2s\ell_n})+0(e^{-2s\ell_n})
\end{aligned}
$$

by Lemma 2.1 (cf. Mallat (1989) and hence

$$B_n^2 = 0(2^{-2s\ell_n}). \tag{3.25}$$

Let us now compute

$$\begin{aligned} J_n^2 &\equiv E\|f_{\ell_n,k_n,d} - \hat{f}_{nd}\|_2^2 \\ &= \sum_{j=-K_n}^{K_n} E(a_{\ell_n,j} - \hat{a}_{\ell_n,j})^2 \\ &= \sum_{j=-K_n}^{K_n} \operatorname{Var}(\hat{a}_{\ell_n,j}) \\ &= \frac{1}{n} \sum_{j=-K_n}^{K_n} \operatorname{Var}[\phi_{\ell_n,j}^{(d)}(X_1)] \end{aligned} \tag{3.26}$$

and hence

$$\begin{aligned} \frac{n}{2^{\ell_n}} J_n^2 &= \frac{n}{2^{\ell_n}} \sum_{j=-K_n}^{K_n} \frac{1}{n} \operatorname{Var}[\phi_{\ell_n,j}^{(d)}(X_1)] \\ &= \frac{1}{2^{\ell_n}} \sum_{j=-K_n}^{K_n} \left[\int_{-\infty}^{\infty} \phi_{\ell_n,j}^{(d)^2}(u) f(u)du - \left\{ \int_{-\infty}^{\infty} \phi_{\ell_n,j}^{(d)}(u) f(u) du \right\}^2 \right] \\ &= \frac{1}{2^{\ell_n}} \sum_{j=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \phi_{\ell_n,j}^{(d)^2}(u) f(u) du \right] \\ &\quad - \frac{1}{2^{\ell_n}} \sum_{|j|>K_n} \int_{-\infty}^{\infty} \phi_{\ell_n,j}^{(d)^2}(u) f(u) du \\ &\quad - \frac{1}{2^{\ell_n}} \sum_{|j|\le K_n} \{(-1)^d a_{\ell_n,j}\}^2 \\ &= S_1 + S_2 + S_3 \quad \text{(say)}. \end{aligned} \tag{3.27}$$

Suppose that f is of bounded variation on $(-\infty, \infty)$. Note that

$$\begin{aligned} S_1 &= \frac{2^{\ell_n(1+2d)}}{2^{\ell_n}} \int_{-\infty}^{\infty} \phi^{(d)^2}(u) \left\{ \frac{1}{2^{\ell_n}} \sum_{j=-\infty}^{\infty} f\left(\frac{u+j}{2^{\ell_n}}\right) \right\} du \\ &= 2^{2\ell_n d} \int_{-\infty}^{\infty} \phi^{(d)^2}(u) \left\{ \frac{1}{2^{\ell_n}} \sum_{j=-\infty}^{\infty} f\left(\frac{u+j}{2^{\ell_n}}\right) \right\} du \\ &= 2^{2\ell_n d} \int_{-\infty}^{\infty} \phi^{(d)^2}(u) \left\{ \int_{-\infty}^{\infty} f(u)du + O(2^{-\ell_n}) \right\} du \end{aligned}$$

$$
\begin{aligned}
& \text{(by Lemma A.1 of Masry (1994))} \\
= \;& 2^{2\ell_n d}\left\{\int_{-\infty}^{\infty} \phi^{(d)^2}(v)dv\right\}(1+O(2^{-\ell_n})). \qquad (3.28)
\end{aligned}
$$

Furthermore

$$
S_2 = -\frac{1}{2^{\ell_n}} \sum_{|j|>K_n} E\left[\phi^{(d)^2}_{\ell_n,j}(X))\right]. \qquad (3.29)
$$

But

$$
\begin{aligned}
E[\phi^{(d)^2}_{\ell_n,j}(X)] &= \int_{-\infty}^{\infty} \phi^{(d)^2}_{\ell_n,j}(u)f(u)du \\
&= 2^{\ell_n(2d+1)} \int_{-\infty}^{\infty} \phi^{(d)^2}(2^\ell u - j)f(u)du \\
&= 2^{2\ell_n d} \int_{-\infty}^{\infty} \phi^{(d)^2}(u) f\left(\frac{u+j}{2^\ell}\right) du \\
&\le 2^{2\ell_n d}\left\{\frac{D_0}{(|j|/2^{\ell_n+1})^{\beta_0}}\|\phi^{(d)^2}\|_1 + \frac{A_m^2}{(1+|j|/2)^{2m}}2^{\ell_n}\right\} \\
&\le 2^{2\ell_n d}\left\{\frac{D_0\|\phi^{(d)^2}\|_1 2^{(\ell_n+1)\beta_0}}{|j|^{\beta_0}} + \frac{2^{\ell_n+2m}A_m^2}{|j|^{2m}}\right\} \qquad (3.30)
\end{aligned}
$$

by methods similar to those used to derive (3.20). Hence

$$
\begin{aligned}
|S_2| &\le \frac{2^{\ell_n d}}{2^{\ell_n}}\left\{\frac{D_0\|\phi^{(d)^2}\|_1 2^{\beta_0(\ell_n+1)}}{(\beta_0-1)K_n^{\beta_0-1}} + \frac{2^{\ell_n+2m}A_m^2}{(2m-1)K_n^{2m-1}}\right\} \\
&= \frac{D_0\|\phi^{(d)^2}\|_1}{(\beta_0-1)} \frac{2^{\beta_0(\ell_n+1)-2\ell_n d-\ell_n}}{K_n^{\beta_0-1}} \\
&\quad + \frac{A_m^2}{(2\beta_0-1)} \frac{2^{\ell_n+2m-2\ell_n d-\ell_n}}{K_n^{2\beta_0-1}} \\
&= \frac{2^{\ell_n(\beta_0-2d-1)}2^{\beta_0}}{(\beta_0-1)K_n^{\beta_0-1}}\left\{\frac{D_0\|\phi^{(d)^2}\|_1}{(\beta_0-1)} + O(2^{\ell_n(1-\beta_0)})\right\} \\
&= O\left(\frac{2^{\ell_n(\beta_0-2d-1)}}{K_n^{\beta_0-1}}\right) \qquad (3.31)
\end{aligned}
$$

for $m > \beta_0 > 1$ as $\ell_n \to \infty$ from (3.29) and (3.30). Furthermore

$$
|S_3| \le \frac{1}{2^{\ell_n}} \sum_{j=-\infty}^{\infty} a^2_{\ell_n,j} \le \frac{1}{2^{\ell_n}}\|f^{(d)}\|_2^2. \qquad (3.32)
$$

We now state and prove the main theorem.

THEOREM 3.1. *Suppose that $f^{(d)} \in L^2(R) \cap H_2^s$ where $0 < s < r, f \in BV(-\infty,\infty)$, the class of functions which are of bounded variation on R and that the condition (3.1) holds. Suppose that $\ell_n \to \infty$ and K_n is as defined by (3.22) viz*

$$K_n = 2^{\{(2d-1)+\beta_0+2s\}(\ell_n/2\beta_0-1))} \log\, n. \tag{3.33}$$

Then

$$\frac{n}{2^{\ell_n(1+2d)}}\varepsilon_n^2 \to \int_{-\infty}^{\infty} \phi^{(d)^2}(v)dv \text{ as } n \to \infty \tag{3.34}$$

where $\varepsilon_n^2 \equiv E\|f^{(d)} - \hat{f}_{n,d}\|_2^2$.

PROOF. Note that

$$\begin{aligned}\frac{n}{2^{\ell_n}}\varepsilon_n^2 &= (E\|f^{(d)} - \hat{f}_{n,d}\|_2^2)\frac{n}{2^{\ell_n}} \\ &= 0(e^{-2sln}n2^{-\ell_n}) + 0(2^{-2sln}n2^{-\ell_n}) \\ &\quad +2^{2\ell_n d}\{\int_{-\infty}^{\infty} \phi^{(d)^2}(v)dv\}\{1 + O(2^{\ell_n})\} \\ &\quad +O\left(\frac{2^{\ell_n(\beta_0-2d-1)}}{K_n^{\beta_0-1}}\right) + O(2^{-\ell_n}).\end{aligned} \tag{3.35}$$

Furthermore $\ell_n \to \infty$ and

$$2^{\ell_n(\beta_0-4d-1)}/K_n^{\beta_0-1} \to 0 \text{ as } n \to \infty \tag{3.36}$$

from the choice of K_n. Hence

$$\frac{n}{2^{\ell_n(1+2d)}}\varepsilon_n^2 \to \int_{-\infty}^{\infty} \phi^{(d)^2}(v)dv \text{ as } n \to \infty \tag{3.37}$$

REMARK. If $d = 0$, relation (3.37) reduces to Theorem 3.2 in Masry (1994). If $\ell_n = n^{1/(2s+1)}$, then it follows that

$$n^{\frac{2(s-d)}{2s+1}}\,\varepsilon_n^2 \to \int_{-\infty}^{\infty} \phi^{(d)^2}(v)dv \text{ as } n \to \infty \tag{3.38}$$

which shows that the IMSE for the wavelet based estimator is of the same order as for the best kernel type estimator for the d-th derivative of a density (cf. Muller and Gasser (1979)).

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