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Orderfield Property of Mixtures of Stochastic Games

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Abstract

We consider certain mixtures, Γ, of classes of stochastic games and provide sufficient conditions for these mixtures to possess the orderfield property. For 2-player zero-sum and non-zero sum stochastic games, we prove that if we mix a set of states S_1 where the transitions are controlled by one player with a set of states S_2 constituting a sub-game having the orderfield property (where $S_1 \cap S_2 = \emptyset$), the resulting mixture Γ with states $S = S_1 \cup S_2$ has the orderfield property if there are no transitions from S_2 to S_1. This is true for discounted as well as undiscounted games. This condition on the transitions is sufficient when S_1 is perfect information or SC (Switching Control) or ARAT (Additive Reward Additive Transition). In the zero-sum case, S_1 can be a mixture of SC and ARAT as well. On the other hand,when S_1 is SER-SIT (Separable Reward - State Independent Transition), we provide a counter example to show that this condition is not sufficient for the mixture Γ to possess the orderfield property. In addition to the condition that there are no transitions from S_2 to S_1, if the sum of all transition probabilities from S_1 to S_2 is independent of the actions of the players, then Γ has the orderfield property even when S_1 is SER-SIT. When S_1 and S_2 are both SER-SIT, their mixture Γ has the orderfield property even if we allow transitions from S_2 to S_1. We also extend these results to some multi-player games namely, mixtures with one player control Polystochastic games. In all the above cases, we can inductively mix many such games and continue to retain the orderfield property.

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1 Introduction

Shapley (1953), in his seminal paper, introduced zero-sum discounted finite stochastic games as a generalization of matrix games. A zero-sum discounted finite stochastic game consists of a finite number of states, finite sets of actions for the players, a payoff matrix in each state and transition probabilities from each state to every other state, for every pair of actions of the players. As in the case of matrix games, one of the players chooses rows and the other player chooses columns. We shall assume that the row chooser is the maximizer and the column chooser is the minimizer. Given a starting state, the players simultaneously choose actions resulting in an immediate payoff (the corresponding entry in the payoff matrix) that is paid to the row chooser by the column chooser and the game moves to a new state depending on the transition probabilities. Now the players choose actions in the new state resulting in a payoff in that state and so on. At each time period, these payoffs are successively discounted by a factor $\beta \in [0, 1)$ and these discounted payoffs are accumulated over the infinite horizon. Starting at different states, we obtain different accumulated discounted payoffs. The aim of the row chooser is to maximize these accumulated discounted payoffs and the aim of the column chooser is to minimize the same. Strategies of players are probability distributions over their action sets, at each time period. Shapley (1953) showed that every zero-sum discounted finite stochastic game has an optimal value and optimal stationary strategies, that is, strategies that depend only on the current state and not on how the state was reached.

The concept of undiscounted (or limiting average) payoffs in stochastic games was introduced by Gillette (1957). Undiscounted payoffs are *lim sup* or *lim inf* of accumulated average payoffs over the infinite run. Mertens and Neyman (1981) showed that every undiscounted stochastic game has a value, though optimal strategies may not exist. "The Big Match" with undiscounted payoffs (Blackwell and Ferguson, 1968) is an example of an undiscounted stochastic game where one of the players does not have an optimal strategy, even when behavioral strategies (strategies that depend on the history) are allowed.

On the one hand, significant research has been done in proving such existence results. Such results have also been extended to non-zero-sum stochastic games (Fink, 1964, Takahashi, 1964, Sobel, 1971) and also to stochastic games with infinite state space and infinite action space (Maitra and Parthasarathy, 1970). Maitra and Sudderth (1996) proposed an alternative proof of the existence of value in the finite, undiscounted, zero-sum case which extends to the case when the state space is uncountable as well. On the other hand, computing the optimal value and optimal strategies (in the zero-sum case), and a Nash equilibrium (in the non-zero-sum case) has triggered a flurry of research activity, computational as well as theoretical.

Weyl (1950) showed that matrix games possess the orderfield property. That is, given payoffs from an ordered field, there exists a pair of optimal strategies whose coordinates lie in the same ordered field. It follows that the optimal value lies in the same ordered field as well. Bimatrix games also have the orderfield property when restricted to the rational field. Nash (1951) gave an example of a 3-player non-cooperative game with rational payoffs but a unique irrational equilibrium. Unlike matrix games, stochastic games may not possess the orderfield property even in the discounted zero-sum case as pointed out by Shapley (1953). For an explicit example, see Parthasarathy and Raghavan (1981).

In this paper, whenever we talk of the orderfield property, we restrict ourselves to the field of rationals.

Definition 1.1. Stochastic Game with the Orderfield Property: A zero-sum stochastic game with rational inputs (that is, rational payoffs, rational transition probabilities and a rational discount factor (in case of discounted stochastic games) is said to possess the orderfield property if it has a pair of rational optimal strategies (that is, optimal strategies whose coordinates are rational). It follows that the value of the game is rational too.

A non-zero-sum stochastic game with rational inputs is said to possess the orderfield property if it has a pair of Nash equilibrium strategies whose coordinates are rational. It follows that the corresponding equilibrium payoffs are rational as well.

Some classes of stochastic games have been shown to satisfy the orderfield property owing to their special structures. As at least one rational solution is guaranteed for these games, there is hope for finding a finite arithmetic-step algorithm to solve them. Examples of finite 2-player stochastic games with the orderfield property are discounted and undiscounted, zero-sum and non-zero-sum one player control stochastic games, perfect information stochastic games, switching control stochastic games, SER-SIT (Separable Reward - State Independent Transition) games and ARAT (Additive Reward Additive Transition) games. We discuss these classes in detail in the next section. Researchers have been successful in designing algorithms for some of these classes. Refer Filar et al. (1991), Nowak and Raghavan (1993), Raghavan and Syed (2002, 2003), Vrieze (1981) for some of these algorithms. On the other hand, there are classes where the problem is open. For some classes of stochastic games, though algorithms for solving them are known, search is on for efficient algorithms to solve them. For example, there is no efficient algorithm (yet) to solve switching control stochastic games.

In this paper, we show that the orderfield property extends to mixtures of these classes with some restrictions on the transition probabilities. By a mixture, we mean the following. We mix different classes by allowing transitions among states of different classes. It is known that such mixtures may lead to the breakdown of the orderfield property. For example, Sinha (1989) shows an example where a mixture of two zero-sum SER-SIT games does not possess the orderfield property. In this mixture, the set of states S is partitioned to two subsets S_1 and S_2, such that S_1 and S_2 are SER-SIT but the whole game S is not SER-SIT. On the other hand, some mixtures are known to possess the orderfield property. An example of a mixture class that has the orderfield property is a mixture of SC (Switching Control) and ARAT classes (Sinha, 1989). A game belonging to this mixture class has some states satisfying the switching control property and the remaining states being ARAT. Neogy et al. (2008) provide a constructive proof and hence an algorithm to solve SC-ARAT mixtures. In this paper, we propose sufficient conditions for mixture classes to possess the orderfield property. For example, whenever we mix states from two different classes C_1 and C_2, the resulting mixture Γ has the orderfield property if C_1 is one player control, perfect information, switching control or ARAT, states in C_2 constitute a sub-game which can be *any* game with the orderfield property and there are no transitions from states in C_2 to those in C_1. On the other hand, if C_1 is SER-SIT, we provide a counter example where the mixture Γ does not possess the orderfield property. When C_1 is SER-SIT, we place additional restrictions on the transition probabilities to ensure that the resulting mixture has the orderfield property. We highlight the fact that C_2 can constitute any game with the orderfield property, not necessarily from a known class. C_2 can be SER-SIT as well. Inductively mixing such classes leads to a mixture of multiple classes and this mixture has the orderfield property as well.

Some multi-player stochastic games have been shown to possess the orderfield property as well. For example, Mohan, Neogy and Parthasarathy (1997) show that one player control polystochastic games have the orderfield property in the discounted case. Raghavan and Syed (2002) extend this result to undiscounted polystochastic games as well. Till date, research on mixtures of classes of stochastic games has been restricted to the 2-player case. In section 5, we look at mixtures of 2-player games with one player control polystochastic games.

For a detailed discussion on stochastic games, the reader may refer to the book by Filar and Vrieze (1997), and to the chapter on Stochastic Games by Mertens (2002) in the Handbook on Game Theory with Economic Applications, volume 3.

2 Background and preliminaries

2.1 Stochastic games. In this section, we define 2-player zero-sum and non-zero-sum finite stochastic games with discounted as well as undiscounted payoffs and we state Shapley's theorem for discounted zero-sum stochastic games.

DEFINITION 2.1. 2-Player, Finite Stochastic Game: A 2-player, finite state space, finite action space stochastic game consists of

1. Two players P_1 and P_2. We shall sometimes refer to them as players 1 and 2.

2. A finite, non-empty set of states, $S = \{1, 2, \ldots, N\}$.

3. For each state $s \in S$, finite, non-empty sets $A_1(s) = \{1, 2, \ldots, m_1(s)\}$ and $A_2(s) = \{1, 2, \ldots, m_2(s)\}$ of actions for players P_1 and P_2 respectively. Without loss of generality, we may assume $A_1(s) = A_1$ and $A_2(s) = A_2$, $\forall s \in S$.

4. Immediate rewards $r_1(s, i, j)$ for player 1 and $r_2(s, i, j)$ for player 2, where $s \in S, i \in A_1, j \in A_2$, when the game is in state s and the players choose actions i and j respectively. If $r_2 = -r_1$ then, we have a zero-sum game. Otherwise, the game is a non-zero-sum game. We will use r in place of r_1 in case of zero-sum games. We will denote the matrix of immediate rewards in state s by $R_1(s)$ and $R_2(s)$ for players 1 and 2 respectively (and by $R(s)$ in case of zero-sum games).

5. Transition probabilities $(q(s'|s, i, j) : (s, s') \in S \times S, i \in A_1, j \in A_2)$ where $q(s'|s, i, j)$ is the probability of transition from state s to state s' given that players 1 and 2 choose actions $i \in A_1, j \in A_2$ respectively. These transition probabilities constitute the "law of motion" of the game. We will use $q(s, i, j)$ to denote the corresponding probability

distribution. Given, $i \in A_1, j \in A_2$, we denote the $N \times N$ transition matrix by $Q(i,j)$.

The game proceeds as follows. Given a starting state $s_0 \in S$, the players simultaneously choose actions $i_0 \in A_1$ and $j_0 \in A_2$ resulting in payoffs of $r_1(s_0, i_0, j_0)$ and $r_2(s_0, i_0, j_0)$ to players 1 and 2 respectively. The game moves to a new state s_1 according to the law of motion $q(s_0, i_0, j_0)$, the players choose actions $i_1 \in A_1$ and $j_1 \in A_2$ resulting in payoffs of $r_1(s_1, i_1, j_1)$ and $r_2(s_1, i_1, j_1)$ and so on.

In general, strategies can depend on complete histories of the game until the current stage. Such strategies are called behavioral strategies. We shall look at the simpler class of strategies called stationary strategies which depend only on the current state s and not on how s was reached.

For player 1, a stationary strategy f is a function from S to P_{A_1} where S is the state space and P_{A_1} is the set of probability distributions on player 1's action set A_1. Similarly, we define a stationary strategy g for player 2 as a function from S to P_{A_2}. Alternatively, we can look at f as an N-tuple of probability distributions from P_{A_1}, one distribution per state. That is, $f \in P_{A_1}^N$ where N is the number of states. Similarly $g \in P_{A_2}^N$. Pure stationary strategies are simply a set of actions, one per state.

DEFINITION 2.2. β-Discounted Payoffs: In the non-zero-sum case, given an initial state s_0, a pair of stationary strategies (f, g) of players 1 and 2, and a discount factor $\beta \in (0,1)$, we define β-discounted payoffs as follows:

$$[I_\beta^{(1)}(f,g)](s_0) = \sum_{t=0}^{\infty} \beta^t r_t^{(1)}(s_0, f, g)) \text{ for } P_1$$

and

$$[I_\beta^{(2)}(f,g)](s_0) = \sum_{t=0}^{\infty} \beta^t r_t^{(2)}(s_0, f, g)) \text{ for } P_2.$$

DEFINITION 2.3. Undiscounted payoffs: In the non-zero-sum case, given an initial state s_0, and a pair of stationary strategies (f, g), of players 1 and 2, we define the undiscounted (or limiting average) payoffs as follows:

$$[\Phi^{(1)}(f,g)](s_0) = \liminf_{T \uparrow \infty} \left[\frac{1}{T+1} \sum_{t=0}^{T} r_t^{(1)}(s_0, f, g) \right] \text{ for } P_1$$

and

$$[\Phi^{(2)}(f,g)](s_0) = \liminf_{T\uparrow\infty} \left[\frac{1}{T+1} \sum_{t=0}^{T} r_t^{(2)}(s_0,f,g) \right] \text{ for } P_2.$$

In the above definitions, $r_t^{(1)}(s_0,f,g)$ and $r_t^{(2)}(s_0,f,g)$ are the expected immediate rewards at the t-th stage, to players P_1 and P_2 respectively.

In the zero-sum β-discounted case,

$$[I_\beta^{(1)}(f,g)](s_0) = -[I_\beta^{(2)}(f,g)](s_0).$$

(We shall denote $I_\beta^{(1)}$ by I_β in this case). Similarly, for the zero-sum undiscounted case,

$$[\Phi^{(1)}(f,g)](s_0) = -[\Phi^{(2)}(f,g)](s_0).$$

(We shall denote $\Phi^{(1)}$ by Φ in this case).

We write $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$, $(\Gamma = (S, A_1, A_2, r_1, r_2, q))$ to denote a β-discounted (undiscounted) non-zero-sum stochastic game with set of states S, sets of actions A_1 and A_2 and rewards r_1 and r_2 as per Definition 2.1 above. Whenever $r_1 = -r_2$, we have a zero-sum game.

Definition 2.4. Optimal Strategies and Optimal Value: A pair of stationary strategies (f^*, g^*) is optimal in the zero-sum discounted case, if for all $s \in S$,

$$[I_\beta(f,g^*)](s) \le [I_\beta(f^*,g^*)](s) \le [I_\beta(f^*,g)](s) \ \forall f \in P_{A_1}^N, \ \forall g \in P_{A_2}^N,$$

(assuming player 1 is the maximizer and player 2 is the minimizer). The vector $I_\beta(f^*,g^*)$ as a function of the starting state s is unique (Shapley, 1953) and we denote this optimal value vector by v_β.

A pair of stationary strategies (f^*, g^*) is optimal for the undiscounted zero-sum game, if for all $s \in S$,

$$[\Phi(f,g^*)](s) \le [\Phi(f^*,g^*)](s) \le [\Phi(f^*,g)](s) \ \forall f \in P_{A_1}^N, \ g \in P_{A_2}^N,$$

(assuming player 1 is the maximizer and player 2 is the minimizer).

We write v to denote the optimal value vector $\Phi(f^*, g^*)$ of the undiscounted stochastic game which is a function of the initial state s.

DEFINITION 2.5. Nash equilibrium: A pair of stationary strategies (f^*, g^*) constitutes a Nash equilibrium in the discounted case if for all $s \in S$

$$[I_\beta^{(1)}(f^*, g^*)](s) \geq [I_\beta^{(1)}(f, g^*)](s) \text{ for each } f \in P_{A_1}^N$$

and

$$[I_\beta^{(2)}(f^*, g^*)](s) \geq [I_\beta^{(2)}(f^*, g)](s) \text{ for each } g \in P_{A_2}^N.$$

(assuming that both players want to maximize their payoffs).

Similarly, in the undiscounted case, a pair of stationary strategies (f^*, g^*) constitutes a Nash equilibrium, if for all $s \in S$

$$[\Phi^{(1)}(f^*, g^*)](s) \geq [\Phi^{(1)}(f, g^*)](s) \text{ for each } f \in P_{A_1}^N$$

and

$$[\Phi^{(2)}(f^*, g^*)](s) \geq [\Phi^{(2)}(f^*, g)](s) \text{ for each } g \in P_{A_2}^N.$$

(assuming that both players want to maximize their payoffs).

THEOREM 2.1 (Shapley, 1953). *A 2-player zero-sum discounted stochastic game $\Gamma_\beta = (S, A_1, A_2, r, q, \beta)$ has an optimal value vector v_β which is the unique solution of the following system of equations*

$$v(s) = \text{val}[R(s, v)]$$

for all $s \in S$, where $R(s, v)$ is the auxiliary matrix game with $(i, j)^{th}$ entry given by $r(s, i, j) + \beta \sum_{s' \in S} q(s'|s, i, j)v(s')$ and the minmax value of the matrix game given by val$[R(s, v)]$.

For each state $s \in S$, if $(f^*(s), g^*(s))$ is a pair of optimal strategies of the matrix game $R(s, v_\beta)$, then (f^*, g^*) is a pair of optimal strategies for the discounted stochastic game Γ_β, where $f^* = (f^*(1), f^*(2), \ldots, f^*(N))$ and $g^* = (g^*(1), g^*(2), \ldots, g^*(N))$.

2.2 Orderfield property of stochastic games. We describe some classes of stochastic games that are known to possess the orderfield property. We refer the reader to a survey by Raghavan and Filar (1991), a survey by Mohan, Neogy and Parthasarathy (2001) and a survey by Raghavan (2003) for more details on these classes and on algorithms for solving some of them.

Definition 2.6. Stochastic Games with Perfect Information (Shapley, 1953) are stochastic games in which in every state, the action space of (at least) one of the players is a singleton. In other words, we can partition the set of states S into S_1 and S_2 where S_1 is the set of player 1 states (where player 2 has just one action and hence no choice) and S_2 is the set of player 2 states (where player 1 has just one action).

Definition 2.7. One Player Control Stochastic Games (Parthasarathy and Raghavan, 1981) are stochastic games in which only one of the players controls the transitions. For example, when player 2 controls transitions, $q(s'|s,i,j) = q(s'|s,j)$ for all $i \in A_1, j \in A_2, s, s' \in S$.

Definition 2.8. Simple Stochastic Games (Condon, 1992) are stochastic games with reachability objectives and no immediate rewards. There are 2-special absorbing states, the 0-sink and the 1-sink. Player 1 (the maximizer) wins if the 1-sink is reached, player 2 wins otherwise. Leaving out these special sink states, we can partition the remaining states into S_1, S_2 and S_3, where S_1 is the set of player 1 states, (where player 2 has just one action and hence no choice), S_2 is the set of player 2 states (where player 1 has just one action) and S_3 is the set of nature (or average) states where the players do not have a choice and nature chooses the next state according to some probability distribution.

Remark 2.1. SSGs form a subclass of Perfect Information Stochastic Games where the transition probabilities are 0 or 1 for all action-pairs in all states, S_1 and S_2 are player 1 and player 2 states respectively and in S_3, both players have just one action.

Definition 2.9. SC (Switching Control) Stochastic Games (Filar, 1981) are games where the law of motion is controlled by player 1 alone when the game is played in a certain subset of states and by player 2 alone when the game is played in other states. In other words, a switching control game is a stochastic game in which the set of states are partitioned into sets S_1 and S_2 where the transition probabilities are given by

$$\begin{aligned} q(s'|s,i,j) &= q(s'|s,i), \text{ for all } s' \in S, s \in S_1, i \in A_1, j \in A_2, \\ q(s'|s,i,j) &= q(s'|s,j), \text{ for all } s' \in S, s \in S_2, j \in A_2, i \in A_1. \end{aligned}$$

Remark 2.2. Switching Control Stochastic Games are a superclass of Perfect Information Stochastic Games, One Player Control Stochastic Games and SSGs.

DEFINITION 2.10. SER-SIT (Separable Reward-State Independent Transition) Games (Parthasarathy, Tijs and Vrieze, 1984) are stochastic games in which

1. the rewards can be written as the sum of a function that depends on the state alone and another function that depends on the actions alone. That is, $r(s,i,j) = c(s) + a(i,j)$ for all $s \in S, i \in A_1, j \in A_2$, for the zero-sum case. For the non-zero sum case, $r_1(s,i,j) = c(s) + a(i,j)$ and $r_2(s,i,j) = d(s) + b(i,j)$ for P_1 and P_2 respectively, for all states $s \in S$, and actions $i \in A_1, j \in A_2$.

2. the transitions are independent of the state from which the game transitions. That is, $q(s'|s,i,j) = q(s'|i,j)$ for all $i \in A_1, j \in A_2, s, s' \in S$.

DEFINITION 2.11. ARAT (Additive Reward Additive Transition) Games (Raghavan, Tijs and Vrieze, 1986) are stochastic games where

1. the reward function can be written as the sum of two functions, one depending on player 1 and the other on player 2. For the zero-sum case, $r(s,i,j) = r'(s,i) + r''(s,j)$, for all $i \in A_1, j \in A_2, s \in S$. We can similarly write down r_1 and r_2 in the case of non-zero-sum games.

2. the transition probabilities can be written as a sum of two functions, one depending on player 1 and the other on player 2. That is, for all $i \in A_1, j \in A_2, s, s' \in S$,

$$q(s'|s,i,j) = q_1(s'|s,i) + q_2(s'|s,j).$$

Different algorithms have been proposed for solving many of these classes of stochastic games. Some of these algorithms reduce these stochastic games to a Linear Program, some of them reduce them to matrix or bimatrix games, some of them use policy-improvement techniques similar to those for Markov Decision Processes (MDP) and others use Linear Complementarity Problems (LCP) and Vertical LCPs (Refer Cottle, Pang and Stone, 1992). For related results on the orderfield property of stochastic games and algorithms for various classes, apart from the citations listed above, we refer the reader to Flesch, Thuijsman and Vrieze (2007), Mohan et al. (1999), Schultz (1992), Sobel (1981), Syed (1999), Thuijsman and Raghavan (1997), Vrieze et al. (1983).

There are interesting examples of stochastic games that have the orderfield property and that do not belong to any of the above classes. The Big

Match (Blackwell and Ferguson, 1968) with undiscounted payoffs does not have the orderfield property. However, the discounted version of the Big Match has the orderfield property.

We now look at mixtures of classes of stochastic games. Each class, C, defined above has some structure and we shall say a state $s_0 \in S$ has property C, or simply $s_0 \in C$ if the structure of C holds (locally) in s_0. For example, s_0 is a one player control state if $q(s'|s_0, i, j) = q(s'|s_0, i)$ for all $s' \in S, i \in A, j \in B$. Here, s_0 is controlled by player 1. Note that the whole game may not be a one player control game. We can extend this notion from states to sets of states as well.

Definition 2.12. Mixture Class / Game: A stochastic game Γ with set of states S is a mixture of classes C_1 and C_2 if $S = S_1 \cup S_2, (S_1 \cap S_2 = \emptyset)$ such that $S_1 \in C_1$ and $S_2 \in C_2$.

For example, $\Gamma_\beta = (S, A_1, A_2, r, q, \beta)$ is a mixture of a SER-SIT and a one player control game if $S = S_1 \cup S_2, (S_1 \cap S_2 = \emptyset)$ such that

$$\begin{aligned} r(s,i,j) &= c(s) + a(i,j) \text{ for all } i \in A_1, j \in A_2, s \in S_1, \\ q(s'|s,i,j) &= q(s'|i,j) \text{ for all } i \in A_1, j \in A_2, s \in S_1, s' \in S, \\ \text{and } q(s'|s,i,j) &= q(s'|s,i) \text{ for all } i \in A_1, j \in A_2, s \in S_2, s' \in S. \end{aligned}$$

Mixtures of some classes of stochastic games have been shown to have the orderfield property as well. For example, Sinha (1989) shows that a mixture of SC and ARAT Games (denoted SC/ARAT) has the orderfield property. On the other hand, it is also known that given classes of stochastic games having the orderfield property, mixtures of these classes may not have the orderfield property. For example, Sinha (1989) discusses the following example where a mixture of two (zero-sum) SER-SIT games does not possess the orderfield property.

Example 2.1. Mixture of two SER-SIT Games that does not have the Orderfield Property:

Consider the following zero-sum β-discounted stochastic game with 4 states s_1, s_2, s_3, s_4 and the payoffs and transitions as given below:

$$s_1 : \begin{bmatrix} \begin{matrix} 0 \\ (1,0,0,0) \end{matrix} & \begin{matrix} 0 \\ (0,0,1/2,1/2) \end{matrix} \\ \\ \begin{matrix} 0 \\ (1,0,0,0) \end{matrix} & \begin{matrix} 0 \\ (0,0,1/2,1/2) \end{matrix} \end{bmatrix}, \quad s_2 : \begin{bmatrix} \begin{matrix} -1 \\ (1,0,0,0) \end{matrix} & \begin{matrix} -1 \\ (0,0,1/2,1/2) \end{matrix} \\ \\ \begin{matrix} -1 \\ (1,0,0,0) \end{matrix} & \begin{matrix} -1 \\ (0,0,1/2,1/2) \end{matrix} \end{bmatrix},$$

$$s_3 : \begin{bmatrix} -1 & -1 \\ (1,0,0,0) & (0,0,1/2,1/2) \\ & \\ 0 & 0 \\ (1,0,0,0) & (0,0,1/2,1/2) \end{bmatrix}, \quad s_4 : \begin{bmatrix} 0 & 0 \\ (1,0,0,0) & (0,0,1/2,1/2) \\ & \\ 1 & 1 \\ (1,0,0,0) & (0,0,1/2,1/2) \end{bmatrix}.$$

Here, we represent the $(i, j)^{th}$ entry in the matrix corresponding to state s_k as follows:

$$\begin{bmatrix} r(s_k, i, j) \\ (q(s_1|s_k, i, j), q(s_2|s_k, i, j), q(s_3|s_k, i, j), q(s_4|s_k, i, j)) \end{bmatrix}.$$

Note that states s_1 and s_2 are SER-SIT, and states s_3 and s_4 are SER-SIT, but the mixture is not SER-SIT as it violates the Separable Reward property.

Let $\beta = \frac{3}{4}$. The value of the above stochastic game starting at state s_1 is $v_\beta(1) = -4 + \sqrt{13}$.

3 Sufficient conditions for the orderfield property of mixture classes

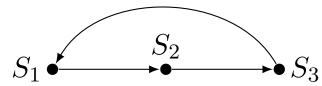
Given different classes of stochastic games, we look at the transitions among states in these classes to define cycles and classes that are cycle-free. We also define a sink class and use these concepts to derive sufficient conditions for the orderfield property of mixtures of stochastic game classes.

Definition 3.1. Cycle, Length of a cycle: Given a stochastic game Γ with set of states S and a partition of S into S_1 and S_2 (that is, $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$), we say S_1 and S_2 are in a cycle if there exist pairs of states $(s_1, s_2) \in S_1 \times S_2$ and $(s'_1, s'_2) \in S_1 \times S_2$, and pairs of actions $(i, j) \in A_1 \times A_2$ and $(i', j') \in A_1 \times A_2$ such that

$$q(s_2|s_1, i, j) > 0 \quad \text{and} \quad q(s'_1|s'_2, i', j') > 0.$$

We can also extend this definition to a partition of S into more than two subsets where we can talk of cycles between pairs of subsets as well as cycles involving more than two subsets.

Cycles between pairs of subsets are cycles of length 2. Absorbing classes correspond to self-loops or cycles of length 1. Following is an example of a cycle of length 3.



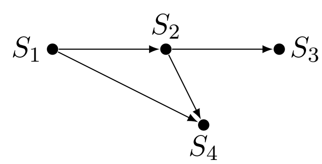
DEFINITION 3.2. Cycle-free classes, Sink and Sub-game restricted to a subset of states: Given a stochastic game Γ with set of states $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$, we say S_1 and S_2 are cycle-free if they are not in a cycle. That is,

$$\begin{aligned} \text{either} \quad & q(s_1|s_2, i, j) = 0, \forall s_1 \in S_1, \forall s_2 \in S_2, \forall i \in A_1, \forall j \in A_2, \\ \text{or} \quad & q(s_2|s_1, i, j) = 0, \forall s_1 \in S_1, \forall s_2 \in S_2, \forall i \in A_1, \forall j \in A_2. \end{aligned}$$

We can extend this definition to a partition of S into more than two subsets as well. If $S = S_1 \cup S_2 \cup \ldots \cup S_k$, ($S_{k_1} \cap S_{k_2} = \emptyset$ for $k_1 \neq k_2$), S_1, S_2, …, S_k are cycle-free if there exists no cycle of any length $l \geq 2$ among them.

If S_1, S_2, …, S_k are cycle-free, at least one of them is an absorbing subset with no transitions going out of the subset. We call an absorbing subset a sink. Without loss of generality, we may order the subsets such that there are no transitions from S_{k_2} to S_{k_1} whenever $k_2 > k_1$. It follows that S_k is a sink. That is, $\sum_{s'_k \in S_k} q(s'_k|s_k, i, j) = 1, \forall s_k \in S_k, \forall i \in A_1, \forall j \in A_2$. Once the game reaches states in S_k, the game remains in S_k. In particular, if the starting state $s_0 \in S_k$, the whole game is within S_k. We can, hence, talk of the sub-game restricted to S_k, which is an independent stochastic game. We denote this by $\Gamma|S_k$.

There may be two or more sinks as well. In the following example, S_1, S_2, S_3 and S_4 are cycle-free and S_3 and S_4 are sinks.



When k = 2, S_1 and S_2 are cycle free implies either S_1 is a sink or S_2 is a sink. If we order the subsets as done above, then S_2 is a sink.

3.1 Mixtures of 2-player β-discounted stochastic games. We state and prove the following sufficient condition for mixtures of 2-player zero-sum β-discounted stochastic games to possess the orderfield property. We prove the theorem when a subset S_1 of the set of states S is one player control. A similar proof works when S_1 is perfect-information, SC, ARAT or a mixture of SC and ARAT states (SC/ARAT). Note that SC/ARAT encompasses all the other classes we have mentioned and has the orderfield property (Sinha, 1989).

THEOREM 3.1. *Let $\Gamma_\beta = (S, A_1, A_2, r, q, \beta)$ be a finite zero sum β-discounted stochastic game where $S = S_1 \cup S_2$, $(S_1 \cap S_2 = \emptyset)$. Assume the following conditions:*

(i) The inputs to Γ_β are rational. That is, r, q and β are rational.

(ii) $\sum_{s_2 \in S_2} q(s_2|s, i, j) = 1$ for all $s \in S_2, i \in A_1, j \in A_2$. In other words, S_2 is a sink. The independent sub-game restricted to the states S_2, $\Gamma_\beta|S_2$, has the orderfield property.

(iii) S_1 belongs to {One Player Control, Perfect Information, SC, ARAT, SC/ARAT}.

Then the mixed stochastic game Γ_β has the orderfield property.

PROOF. We prove the theorem when S_1 is one player control. Without loss of generality, let S_1 be controlled by player 1, that is, $q(s|s_1, i, j) = q(s|s_1, i)$, for all $s_1 \in S_1, s \in S, i \in A_1, j \in A_2$.

Define a new game $\Gamma'_\beta = (S' = S \cup \{s^*\}, A_1, A_2, r', q', \beta)$ where s^* is a new absorbing state such that

$$r'(s^*, i, j) = 0, \forall i \in A_1, \forall j \in A_2.$$

$$r'(s, i, j) = r(s, i, j) + \beta \sum_{s_2 \in S_2} q(s_2|s, i) v_\beta(s_2), \forall s \in S_1, \forall i \in A_1, \forall j \in A_2. \tag{3.1}$$

$$r'(s, i, j) = r(s, i, j), \forall s \in S_2, \forall i \in A_1, \forall j \in A_2.$$

$$q'(s^*|s, i) = 1 - \sum_{s_1 \in S_1} q(s_1|s, i), \forall s \in S_1, \forall i \in A_1.$$

$$q'(s_1|s, i) = q(s_1|s, i), \forall s \in S_1, \forall s_1 \in S_1, \forall i \in A_1.$$

$$q'(s_2|s, i) = 0, \forall s \in S_1, \forall s_2 \in S_2, \forall i \in A_1.$$

$$q'(s_2|s, i, j) = q(s_2|s, i, j), \forall s \in S_2, \forall s_2 \in S_2, \forall i \in A_1, \forall j \in A_2.$$

$$q'(s^*|s^*, i, j) = 1, \forall i \in A_1, \forall j \in A_2.$$

It is easy to see that the sub-game restricted to S_2 is the same independent stochastic game in both Γ_β and Γ'_β. Therefore, for all $s_2 \in S_2$, optimal values and optimal strategies in Γ_β are precisely those in Γ'_β .

Now, for all $s \in S_1$, Shapley equations for Γ'_β are

$$v'_\beta(s) = \mathrm{val}(r'(s,i,j) + \beta \sum_{s_1 \in S_1} q(s_1|s,i)v'_\beta(s_1))$$

and Shapley equations for the game Γ_β are

$$v_\beta(s) = \mathrm{val}(r(s,i,j) + \beta \sum_{s' \in S} q(s'|s,i)v_\beta(s')).$$

Since $r'(s,i,j) = r(s,i,j) + \beta \sum_{s_2 \in S_2} q(s_2|s,i)v_\beta(s_2)$ (from (3.1)), the auxiliary games above for Γ_β and Γ'_β coincide. Hence, optimal value and optimal strategies are the same in both the games.

Further, note that the sub-game of Γ'_β restricted to $S_1 \cup \{s^*\}$ is an independent one player control game and hence, has the orderfield property. (Since the inputs r' and q' are rationals. The fact that r' is rational follows as r is rational and as $v_\beta(s_2)$ is rational for all $s_2 \in S_2$).

Hence, given rational payoffs, transition probabilities and β, there exists rational optimal value and optimal strategies in S_1 as well as in S_2 in both the games (Γ_β and Γ'_β), proving the theorem for the one player control case.

Similarly, we can prove the existence of orderfield property in the mixture Γ_β when S_1 is perfect information, SC, ARAT or an SC/ARAT mixture. In particular, when S_1 is ARAT, note that we can construct a new game Γ'_β where the rewards and transition probabilities are both additive. $\square$

In fact, S_2 can be any finite zero sum β-discounted stochastic game with the orderfield property. For example, it can be the discounted Big Match which does not belong to any known class.

The following example contains cycles between two classes and has the orderfield property. This shows that cycle-free mixtures are not necessary for the orderfield property to hold.

Example 3.1. Mixture of SER-SIT and one player control states, with cycles, but having the orderfield property:

$$s_1 : \begin{bmatrix} 3 & 0 \\ (\frac{1}{4},\frac{1}{4},\frac{1}{2}) & (\frac{1}{4},\frac{1}{4},\frac{1}{2}) \\ 0 & 1 \\ (\frac{1}{4},\frac{1}{4},\frac{1}{2}) & (\frac{1}{4},\frac{1}{4},\frac{1}{2}) \end{bmatrix}, \quad s_2 : \begin{bmatrix} 2 & -1 \\ (\frac{1}{4},\frac{1}{4},\frac{1}{2}) & (\frac{1}{4},\frac{1}{4},\frac{1}{2}) \\ -1 & 0 \\ (\frac{1}{4},\frac{1}{4},\frac{1}{2}) & (\frac{1}{4},\frac{1}{4},\frac{1}{2}) \end{bmatrix}, \quad s_3 : \begin{bmatrix} 0 \\ (1,0,0) \end{bmatrix}.$$

This game is a mixture of a set SER-SIT states $S_1 = \{s_1, s_2\}$ and a set consisting of a one player control state $S_2 = \{s_3\}$. Clearly, this game has cycles between S_1 and S_2. We shall show that this game has the orderfield property.

Using Shapley's theorem, value of the above stochastic game starting at state s_1 is

$$v_1 = \text{val} \begin{bmatrix} 3 + \beta(\frac{v_1}{4} + \frac{v_2}{4} + \frac{v_3}{2}) & \beta(\frac{v_1}{4} + \frac{v_2}{4} + \frac{v_3}{2}) \\ \\ \beta(\frac{v_1}{4} + \frac{v_2}{4} + \frac{v_3}{2}) & 1 + \beta(\frac{v_1}{4} + \frac{v_2}{4} + \frac{v_3}{2}) \end{bmatrix},$$

where we have written v_s in place of $v_\beta(s)$ for the sake of brevity.

Here, $v_2 = v_1 - 1$ and $v_3 = \beta v_1$.

It is easy to see that the auxiliary game corresponding to s_1 does not have a pure optimal strategy pair. Therefore, using Kaplansky's theorem (1945) we get $v_1 = (3 - \beta)/[2(2 + \beta)(1 - \beta)]$ which is rational whenever β is rational. $(\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{4}, \frac{3}{4})$ constitute a pair of optimal strategies for the players in states s_1 and s_2.

In the above Theorem 3.1, S_2 can be SER-SIT. If S_1 is SER-SIT with the other conditions of the theorem remaining the same, the orderfield property may not hold. We provide a counter example in the next section. Further, for mixtures with SER-SIT to possess the orderfield property, we provide a sufficient condition by placing an additional restriction on the transition probabilities. When we mix two SER-SIT games, given this additional restriction on the transitions, we show that we do not require the classes to be cycle-free.

We now extend Theorem 3.1 to a mixture of more than 2 classes and the proof involves inductively applying Theorem 3.1.

THEOREM 3.2. *Let $\Gamma_\beta = (S, A_1, A_2, r, q, \beta)$ be a finite zero-sum β-discounted stochastic game, where $S = S_1 \cup S_2 \cup \ldots \cup S_k$ ($S_{k_1} \cap S_{k_2} = \emptyset$ for $k_1 \neq k_2$). Assume the following conditions:*

(i) The inputs to Γ_β are rational. That is, r, q and β are rational.

(ii) $S_1, S_2, \ldots, S_k$ are cycle-free. That is, for all k_1, k_2 ($1 \leq k_1 < k_2 \leq k$),

$$q(s_{k_1} | s_{k_2}, i, j) = 0 \text{ for all } i \in A_1, j \in A_2, s_{k_1} \in S_{k_1}, s_{k_2} \in S_{k_2}.$$

(iii) For each l, $1 \leq l \leq k$, either S_l is an SC/ARAT mixture or S_l is a sink such that $\Gamma_\beta | S_l$ has the orderfield property.

Then the mixed stochastic game Γ_β has the orderfield property.

We now prove the following sufficient condition for mixtures of 2-player non-zero-sum β-discounted stochastic games to possess the orderfield property.

THEOREM 3.3. *Let $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ where the set of states S is partitioned into S_1 and S_2, (that is, $S = S_1 \cup S_2$, $(S_1 \cap S_2 = \emptyset)$), be a finite, 2-person, non-zero sum stochastic game with β-discounted payoffs. Assume the following conditions:*

(i) The inputs to Γ_β are rational. That is, r_1, r_2, q and β are rational.

(ii) $\sum_{s_2 \in S_2} q(s_2|s, i, j) = 1$, $\forall\ s \in S_2$, $i \in A_1, j \in A_2$. That is, S_2 is a sink.

(iii) The sub-game restricted to S_2, $\Gamma_\beta|S_2$ has a pair of equilibrium strategies, (f^*, g^*), whose coordinates are rational.

(iv) $S_1 \in \{$One Player Control, Perfect Information, SC, ARAT$\}$.

Then the mixed stochastic game Γ_β has the orderfield property.

PROOF. We shall prove the theorem when S_1 is one player control. Without loss of generality, let player 1 be the controlling player, that is, $q(s_1|s, i, j) = q(s_1|s, i)$ for all $s \in S_1, i \in A_1, j \in A_2$.

We are given a pair of rational equilibrium strategies, (f^*, g^*), for $\Gamma_\beta|S_2$. We shall show that the corresponding equilibrium payoffs in S_2, given by $I_\beta^{(1)}(f^*, g^*)(s)$ for player 1 and $I_\beta^{(2)}(f^*, g^*)(s)$ for player 2, are rational as well, that is, for player 1, $I_\beta^{(1)}(f^*, g^*) : S_2 \to R$ (the set of reals). We will show that $I_\beta^{(1)}(f^*, g^*)(s) \in \mathbb{Q}$ (the set of rationals), for all $s \in S_2$.

Note that the strategies f^* and g^* are restricted to S_2. That is,

$$f^* : S_2 \to P_{A_1}, \quad g^* : S_2 \to P_{A_2}.$$

Without loss of generality, let $S_1 = \{1, 2, \ldots, k\}$ and $S_2 = \{k+1, k+2, \ldots, n\}$.

$$I_\beta^{(1)}(f^*, g^*) = (I_\beta^{(1)}(f^*, g^*)(k+1), \ldots, I_\beta^{(1)}(f^*, g^*)(n))$$

$$= (I - \beta Q(f^*, g^*))^{-1} \begin{bmatrix} r_1(k+1, f^*(k+1), g^*(k+1)) \\ r_1(k+2, f^*(k+2), g^*(k+2)) \\ \vdots \\ r_1(n, f^*(n), g^*(n)) \end{bmatrix},$$

where

$$r_1(l, f^*(l), g^*(l)) = \sum_{i,j} r_1(l, i, j) f_i^*(l) g_j^*(l), k+1 \le l \le n$$

and $Q(f^*, g^*)$ is the $N \times N$ matrix of transitions among states, when the players play (f^*, g^*). $(s, s')th$ element of $Q(f^*, g^*)$ is

$$q(s'|s, f^*(s), g^*(s)) = \sum_{i,j} f_i^*(s) q(s'|s, i, j) g_j^*(s).$$

As $f_i^*(s), q(s'|s, i, j)$ and $g_j^*(s)$ are rational for all $s \in S_2, s' \in S, i \in A_1, j \in A_2$, $Q(f^*, g^*)$ is rational too. Thus, $I - \beta Q$ is rational. Further, it is easy to show that $I - \beta Q$ is invertible. Thus, $I_\beta^{(1)}(f^*, g^*)$ is rational. Similarly, $I_\beta^{(2)}(f^*, g^*)$ is rational too.

[Note that, unlike the zero-sum case where the optimal value is unique, the non-zero-sum case may have different equilibrium payoffs corresponding to different equilibrium strategies. We have shown that whenever we have a pair of rational equilibrium strategies, the "corresponding" payoffs are rational as well.]

Now, as in the proof of Theorem 3.1 for the zero-sum case, define a new game $\Gamma' = (S' = S_1 \cup S_2 \cup \{s^*\}, A_1, A_2, r_1', r_2', q', \beta)$ where s^* is a new absorbing state with immediate rewards that are 0 for both players no matter what they play.

In the non-zero-sum case as well, it suffices to define r_1' in terms of $I_\beta^{(1)}$ and keep r_2' unchanged (as player 1 is the controlling player). That is

$$\begin{aligned} r_1'(s, i, j) &= r_1(s, i, j) \\ &\quad + \beta \sum_{s_2 \in S_2} q(s_2|s, i) I_\beta^{(1)}(f^*, g^*)(s_2), \forall s \in S_1, \forall i \in A_1, \forall j \in A_2. \\ r_1'(s, i, j) &= r_1(s, i, j), \forall s \in S_2, \forall i \in A_1, \forall j \in A_2. \\ r_2'(s, i, j) &= r_2(s, i, j), \forall s \in S, \forall i \in A_1, \forall j \in A_2. \end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1 for the zero-sum case. □

We can inductively extend this theorem in the non-zero-sum case to mixtures of more than 2 classes of stochastic games too.

3.2 Two-person undiscounted stochastic games. For the undiscounted zero-sum case, we just state the following theorem and skip the proof.

THEOREM 3.4. *Let $\Gamma = (S, A_1, A_2, r, q)$ where $S = S_1 \cup S_2$, $(S_1 \cap S_2 = \emptyset)$, be a finite, zero sum stochastic game with undiscounted payoffs. Assume the following conditions:*

(i) The inputs to Γ are rational. That is, r and q are rational.

(ii) $\sum_{s_2 \in S_2} q(s_2|s,i,j) = 1$, $\forall s \in S_2, i \in A_1, j \in A_2$.

(iii) For all β rational, $v_\beta(s)$ is rational, $\forall s \in S_2$.

(iv) The undiscounted sub-game restricted to S_2, $\Gamma|S_2$, has the orderfield property, that is, $\exists$ a pair of rational optimal strategies (f^*, g^*) and $v(s) = lim_{\beta\uparrow 1}(1-\beta)v_\beta(s) \in Q, \forall s \in S_2$. (Bewley and Kohlberg, 1976)

(v) S_1 is an SC/ARAT mixture.

Then the undiscounted stochastic game, Γ, has the orderfield property.

We can write down a similar sufficient condition for the undiscounted non-zero-sum case too and in all the above cases, we can extend to mixtures of more than 2 classes as well.

4 SER-SIT mixtures

For mixtures with SER-SIT games, the conditions in the above theorems are not sufficient for the mixture to have the orderfield property. While the sinks can be SER-SIT, if non-sinks are SER-SIT, the orderfield property may breakdown. We provide a counter example below.

EXAMPLE 4.1. Cycle-free mixture of a SER-SIT Game and an independent perfect information stochastic game, that does not have the Orderfield

Property:

$$s_1 : \begin{bmatrix} 3 & 0 \\ (\frac{1}{4},\frac{1}{4},\frac{1}{2}) & (\frac{1}{2},\frac{1}{4},\frac{1}{4}) \\ 0 & 1 \\ (\frac{1}{4},\frac{1}{2},\frac{1}{4}) & (\frac{1}{4},\frac{1}{4},\frac{1}{2}) \end{bmatrix}, \ s_2 : \begin{bmatrix} 3 & 0 \\ (\frac{1}{4},\frac{1}{4},\frac{1}{2}) & (\frac{1}{2},\frac{1}{4},\frac{1}{4}) \\ 0 & 1 \\ (\frac{1}{4},\frac{1}{2},\frac{1}{4}) & (\frac{1}{4},\frac{1}{4},\frac{1}{2}) \end{bmatrix}, \ s_3 : \begin{bmatrix} 0 \\ (0,0,1) \end{bmatrix}.$$

This game is a mixture of a set of SER-SIT states $S_1 = \{s_1, s_2\}$ and a set consisting of perfect information state $S_2 = \{s_3\}$. It is easy to see that S_1 and S_2 are cycle-free, and S_2 is a sink.

Let $\beta = \frac{1}{2}$. Then $v_1 = v_2 = \frac{96-8\sqrt{111}}{11}$.

Now, we provide a sufficient condition for cycle-free mixtures with non-sink SER-SIT states to possess the orderfield property. We state and prove the following theorem for the zero-sum discounted case. This sufficient condition can be proved for the non-zero-sum case and the undiscounted case as well.

THEOREM 4.1. *Let $\Gamma_\beta = (S, A_1, A_2, r, q, \beta)$ where the set of states S is partitioned into S_1 and S_2, (that is, $S = S_1 \cup S_2$, ($S_1 \cap S_2 = \emptyset$)), be a finite 2-person zero sum stochastic game with β-discounted payoffs. Assume the following conditions:*

- *(i) The inputs to Γ_β are rational. That is, r, q and β are rational.*
- *(ii) $\sum_{s_2 \in S_2} q(s_2|s,i,j) = 1$ for all $s \in S_2, i \in A_1, j \in A_2$. In other words, S_2 is a sink. The independent sub-game restricted to the states S_2, $\Gamma_\beta|S_2$, has the orderfield property.*
- *(iii) All states in S_1 are SER-SIT, that is, $r(s_1,i,j) = c(s_1) + a(i,j)$ for all $s_1 \in S_1$, $i \in A_1$, $j \in A_2$ and $q(s|s_1,i,j) = q(s|i,j)$ for all $s \in S$, $s_1 \in S_1$, $i \in A_1$, $j \in A_2$.*
- *(iv) For all $s \in S_1$, $\sum_{s_1 \in S_1} q(s_1|i,j) = q_0$ for all $i \in A_1, j \in A_2$ where $q_0 \in [0,1]$ is a constant.*

Then the mixed stochastic game Γ_β has the orderfield property.

PROOF. Without loss of generality, let $S_1 = \{1, 2, ..., k\}$ and $S_2 = \{k + 1, k + 2, ..., N\}$. By Shapley's theorem, the value of the stochastic game starting at state 1 is as follows:

$$v_1 = \text{val}\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1m_2} \\ r_{21} & r_{22} & \dots & r_{2m_2} \\ \vdots & \vdots & & \vdots \\ r_{m_1 1} & r_{m_1 2} & \dots & r_{m_1 m_2} \end{bmatrix},$$

where

$$r_{ij} = r(1,i,j) + \beta \sum_{s \in S} q(s|i,j) v_s. \tag{4.1}$$

(As state $1 \in S_1$ is SER-SIT) that is,

$$r_{ij} = r(1,i,j) + \beta \sum_{s' \in S_1} q(s'|i,j) v_{s'} + \beta \sum_{s'' \in S_2} q(s''|i,j) v_{s''}.$$

As S_1 is SER-SIT, we can write $v_{s_1} = v_1 + c_{s_1}$ for all $s_1 \in S_1$ where $c_{s_1} = c(s_1) - c(1)$. Therefore,

$$r_{ij} = r(1,i,j) + \beta \sum_{s' \in S_1} q(s'|i,j)(v_1 + c_{s'}) + \beta \sum_{s'' \in S_2} q(s''|i,j) v_{s''} = m_{ij} + w,$$

where

$$m_{ij} = r(1,i,j) + \beta \sum_{s' \in S_1} q(s'|i,j) c_{s'} + \beta \sum_{s'' \in S_2} q(s''|i,j) v_{s''}$$

and $w = \beta v_1 \sum_{s' \in S_1} q(s'|i,j)$. Due to assumption (iv),

$$w = \beta v_1 q_0, \tag{4.2}$$

which is independent of the choice of actions of the players.

For all $i \in A_1, j \in A_2$, m_{ij} is rational. (As r, q and β are rational. Moreover, $v_{s''}$ is rational as the sub-game restricted to S_2 has the orderfield property). Therefore (4.1) becomes

$$v_1 = \text{val}(M) + w, \tag{4.3}$$

where $M = [m_{ij}]_{1 \le i \le m_1, 1 \le j \le m_2}$. M is a matrix game with rational entries and hence has the orderfield property. The coefficient of v_1 in (4.2) is rational. Hence (4.3) is a linear equation in v_1 with rational coefficients. Thus, v_1 is rational. It follows that v_{s_1} is rational for all $s_1 \in S_1$. (Since

$v_{s_1} = v_1 + c_{s_1}$). Optimal strategies of the players in M are optimal strategies in the stochastic game as well, for all initial states $s_1 \in S_1$. □

The following example contains non-sink SER-SIT states but the total transitions out of each of these SER-SIT states is a constant across action pairs (assumption (iv) in the above theorem).

EXAMPLE 4.2. Non-sink SER-SIT states mixed with one player control states, and having the orderfield property:

$$s_1 : \begin{bmatrix} \begin{matrix} 1 \\ (\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}) \end{matrix} & \begin{matrix} 0 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0) \end{matrix} \\ & \\ \begin{matrix} 0 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0) \end{matrix} & \begin{matrix} 1 \\ (0, \frac{1}{2}, 0, \frac{1}{2}) \end{matrix} \end{bmatrix}, \quad s_2 : \begin{bmatrix} \begin{matrix} 2 \\ (\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}) \end{matrix} & \begin{matrix} 1 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0) \end{matrix} \\ & \\ \begin{matrix} 1 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0) \end{matrix} & \begin{matrix} 2 \\ (0, \frac{1}{2}, 0, \frac{1}{2}) \end{matrix} \end{bmatrix},$$

$$s_3 : \begin{bmatrix} \begin{matrix} 1 \\ (0,0,1,0) \end{matrix} & \begin{matrix} 0 \\ (0,0,1,0) \end{matrix} \\ & \\ \begin{matrix} 0 \\ (0,0,0,1) \end{matrix} & \begin{matrix} 1 \\ (0,0,0,1) \end{matrix} \end{bmatrix}, \quad s_4 : \begin{bmatrix} \begin{matrix} 1 \\ (0,0,0,1) \end{matrix} \end{bmatrix}$$

Let β be a rational discount factor.

This game is a mixture of a set of SER-SIT states $S_1 = \{s_1, s_2\}$ and a set of one player control states $S_2 = \{s_3, s_4\}$. Though the SER-SIT class S_1 is not a sink, this game has the orderfield property. Note that the sum of the transition probabilities from s_1 to s_3 and s_4 is always $\frac{1}{2}$, independent of the actions of the players. Similarly for s_2. S_2 constitutes an independent one player control game whose value vector is ($v_3 = \frac{1}{(1-\beta)(2-\beta)}, v_4 = \frac{1}{1-\beta}$) and optimal strategies are $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2-\beta}, \frac{1-\beta}{2-\beta})$ for players 1 and 2 respectively in state 3. (In state 4, both players have just 1 action). Now v_2 can be written in terms of v_1. Here, $v_2 = v_1 + 1$.

The above facts enable us to "separate out" the variables v_1 and v_2 from the auxiliary matrix games for states s_1 and s_2 as described below. Value of the stochastic game starting at state s_1 is $v_1 = \mathrm{val}(M) + w$, where

$$M = \begin{bmatrix} 1 + \frac{\beta}{4}(v_3 + v_4) & \frac{\beta}{4} + \frac{\beta v_3}{2} \\ & \\ \frac{\beta}{4} + \frac{\beta v_3}{2} & 1 + \frac{\beta}{2} + \frac{\beta v_4}{2} \end{bmatrix}$$

and $w = \frac{\beta v_1}{2}$. M is a matrix game with rational entries as v_3 and v_4 are rational. Therefore, val(M) is rational, giving v_1 and v_2 are rational. Further the matrix game M has a pair of rational optimal strategies which are also optimal strategies in s_1 and s_2 for both the players.

In the following example, we consider two sets of SER-SIT states such that the mixture has the orderfield property, though there are cycles between the two sets. This example illustrates a sufficient condition for a mixture of two SER-SIT classes to possess the orderfield property, which we formally state and prove in Theorem 4.2 (after the example).

Example 4.3. Mixture of two SER-SIT Games that has the Orderfield Property: Consider the following zero-sum stochastic game with 4 states s_1, s_2, s_3, s_4 and the payoffs and transitions as given below:

$$s_1 : \begin{bmatrix} \begin{matrix} 0 \\ \left(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}\right) \end{matrix} & \begin{matrix} 0 \\ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \end{matrix} \\ & \\ \begin{matrix} 0 \\ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \end{matrix} & \begin{matrix} 0 \\ \left(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}\right) \end{matrix} \end{bmatrix}, \quad s_2 : \begin{bmatrix} \begin{matrix} -1 \\ \left(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}\right) \end{matrix} & \begin{matrix} -1 \\ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \end{matrix} \\ & \\ \begin{matrix} -1 \\ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \end{matrix} & \begin{matrix} -1 \\ \left(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}\right) \end{matrix} \end{bmatrix},$$

$$s_3 : \begin{bmatrix} \begin{matrix} -1 \\ \left(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}\right) \end{matrix} & \begin{matrix} -1 \\ \left(\frac{1}{4}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}\right) \end{matrix} \\ & \\ \begin{matrix} 0 \\ \left(\frac{1}{4}, \frac{1}{6}, \frac{1}{2}, \frac{1}{12}\right) \end{matrix} & \begin{matrix} 0 \\ \left(\frac{1}{6}, \frac{1}{4}, \frac{5}{12}, \frac{1}{6}\right) \end{matrix} \end{bmatrix}, \quad s_4 : \begin{bmatrix} \begin{matrix} 0 \\ \left(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}\right) \end{matrix} & \begin{matrix} 0 \\ \left(\frac{1}{4}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}\right) \end{matrix} \\ & \\ \begin{matrix} 1 \\ \left(\frac{1}{4}, \frac{1}{6}, \frac{1}{2}, \frac{1}{12}\right) \end{matrix} & \begin{matrix} 1 \\ \left(\frac{1}{6}, \frac{1}{4}, \frac{5}{12}, \frac{1}{6}\right) \end{matrix} \end{bmatrix}.$$

Here, states 1 and 2 are SER-SIT, and states 3 and 4 are SER-SIT, but the mixture is not SER-SIT.

$v_1 = \text{val}(M_1) + w_1$ where $w_1 = \frac{\beta}{2}v_1 + \frac{\beta}{2}v_3$ and $M_1 = \begin{bmatrix} \frac{\beta}{4} & 0 \\ 0 & \frac{\beta}{4} \end{bmatrix}$.

val$(M_1) = \frac{\beta}{8}$ and the strategy $(\frac{1}{2}, \frac{1}{2})$ is optimal for both players.

$v_3 = \text{val}(M_2) + w_2$ where $w_2 = \frac{5}{12}\beta v_1 + \frac{7}{12}\beta v_3$ and $M_2 = \begin{bmatrix} -1 & -1+\frac{\beta}{6} \\ -\frac{\beta}{12} & -\frac{\beta}{12} \end{bmatrix}$.

val$(M_2) = -\frac{\beta}{12}$ and (2^{nd} row, 1^{st}column) is a pair of optimal strategies.

We get the following linear equations with rational coefficients, solving which we obtain v_1 and v_3:

$$v_1 = \frac{\beta}{8} + \frac{\beta}{2}v_1 + \frac{\beta}{2}v_3,$$

$$v_3 = -\frac{\beta}{12} + \frac{5}{12}\beta v_1 + \frac{7}{12}v_3.$$

Now, we state and prove the following sufficient condition for a mixture of two SER-SIT sets to possess the orderfield property. In this case, there may be cycles between S_1 and S_2 and the mixture has the orderfield property as long the total transitions going out of S_1 and those out of S_2 are constants.

THEOREM 4.2. *Let* $\Gamma_\beta = (S,\ A_1,\ A_2,\ r,\ q,\ \beta)$ *where the set of states* S *is partitioned into* S_1 *and* S_2*, (that is,* $S = S_1 \cup S_2$*,* $(S_1 \cap S_2 = \emptyset)$*), be a finite 2-person zero sum stochastic game with* β*-discounted payoffs. Assume the following conditions:*

(i) The inputs to Γ_β *are rational. That is,* r*,* q *and* β *are rational.*

(ii) All states in S_1 *are SER-SIT, that is,* $r(s_1,i,j) = c(s_1) + a(i,j)$ *for all* $s_1 \in S_1$*,* $i \in A_1$*,* $j \in A_2$ *and* $q(s|s_1,i,j) = q(s|i,j)$ *for all* $s \in S$*,* $s_1 \in S_1$*,* $i \in A_1$*,* $j \in A_2$*.*

(iii) For all $s \in S_1$*,* $\sum_{s_1 \in S_1} q(s_1|i,j) = q_0$ *for all* $i \in A_1, j \in A_2$ *where* $q_0 \in [0,1]$ *is a constant.*

(iv) All states in S_2 *are SER-SIT, that is,* $r(s_2,i,j) = c'(s_2) + a'(i,j)$ *for all* $s_2 \in S_2$*,* $i \in A_1$*,* $j \in A_2$ *and* $q(s|s_2,i,j) = q(s|i,j)$ *for all* $s \in S$*,* $s_2 \in S_2$*,* $i \in A_1$*,* $j \in A_2$*.*

(v) For all $s \in S_2$*,* $\sum_{s_2 \in S_2} q(s_2|i,j) = {q_0}'$ *for all* $i \in A_1, j \in A_2$ *where* ${q_0}' \in [0,1]$ *is a constant.*

Then the mixed stochastic game Γ_β *has the orderfield property.*

PROOF. Without loss of generality, let $S_1 = \{1,2,...,k\}$ and $S_2 = \{k+1, k+2, ..., N\}$. The value of the stochastic game starting at state 1 is

$$v_1 = \text{val}\left[\ r(1,i,j) + \beta \textstyle\sum_{s \in S} q(s|i,j) v_s\ \right]_{1 \le i \le m_1, 1 \le j \le m_2}.$$

Here S_1 and S_2 are both SER-SIT. Hence we can write $v_{s_1} = v_1 + c_{s_1}$ for all $s_1 \in S_1$ where $c_{s_1} = c(s_1) - c(1)$, and $v_{s_2} = v_{k+1} + {c'}_{s_2}$ for all $s_2 \in S_2$ where ${c'}_{s_2} = c'(s_2) - c'(k+1)$.

In lines similar to the proof of Theorem 4.1,

$$v_1 = \text{val}(M) + w, \tag{4.4}$$

where $w = \beta v_1 q_0 + \beta v_{k+1}(1 - q_0)$

$$M = \left[r(1,i,j) + \beta \sum_{s_1 \in S_1} q(s_1|i,j) c_{s_1} + \beta \sum_{s_2 \in S_2} q(s_2|i,j) {c'}_{s_2} \right]_{1 \le i \le m_1, 1 \le j \le m_2}$$

and w is linear in v_1 and v_{k+1} with rational coefficients and is independent of the choice of actions of the players. Note that M is free of variables and all entries are rational. Hence val(M) is rational and there exists rational optimal strategies which are optimal strategies for all $s_1 \in S_1$ in the stochastic game as well.

Now, the value of the stochastic game starting at state $k+1$ is

$$v_{k+1} = \text{val}(M') + w', \tag{4.5}$$

where $w' = \beta v_1(1 - {q_0}') + \beta v_{k+1} {q_0}'$ and M' is given by

$$\left[r(k+1,i,j) + \beta \sum_{s_1 \in S_1} q(s_1|i,j) c_{s_1} + \beta \sum_{s_2 \in S_2} q(s_2|i,j) {c'}_{s_2} \right]_{1 \le i \le m_1, 1 \le j \le m_2}.$$

There exists rational optimal strategies for all $s_2 \in S_2$ as well, since M' is a matrix game with rational entries.

Now, equations (4.4) and (4.5) constitute a system of linear equations in v_1 and v_{k+1} with rational coefficients. It follows that v_s is rational for all $s \in S$. □

We can extend the above sufficient conditions for mixtures with SER-SIT states to the discounted non-zero-sum case, undiscounted zero-sum case and to the undiscounted non-zero-sum case as well. Similar to Theorem 3.2, we have sufficient conditions for mixtures with SER-SIT states and involving more than two sets of states.

5 Orderfield property of mixtures of 2-player games with (multi-player) polystochastic games

Janovskaya (1968) introduced polymatrix games and Howson (1972) gave an algorithm by formulating these games as an LCP. Polystochastic games were introduced by Mohan, Neogy and Parthasarathy (1997) as a generalization of polymatrix games. They showed that one player control polystochastic games have the orderfield property by formulating them as an LCP that can be processed by Lemke's algorithm (1965). In this section, we look at mixtures of 2-player games with polystochastic games.

DEFINITION 5.1. A non-zero-sum, polystochastic game is a multi-player stochastic game defined as follows.

1. There are n players, n is finite.
2. $S = \{1, 2, \dots, N\}$ is a non-empty, finite set of states.
3. $A_k = \{1, 2, \dots, m_k\}$ is the set of actions for player $k, 1 \leq k \leq n$.
4. At each state s, each pair of players, (k_1, k_2), has associated partial rewards, $r_{k_1 k_2}(s, i, j)$. These partial rewards depend only on the actions i and j of players k_1 and k_2 and are independent of the actions chosen by the other players. Total immediate reward of player k_1 in state s $r_{k_1}(s, i, j) = \sum_{k_2 \neq k_1}(r_{k_1 k_2}(s, i, j_{k_2}))$ where $i = i(k_1)$ is the action chosen by player k_1 in state s and $j = (j_{k_2})$ is the vector of actions chosen by all other players k_2. Let $R_{k_1 k_2}(s)$ be the corresponding matrix of partial immediate rewards to player k_1 depending on actions of k_2.
5. Transition probabilities $q(s'|s, i_1, i_2, ..., i_n)$ are the probabilities of transition from state s to state s' when the players respectively choose actions $i_1, i_2, ..., i_n$.

We can extend the definitions of different types of strategies and definitions of classes of stochastic games to polystochastic games as well.

Now, we look at mixtures of one player control polystochastic states with 2-player states of known classes. We state the following theorem for 3-player polystochastic games and this can be extended to the general n-player case as well.

THEOREM 5.1. *Let $\Gamma_\beta = (S, A_1, A_2, A_3, r_1, r_2, r_3, q, \beta)$ be a finite, 3-person, non-zero sum stochastic game with β-discounted payoffs, where $S = S_1 \cup S_2$, $(S_1 \cap S_2 = \emptyset)$. Assume the following conditions:*

(i) The inputs to Γ_β are rational. That is, r_1, r_2, r_3, q and β are rational.

(ii) S_2 is a sink. Furthermore, $\Gamma_\beta | S_2$ is a one player control polystochastic game.

(iii) States in S_1 involve only two players, say player 1 and player 2. Further, $S_1 \in \{$One Player Control, Perfect Information, SC, ARAT$\}$.

Then the mixed stochastic game S has the orderfield property.

The proof is similar to the proof of the 2-player discounted non-zero sum case. A similar theorem holds for the undiscounted case as well. When S_1 is SER-SIT, we have sufficient conditions similar to that for the 2-player case.

6 Conclusion and future work

We extend known classes of stochastic games with the orderfield property to mixtures of some of these classes. In the 2-player case, if the set of states S can be partitioned into cycle-free subsets $S_1, S_2, \ldots S_k$ where each S_l has the orderfield property and any non-sink S_h is either one player control or perfect information or switching control or ARAT or SC/ARAT (in the zero-sum case), then the mixture has the orderfield property. If there are non-sink sets S_h of SER-SIT states, then we additionally require the sum of the transition probabilities going out of S_h to be a constant, for each such S_h. If all these sets are SER-SIT, we can drop the cycle-free requirement. We extend these theorems to multi-player stochastic games by allowing sinks to be one player control polystochastic games.

We believe that cycle-free mixtures of different one player control polystochastic games have the orderfield property. In other words, switching control polystochastic games where there are no cycles between states of both players, have the orderfield property. We can also extend multi-player mixtures to mixtures of 2-player switching control or ARAT games with multi-player perfect information or ARAT games. Further, we can also mix different multi-player perfect information or ARAT classes among themselves or with one player control polystochastic games such that the polystochastic games are sinks.

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