

3D-4D Interlinkage Of qqq Wave Functions Under 3D Support For Pairwise Bethe-Salpeter Kernels

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Abstract

Using the method of Green's functions within the framework of a Bethe-Salpeter formalism characterized by a pairwise qq interaction with a Lorentz-covariant 3D support to its kernel, the 4D BS wave function for a system of three identical relativistic spinless quarks is reconstructed from the corresponding 3D quantities which satisfy a *fully connected* 3D BSE. This result is a 3-body generalization of a similar 3D-4D interconnection for the corresponding 2-body wave functions found earlier under identical conditions of a Lorentz-covariant 3D support to the corresponding BS kernel, ('CIA' for short), for the $q\bar{q}$ interaction. (The generalization from spinless to fermion quarks is straightforward).

To set 'CIA' for 3D Kernel support in the context of contemporary approaches to the qqq baryon problem, a model scalar 4D qqq BSE with pairwise contact interactions simulating the NJL-Faddeev equations, is worked out fully and compared with the 'CIA' vertex function which reduces exactly to the 4D NJL-Faddeev form in the limit of zero spatial range. This consistency check is part of a detailed accounting of the CIA vertex structure whose physical motivation stems from the role of spectroscopy when considered as an integral part of any QCD-motivated dynamical investigation.

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1 Introduction: The Relativistic qqq Problem

The qqq problem must obey the mathematical requirement of *connectedness* [1-4], signalled by the absence of δ -functions in its defining equation, Schroedinger (in 3D) or BSE (in 4D). Further, for a *relativistic* 2- or 3-body problem, the historical issue of 3D reduction from a 4D BSE has generated much interest from the outset [5-8] to more recent times [9-10], motivated by the Markov-Yukawa transversality condition [11]. Physically, a 3D reduction of the BSE is linked to the problem of *observed* O(3)-like hadron spectra [12], while without 3D reduction, a 4D BSE in Wick-rotated form gives O(4)-like spectra [13]. In this paper we shall address both the issues of *connectedness* and ‘3D-4D interlinkage’ of the qqq BS wave functions under 3D support for the pairwise kernels, termed ‘CIA’ for short, as a 3-body extension of an earlier $q\bar{q}$ investigation [9].

Our principal result is mathematical: Derivation of an explicit interconnection between the 3D and 4D forms of the qqq BS wave functions, via Green’s function techniques for the bound qqq state. However, an apparent lack of familiarity with the 3D support ansatz, in the quantum mechanics literature, necessitates a prior exposition of its tenets vis-a-vis the more familiar ones, which we shall outline in the following sub-sections, keeping in mind the overall requirement of connectedness in a 3-particle amplitude [1-4], signalled by the *absence of any delta function* in its structure, either explicitly or through its defining equation. Originally derived within a 3D framework [1-4] whose prototype dynamics is the Schroedinger equation, its basic logic applies to the 4D BSE framework with pairwise kernels, with or without 3D reduction.

1.1 3D Reduction of 4D BSE

The problem of 3D reduction of the BSE itself has had a long history [5-8], born out of certain intuitive compulsions to keep to a common time for the internal components of a bound state composite. These have ranged from the instantaneous approximation [5] through the quasi-potential approach [6], to other variants [7,8] in which the interaction of the constituents can be given a physical meaning in their respective mass shells, a result that can also be justified within the tenets of *local* field theory [8]. In all these methods the starting BSE is formally 4D in all details, *including its kernel*, but the associated propagators are manipulated in various subtle ways ranging from an ‘overall instantaneity’ (the quasi-potential approach [6]) to their individual ‘on-shellness’ in varying degrees [7,8]. In all these approaches, a common feature is that once the initial BSE is thus reduced to a 3D form, there is no getting back to its original 4D form.

An alternative approach which is of more recent origin [9,10], is based on the Markov-Yukawa Transversality Condition [11] wherein a 3D support is postulated at the outset to the *kernel* of an otherwise 4D BSE, albeit in a Lorentz-covariant form, but the propagators are left untouched in their original 4D form. This is achieved by simply demanding that the kernel K of the pairwise interaction be a function of only the component \hat{q} of the relative 4-momentum q which is *transverse* to the total 4-momentum P of the composite, viz., $\hat{q}_\mu = q - q.PP_\mu/P^2$, so that $\hat{q}.P = 0$ identically. This may be regarded as a sort of complementary strategy to the approaches [6-8] wherein the propagators are manipulated but the kernel is left untouched. The main difference between the two approaches is that while in [6-8] the original 4D BSE must be given up for good in favour of the reduced 3D BSE as a fresh starting point of the dynamics, the alternative approach [9,10] based on the

Transversality Condition [11] allows both forms to be used interchangeably according to the demands of physics. For, not only does this ansatz of ‘Covariant Instanteity’ (termed CIA for short [9]) lead to an exact 3D reduction of the BSE, but also facilitates an equally exact reconstruction of the original 4D BSE form without extra charge [9], so that both forms are completely equivalent. For a 2-quark problem this was shown explicitly some time ago [9] under ‘CIA’ which gives a concrete realization of a ‘two-tier’ approach [14] wherein the 3D reduction of the original 4D BSE serves for the dynamics of the observed O(3)-like spectra [12], while the reconstructed 4D BS wave function provides a natural language for applications to various transition amplitudes as 4D quark loop integrals [14,9]. The exact interconnection that CIA yields between the 3D and 4D forms of the BSE shows up through the expression of the 4D BS vertex function Γ as a simple product of only 3D quantities, viz., $D \times \phi$, where D and ϕ are the 3D denominator and wave functions respectively, satisfying a relativistic Schroedinger-like equation [9]. We next turn to the physical basis of this method [15], termed 3D-4D-BSE in the following.

1.2 3D-4D BSE vis-a-vis other QCD Motivated Methods

The physical basis of 3D-4D-BSE [15] vis-a-vis other 4D BSE-cum-SDE approaches [16], is dynamical breaking of chiral symmetry, or $DB\chi S$ for short, a la NJL [17]. While the original NJL model [17] was conceived as a contact interaction, its vital feature of chiral symmetry and its breaking, which it shares with other effective theories [18,19] nevertheless is a key ingredient which various workers have attempted to bring out [20-23] from QCD premises in the low frequency limit to simulate its non-perturbative features. Of particular interest to the present investigation is a non-local 4-quark interaction mediated by vector exchange [21], as a prototype of the non-perturbative gluon propagator, which offers a generalized $DB\chi S$ mechanism for generating a momentum-dependent mass-function $m(p)$ via the Schwinger-Dyson equation (SDE). As is well known, this mechanism accounts for the bulk of the ‘constituent’ mass of ud quarks via Politzer additivity [24]. This formalism leads to the conventional 4D BSE-SDE type framework [21,16] which was discussed in [15] vis-a-vis 3D-4D-BSE [9].

A related aspect of QCD concerns its structure in the large N_C limit, when it reduces to an effective theory of weakly interacting mesons and glueballs [25], while baryons emerge as solitonic configurations in the background meson fields [26]. These ideas were later concretized by many authors [22,23] using functional integration techniques [27], which combined both the $DB\chi S$ and large N_C aspects of QCD in a systematic reduction procedure to give rise to an effective action involving local meson fields. While the large N_C limit motivates an effective 4-quark interaction of the non-local NJL-type [21], the vacuum degeneracy of the effective action so derived [22,23] exhibits a complex structure arising from $DB\chi S$, viz., after integrating out over the quark d.o.f.’s, the effective action of the chiral π, σ - field (treated classically) is approximately a sum of bosonic and fermionic parts, of which the former gives rise to mesonic excitations [28], while the latter provides the solitonic solution corresponding to ‘baryon-number-one’[29].

Despite a common QCD basis for the 4D BSE-SDE [16] and the solitonic [26] pictures, their actual technologies are so different that only the former [16], and not the latter [26], is directly relevant to our discussion. In particular, the Lagrangian formulation [21] to which our 3D-4D BSE comes closest, corresponds to an effective $q\bar{q}/qq$ interaction with a gluon-like propagator to simulate both the perturbative and non-perturbative regimes,

as originally envisaged [30] in the instantaneous approximation [5]. Subsequently this kernel was given a covariant 3D basis [9] in the spirit of [11], and the reduced 3D BSE found to agree with the spectra of mesons [31], as well as of baryons [32], under a common parametrization for the gluon propagator with 2 basic constants plus the constituent quark mass $m_q \approx 270 MeV$. This last turned out to be consistent with the chiral symmetry breaking SDE solution in the low momentum limit [15], so that this quantity is no longer a free parameter. Thus this BSE-cun-SDE approach [15] with an effective gluon-like propagator [30] produces the usual $DB\chi S$ scenario, in harmony with the standard 4D BSE-SDE picture [21,16], except for its hybrid 3D-4D structure, designed to unify the 3D hadron spectra [31-32] with the 4D quark-loop integrals [9,15]. With this physical background for 3D-4D BSE, we now turn to the qqq sector.

1.3 The qqq BSE Structure under 3D Kernel Support

Does an interconnection similar to the $q\bar{q}$ case [9] exist in the corresponding BS amplitudes for a qqq system under identical conditions of 3D support to the pairwise BS kernel? This question is of great practical value since the 3D reduction of the 4D BSE already produces *fully connected* integral equations for both the $q\bar{q}$ [31] and qqq [32] systems with explicit solutions for the 3D wave functions. Therefore a reconstruction [33] of the 4D qqq wave (vertex) function in terms of the corresponding 3D quantities is a vital ingredient for applications to various types of *transition amplitudes* involving qqq baryons, analogously to the $\bar{q}q$ case [9,15]. To keep the technicalities to the barest minimum, we shall consider 3 identical spinless particles for simplicity and definiteness, which however need not detract from the generality of the ensuing singularity structures. The answer is found to be in the affirmative, except for the recognition that a 3D support to the pairwise BS kernel implies a truncation of the Hilbert space. Such truncation, while still allowing an unambiguous reduction of the BSE from the 4D to the 3D level, nevertheless leaves an information gap in the *reverse direction*, viz., from 3D to 4D [33] for any n -body system except for $n = 2$ where both transitions are exactly reversible (a sort of degenerate situation) [9]. The extra charge needed to complete the reverse transition comes in the form of a 1D delta-function which however has nothing to do with connectedness [3,4] of an n -body amplitude, as will be formally demonstrated in Sects.2-4, and fully accounted for in Sect.5 through a detailed comparison with a model NJL-Faddeev problem which has proved popular in recent years [34]. (Other methods include i) non-topological solitons [29,35]; and ii) QCD bosonization with diquarks [36], but these will not be considered).

Regarding 3D-4D-BSE [33] versus 4D NJL-Faddeev [34], the following perspective is in order: The 3D-4D-BSE pairwise kernel with covariant instantaneity (CIA) has a contact interaction only in the 1D time-like variable, while the corresponding 4D-NJL kernel exhibits contact behaviour in all the 4D space-time variables. The consistency of the two approaches is ensured by checking that the latter is the (expected) $K = const$ limit of the former, and further that the Hilbert space information gap in the way of reconstruction of the 4D BS wave function in terms of 3D quantities [33] vanishes in this limit, so that the full 4D structure of NJL-Faddeev [34] is automatically recovered.

The paper which makes use of Green's function techniques, is organized as follows. In Sec.2 we rederive the 3D-4D interconnection [9] for a two-body system, now by the Green's function method, whence we reproduce the previously derived result [9] for the corresponding BS wave functions in 3D and 4D forms. In Sec.3, the BSE for the 4D

Green's function for 3 identical spinless quarks (q), is reduced to the 3D form by integrating w.r.t. *two* internal time-like momenta, and in so doing, introducing 3D Green's functions satisfying a fully connected 3D BSE, free from δ -functions, as anticipated from BS wave function studies [32,37]. With this 3D BSE as the check point, Sec.4 gives a reconstruction of the full 4D Green's function in terms of its (partial) 3D counterparts, so as to satisfy exactly the above 3D BSE, *after* integration w.r.t. the relevant time-like momenta. The resultant vertex function, eq.(4.10), has a 1D δ -function singularity which admits a (Fermi-like) 'pseudopotential' interpretation [38].

Sec.5 is devoted to the solution of a 4D NJL-Faddeev problem with scalar-isoscalar quarks in pairwise contact interaction, leading to a 4D baryon-quark-diquark vertex function (5.9), and subjected to a pointwise comparison with (4.10) [sec.(5.4)], and its structure accounted for. Sec.6 sums up our main results as obtained in Secs.3-4, vis-a-vis the corresponding results of NJL-Faddeev [34] as obtained in Sec.5, and briefly indicates the technical issues arising from the inclusion of spin. It concludes with a brief comparison with some contemporary approaches, [6-8,16,34-36], especially in relation to their respective roles in treating the spectroscopy sector as an integral part of the qqq dynamics.

2 3D-4D Interconnection For $\bar{q}q$ System

If the BSE for a spinless $\bar{q}q$ system has a 3D support for its kernel K as $K(\hat{q}, \hat{q}')$ where \hat{q} is the component of the relative momentum $q = (p_1 - p_2)/2$ *orthogonal* to the total hadron 4-momentum $P = p_1 + p_2$, then the 4D hadron-quark vertex function Γ is a function of \hat{q} only [9]. For this 2-body case the 4D and 3D forms of the BSE are exactly reversible without further assumptions. For the 3-body case a corresponding 3D-4D connection was obtained on the basis of semi-intuitive arguments [37] whose formal derivation is the central aim of this paper. To that end we shall formulate the 4D and 3D BSE's in terms of Green's functions to derive the 3D-4D connection for a two-body system in terms of their respective Green's functions, in preparation for the generalization to the three-body case in the next two sections.

We shall mostly use the notation and phase conventions of [8,12] for the various quantities (momenta, propagators, etc). The 4D qq Green's function $G(p_1 p_2; p_1' p_2')$ near a *bound* state satisfies a 4D BSE without the inhomogeneous term, viz. [9,37],

$$i(2\pi)^4 G(p_1 p_2; p_1' p_2') = \Delta_1^{-1} \Delta_2^{-1} \int dp_1'' dp_2'' K(p_1 p_2; p_1'' p_2'') G(p_1'' p_2''; p_1' p_2') \quad (2.1)$$

where

$$\Delta_1 = p_1^2 + m_q^2, \quad (2.2)$$

and m_q is the mass of each quark. Now using the relative 4-momentum $q = (p_1 - p_2)/2$ and total 4-momentum $P = p_1 + p_2$ (similarly for the other sets), and removing a δ -function for overall 4-momentum conservation, from each of the G - and K - functions, eq.(2.1) reduces to the simpler form

$$i(2\pi)^4 G(q, q') = \Delta_1^{-1} \Delta_2^{-1} \int d\hat{q}'' M d\sigma'' K(\hat{q}, \hat{q}'') G(q'', q') \quad (2.3)$$

where $\hat{q}_\mu = q_\mu - \sigma P_\mu$, with $\sigma = (q \cdot P)/P^2$, is effectively 3D in content (being orthogonal to P_μ). Here we have incorporated the ansatz of a 3D support for the kernel K

(independent of σ and σ'), and broken up the 4D measure dq'' arising from (2.1) into the product $d\hat{q}''Md\sigma''$ of a 3D and a 1D measure respectively. We have also suppressed the 4-momentum P_μ label, with ($P^2 = -M^2$), in the notation for $G(q, q')$.

Now define a fully 3D Green's function $\hat{G}(\hat{q}, \hat{q}')$ as [9,37]

$$\hat{G}(\hat{q}, \hat{q}') = \int \int M^2 d\sigma d\sigma' G(q, q') \quad (2.4)$$

and two (hybrid) 3D-4D Green's functions $\tilde{G}(\hat{q}, q')$, $\tilde{G}(q, \hat{q}')$ as

$$\tilde{G}(\hat{q}, q') = \int Md\sigma G(q, q'); \quad \tilde{G}(q, \hat{q}') = \int Md\sigma' G(q, q'); \quad (2.5)$$

Next, use (2.5) in (2.3) to give

$$i(2\pi)^4 \tilde{G}(q, \hat{q}') = \Delta_1^{-1} \Delta_2^{-1} \int dq'' K(\hat{q}, \hat{q}'') \tilde{G}(q'', \hat{q}') \quad (2.6)$$

Now integrate both sides of (2.3) w.r.t. $Md\sigma$ and use the result [9]

$$\int Md\sigma \Delta_1^{-1} \Delta_2^{-1} = 2\pi i D^{-1}(\hat{q}); \quad D(\hat{q}) = 4\hat{\omega}(\hat{\omega}^2 - M^2/4); \quad \hat{\omega}^2 = m_q^2 + \hat{q}^2 \quad (2.7)$$

to give a 3D BSE w.r.t. the variable \hat{q} , while keeping the other variable q' in a 4D form:

$$(2\pi)^3 \tilde{G}(\hat{q}, q') = D^{-1} \int d\hat{q}'' K(\hat{q}, \hat{q}'') \tilde{G}(\hat{q}'', q') \quad (2.8)$$

Now a comparison of (2.3) with (2.8) gives the desired connection between the full 4D G -function and the hybrid $\tilde{G}(\hat{q}, q')$ -function:

$$2\pi i G(q, q') = D(\hat{q}) \Delta_1^{-1} \Delta_2^{-1} \tilde{G}(\hat{q}, q') \quad (2.9)$$

which is the Green's function counterpart, *near the bound state*, of the same result [9] connecting the corresponding BS wave functions. Again, the symmetry of the left hand side of (2.9) w.r.t. q and q' allows us to write the right hand side with the roles of q and q' interchanged. This gives the dual form

$$2\pi i G(q, q') = D(\hat{q}') \Delta_1'^{-1} \Delta_2'^{-1} \tilde{G}(q, \hat{q}') \quad (2.10)$$

which on integrating both sides w.r.t. $Md\sigma$ gives

$$2\pi i \tilde{G}(\hat{q}, q') = D(\hat{q}') \Delta_1'^{-1} \Delta_2'^{-1} \hat{G}(\hat{q}, \hat{q}'). \quad (2.11)$$

Substitution of (2.11) in (2.9) then gives the symmetrical form

$$(2\pi i)^2 G(q, q') = D(\hat{q}) \Delta_1^{-1} \Delta_2^{-1} \hat{G}(\hat{q}, \hat{q}') D(\hat{q}') \Delta_1'^{-1} \Delta_2'^{-1} \quad (2.12)$$

Finally, integrating both sides of (2.8) w.r.t. $Md\sigma'$, we obtain a fully reduced 3D BSE for the 3D Green's function:

$$(2\pi)^3 \hat{G}(\hat{q}, \hat{q}') = D^{-1}(\hat{q}) \int d\hat{q}'' K(\hat{q}, \hat{q}'') \hat{G}(\hat{q}'', \hat{q}') \quad (2.13)$$

Eq.(2.12) which is valid near the bound state pole (since the inhomogeneous term has been dropped for simplicity) expresses the desired connection between the 3D and 4D forms of the Green's functions; and eq(2.13) is the determining equation for the 3D form. A spectral analysis can now be made for either of the 3D or 4D Green's functions in the standard manner, viz.,

$$G(q, q') = \sum_n \Phi_n(q; P) \Phi_n^*(q'; P) / (P^2 + M^2) \quad (2.14)$$

where Φ is the 4D BS wave function. A similar expansion holds for the 3D G -function \hat{G} in terms of $\phi_n(\hat{q})$. Substituting these expansions in (2.12), one immediately sees the connection between the 3D and 4D wave functions in the form:

$$2\pi i \Phi(q, P) = \Delta_1^{-1} \Delta_2^{-1} D(\hat{q}) \phi(\hat{q}) \quad (2.15)$$

whence the BS vertex function becomes $\Gamma = D \times \phi / (2\pi i)$ as found in [9]. We shall make free use of these results, taken as qq subsystems, for our study of the qqq G -functions in Sections 3 and 4.

3 qqq Green's Function: 3D Reduction of BSE

As in the two-body case, and in an obvious notation for various 4-momenta (without the Greek suffixes), we consider the most general Green's function $G(p_1 p_2 p_3; p_1' p_2' p_3')$ for 3-quark scattering *near the bound state pole* (for simplicity) which allows us to drop the various inhomogeneous terms from the beginning. Again we take out an overall delta function $\delta(p_1 + p_2 + p_3 - P)$ from the G -function and work with *two* internal 4-momenta for each of the initial and final states defined as follows [37]:

$$\sqrt{3} \xi_3 = p_1 - p_2 ; \quad 3\eta_3 = -2p_3 + p_1 + p_2 \quad (3.1)$$

$$P = p_1 + p_2 + p_3 = p_1' + p_2' + p_3' \quad (3.2)$$

and two other sets ξ_1, η_1 and ξ_2, η_2 defined by cyclic permutations from (3.1). Further, as we shall be considering pairwise kernels with 3D support, we define the effectively 3D momenta \hat{p}_i , as well as the three (cyclic) sets of internal momenta $\hat{\xi}_i, \hat{\eta}_i$, ($i = 1, 2, 3$) by [37]:

$$\hat{p}_i = p_i - \nu_i P ; \quad \hat{\xi}_i = \xi_i - s_i P ; \quad \hat{\eta}_i = \eta_i - t_i P \quad (3.3)$$

$$\nu_i = (P \cdot p_i) / P^2 ; \quad s_i = (P \cdot \xi_i) / P^2 ; \quad t_i = (P \cdot \eta_i) / P^2 \quad (3.4)$$

$$\sqrt{3} s_3 = \nu_1 - \nu_2 ; \quad 3t_3 = -2\nu_3 + \nu_1 + \nu_2 \quad (+\text{cyclic permutations}) \quad (3.5)$$

The space-like momenta \hat{p}_i and the time-like ones ν_i satisfy [37]

$$\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 0 ; \quad \nu_1 + \nu_2 + \nu_3 = 1 \quad (3.6)$$

Strictly speaking, in the spirit of covariant instantaneity, we should have taken the relative 3D momenta $\hat{\xi}, \hat{\eta}$ to be in the instantaneous frames of the concerned pairs, i.e., w.r.t. the rest frames of $P_{ij} = p_i + p_j$; however the difference between the rest frames of P and P_{ij} is small and calculable [37], while the use of a common 3-body rest frame ($P = 0$) lends considerable simplicity and elegance to the formalism.

We may now use the foregoing considerations to write down the BSE for the 6-point Green's function in terms of relative momenta, on closely parallel lines to the 2-body case. To that end note that the 2-body relative momenta are $q_{ij} = (p_i - p_j)/2 = \sqrt{3}\xi_k/2$, where (ijk) are cyclic permutations of (123). Then for the reduced qqq Green's function, when the *last* interaction was in the (ij) pair, we may use the notation $G(\xi_k\eta_k; \xi_k'\eta_k')$, together with 'hat' notations on these 4-momenta when the corresponding time-like components are integrated out. Further, since the pair ξ_k, η_k is *permutation invariant* as a whole, we may choose to drop the index notation from the complete G -function to emphasize this symmetry as and when needed. The G -function for the qqq system satisfies, in the neighbourhood of the bound state pole, the following (homogeneous) 4D BSE for pairwise qq kernels with 3D support:

$$i(2\pi)^4 G(\xi\eta; \xi'\eta') = \sum_{123} \Delta_1^{-1} \Delta_2^{-1} \int d\hat{q}_{12}'' M d\sigma_{12}'' K(\hat{q}_{12}, \hat{q}_{12}'') G(\xi_3''\eta_3''; \xi_3'\eta_3') \quad (3.7)$$

where we have employed a mixed notation (q_{12} versus ξ_3) to stress the two-body nature of the interaction with one spectator at a time, in a normalization directly comparable with eq.(2.3) for the corresponding two-body problem. Note also the connections

$$\sigma_{12} = \sqrt{3}s_3/2; \quad \hat{q}_{12} = \sqrt{3}\hat{\xi}_3/2; \quad \hat{\eta}_3 = -\hat{p}_3, \quad etc \quad (3.8)$$

The next task is to reduce the 4D BSE (3.7) to a fully 3D form through a sequence of integrations w.r.t. the time-like momenta s_i, t_i applied to the different terms on the right hand side, *provided both* variables are simultaneously permuted. We now define the following fully 3D as well as mixed (hybrid) 3D-4D G -functions according as one or more of the time-like ξ, η variables are integrated out:

$$\hat{G}(\hat{\xi}\hat{\eta}; \hat{\xi}'\hat{\eta}') = \int \int \int \int ds dt ds' dt' G(\xi\eta; \xi'\eta') \quad (3.9)$$

which is S_3 -symmetric.

$$\tilde{G}_{3\eta}(\xi\hat{\eta}; \xi'\hat{\eta}') = \int \int dt_3 dt_3' G(\xi\eta; \xi'\eta'); \quad (3.10)$$

$$\tilde{G}_{3\xi}(\hat{\xi}\eta; \hat{\xi}'\eta') = \int \int ds_3 ds_3' G(\xi\eta; \xi'\eta'); \quad (3.11)$$

The last two equations are however *not* symmetric w.r.t. the permutation group S_3 , since both the variables ξ, η are not simultaneously transformed; this fact has been indicated in eqs.(10,11) by the suffix "3" on the corresponding (hybrid) \tilde{G} -functions, to emphasize that the 'asymmetry' is w.r.t. the index "3". We shall term such quantities " S_3 -indexed", to distinguish them from S_3 -symmetric quantities as in eq.(3.9). The full 3D BSE for the \hat{G} -function is obtained by integrating out both sides of (3.7) w.r.t. the st -pair variables $ds_i ds_j' dt_i dt_j'$ (giving rise to an S_3 -symmetric quantity), and using (3.9) together with (3.8) as follows:

$$(2\pi)^3 \hat{G}(\hat{\xi}\hat{\eta}; \hat{\xi}'\hat{\eta}') = \sum_{123} D^{-1}(\hat{q}_{12}) \int d\hat{q}_{12}'' K(\hat{q}_{12}, \hat{q}_{12}'') \hat{G}(\hat{\xi}''\hat{\eta}''; \hat{\xi}'\hat{\eta}') \quad (3.12)$$

This integral equation for \hat{G} which is the 3-body counterpart of (2.13) for a qq system in the neighbourhood of the bound state pole, is the desired 3D BSE for the qqq system in a

fully connected form, i.e., free from delta functions. Now using a spectral decomposition for \hat{G}

$$\hat{G}(\hat{\xi}\hat{\eta}; \hat{\xi}'\hat{\eta}') = \sum_n \phi_n(\hat{\xi}\hat{\eta}; P) \phi_n^*(\hat{\xi}'\hat{\eta}'; P) / (P^2 + M^2) \quad (3.13)$$

on both sides of (3.12) and equating the residues near a given pole $P^2 = -M^2$, gives the desired equation for the 3D wave function ϕ for the bound state in the connected form:

$$(2\pi)^3 \phi(\hat{\xi}\hat{\eta}; P) = \sum_{123} D^{-1}(\hat{q}_{12}) \int d\hat{q}_{12}'' K(\hat{q}_{12}, \hat{q}_{12}'') \phi(\hat{\xi}''\hat{\eta}''; P) \quad (3.14)$$

The solution of this equation for the ground state was found in [32] in a *gaussian* form which implies that $\phi(\hat{\xi}\hat{\eta}; P)$ is an S_3 -invariant function of $\hat{\xi}_i^2 + \hat{\eta}_i^2$, valid for *any* index ‘i’. While the gaussian form may prove too restrictive for more general applications, the mere S_3 -symmetry of ϕ in the $(\hat{\xi}_i, \hat{\eta}_i)$ pair may prove adequate in practice, and hence useful for both the solution of (3.14) *and* for the reconstruction of the 4D BS wave function in terms of the 3D wave function (3.14), as is done in Sec.4 below.

4 Reconstruction of the 4D BS Wave Function

To *re-express* the 4D G -function (3.7) in terms of the 3D \hat{G} -function (3.12), we first adapt the result (2.12) to the hybrid Green’s function of the (12) subsystem given by $\tilde{G}_{3\eta}$, eq.(3.10), in which the 3-momenta $\hat{\eta}_3, \hat{\eta}'_3$ play a parametric role reflecting the spectator status of quark #3, while the *active* roles are played by $q_{12}, q_{12}' = \sqrt{3}(\xi_3, \xi_3')/2$, for which the analysis of Sec.2 applies directly. This gives

$$(2\pi i)^2 \tilde{G}_{3\eta}(\xi_3 \hat{\eta}_3; \xi_3' \hat{\eta}'_3) = D(\hat{q}_{12}) \Delta_1^{-1} \Delta_2^{-1} \hat{G}(\hat{\xi}_3 \hat{\eta}_3; \hat{\xi}'_3 \hat{\eta}'_3) D(\hat{q}'_{12}) \Delta_1'^{-1} \Delta_2'^{-1} \quad (4.1)$$

where on the right hand side, the ‘hatted’ G -function has full S_3 -symmetry, although (for purposes of book-keeping) we have not shown this fact explicitly by deleting the suffix ‘3’ from its arguments. A second relation of this kind may be obtained from (3.7) by noting that the 3 terms on its right hand side may be expressed in terms of the hybrid $\tilde{G}_{3\xi}$ functions vide their definitions (3.11), together with the 2-body interconnection between (ξ_3, ξ_3') and $(\hat{\xi}_3, \hat{\xi}'_3)$ expressed once again via (4.1), but without the ‘hats’ on η_3 and η_3' . This gives

$$\begin{aligned} (\sqrt{3}\pi i)^2 G(\xi_3 \eta_3; \xi_3' \eta_3') &= (\sqrt{3}\pi i)^2 G(\xi \eta; \xi' \eta') \\ &= \sum_{123} \Delta_1^{-1} \Delta_2^{-1} (\pi i \sqrt{3}) \int d\hat{q}_{12}'' M d\sigma_{12}'' K(\hat{q}_{12}, \hat{q}_{12}'') G(\xi_3'' \eta_3''; \xi_3' \eta_3') \\ &= \sum_{123} D(\hat{q}_{12}) \Delta_1^{-1} \Delta_2^{-1} \tilde{G}_{3\xi}(\hat{\xi}_3 \eta_3; \hat{\xi}'_3 \eta_3') \Delta_1'^{-1} \Delta_2'^{-1} \end{aligned} \quad (4.2)$$

where the second form exploits the symmetry between ξ, η and ξ', η' .

This is as far as we can go with the qqq Green’s function, using the 2-body techniques of Sec.2. However,, unlike the 2-body case where the reconstruction of the 4D G -function in terms of the corresponding 3D quantity was complete at this stage, the process is not yet complete for the 3-body case, as eq.(4.2) clearly shows. This is due to the *truncation* of Hilbert space implied in the ansatz of 3D support to the pairwise BSE kernel K which, while facilitating a 4D to 3D BSE reduction without extra charge, does *not* have the

complete information to permit the *reverse* transition (3D to 4D) without additional assumptions. This limitation of the 3D support ansatz for the BSE kernel affects all n -body systems except $n = 2$ (which may be regarded as a sort of degenerate situation).

Now as a purely mathematical problem, we must look for a suitable ansatz for the quantity $\tilde{G}_{3\xi}$ on the right hand side of (4.2) in terms of *known* quantities, so that the reconstructed 4D G -function satisfies the 3D equation (3.12) exactly, which may be regarded as a “check-point” for the entire exercise. We therefore seek a structure of the form

$$\tilde{G}_{3\xi}(\hat{\xi}_3\eta_3; \hat{\xi}'_3\eta'_3) = \hat{G}(\hat{\xi}_3\hat{\eta}_3; \hat{\xi}'_3\hat{\eta}'_3) \times F(p_3, p_3') \quad (4.3)$$

where the unknown function F must involve only the momentum of the spectator quark #3. A part of the η_3, η'_3 dependence has been absorbed in the \hat{G} function on the right, so as to satisfy the requirements of S_3 -symmetry for this 3D quantity, whether it has a gaussian structure [32] (where it is explicit), or a more general one (where it is not so explicit); see the discussion below eq.(3.14).

As to the remaining factor F , it is necessary to choose its form in a careful manner so as to conform to the conservation of 4-momentum for the *free* propagation of the spectator between two neighbouring vertices, consistently with the symmetry between p_3 and p_3' . A possible choice consistent with these conditions is:

$$F(p_3, p_3') = C_3\Delta_3^{-1}\delta(\nu_3 - \nu_3') \quad (4.4)$$

where we have taken only the time-component of the 4-momentum p_3 in the δ -function since the effect of its space component has already been absorbed in the “connected” (3D) Green’s function \hat{G} . Next, Δ_3^{-1} represents the “free” propagation of quark #3 between successive vertices, while C_3 represents some residual effects which may at most depend on the 3-momentum \hat{p}_3 , but must satisfy the main constraint that the 3D BSE, eq.(3.12), be *explicitly* satisfied.

To check the self-consistency of the ansatz (4.4), integrate both sides of (4.2) w.r.t. $ds_3ds_'dt_3dt_3'$ to recover the 3D S_3 -invariant \hat{G} -function on the left hand side. Next, in the first form on the right hand side, integrate w.r.t. ds_3ds_3' on the G -function which alone involves these variables. This yields the quantity $\tilde{G}_{3\xi}$. At this stage, employ the ansatz (4.4) to integrate over dt_3dt_3' . Consistency with the 3D BSE, eq.(3.12), now demands

$$C_3 \int \int d\nu_3 d\nu_3' \Delta_3^{-1} \delta(\nu_3 - \nu_3') = 1; \text{ (since } dt = d\nu \text{)} \quad (4.5)$$

The 1D integration w.r.t. $d\nu_3$ may be evaluated as a contour integral over the propagator Δ^{-1} , which gives the pole at $\nu_3 = \hat{\omega}_3/M$, (see below for its definition). Evaluating the residue then gives

$$C_3 = i\pi/(M\hat{\omega}_3); \quad \hat{\omega}_3^2 = m_q^2 + \hat{p}_3^2 \quad (4.6)$$

which will reproduce the 3D BSE, eq.(3.12), *exactly!* Substitution of (4.4) in the second form of (4.2) finally gives the desired 3-body generalization of (2.12) in the form

$$3G(\xi\eta; \xi'\eta') = \sum_{123} D(\hat{q}_{12})\Delta_{1F}\Delta_{2F}D(\hat{q}'_{12})\Delta_{1F}'\Delta_{2F}'\hat{G}(\hat{\xi}_3\hat{\eta}_3; \hat{\xi}'_3\hat{\eta}'_3)[\Delta_{3F}/(M\pi\hat{\omega}_3)] \quad (4.7)$$

where for each index, $\Delta_F = -i\Delta^{-1}$ is the Feynman propagator.

From this structure of the 4D Green’s function near the bound state pole, we can infer the corresponding structure of the 4D BS *wave function* $\Phi(\xi\eta; P)$ through a spectral

representation like (3.13) for the 4D Green's function G on the left hand side of (4.2). Equating the residues on both sides gives the desired 4D-3D connection between Φ and ϕ :

$$\Phi(\xi\eta; P) = \sum_{123} D(\hat{q}_{12}) \Delta_1^{-1} \Delta_2^{-1} \phi(\hat{\xi}\hat{\eta}; P) \times \sqrt{\frac{\delta(\nu_3 - \hat{\omega}_3/M)}{M\hat{\omega}_3\Delta_3}} \quad (4.8)$$

From (4.8) we can infer the structure of the baryon- qqq vertex function by rewriting it in the alternative form [37]:

$$\Phi(\xi\eta; P) = \frac{(V_1 + V_2 + V_3)}{\Delta_1\Delta_2\Delta_3} \quad (4.9)$$

where

$$V_3 = D(\hat{q}_{12}) \phi(\hat{\xi}\hat{\eta}; P) \times \sqrt{\frac{\Delta_3\delta(\nu_3 - \hat{\omega}_3/M)}{M\hat{\omega}_3}} \quad (4.10)$$

The quantity V_3 is the baryon- qqq vertex function corresponding to the ‘‘last interaction’’ in the (12) pair, and so on cyclically. This is precisely the form (apart from a constant factor that does not affect the baryon normalization) that had been anticipated in an earlier study in a semi-intuitive fashion [37].

5 A Simplified 4D NJL-Faddeev Model

We now set up a simplified 4D NJL-Faddeev bound state problem, termed NJL-Faddeev, with 3 scalar-isoscalar quarks interacting pairwise in a contact fashion. To facilitate a direct comparison with 3D-4D-BSE, we employ the same notation and phase convention for the various quantities as in Secs.3-4, but in view of the bound state nature of the problem it is enough to work with the 4D BSE for the wave function only, as there is now no special advantage in dealing with the Green's function. We start with the qq problem as a prerequisite for the solution of the qqq problem.

5.1 qq Bound State in NJL Model

The BSE for the 4D wave function Φ for a qq system may be written down for the NJL model, in the notation of Sec.2 as :

$$i(2\pi)^4 \Phi(q_{12}P_{12}) = \Delta_1^{-1} \Delta_2^{-1} \lambda \int d^4 q_{12}' \Phi(q_{12}'P_{12}) \quad (5.1)$$

where λ is the strength of the contact NJL interaction for any pair of (scalar) quarks. The solution of this equation simply reads as [39]

$$\Phi(q_{12}P_{12}) = A \Delta_1^{-1} \Delta_2^{-1} \quad (5.2)$$

When plugged back into (5.1), one gets an ‘eigenvalue’ equation for the invariant mass $M_d^2 = -P_{12}^2$ of an isolated bound qq pair in the implicit form of a determining equation for λ :

$$\lambda^{-1} = -i(2\pi)^{-4} \int d^4 q \Delta_1^{-1} \Delta_2^{-1} \equiv h(M_d) \quad (5.3)$$

where $\Delta_{1,2}$ are given by (2.2) and $p_{1,2} = P_{12}/2 \pm q$, viz., $\Delta_{1,2} = m_q^2 + q^2 - M_d^2/4 \pm q \cdot P_{12}$, and we have indicated the result of integration by a function $h(M_d)$ of the mass M_d of

the composite bound state (diquark). Unfortunately the integral (5.3) is logarithmically divergent, but it can be regularized with a 4D ultraviolet cut-off Λ , together with a Wick rotation, i.e., $q_0 \rightarrow iq_0$, which is allowed by the singularities of the two propagators. The exact result is:

$$16\pi^2\lambda^{-1} = 1 + \ln(4\Lambda^2/M_d^2) - 2\frac{\sqrt{4m_q^2 - M_d^2}}{M_d}\arcsin(M_d/2m_q) \quad (5.4)$$

under the condition $M_d < 2m_q$. A slightly less accurate but much simpler form which is also easier to adapt to the qqq problem to follow, may be obtained by the Feynman method of introducing an auxiliary integration variable u ($0 < u < 1$) to combine the two propagators, followed by a Wick rotation and a translation to integrate over d^4q (ignoring surface terms which formally arise due to the logarithmic divergence) :

$$16\pi^2\lambda^{-1} = \ln\frac{6\Lambda^2}{6m_q^2 - M_d^2} - 1 \equiv 16\pi^2h(M_d), \quad (5.5)$$

thus defining a diquark ‘self-energy’ function $h(M)$ where the ‘on-shell’ value is $M = M_d$. Eq.(5.5) also provides a determining equation for the NJL strength parameter λ in terms of the ‘diquark’ mass M_d and the 4D cut-off parameter Λ . This result is clearly 4D invariant, a feature that characterizes the tenets of the NJL model [17].

5.2 NJL- qqq Bound State Problem

We now set up the corresponding NJL- qqq problem under the same $q - q$ contact interaction strength λ . Using the same notation for the various 4-momenta and propagators as listed in Sec.3, the 4D wave function $\Phi(\xi, \eta; P)$ expressed in terms of any of the S_3 invariant pairs (ξ_i, η_i) of internal 4-momenta satisfies the BSE:

$$i(2\pi)^4\Phi(\xi, \eta; P) = \sum_{123} \lambda\Delta_1^{-1}\Delta_2^{-1} \int d^4q'_{12}\Phi(\xi'_3, \eta_3; P) \quad (5.6)$$

where the arguments of Φ on the LHS are not-indexed since it is S_3 -symmetric as a whole, while those on the RHS are indexed in order to indicate which subsystem is in pairwise interaction (see explanation in Sec.3). The solution of this equation may be read off from the observation that the integration w.r.t. $q'_{12} = \sqrt{3}\xi'_3/2$ leaves the respective integrals as functions of η_i only, where $i = 1, 2, 3$. Thus [2]

$$\Phi(\xi, \eta; P) = \sum_{123} \Delta_1^{-1}\Delta_2^{-1}F(\eta_3) \quad (5.7)$$

where F is a function of a single variable η_i . Next, plugging back the solution (5.7) into the main equation (5.6), gives the following integral equation in a single variable η_3 , as a routine procedure applicable to separable potentials [2] (see also [34b]) :

$$(h(M_d) - h(M_{12}))F(\eta_3) = -i(2\pi)^{-4}\Delta_3^{-1} \int d^4q'_{12}[F(\eta'_2)\Delta_1'^{-1} + (1 \leftrightarrow 2)] \quad (5.8)$$

We note in passing that the cut-off parameter Λ drops out from the LHS, as may be checked by substitution for $h(M)$ from (5.5). This means that the 4D diquark propagator $(h(M_d) - h(M_{12}))^{-1}$ is formally independent of the cut-off Λ , in this simplified NJL model.

Next, the meaning of the function $F(\eta)$ can be inferred from an inspection of eq.(5.8), on similar lines to the 3D [2] or 4D [34b,c] studies: $F(\eta_3)$ is the 4D ‘quark(3)-diquark(12)’ wave function which is generated by an exchange force represented by the propagators $\Delta_1'^{-1}$ and $\Delta_2'^{-1}$ in the first and second terms on the RHS respectively. And the baryon- qqq vertex function V_3 corresponding to a break-up of the baryon into quark(3) and diquark (12) may be identified by multiplying this quantity with the product of the inverse propagators of quark(3) and diquark(12):

$$V_3 \equiv V(\eta_3) = \Delta_3 f(\eta_3) F(\eta_3) \quad (5.9)$$

where the diquark inverse propagator is reexpressed as

$$f(\eta_3) = h(M_d) - h(M_{12}) = (4\pi)^{-2} \ln \frac{6m_q^2 + \eta_3^2 - 4M_B^2/9}{6m_q^2 - M_d^2}, \quad (5.10)$$

making use of eq.(5.5) and the kinematical relation $\Delta_i = m_q^2 + \eta_i^2 - M_B^2/9$, where M_B is the mass of the bound qqq state, and $i = 1, 2, 3$. The quantity V_3 of eq(5.9) may be compared directly (except for normalization) with the corresponding ‘3D-4D-BSE’ quantity (4.10).

5.3 Solution of the Bound qqq State Eq.(5.8)

We now turn to the Lorentz structure of the NJL- qqq equation (5.8), as well as an approximate analytic solution for the energy eigenvalues of the bound qqq states. To that end we substitute (5.9) in (5.8) to give an integral equation for $V(\eta_3)$, with $\eta_2' \equiv \eta$ for short:

$$V(\eta_3) = -2i(2\pi)^{-4} \int d^4\eta V(\eta) f^{-1}(\eta) \times (m_q^2 + \eta^2 - M_B^2/9)^{-1} (m_q^2 + (\eta_3 + \eta)^2 - M_B^2/9)^{-1} \quad (5.11)$$

where the factor 2 on the RHS signifies equal effects of the two terms on the RHS of (5.8). For a bound state solution of this equation, with $M_B < M_d + m_q$, the singularity structures permit a Wick rotation $\eta_0 \rightarrow i\eta_0$ which converts η into a Euclidean variable η_E . This shows without further ado that eq.(5.11) is 4D-invariant just like its qq counterpart eq.(5.3). This is not quite the same thing as the old result [13] on O(4)-like spectra with harmonic confinement in the limit of infinite quark mass [13], since this NJL-Faddeev model of contact interaction, patterned after similar approaches [34], lacks a confining interaction, so that although in principle eq.(5.11) predicts a spectrum of bound states at the qqq level (starting with NJL(contact) pairwise interactions), such spectra cannot be a realistic representation of the *actual* hadron spectra [12]. (Note that the 4D form factors employed in some NJL interactions [34b] for convergence of the integrals, do not have the significance of confinement; see however other attempts at confinement [40]). In the absence of a confining mechanism, most such NJL-Faddeev models [34] have effectively produced only one non-trivial bound state - the nucleon/Delta. We now show how this comes about via Wick rotation in (5.11).

For an approximate analytic solution of eq.(5.11), note that the logarithmic function $f(\eta)$ in the integral appearing on the right is slowly varying, so that not much error is incurred by taking it out of the integral and replacing it with an average value $< f(\eta) >$. It is now possible to ‘match’ both sides with an effectively constant $V(\eta)$, *provided* any further logarithmic dependence on η is also similarly treated for consistency. The

integral is now exactly of the type (5.3), i.e., logarithmically divergent, and can be handled successively by Wick rotation, Feynman auxiliary variable u , and a translation. The result is again of the form (5.5), and after cancelling out the factors $V(\eta_3)$ and $V(\eta)$ from both sides, the eigenvalue equation reads:

$$\langle f(\eta) \rangle = 2(4\pi)^{-2} \left[\ln \frac{\Lambda^2}{\langle \eta_3^2/6 \rangle + m_q^2 - M_B^2/27} - 1 \right] \quad (5.12)$$

To simplify this equation, we express all quantities in terms of the $h(M)$ functions given in (5.5) and (5.10), and ignore the difference between $\eta = \eta'_2$ and η_3 inside the logarithms, to give

$$h(M_d) - h(M_{12}) = 2h(M_{12}); \Rightarrow \lambda^{-1} = h(M_d) = 3h(M_{12}) \quad (5.13)$$

The last equation brings out clearly the fact that the baryon binding comes about from *three* pairs of qq interaction, albeit off-shell, since the function $\langle M_{12}^2 \rangle = \langle \eta_3^2 \rangle - 4M_B^2/9$ still depends on the (average) value of η^2 . The qualitative features are thus on expected lines, but this oversimplified model was not intended to be pushed for an actual fit to the nucleon mass (which at minimum requires the introduction of spin-isospin d.o.f.[34]), beyond the general feature of a quark-diquark structure that characterizes the NJL-Faddeev approach [34], as expected from any separable potential [2], of which the NJL model is a special case. The ‘bosonization’ approaches [36] also bring out the quark-diquark structure, but in a more general (non-linear) fashion [36b].

5.4 Comparison of NJL-Faddeev with 3D-4D-BSE

We end this section with a comparison between the vertex functions (4.10) of 3D-4D-BSE [33], and (5.9) of NJL-Faddeev [34], which reflect the corresponding differences in their respective dynamical premises. The NJL-Faddeev form (5.9) of V_3 is explicitly Lorentz invariant, with full 4D Hilbert space information built into its structure. Its quark-diquark structure merely reflects the ‘separable’ nature of the NJL model [17]. There is no special motivation here for a 3D reduction of the types [5-8] described in Sec.1. The 4D (Lorentz-invariant) structure is in-built in NJL-Faddeev [34], as seen from eqs(5.4-5) for the qq state, and its qqq counterpart in eq.(5.8).

In contrast, the vertex function (4.10) obtained from the 3D-4D-BSE formalism [9-11] is merely Lorentz covariant due to the 3D kernel support, but the derivation is otherwise quite general (much more than NJL-Faddeev) since it is valid for *any* form of the kernel as long as it is 3D in content. This leads to an exact 3D reduction of the (4D) BSE whose formal solution is a 3D wave function $\phi(\hat{\xi}, \hat{\eta})$, a function of *two* independent 3-momenta [32], in contrast to its NJL counterpart $F(\eta_3)$ in (5.9) which is a function of a single 4-momentum η_3 . The denominator function $D(\hat{q}_{12})$ of (4.10) similarly is a 3D counterpart of the corresponding 4D inverse propagator $f(\eta_3)$ in (5.9). Finally the big radical in (4.10) corresponds to the inverse propagator Δ_3 in (5.9), but has a vastly more involved structure which may be traced back to the difference in their respective dynamical premises which we now seek to account for analytically.

While the premises of NJL-Faddeev [34] are traceable to the contact 4-fermion interaction [17], those of 3D-4D-BSE [9-11] are based on an interplay of the 3D and 4D BSE forms with an otherwise general but 3D kernel support. To reconcile the two approaches in a mathematically consistent way, note first that while the ‘zero extension’ in the temporal direction is common to both, NJL-Faddeev has also a zero spatial extension, but

3D-4D-BSE has ‘normal’ spatial extension. To reduce the latter to the former, all that is required is to set its spatial extension to zero ! This is simply achieved (in momentum space) by setting its 3D kernel K equal to a *constant*. For, as is easily checked in this limit, the structures of the various $4D$ equations in Secs.2-4 reduce exactly to their 4D NJL counterparts in Sec.5.(1-3), via the simple identification $K = \lambda$, so that there is no more need to make 3D reductions or 4D reconstructions! [The conversion from the Green’s function form of Secs.2-4 to the wave function language of secs.5.(1-3) is trivial]. Indeed the formulation of NJL-Faddeev in this Section has been made in close enough correspondence with that of 3D-4D-BSE to bring out the transparency of the parallelism.

After this basic check between these two formalisms, it is clear that the ambiguity in the reconstruction of the 4D wave function from the 3D form vanishes in the $K = \text{const}$ limit, so that the same is directly attributable to the (mere Lorentz covariant) 3D form of the BSE kernel. Indeed from the derivation in Sec.4 it is clear that the 1D δ -function in (4.10) fills up an information gap in the reconstruction from a truncated 3D to the full 4D Hilbert space in the simplest possible manner, while satisfying a vital self-consistency check by reproducing the full structure (3.12) of the 3D BSE. This already lends *sufficiency* to the ansatz (4.4) which leads to (4.10). As to its ‘necessity’, this ansatz has certain desirable properties like on-shell propagation of the spectator in between two successive interactions, as well as an explicit symmetry in the p_3 and p_3' momenta. There is a fair chance of its uniqueness within some general constraints, but so far we have not been able to prove this.

The other question concerns the compatibility of the 1D δ - function in (4.10) with the standard requirement of connectedness [3-4]. Both the δ -function and the Δ_{3F} propagator appear in *rational* forms in the 4D Green’s function, eq.(4.7), reflecting a free on-shell propagation of the spectator between two vertex points. The square root feature in the baryon- qqq vertex function (4.10) is a technical artefact resulting from equal distribution of this singularity between the initial and final state vertex points, and has no deeper significance. Furthermore, as the steps in Sec.3 indicate, the three-body connectedness has already been achieved at the 3D level of reduction, so the ‘physics’ of this singularity, generated via eq.(4.4), must be traced to some mechanism other than a lack of connectedness [3,4] in the 3-body scattering amplitude. A plausible analogy is to a sort of (Fermi-like) ‘pseudopotential’ of the type employed to simulate the effect of chemical binding in the coherent scattering of neutrons from a hydrogen molecule in connection with the determination of the *singlet* $n - p$ scattering length [38]. Such δ -function potentials have no deeper significance other than depicting the vast mismatch in the frequency scales of nuclear and molecular interactions. In the present case, the instantaneity in time of the pairwise interaction kernel in an otherwise 4D Hilbert space causes a similar mismatch, needing a 1D δ -function to fill the gap. And just as the ‘pseudo-potential’ in the above example [38] does not have any observable effect, the singularity under radicals in (4.10) will *not* show up in any physical amplitude for hadronic transitions via quark loops, since the Green’s functions (4.7) involve both the δ - function and the propagator Δ_{3F} in *rational* forms before the relevant quark loop integrations over them are performed.

As to the wider ramifications on spectroscopy, the L -excited qqq states that characterize the hadron spectrum, are effectively absent in the NJL-Faddeev eq.(5.11), unlike in 3D-4D-BSE [31-32] where they do. On the other hand, both vertex functions (4.10) and (5.9) are perfectly capable of generating amplitudes for baryonic transitions via quark-diquark loop integrals, as shown with NJL-Faddeev [41] for items such as axial coupling

constants, nucleon magnetic moments and the $\pi - N \sigma$ -term, and with 3D-4D-BSE for similar items [42,43].

6 Discussion and Summary

We first recapitulate the perspective on 3D reduction of the BSE, outlined in sec.(1.2): It involves both conceptual (simultaneity of constituents in a bound state) and observational (O(3)-like spectra [12]) issues. In this respect, the traditional 3D approaches to the BSE [5-8] work with normal 4D kernel support but manipulate the associated propagators in the 4D BSE in various ways [6-8] to reduce it to a 3D form. There is no formal problem of compatibility with the observed O(3) spectra [12] in such qqq approaches, except that, to the author's knowledge, there is little evidence so far of such methods going beyond the 2-quark level. Anyhow, in such methods, the reduced 3D BSE represents a fresh starting point of the dynamics, and there is no going back to its earlier 4D form. Because of this reason, one does not see in these approaches [6-8] such unorthodox radicals as in (4.7-10) which characterizes the alternative approach [9,10] based on the Markov-Yukawa Transversality Condition [11]: It postulates a 3D support to the BSE kernel in a covariant fashion by demanding the internal 4-momenta to be transverse to the total 4-momentum of the composite hadron, and allows an *exact* 3D reduction of the original 4D BSE [9-10], thus automatically ensuring O(3)-spectra. More importantly, it allows an *exact* reconstruction of the 4D BSE amplitude without extra charge, so that both BSE forms are completely equivalent and therefore simultaneously available for both 3D spectroscopy and 4D quark-loop amplitudes in a 'two-tier' fashion [14]. (This was shown for the 2-quark problem some years ago [9], and the present exercise [33] is for the corresponding 3-quark problem).

6.1 Comparison with Other 4D Approaches: Spectroscopy

The principal results of this paper on the structure of the 3D-4D-BSE formalism [9,33,37] for the qqq problem are contained in eqs(4.7-10). For comparison with other contemporary approaches to the qqq problem, we have chosen for explicit display a simplified form of the 4D NJL-Faddeev model [34], which nevertheless retains its principal features, and outlined in Sec.5 the derivation of its main results through a formulation designed to bring out the parallelism with 3D-4D-BSE of Secs.(2-4) in point-wise details. In particular, the equations of 3D-4D-BSE *match* exactly those of NJL-Faddeev by setting the BSE kernel equal to a constant. This calibration ensures that any ambiguity inherent in the reconstruction of the 4D baryon- qqq vertex function in Sec.4 fully disappears in this zero range limit, so that the difference between these 2 approaches may be directly attributed to the finite spatial range of the 3D-4D-BSE formulation. The respective vertex functions also stand a close comparison, except for the 2-body (quark-diquark) structure of NJL-Faddeev (due to the separable interaction [2]), versus the genuine 3-body structure of 3D-4D BSE. The only item of dissimilarity is in respect of the singularity factor under radicals in (4.10), vis-a-vis the inverse propagator Δ_3 of the spectator in (5.9). This has been accounted for in sec.(5.4), mathematically through self-consistency checks, and physically by an analogy with pseudopotentials of the type employed for coherent scattering of slow neutrons from hydrogen molecules [38], with no observable effects. In particular, the 1D δ -function in no way implies a lack of connectedness [3-4] in the qqq amplitude.

As regards spin, the extension of the above formalism to fermion quarks is a straightforward process amounting to the replacement of $\Delta_F = -i\Delta^{-1}$ by the corresponding S_F -functions, as has been described elsewhere [42,37]. In particular, the fermion vertex function has recently been applied to the problem of proton-neutron mass difference [43] via quark loop integrals, to bring out the practicability of its application without parametric uncertainties, since the entire formalism is linked all the way from spectroscopy to hadronic transition amplitudes [14] via quark-loops.

In sec.(1.2) we have attempted briefly to set this 3D-4D BSE-SDE approach in the physical context of other 4D QCD-motivated BSE-SDE approaches [21,16] which are all governed by $DB\chi S$ on the one hand, and the large N_C limit on the other. These are more general than the contact NJL model [17,34] since, unlike in [17,34], their non-local kernel structures allow incorporation of confinement for prediction of L -excited hadrons. Our *vector*-type (gluon-like) kernel [30] for both the perturbative and non-perturbative QCD regimes gives it a chirally invariant look at the 4-quark Lagrangian level, and with the constituent mass m_q ‘understood’ via $DB\chi S$ [15], fits in quite well within a broader BSE-SDE scenario [21,16], except for its hybrid 3D-4D content [9].

This last brings us to the spectroscopy implications of 3D-4D-BSE vis-a-vis other regular 4D BSE models [21,16,34,36]. Thus NJL-Faddeev [34] supports one non-trivial bound state - the nucleon (see Sec.5), in the absence of a confinement mechanism, although a separable qq interaction *per se* does not rule out excited states at the 3-body level. The solitonic baryon models [29,35] also have similar predictions. In contrast, 4D BSE’s [16] with their *local* 4D interaction kernels, have in principle the capacity to predict L -excited states as well, except that their actual applications to $q\bar{q}$ mesons have so far not gone beyond the ground $L = 0$ states, albeit for many flavour/spin combinations [16], while the qqq spectroscopy of such BSE-SDE models [16] is not yet in sight. In this regard, the predictions of such models [16] on genuinely L -excited spectra would be of considerable interest for resolving the old issue of $O(4)$ -like spectra found within a Wick-rotated BSE formalism under harmonic confinement [13].

In contrast, the 3D-4D-BSE formalism [9,33] seems capable of encompassing both 3D spectroscopy [31-32] and 4D matrix elements within a common dynamical framework. The only cost factor is the information gap in the reconstruction of the 4D vertex in terms of 3D ingredients; the 1D δ -function in the baryon- qqq vertex function, eq.(4.10), fills up the gap, and is accounted for in detail in sec.(5.4). Another positive aspect of the 3D-4D formalism lies in the generality and flexibility of its 3D kernel structure, since the 3D-4D interconnection [9,33] does *not* depend on a precise knowledge of its functional form. And with its dynamics already rooted in spectroscopy [31-32], it has so far given parameter-free predictions on magnetic moments [42] as well as the $n - p$ mass difference [43], the latter being 1.28 MeV (vs. expt 1.29).

Yet another type of approach to the qqq problem, as available in the literature, concerns parametric representations attuned to QCD-sum rules [44], effective Lagrangians for hadronic transitions to “constituent” quarks, with ad hoc assumptions on the hadron- qqq form factor [45], similar (parametric) ansatze for the hadron- quark-diquark form factor [46]; or more often simply direct gaussian parametrizations for the qqq wave functions as the starting point of the investigation [47]. Such approaches are often quite effective for the investigations of some well-defined sectors of hadron physics with quark degrees of freedom, but are in general much less predictive than dynamical-equation-based methods like NJL- Faddeev [34] or 3D-4D-BSE [33], when extended beyond their immediate

domains of applicability.

6.2 Summary and Conclusion

To summarise, this work arose out of the need for a formal demonstration [33] of a semi-intuitive ansatz [37] for the reconstruction of the 4D baryon- qqq vertex function in terms of 3D ingredients, as in an earlier work for the 2-quark case case [9], under conditions of a covariant 3D support to the pairwise BS kernel. This has been viewed as a self-contained mathematical problem in its own right for an otherwise general 3D kernel, in the hope that the rich potential of such approaches as the 3D-4D-BSE formalism to access both spectroscopy and quark-loop amplitudes, under a common umbrella for 2- and 3-quark hadrons, will prove useful to others wishing to explore similar variants of the Transversality Principle [11]. The unfinished task in the 3D-4D programme has now been completed via Green's function techniques for both the 2- and 3- body problems, with a prior calibration to the 2-body case. In the process, the loss of Hilbert space information involved in the 4D reconstruction has been made up with the help of certain 1D δ -functions signifying on-shell propagation of the spectator #3 between two successive interactions of the 12-pair, but which in no way violate connectivity [3,4]. The central result, eq.(4.10), for the reconstructed 4D vertex function [33], has been fully accounted for through a detailed comparison with the structure of the NJL-Faddeev model [34].

The entire approach in the programme has been conceived with due emphasis on the need to include the spectroscopy sector as an integral part of any 'dynamical equation based' approach. This perspective is hardly new, having been set more than 25 years ago by none other than Feynman et al [48], but can certainly stand a reiteration.

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