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Multiplicity Formulae for Discrete Series

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Introduction

In a series of papers [20, 21] and [22], Schmid obtained several important results on the discrete series for semisimple Lie groups. The purpose of this paper is to prove Schmid's results by somewhat different methods and to relax as much as possible the restriction imposed on the regularity of the parameters of discrete classes. Though the basic line is similar to Schmid's, the main difference is that our methods do not rely upon complex analysis on the non-compact flag manifold G/T as developed in [20] and [21]. Rather, we just rely on some elementary differential calculus on the symmetric space G/K. This difference gives rise to less restrictive assumptions since the results on the vanishing of L^2-cohomologies in the symmetric space situation seem to be sharper, so far. Our work also leads to some interesting results concerning the multiplicity formula of discrete classes in $L^2(\Gamma \backslash G)$. In our development, we shall give an alternative proof of the key fact (Theorem 1, § 4) which is obtained in [20] using the complex analysis on G/T. Our proof, given in § 5, will be carried out directly in the symmetric space situation, and some standard theory of sheaf cohomology centering the Borel-Weil-Bott theorem on the compact flag manifold K/T will be used as in [20]. In this proof, our methods seem to be quite elementary.

We now give a more precise description of the contents of the paper. Let G be a non-compact real semisimple Lie group with discrete series $\mathscr{E}_2 \neq \phi$. Assume, for simplicity throughout the paper, that G is a connected real form of a simply connected complex semisimple Lie group $G^{\mathbb{C}}$. Harish-Chandra [7] showed that there exists a compact Cartan subgroup T of G and that, if one denotes by $\hat{T}'$ the set of regular characters of T, there exists a distinguished surjection $\omega\colon \hat{T}' \to \mathscr{E}_2$. We fix T and a maximal compact subgroup K containing T once and for all. Let $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{t}^{\mathbb{C}}$ denote the complexifications of the Lie algebras $\mathfrak{g}$, $\mathfrak{t}$ of G, T respectively. In considering the discrete class $\omega(\Lambda)$ for a given regular character $\Lambda \in \hat{T}'$, we always choose a positive root system P for the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ such that Λ is regular dominant with respect to P, i.e., $P = \{\alpha; (\Lambda, \alpha) > 0\}$. Here as

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usual, the character group $\hat{T}$ of T is identified with a lattice in the dual space of $\mathfrak{t}^{\mathbb{C}}$, which is equipped with the inner product $(\ ,\ )$ induced by the Killing form of $\mathfrak{g}$. We denote by P_k (resp. P_n) the set of positive compact (resp. non-compact) roots. For a set $Q \subset P$, we write $\langle Q \rangle = \sum_{\alpha \in Q} \alpha$, and put $\rho = \frac{1}{2}\langle P \rangle$, $\rho_k = \frac{1}{2}\langle P_k \rangle$ and $\rho_n = \frac{1}{2}\langle P_n \rangle$.

The central results may be stated as follows.

Theorem. *For a regular character Λ, choose a positive root system P as above and put $\lambda = \Lambda - \rho$. Assume that*

(i) $(\lambda, \alpha) > 0$ *for every* $\alpha \in P_n$, *and*

(ii) $(\lambda, \alpha) \geqq \max_{Q \subset P_n} (\rho_n - \langle Q \rangle, \alpha)$ *for every* $\alpha \in P_k$.

Then

(I) *(Realization). The discrete class $\omega(\Lambda) = \omega(\lambda + \rho) \in \mathscr{E}_2$ is realized by the left regular representation on the Hilbert space consisting of all $V_{\lambda + 2\rho_n}$-valued square-integrable functions f on G such that $f(gk) = k^{-1} f(g)$ $(g \in G, k \in K)$ and $v(\Omega) f = (|\Lambda|^2 - |\rho|^2) f$. In the above, $V_{\lambda + 2\rho_n}$ denotes the irreducible K-module with highest weight $\lambda + 2\rho_n$ and $v(\Omega)$ denotes the ordinary action of the Casimir operator Ω of G.*

(II) *(Lowest K-type theorem). Let π be any irreducible unitary representation of G, and consider the restriction $\pi|K$ to the maximal compact subgroup K. Suppose that $\pi|K$ contains the irreducible representation of K with highest weight $\lambda + 2\rho_n$ but does not contain those with highest weight $\lambda + 2\rho_n - \alpha$ for any $\alpha \in P_n$. Then π belongs to $\omega(\lambda + \rho)$ and the multiplicity of the representation of K with highest weight $\lambda + 2\rho_n$ in $\pi|K$ is one.*

(III) *(One part of Blattner's conjecture). The restriction to K of an irreducible representation belonging to $\omega(\lambda + \rho)$ contains the irreducible representation of K with highest weight μ with multiplicity not greater than $b_\lambda(\mu)$ where $b_\lambda(\mu)$ is the integer defined for λ, μ by Blattner* (see § 4).

(IV) *(Dimension of spaces of automorphic forms). For a discrete subgroup Γ of G such that $\Gamma \backslash G$ is compact, the multiplicity $N_{\omega(\lambda+\rho)}(\Gamma)$ of $\omega(\lambda + \rho)$ in the right regular representation of G on $L^2(\Gamma \backslash G)$ can be explicitly computed (Theorem 4, § 8). If Γ has no elliptic elements other than the identity, then the formula is*

$$N_{\omega(\lambda+\rho)}(\Gamma) = d_{\omega(\lambda+\rho)}\, v(\Gamma \backslash G),$$

where $d_{\omega(\lambda+\rho)}$ denotes the formal degree of $\omega(\lambda + \rho)$ and $v(\Gamma \backslash G)$ is the volume of the fundamental domain $\Gamma \backslash G$.

It should be noticed that the assumption (ii) in the theorem can be replaced by the following two more complicated but weaker ones:

(ii)^a $(\lambda + \rho, \alpha) \geqq \max_{Q \subset P_n} (\rho_n - \langle Q \rangle, \alpha)$ *for every* $\alpha \in P_k$ *and* (ii)[b] *for every $Q \subset P_n$ such that $\lambda + 2\rho_n - \langle Q \rangle$ is dominant with respect to P_k, $\lambda + \rho_n - \langle Q \rangle$ is also dominant with respect to P_k.*

Furthermore, we shall see that if the positive root system P satisfies a certain condition ("admissibility" defined in the beginning part of § 9), the assumption (ii)[a] can be removed (Propisition 9.1, § 9).

Under more restrictive assumptions on λ, (I) has been shown in [9, Theorem 2]; (II) in [21, Lemma 9]; (III) in [22, Theorem 2]; (IV) in [21, Theorems 2 and 3]. Further, there have been several other works [16, 18, 21] and [22] on realization of discrete series. In [18], $\omega(\lambda+\rho)$ has been realized on the space of harmonic spinors only under the assumption (i).

For a moment, we concentrate on (IV) which gives us some new information on the multiplicity of certain discrete series in $L^2(\Gamma \backslash G)$. In the case of $G = SL(2, \mathbb{R})$, the following is classically known. If $\hat{T}'$ is identified with the set of non-zero integers in a standard way, $N_{\omega(n)}(\Gamma)$ $(n \in \mathbb{Z} - \{0\})$ equals the dimension of the space of automorphic forms of weight $|n|+1$, which turns out to be $|n|(g-1)$ for $|n| \neq 1$, or g for $|n|=1$, when Γ has no elliptic elements other than the identity. Here g denotes the genus of the compact Riemann surface with fundamental group Γ. From the viewpoint of representation theory, $\omega(n)$ is an "integrable" discrete class in the sense of [6, VI] if and only if $|n| \neq 1$. Since $|n|(g-1) = d_{\omega(n)}\, v(\Gamma \backslash G)$, one sees, in this case, that the formula in (IV) holds if and only if $\omega(n)$ is integrable. In general, by means of the Selberg trace formula with some results of Harish-Chandra [7], Langlands showed in [13, 14] that if $\omega(\lambda+\rho)$ is integrable, then the formula in (IV) is valid.

However, as was noticed in [10, § 4], even for the "holomorphic" discrete series such an "integrability" condition has nothing to do with the validity of the above type of simple formulae (see also [4]). Our results indicate that the same phenomenon occurs in general, since one knows at least a necessary condition in order that $\omega(\lambda+\rho)$ is integrable by the result of Trombi and Varadarajan [23]. In fact, though the conditions (i) and (ii) may not cover all integrable classes, one can see that in many cases the formulae in (IV) seem to be valid for infinitely many non-integrable discrete classes (for more details, see Remark 2 at the end of § 8). Actually, for $G = SU(m, 1)$ or $\mathrm{Spin}(2m, 1)$ $\big(=$the two-sheeted covering of the identity component of $SO(2m, 1)\big)$, we shall see in § 9 that the assumption (ii) can be removed and, if $m \neq 1$, then every Weyl chamber contains infinitely many Λ such that Theorem is true for $\omega(\Lambda) \in \mathscr{E}_2$ though $\omega(\Lambda)$ is not integrable (Proposition 9.2, § 9).

In concluding the remarks, we understand that the method of the Selberg trace formula may be still effective to get information about the multiplicity of non-integrable discrete classes (see [4]). For this, there is an unpublished work of Langlands for $SL(2, \mathbb{R})$ and recently Sally, Jr. has obtained some results for the covering group of $SL(2, \mathbb{R})$[1].

We finally sketch the outline of the proofs, introducing what will be discussed in each section. Our first aim is to construct a certain elliptic complex $\mathbb{E}_{\lambda+2\rho_n}$:

$$0 \longrightarrow C^\infty(E_{V_{\lambda+2\rho_n}}) \xrightarrow{\mathscr{D}} C^\infty(E_{V^1_{\lambda+2\rho_n}}) \xrightarrow{\mathscr{D}} \cdots \xrightarrow{\mathscr{D}} C^\infty(E_{V^m_{\lambda+2\rho_n}}) \longrightarrow 0$$

over the symmetric space G/K whose first term $C^\infty(E_{V_{\lambda+2\rho_n}})$ is the space of C^∞ sections of the vector bundle over G/K associated to the irreducible K-module $V_{\lambda+2\rho_n}$ with highest weight $\lambda+2\rho_n$. This $\mathbb{E}_{\lambda+2\rho_n}$ should have the following property: the elliptic operator $\mathscr{D}+\mathscr{D}^*$ reduces to the Dirac operator considered in [18] where $\mathscr{D}^*$ is the formal adjoint of $\mathscr{D}$ under a suitable metric on each vector bundle (Lemma 3.3, § 3). For this, we shall introduce several facts about spinors in § 2. We note that observations similar to some of those in § 2 were made independently by Ozeki. The elliptic complex $\mathbb{E}_{\lambda+2\rho_n}$ turns out to coincide with that obtained in [8] under a condition like (ii)^a stated after theorem (Lemma 3.4). This fact will be used for the proof of Theorem 1 in § 4.

Secondly, we shall need the K-types of the G-module $H^0(\mathbb{E}_{\lambda+2\rho_n})$ which is defined as the kernel of the first $\mathscr{D}$: $C^\infty(E_{V_{\lambda+2\rho_n}}) \to C^\infty(E_{V^1_{\lambda+2\rho_n}})$ (Theorem 1, § 4). This has already been obtained by Schmid in [20] and will play a key role afterward. Roughly speaking, the K-types of $H^0(\mathbb{E}_{\lambda+2\rho_n})$ satisfy Blattner's conjecture as in Theorem (III) under the assumption (ii)^a. We shall give a different proof of Theorem 1 in § 5. Our method is to look at the first terms of Taylor expansions of sections in $H^0(\mathbb{E}_{\lambda+2\rho_n})$. For this, we develop the elementary framework of jets in § 1, and, in § 4, recall some lemmas of Schmid in [20] on K-modules constructed out of sheaf cohomologies over the compact flag manifold K/T. Since Schmid's proofs are short, we shall reproduce them for the sake of completeness.

Thirdly, considering the L^2-cohomologies $H^q_{(2)}(\Gamma; \mathbb{E}_{\lambda+2\rho_n})$ of $\mathbb{E}_{\lambda+2\rho_n}$ over $\Gamma \backslash G/K$ for a discrete subgroup Γ of G, we prove the vanishing theorem

$$H^q_{(2)}(\Gamma; \mathbb{E}_{\lambda+2\rho_n})=0 \quad \text{for } q>0$$

under the assumptions (i) and (ii)^b (Theorem 2, § 6). The idea is similar to that in [18]. In view of the alternating sum formula of Narasimhan and Okamoto [16], the unitary representation on the space $H^0_{(2)}(\{1\}; \mathbb{E}_{\lambda+2\rho_n})$ then belongs to the discrete class $\omega(\lambda+\rho)$ (Theorem 3,

[1] We are grateful to Professor P. Sally who informed us of these results, and kindly gave valuable linguistic advice to one of us (Hotta) during the preparation of this manuscript.

§ 7). This proves Theorem (I), and, with the aid of Theorem 1, (II) and (III) are completed in § 7 by arguments similar to those in [21, 22].

Also, those arguments verify the equality

$$N_{\omega(\lambda+\rho)}(\Gamma)=\dim H^0_{(2)}(\Gamma; \mathbb{E}_{\lambda+2\rho_n}) \qquad \text{for } \Gamma$$

with compact quotient $\Gamma \backslash G$ (Lemma 8.1, § 8), where the right-hand side can actually be computed by the index theorem of Atiyah and Singer as in [10] combined with the vanishing Theorem 2 (Theorem 4, § 8). We will thus complete the proofs of (I) ~ (IV).

In § 9, we illustrate a few examples for which better information is obtained.

Before starting the discussion, for the convenience of the reader, we will make notational remarks most of which will be repeated when they first appear.

§ 0. Notations

We first collect symbols some of which will appear without definitions throughout the paper. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers. For a set Q, $|Q|$ denotes the number of elements in Q; when Q is a finite set of vectors, we write

$$\langle Q\rangle = \sum_{\alpha\in Q}\alpha,$$

as already introduced.

When E is a C^∞ vector bundle over a C^∞ manifolds X, $C^\infty(E)$ (resp. $C_0^\infty(E)$) denotes the space of C^∞ sections of E (resp. with compact support). Usually, $\mathbb{1}_X$ will denote the trivial bundle over X with fibres $\mathbb{C}$; hence the space of complex valued C^∞ functions on X will be denoted by $C^\infty(\mathbb{1}_X)$. When E is a complex vector bundle equipped with hermitian metrics on the fibres and X has a volume element, $L^2(E)$ will denote the Hilbert space consisting of square-integrable sections under those metrics. (We, however, preserve the usual convention and denote by $L^2(\Gamma \backslash G)$ the space of square-integrable functions on the manifold $\Gamma \backslash G$.)

When a vector space V is defined over $\mathbb{R}$, $V^{\mathbb{C}}$ denotes the complexification $V\otimes_{\mathbb{R}}\mathbb{C}$ of V. In this case, if a complex vector space W is given, $V\otimes W$ will denote the tensor product over $\mathbb{R}$, but will often be regarded as a vector space over $\mathbb{C}$ through the identification $V\otimes_{\mathbb{R}} W = V^{\mathbb{C}}\otimes_{\mathbb{C}} W$. These conventions will be adopted also for vector bundles.

For the convenience of the reader, we collect here the basic notations which occur throughout the paper. Some of these will be repeated when they first appear. Fix $G\supset K\supset T$ as in the introduction, and let $\mathfrak{g}\supset\mathfrak{k}\supset\mathfrak{t}$ be the corresponding Lie algebras. Except in § 2, we assume that the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is of non-compact type. The letter X will stand for the symmetric space G/K except in § 1. We denote by W the Weyl group for the pair (G, T), i.e., the quotient group of the normalizer of T in G modulo T. Note that W is then isomorphic to the Weyl group for (K, T) or $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. We denote by $\mathfrak{p}$ the subspace of $\mathfrak{g}$ in the Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, and often consider $\mathfrak{p}$ as a K-module through the adjoint action. The root system for $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ will be denoted by Σ (which also stands for the summation). For a root $\alpha\in\Sigma$, $\mathfrak{g}^{\alpha}$ denotes the one-dimensional eigenspace of α in $\mathfrak{g}^{\mathbb{C}}$. When a positive root system P is fixed, we set

$$P_k=\{\alpha\in P;\ \mathfrak{g}^{\alpha}\subset\mathfrak{k}^{\mathbb{C}}\},$$

and

$$P_n=\{\alpha\in P;\ \mathfrak{g}^{\alpha}\subset\mathfrak{p}^{\mathbb{C}}\},$$

and call roots in P_k (resp. P_n) positive compact (resp. non-compact). We put

$$\rho=\tfrac{1}{2}\langle P\rangle,\quad \rho_k=\tfrac{1}{2}\langle P_k\rangle \quad\text{and}\quad \rho_n=\tfrac{1}{2}\langle P_n\rangle \tag{0.1}$$

as in the introduction; $m=|P_n|\,(=\tfrac{1}{2}\dim X)$ and $s=|P_k|$ (the letter s may be used also for a section of vector bundles).

On the dual space $\mathrm{Hom}(\sqrt{-1}\,\mathfrak{t}, \mathbb{R})$ of $\sqrt{-1}\,\mathfrak{t}$, one can define the inner product (,) in a usual way by the Killing form of $\mathfrak{g}^{\mathbb{C}}$. We then set

$$\begin{aligned}\mathscr{F}&=\{\mu\in\mathrm{Hom}(\sqrt{-1}\,\mathfrak{t}, \mathbb{R});\ 2(\mu,\alpha)/(\alpha,\alpha)\in\mathbb{Z}\ (\alpha\in\Sigma)\},\\ \mathscr{F}_0&=\{\mu\in\mathscr{F};\ (\mu,\alpha)\geqq 0\ (\alpha\in P_k)\}\end{aligned} \tag{0.2}$$

and

$$\mathscr{F}_0^k=\{\mu\in\mathrm{Hom}(\sqrt{-1}\,\mathfrak{t}, \mathbb{R});\ 2(\mu,\alpha)/(\alpha,\alpha)\in\mathbb{Z}\text{ and }\geqq 0\ (\alpha\in P_k)\}.$$

By our assumption in the introduction, $\mathscr{F}$ is isomorphic to the character group $\hat{T}$ of T by the map which takes $\mu\in\mathscr{F}$ to e^{μ}. For $\mu\in\mathscr{F}_0$, V_μ stands for the irreducible K-module with highest weight μ, and for $\mu\in\mathscr{F}_0^k$, V_μ stands for the irreducible $\mathfrak{k}$-module with highest weight μ.

In general, when two K-modules V, W are given, $\mathrm{Hom}_K(V, W)$ denotes the subspaces of $\mathrm{Hom}(V, W)$ consisting of linear maps which commute with the K-actions, and we set $(V\colon W)=\dim\mathrm{Hom}_K(V, W)$, the intertwining number. For the compact group K, one can consider the complex reductive group $K^{\mathbb{C}}$ where K is a real form of $K^{\mathbb{C}}$. Then for a finite dimensional K-module V over $\mathbb{C}$, we shall often regard V as a holomorphic $K^{\mathbb{C}}$-module by extending naturally. Adopting the convention stated before, for a K-module W defined over $\mathbb{R}$, we may consider $W\otimes V=W^{\mathbb{C}}\otimes_{\mathbb{C}}V$ as a holomorphic $K^{\mathbb{C}}$-module.

For a finite dimensional K-module V, E_V will denote the vector bundle over $X=G/K$ associated to V. We shall often identify $C^\infty(E_V)$ with the space of V-valued C^∞ functions f on G such that $f(gk)=k^{-1}f(g)$ $(g\in G,\ k\in K)$. For $C_0^\infty(E_V)$ and $L^2(E_V)$, we adopt similar identifications.

From § 2 until § 5, we shall consider the Borel subgroup B of $K^{\mathbb{C}}$ whose Lie algebra is

$$\mathfrak{b}=\mathfrak{t}^{\mathbb{C}}\oplus\sum_{\alpha\in P_k}\mathfrak{g}^{\alpha}.$$

For a linear form $\mu \in \mathrm{Hom}(\mathfrak{t}^{\mathbb{C}}, \mathbb{C})$, we denote by l_μ the one-dimensional $\mathfrak{b}$-module obtained by extending trivially on the nilpotent radical $\sum_{\alpha \in P_k} \mathfrak{g}^\alpha$. When $\mu \in \mathscr{F}$, l_μ also stands for the B-module given by it. Since $\mathfrak{p}^{\mathbb{C}}$ has the structure of a $K^{\mathbb{C}}$-module, $\mathfrak{p}^{\mathbb{C}}$ also stands for the B-module obtained by restriction. Then $\mathfrak{p}_+ = \sum_{\alpha \in P_n} \mathfrak{g}^\alpha$ is a B-submodule of $\mathfrak{p}^{\mathbb{C}}$. For the subspace $\mathfrak{p}_- = \sum_{\alpha \in P_n} \mathfrak{g}^{-\alpha}$, we shall regard $\mathfrak{p}_-$ as the quotient B-module $\mathfrak{p}_- = \mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_+$. For a holomorphic B-module $\mathfrak{m}$, $H^i(\mathfrak{m})$ stands for the i-th cohomology space with coefficients in the sheaf of germs of holomorphic sections of the vector bundle over $K^{\mathbb{C}}/B$ associated to $\mathfrak{m}$.

§ 1. Elementary Differential Calculus

For later use, we fix some generalities about differential operators on vector bundles. For the standard language of jets of vector bundles, more detailed accounts are found in [17, Chap. IV].

Let X be a paracompact C^∞ manifold, E a C^∞ vector bundle over X. For a point $o \in X$, we consider the filtration of the space $C^\infty(E)$ of C^∞ sections of E,

$$C^\infty(E) = I_o^0(E) \supset I_o^1(E) \supset \cdots \supset I_o^l(E) \supset \cdots, \tag{1.1}$$

where $I_o^l(E)$ is the space of C^∞ sections of E whose derivatives up to the $(l-1)$-th order vanish at o $(l = 0, 1, 2, \ldots)$. More precisely, take a small open neighborhood $\mathscr{U}$ of o with coordinate system $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ ($n = \dim X$, and $x(o) = o$ the origin of $\mathbb{R}^n$) such that the restriction $E|\mathscr{U}$ of E to $\mathscr{U}$ can be trivialized. Choosing a trivialization $E|\mathscr{U} \xrightarrow{\sim} \mathscr{U} \times E_o$, where E_o is the fibre of E at o, one can regard a section $s \in C^\infty(E|\mathscr{U})$ as an E_o-valued function $\tilde{s} \in C^\infty(\mathbb{1}_{\mathscr{U}}) \otimes E_o$. We adopt the multi-index notation:

$$\alpha = (\alpha_1, \ldots, \alpha_n) \qquad (\alpha_i \geqq 0 \text{ and } \in \mathbb{Z}),$$

$$|\alpha| = \sum_{i=1}^n \alpha_i,$$

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

The property

$$\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} \tilde{s}\right)(o) = 0 \qquad \text{for } |\alpha| \leqq l-1,\ s \in C^\infty(E|\mathscr{U}),$$

is independent of choice of coordinate system and trivialization on $\mathscr{U}$, and $I_o^l(E)$ consists of $s \in C^\infty(E)$ whose restriction to $\mathscr{U}$ satisfies this property.

By definition, the space of l-jets of E at o is $J_o^l(E)=C^\infty(E)/I_o^{l+1}(E)$, which turns out to be a finite dimensional complex vector space. The natural projection $C^\infty(E)$ onto $J_o^l(E)$ is denoted by j_o^l. Let T_o be the tangent space of X at o and T_o^* the cotangent space at o. Under the coordinate system chosen above, denoting by $dx=(dx_1, \ldots, dx_n)$ the dual basis in T_o^* to $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, one has the map

$$I_o^l(E) \ni s \mapsto \sum_{|\alpha|=l} \frac{1}{\alpha!}(dx)^\alpha \otimes \left(\frac{\partial^l}{\partial x^\alpha}\tilde{s}\right)(o) \in S^l(T_o^*)\otimes E_o, \tag{1.2}$$

where $\alpha!=\alpha_1!\ldots\alpha_n!$ and $(dx)^\alpha=dx_1^{\alpha_1}\ldots dx_n^{\alpha_n}\in S^l(T_o^*)$ for $\alpha=(\alpha_1, \ldots, \alpha_n)$. Here $S^l(T_o^*)$ denotes the l-th symmetric power of T_o^*, which will be also regarded as the space of polynomial functions of homogeneous degree l on T_o. By this map, one then has an exact sequence

$$0 \to I_o^{l+1}(E) \to I_o^l(E) \to S^l(T_o^*)\otimes E_o \to 0 \tag{1.3}$$

([17], Chap. IV, Lemma 3]). On the other hand, by the canonical map $j_o^l\colon I_o^l(E) \to J_o^l(E)$, one has an inclusion

$$I_o^l(E)/I_o^{l+1}(E) \hookrightarrow J_o^l(E)$$

since $\operatorname{Ker} j_o^l = I_o^{l+1}(E)$. Composing this map with that obtained by the exact sequence (1.3), one has a natural inclusion $S^l(T_o^*)\otimes E_o \hookrightarrow J_o^l(E)$. It can be seen that

$$0 \to S^l(T_o^*)\otimes E_o \to J_o^l(E) \to J_o^{l-1}(E) \to 0$$

is exact (the jet exact sequence) where $J_o^l(E) \to J_o^{l-1}(E)$ is the projection obtained by the natural inclusion $I_o^{l+1}(E) \hookrightarrow I_o^l(E)$. Making the l-jet bundle $J^l(E)=\bigcup_{o\in X} J_o^l(E)$, one has the jet bundle exact sequence

$$0 \to S^l(T^*)\otimes E \to J^l(E) \to J^{l-1}(E) \to 0 \tag{1.4}$$

where T^* is the cotangent bundle of X ([17], Chap. IV, Theorem 1]). For $J^l(E)$, one also has the l-jet extension map

$$j^l\colon\ C^\infty(E) \to C^\infty(J^l(E))$$

obtained from j_o^l where o runs through points of X.

Let F be another C^∞ vector bundle over X. An l-th order differential operator

$$\mathscr{D}\colon\ C^\infty(E) \to C^\infty(F)$$

is by definition a linear map given by the composite $\mathscr{D}=\bar{A}\circ j^l$ for some linear map

$$\bar{A}\colon\ C^\infty(J^l(E)) \to C^\infty(F)$$

induced from a vector bundle map

$$A\colon\ J^l(E) \to F.$$

When A is composed with the map $S^l(T^*)\otimes E \to J^l(E)$ in (1.4), the vector bundle map

$$\sigma = \sigma(\mathscr{D}):\; S^l(T^*)\otimes E \to F$$

is then called the symbol of $\mathscr{D}$.

Now assume we have a first-order differential operator

$$\mathscr{D}:\; C^\infty(E) \to C^\infty(F).$$

Fix a point $o \in X$ once and for all, and consider the filtrations (1.1) of $C^\infty(E)$ and $C^\infty(F)$ by $I_o^l(E)$ and $I_o^l(F)$. Since $\mathscr{D}$ is of first-order, clearly

$$\mathscr{D}\big(I_o^l(E)\big) \subset I_o^{l-1}(F);$$

hence one has a map

$$I_o^l(E)/I_o^{l+1}(E) \to I_o^{l-1}(F)/I_o^l(F) \tag{1.5}$$

for each $l = 0, 1, 2, \dots$ $\big(I_o^{-1}(F) = I_o^0(F)\big)$. Let

$$\sigma_o:\; T_o^* \otimes E_o \to F_o$$

be the symbol of $\mathscr{D}$ at o. Denoting by $\mathbb{1}_{T_o}$ the trivial bundle over the tangent space T_o at o, one has an exterior differential

$$d:\; C^\infty(\mathbb{1}_{T_o}) \to C^\infty(\mathbb{1}_{T_o}) \otimes T_o^*$$

where

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes dx_i \qquad \text{for } f \in C^\infty(\mathbb{1}_{T_o});$$

in particular,

$$d:\; S^l(T_o^*) \to S^{l-1}(T_o^*) \otimes T_o^*$$

where elements of $S^l(T_o^*)$ are regarded as polynomial functions on T_o. Composing the maps

$$S^l(T_o^*)\otimes E_o \xrightarrow{d\otimes 1} S^{l-1}(T_o^*)\otimes T_o^* \otimes E_o \xrightarrow{1\otimes\sigma_o} S^{l-1}(T_o^*)\otimes F_o,$$

we have the map

$$\mathscr{D}^{(l)}:\; S^l(T_o^*)\otimes E_o \to S^{l-1}(T_o^*)\otimes F_o,$$

which will be called the *polynomialization of $\mathscr{D}$ at o.*

Lemma 1.1. *The diagram*

$$\begin{array}{ccc} I_o^l(E)/I_o^{l+1}(E) & \xrightarrow{\ \sim\ } & S^l(T_o^*)\otimes E_o \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathscr{D}^{(l)}} \\ I_o^{l-1}(F)/I_o^l(F) & \xrightarrow{\ \sim\ } & S^{l-1}(T_o^*)\otimes F_o \end{array}$$

commutes for each $l=0, 1, 2, \ldots$, where the two horizontal isomorphisms are those induced from the exact sequences (1.3), *and the left-hand side vertical map is* (1.5) *induced from $\mathscr{D}$.*

Proof. Take a small neighborhood $\mathscr{U}$ of o in X with coordinate $x=(x_1, \ldots, x_n)$ $(x(o)=o)$ and trivializations of $E|\mathscr{U}$ and $F|\mathscr{U}$. Let

$$\tilde{\mathscr{D}} = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} + b(x)$$

be a local expression of $\mathscr{D}$ on $\mathscr{U}$ under them, where

$$a_i(x), b(x) \in C^\infty(\mathbb{1}_{\mathscr{U}}) \otimes \operatorname{Hom}(E_o, F_o).$$

Then the symbol σ_o at o is expressed as

$$\sigma_o(\xi \otimes e) = \sum_{i=1}^{n} \xi_i a_i(o) e$$

for

$$\xi = \sum_{i=1}^{n} \xi_i dx_i \in T_o^* \qquad (\xi_i \in \mathbb{R}),\ e \in E_o.$$

Under the map $I_o^l(E)/I_o^{l+1}(E) \xrightarrow{\sim} S^l(T_o^*) \otimes E_o$, $s \in I_o^l(E)$ is mapped by (1.2) to

$$\sum_{|\alpha|=l} \frac{1}{\alpha!} (dx)^\alpha \otimes \left(\frac{\partial^l}{\partial x^\alpha} \tilde{s}\right)(o),$$

which goes to

$$\sum_{i=1}^{n} \sum_{|\alpha|=l} \frac{\alpha_i}{\alpha!} (dx)^{\alpha(i)} \otimes a_i(o) \left(\left(\frac{\partial^l}{\partial x^\alpha} \tilde{s}\right)(o)\right)$$

under $S^l(T_o^*) \otimes E_o \to S^{l-1}(T_o^*) \otimes F_o$, where $\alpha(i) = (\alpha_1, \ldots, a_i - 1, \ldots, \alpha_n)$. On the other hand, under the map

$$I_o^l(E) \xrightarrow{\mathscr{D}} I_o^{l-1}(F) \to S^{l-1}(T_o^*) \otimes F_o,$$

s goes to

$$\sum_{|\beta|=l-1} \sum_{i=1}^{n} \frac{1}{\beta!} (dx)^\beta \otimes a_i(o) \left(\left(\frac{\partial^l}{\partial x^\beta \partial x_i} \tilde{s}\right)(o)\right),$$

which proves the lemma. q.e.d.

Among differential operators we shall later meet, specific first-order operators of the following type will often appear. Consider the jet bundle exact sequence (1.4) for $l=1$

$$0 \to T^* \otimes E \to J^1(E) \to E \to 0.$$

A splitting of the sequence gives a connection on E, which induces the covariant differential operator

$$\nabla\colon\ C^\infty(E) \to C^\infty(T^* \otimes E),$$

by composing

$$C^\infty(E) \xrightarrow{j^1} C^\infty(J^1(E)) \xrightarrow{\text{connection}} C^\infty(T^* \otimes E).$$

When E has a connection and a bundle map σ: $T^* \otimes E \to F$ is given for F, one has a first-order differential operator

$$\mathscr{D}\colon\ C^\infty(E) \to C^\infty(F)$$

whose symbol is the given σ, by $\mathscr{D} = \bar{\sigma} \circ \nabla$, where $\bar{\sigma}$: $C^\infty(T^* \otimes E) \to C^\infty(F)$ is induced from the bundle map σ. In this situation, we shall call $\mathscr{D}$ the *differential operator associated to the symbol* σ.

§ 2. Spinors Associated to Symmetric Pairs

In this section, we shall recall and develop several facts on spinors arising in the symmetric space situation.[2] For those results which appear without proofs, we recommend that the reader consult the references [1, 3, 11] and [18].

Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair; i.e., $\mathfrak{g}$ a real semisimple Lie algebra, $\mathfrak{k}$ a subalgebra corresponding to a stabilizer in a group of isometries of a symmetric space whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition for $(\mathfrak{g}, \mathfrak{k})$; i.e., $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. We henceforth assume that rank $\mathfrak{g}$ = rank $\mathfrak{k}$; hence we have a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ in $\mathfrak{k}$. For the complexifications $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ we have a root system Σ. Denoting by $\mathfrak{g}^\alpha$ the one-dimensional eigenspace for a root $\alpha \in \Sigma$, we set

$$\Sigma_k = \{\alpha \in \Sigma,\ \mathfrak{g}^\alpha \subset \mathfrak{k}^{\mathbb{C}}\}$$

and

$$\Sigma_n = \{\alpha \in \Sigma;\ \mathfrak{g}^\alpha \subset \mathfrak{p}^{\mathbb{C}}\}$$

where $\mathfrak{p}^{\mathbb{C}}$ is the complexification of $\mathfrak{p}$. Then clearly $\Sigma = \Sigma_k \cup \Sigma_n$ (disjoint union). Fix a positive root system P in Σ once and for all, and put $P_k = P \cap \Sigma_k$ and $P_n = P \cap \Sigma_n$. Under this positive root system, we consider ρ, ρ_k and ρ_n as in (0.1) in § 0. Notice that $\rho \in \mathscr{F}$ and $(\rho, \alpha) > 0$ $(\alpha \in P)$; $\rho_k, \rho_n \in \mathscr{F}_0^k$ and $(\rho_k, \alpha) > 0$ $(\alpha \in P_k)$ in the notation (0.2) in § 0.

We have a non-degenerate bilinear form (,) on $\mathfrak{p}$ obtained by restricting the Killing form to $\mathfrak{p}$. We then have the Clifford algebra Cliff($\mathfrak{p}$) of $\mathfrak{p}$ with this form (,); i.e., the quotient algebra of the tensor algebra over $\mathfrak{p}$ modulo the two-sided ideal generated by all elements $x \otimes x + (x, x)\,1$ where $x \in \mathfrak{p}$. The complexification Cliff$(\mathfrak{p})^{\mathbb{C}}$ is regarded as the Clifford algebra of $\mathfrak{p}^{\mathbb{C}}$ with the complex-linear extension of (,). It is well-known that Cliff$(\mathfrak{p})^{\mathbb{C}}$ is a simple algebra central over $\mathbb{C}$; hence

[2] Results similar to some of those in this section have been obtained also by Ozeki independently.

isomorphic to a matric algebra of rank 2^m where $m=|P_n|=\frac{1}{2}\dim\mathfrak{p}$. Put $\mathfrak{p}_\pm=\sum_{\alpha\in P_n}\mathfrak{g}^{\pm\alpha}$. Then we have a direct sum $\mathfrak{p}^{\mathbb{C}}=\mathfrak{p}_+\oplus\mathfrak{p}_-$ as linear subspaces. Under the form (,) on $\mathfrak{p}^{\mathbb{C}}$, these two subspaces are totally isotropic. Letting C^P (resp. C^N) be the subalgebra generated by $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$), we therefore have algebra isomorphisms

$$C^P\simeq\bigwedge\mathfrak{p}_+\quad\text{and}\quad C^N\simeq\bigwedge\mathfrak{p}_-,\tag{2.1}$$

where $\bigwedge\mathfrak{p}_\pm$ are the exterior algebra generated by $\mathfrak{p}_\pm$. Let e_P be the non-zero element in C^P whose corresponding element in $\bigwedge\mathfrak{p}_+$ spans the top degree $\bigwedge^m\mathfrak{p}_+$ (unique up to scalar multiples). Then the left ideal L generated by e_P is minimal and

$$L=C^N\cdot e_P=\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}\cdot e_P\tag{2.2}$$

([3, II.2.2]). We observe that the map $C^N\ni x\mapsto x\cdot e_P\in L$ is a linear bijection. For an element $x\in\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}$, define $l_x\in\mathrm{End}\,L$ by $l_x y=xy$ $(y\in L)$. It can be seen that $\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}\ni x\mapsto l_x\in\mathrm{End}\,L$ is an algebra isomorphism.

The Lie algebra $\mathfrak{o}(\mathfrak{p})$ of the orthogonal group on $\mathfrak{p}$ with respect to (,) can be embedded in Cliff($\mathfrak{p}$) as a Lie subalgebra as follows. Let $\{x_i\}_{i=1}^{2m}$ be an orthonormal basis of $\mathfrak{p}$ and consider the subspace $\mathfrak{p}^{(2)}$ of Cliff($\mathfrak{p}$) spanned by elements $x_i x_j$ $(i<j)$. It is easy to see that $\mathfrak{p}^{(2)}$ is a Lie subalgebra of Cliff($\mathfrak{p}$) and for $x\in\mathfrak{p}\subset\mathrm{Cliff}(\mathfrak{p})$ and $y\in\mathfrak{p}^{(2)}$, $\varphi(y)x=yx-xy\in\mathfrak{p}$ in Cliff($\mathfrak{p}$). Here the map $\mathfrak{p}^{(2)}\ni y\mapsto\varphi(y)\in\mathrm{End}\,\mathfrak{p}$ gives an isomorphism from $\mathfrak{p}^{(2)}$ onto $\mathfrak{o}(\mathfrak{p})$. Hence one has the spin representation of $\mathfrak{o}(\mathfrak{p})$ on L.

We now consider the following filtration of L. Using the isomorphism (2.1), we define the subspace $C_q^N\simeq\bigwedge^q\mathfrak{p}_-$ for $0\leqq q\leqq m$. By (2.2), set

$$L^q=\sum_{i\leqq q}C_i^N\cdot e_P,$$

and

$$L^\pm=\sum_{(-1)^q=\pm1}C_q^N\cdot e_P.\tag{2.3}$$

We first notice that $L^m=L=L^+\oplus L^-$ where $L^\pm$ are known to be irreducible $\mathfrak{o}(\mathfrak{p})$-module (the half-spin modules). Since $\mathfrak{k}$ is a subalgebra of $\mathfrak{o}(\mathfrak{p})$, L and $L^\pm$ have $\mathfrak{k}$- (or $\mathfrak{k}^{\mathbb{C}}$-)module structures by restriction, which are also denoted by the same letters. It can be seen that the set of weights of the $\mathfrak{k}$-module $L^\pm$ is respectively

$$\{\rho_n-\langle Q\rangle;\ Q\subset P_n,\ (-1)^{|Q|}=\pm1\}.$$

Though L^q may not be a $\mathfrak{k}$-module, we do have the following result.

Lemma 2.1. *Let* $\mathfrak{b}$ *be the Borel subalgebra of* $\mathfrak{k}^{\mathbb{C}}$ *defined by*

$$\mathfrak{b}=\mathfrak{t}^{\mathbb{C}}\oplus\sum_{\alpha\in P_k}\mathfrak{g}^\alpha.$$

Then L^q *is a* $\mathfrak{b}$*-submodule for* $0\leqq q\leqq m$.

Proof. Consider $\mathfrak{b}$ as a Lie subalgebra of the complexification $\mathfrak{o}(\mathfrak{p})^{\mathbb{C}}$ which is isomorphic to $\mathfrak{p}^{(2)}\otimes\mathbb{C}$ in $\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}$. Embed $\mathfrak{p}_+$ in $\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}$. Then for $x\in\mathfrak{b}$,

$$[x, y]=xy-yx\in\mathfrak{p}_+ \quad \text{for any } y\in\mathfrak{p}_+. \tag{2.4}$$

By this, we first show that L^0 is stable under the $\mathfrak{b}$-action. In fact, it is easily seen by (2.4) that for $x\in\mathfrak{b}$, $xe_p=e_px+ce_p$ for some $c\in\mathbb{C}$; which implies $e_px\in L=\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}\cdot e_p$. On the other hand, $e_px\in e_p\cdot\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}$ which is also a minimal right ideal of $\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}$. Since the intersection of a minimal left ideal with a minimal right one is of dimension one ([3, III.1.1]), $e_p\,\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}\cap L=\mathbb{C}e_p=L^0$. Hence $e_px\in L^0$, which implies $xe_p\in L^0$.

For L^q, it can be seen that $x\cdot C_q^N\cdot e_p\subset L^q+\big(\sum_{i\leqq q} C_i^N\big)\cdot xe_p$ for $x\in\mathfrak{b}$ since $[x, y]\in\mathfrak{p}^{\mathbb{C}}$ for $y\in\mathfrak{p}^{\mathbb{C}}$. By the above argument, $xe_p\in\mathbb{C}e_p$. Hence $x\cdot C_q^N\cdot e_p\subset L^q$, which proves the lemma. q.e.d.

By the Clifford multiplication $\mathfrak{p}\otimes L\to L$, we mean the multiplication of elements in $\mathfrak{p}$ with those of L in the algebra $\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}$ via the natural embedding $\mathfrak{p}\subset\mathrm{Cliff}(\mathfrak{p})^{\mathbb{C}}$. By the definition, we easily see that

$$\mathfrak{p}\otimes L^q \text{ maps into } L^{q+1},$$

and (2.5)

$$L^{q+1}=L^q+\mathfrak{p}\cdot L^q$$

where $\mathfrak{p}\cdot L^q$ denotes the image of $\mathfrak{p}\otimes L$ in L and $L^q+\mathfrak{p}\cdot L^q$ denotes the subspace in L spanned by L^q and $\mathfrak{p}\cdot L^q$.

We now look into the $\mathfrak{b}$-module structures of those L^q. First we notice that the one dimensional $\mathfrak{b}$-module L^0 has a weight ρ_n. This can be directly checked by computation of the action of the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ using root vectors of $\mathfrak{p}^{\mathbb{C}}$. We denote by l_{ρ_n} this one-dimensional $\mathfrak{b}$-module with weight ρ_n.

Next $\mathfrak{p}_+\subset\mathfrak{p}^{\mathbb{C}}$ is, by the choice of $\mathfrak{b}$, a $\mathfrak{b}$-submodule of $\mathfrak{p}^{\mathbb{C}}$, and we regard $\mathfrak{p}_-$ as the quotient $\mathfrak{b}$-module of $\mathfrak{p}^{\mathbb{C}}$ modulo $\mathfrak{p}_+$, as introduced in §0; i.e., we have a $\mathfrak{b}$-module exact sequence

$$0\to\mathfrak{p}_+\to\mathfrak{p}^{\mathbb{C}}\to\mathfrak{p}_-\to 0.$$

Define the map

$$L^q\to\textstyle\bigwedge^q\mathfrak{p}_-\otimes l_{\rho_n}$$

by $L^q=\sum_{i\leqq q} C_i^N\cdot e_p\to C_q^N\cdot e_p\xrightarrow{\sim}\bigwedge^q\mathfrak{p}_-\otimes l_{\rho_n}$. This depends only upon the choice of the isomorphism $L^0=\mathbb{C}e_p\xrightarrow{\sim} l_{\rho_n}$ (up to scalar multiples).

Lemma 2.2. *In the above situation,*

$$0\to L^{q-1}\to L^q\to\textstyle\bigwedge^q\mathfrak{p}_-\otimes l_{\rho_n}\to 0$$

is a $\mathfrak{b}$*-module exact sequence.*

Proof. The exactness as linear spaces is clear. We shall show the map $L^q \to \bigwedge^q \mathfrak{p}_- \otimes l_{\rho_n}$ commutes with the $\mathfrak{b}$-action. Let $x \in \mathfrak{b}$ and $x_1, \ldots, x_q \in \mathfrak{p}_\pm$. Then the action of x on $x_1 \ldots x_q e_P \in L^q$ is given by

$$x(x_1 \ldots x_q e_P) = \sum_{i=1}^{q} x_1 \ldots [x, x_i] \ldots x_q e_P + x_1 \ldots x_q x e_P,$$

while the action of x on $x_1 \wedge \cdots \wedge x_q \otimes v \in \bigwedge^q \mathfrak{p}_- \otimes l_{\rho_n}$ is given by

$$x(x_1 \wedge \cdots \wedge x_q \otimes v) = \sum_{i=1}^{n} x_1 \wedge \cdots \wedge [x, x_i]^- \wedge \cdots \wedge x_q \otimes v + x_1 \wedge \cdots \wedge x_q \otimes x v,$$

where $[x, x_i]^-$ denotes the image of $[x, x_i] \in \mathfrak{p}^{\mathbb{C}}$ by the projection $\mathfrak{p}^{\mathbb{C}} \to \mathfrak{p}_-$ and v is the image of e_P under the isomorphism $L^0 \xrightarrow{\sim} l_{\rho_n}$. When we write $[x, x_i] = [x, x_i]^+ + [x, x_i]^-$ where $[x, x_i]^+ \in \mathfrak{p}_+$, then $x_1 \ldots [x, x_i]^+ \ldots x_q e_P \in L^{q-1}$. Hence

$$\sum_{i=1}^{q} x_1 \ldots [x, x_i] \ldots x_q e_P = \sum_{i=1}^{q} x_1 \ldots [x, x_i]^- \ldots x_q \mod L^{q-1}$$

which shows our assertion. q.e.d.

Lemma 2.3. *The diagram*

$$\begin{array}{ccc} \mathfrak{p} \otimes L^q & \longrightarrow & L^{q+1} \\ \downarrow & & \downarrow \\ \mathfrak{p} \otimes \bigwedge^q \mathfrak{p}_- \otimes l_{\rho_n} & \longrightarrow & \bigwedge^{q+1} \mathfrak{p}_- \otimes l_{\rho_n} \end{array}$$

commutes, where the two vertical maps are given by the maps in Lemma 2.2, *the upper horizontal one is the Clifford multiplication in* (2.5), *and the lower one is given by the exterior multiplication*

$$\mathfrak{p} \otimes \bigwedge^q \mathfrak{p}_- = \mathfrak{p}^{\mathbb{C}} \otimes \bigwedge^q \mathfrak{p}_- \to \mathfrak{p}_- \otimes \bigwedge^q \mathfrak{p}_- \to \bigwedge^{q+1} \mathfrak{p}_-.$$

Moreover when we regard $\mathfrak{p} \otimes L^q$, $\mathfrak{p} \otimes \bigwedge^q \mathfrak{p}_- \otimes l_{\rho_n}$ *as* $\mathfrak{b}$-*modules, as noted in* § 0, *then all the maps are* $\mathfrak{b}$-*module homomorphisms.*

Proof. Let $x \in \mathfrak{p}$ and $y \in C_q^N \cdot e_P$. Write $x = x^+ + x^-$ where $x^\pm \in \mathfrak{p}_\pm$. Then $xy \equiv x^- y \mod L^q$. Hence all the assertions of the lemma are clear from the definitions combined with Lemma 2.2. q.e.d.

§ 3. Dirac Operators and Certain Elliptic Complexes

In this section we shall first recall some facts about the Dirac operators on a symmetric space discussed in [18], and next investigate certain elliptic complexes obtained from the Dirac operators which are closely related to those in [8].

As in the introduction, let G be a semisimple connected Lie group of non-compact type, which is, for simplicity, assumed to be a real form of a simply connected complex Lie group $G^{\mathbb{C}}$. We assume that rank $G = \operatorname{rank} K$ for a maximal compact subgroup K; hence we have a compact Cartan subgroup T of G contained in K, which will be fixed once and for all. The corresponding Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$ then satisfy the assumption of § 2 and the notation associated to them will be kept as in § 2; i.e., fix a positive root system P in the root system Σ, etc. As noted in § 0, the character group $\hat{T}$ of T will be identified with the lattice $\mathscr{F}$ in $\operatorname{Hom}(\sqrt{-1}\,\mathfrak{t}, \mathbb{R})$, and the other notations $\mathscr{F}_0$, $\mathscr{F}_0^k$ are defined as in (0.2) in § 0.

In general, for a K-module V, consider the homogeneous vector bundle $E_V = G \times_K V$ over the symmetric space $X = G/K$ associated to V. For the K-module $\mathfrak{p}$ in the Cartan decomposition in § 2, one has a canonical covariant differential operator

$$\nabla\colon\ C^\infty(E_V) \to C^\infty(E_{\mathfrak{p}\otimes V})$$

defined by

$$\nabla = \sum_{i=1}^{2m} x_i \otimes \nu(x_i) \tag{3.1}$$

where $\{x_i\}_{i=1}^{2m}$ is an orthonormal basis of $\mathfrak{p}$ and $\nu(x_i)$ denotes the left invariant vector field generated by x_i which acts on elements in $C^\infty(E_V)$ as differentiation of V-valued functions on G. We notice that $E_{\mathfrak{p}}$ can be regarded as the cotangent bundle over X since $\mathfrak{p}$ is self-dual under the restricted Killing form. If another K-module W and a K-map $\mathfrak{p}\otimes V \to W$ are given, one then obtains a first order differential operator

$$C^\infty(E_V) \to C^\infty(E_W)$$

by composing ∇ with the map $C^\infty(E_{\mathfrak{p}\otimes V}) \to C^\infty(E_W)$ induced by $\mathfrak{p}\otimes V \to W$. This is a differential operator associated to the "symbol" $\mathfrak{p}\otimes V \to W$, in the sense of the last paragraph in § 1.

For $\lambda \in \mathscr{F}$ such that $\lambda + \rho_n \in \mathscr{F}_0^k$, we have the irreducible $\mathfrak{k}$-module $V_{\lambda+\rho_n}$ with highest weight $\lambda + \rho_n$. Let L be the spin $\mathfrak{k}$-module given in § 2. Then the $\mathfrak{k}$-action on $L \otimes V_{\lambda+\rho_n}$ can be lifted to a K-action ([18, Remark 3.2]). As in § 2, we have the Clifford multiplication

$$\mathfrak{p} \otimes L \otimes V_{\lambda+\rho_n} \to L \otimes V_{\lambda+\rho_n}.$$

The Dirac operator

$$D\colon\ C^\infty(E_{L\otimes V_{\lambda+\rho_n}}) \to C^\infty(E_{L\otimes V_{\lambda+\rho_n}})$$

is by definition the differential operator associated to the above Clifford multiplication as its symbol, with ∇ as in (3.1). We notice that D maps $C^\infty(E_{L^\pm\otimes V_{\lambda+\rho_n}})$ into $C^\infty(E_{L^\mp\otimes V_{\lambda+\rho_n}})$ where $L\otimes V_{\lambda+\rho_n}$ breaks up into $L\otimes V_{\lambda+\rho_n} = (L^+ \otimes V_{\lambda+\rho_n}) \oplus (L^- \otimes V_{\lambda+\rho_n})$.

Lemma 3.1. *D is elliptic and*

$$D^2 = -\nu(\Omega) + (|\lambda+\rho|^2 - |\rho|^2)$$

where Ω is the Casimir operator of G whose action on $C^\infty(E_{L\otimes V_{\lambda+\rho_n}})$ is denoted by $\nu(\Omega)$.

Proof. See [18, Proposition 3.2].

We shall consider an elliptic complex whose "bootstrap" turns out to be the Dirac operator of the above type; i.e., we will untie the Dirac operator.

Let V_μ be the irreducible K-module with highest weight $\mu \in \mathscr{F}_0$. Assume $\mu - \rho_n \in \mathscr{F}_0^k$. One can then consider the irreducible $\mathfrak{k}$-module $V_{\mu-\rho_n}$ and the Dirac operator

$$D\colon\ C^\infty(E_{L\otimes V_{\mu-\rho_n}}) \to C^\infty(E_{L\otimes V_{\mu-\rho_n}}).$$

Note that one of the components of L is the $\mathfrak{k}$-module V_{ρ_n} with highest weight ρ_n and that V_μ is contained in $L^+ \otimes V_{\mu-\rho_n} \subset L \otimes V_{\mu-\rho_n}$ with multiplicity one; V_μ can be regarded as the irreducible component of $L^+ \otimes V_{\mu-\rho_n}$ generated by $L^0 \otimes v_{\mu-\rho_n}$, where L^0 is as in (2.3), and $v_{\mu-\rho_n}$ is a highest weight vector of $V_{\mu-\rho_n}$. Define the K-submodules in $L\otimes V_{\mu-\rho_n}$ successively as

$$\begin{aligned} &L^0(\mu) = V_\mu, \\ &L^1(\mu) = L^0(\mu) + \mathfrak{p}\cdot L^0(\mu) \\ &\dots \\ &L^q(\mu) = L^{q-1}(\mu) + \mathfrak{p}\cdot L^{q-1}(\mu) \\ &\dots \end{aligned}$$

where $\mathfrak{p}\cdot L^{q-1}(\mu)$ is the K-submodule of $L\otimes V_{\mu-\rho_n}$ which is the image of the Clifford multiplication

$$\mathfrak{p}\otimes L^{q-1}(\mu) \to L\otimes V_{\mu-\rho_n}.$$

We thus have a filtration of the K-submodules

$$V_\mu = L^0(\mu) \subset L^1(\mu) \subset \cdots \subset L^q(\mu) \subset \dots$$

$(q = 0, 1, 2, \dots)$.

Lemma 3.2. (i) *$L^m(\mu) = L\otimes V_{\mu-\rho_n}$ where $m = |P_n|$.*

(ii) *If $L^{q_0}(\mu) = L^{q_0+1}(\mu)$ for some q_0, $0 < q_0 \leqq m$, then $L^{q_0}(\mu) = L\otimes V_{\mu-\rho_n}$.*

(iii) *Put $L^q_\pm(\mu) = L^q(\mu) \cap L^\pm \otimes V_{\mu-\rho_n}$. Then $L^q(\mu) = L^q_+(\mu) \oplus L^q_-(\mu)$, $L^q_+(\mu) = L^{q-1}_+(\mu)$ for odd q, and $L^q_-(\mu) = L^{q-1}_-(\mu)$ for even q.*

Proof. By the definition and (2.5) in § 2, $L^q(\mu)$ contains $L^q\otimes v_{\mu-\rho_n}$. Therefore $L^m(\mu)$ contains $L^m\otimes v_{\mu-\rho_n} = L\otimes v_{\mu-\rho_n}$, hence $L\otimes x v_{\mu-\rho_n}$ for any $x\in\mathfrak{k}$, hence $L\otimes V_{\mu-\rho_n}$, which shows (i). The assertion (ii) follows immediately from (i) since $\mathfrak{p}\cdot L^{q_0}(\mu)\subset L^{q_0}(\mu)$. As for (iii), $L^0(\mu)=L^0_+(\mu)$; hence $\mathfrak{p}\cdot L^0(\mu)=L^1_-(\mu)$, and $L^1_+(\mu)=L^0(\mu)$ by the definition since $\mathfrak{p}\cdot L^{\pm}\otimes V_{\mu-\rho_n}\subset L^{\mp}\otimes V_{\mu-\rho_n}$. Assume that (iii) holds for $q-1$. Then $L^q_{\pm}(\mu)=L^{q-1}_{\pm}(\mu)+\mathfrak{p}\cdot L^{q-1}_{\mp}(\mu)$ and $L^{q-1}(\mu)=L^{q-1}_+(\mu)\oplus L^{q-1}_-(\mu)$. If q is odd, then $L^{q-1}_-(\mu)=L^{q-2}_-(\mu)$ by the assumption. Hence

$$L^q_+(\mu)=L^{q-1}_+(\mu)+\mathfrak{p}\cdot L^{q-1}_-(\mu)\subset L^{q-1}_+(\mu).$$

If q is even, we have a similar argument, which shows (iii) for every q. q.e.d.

We now define the K-module

$$V^q_\mu=L^q(\mu)/L^{q-1}(\mu) \qquad (0\leqq q\leqq m).$$

If $V^{q_0}_\mu=0$, then $V^{q_0+1}_\mu=\cdots=V^m_\mu=0$ by Lemma 3.2, (ii). Since under the Clifford multiplication $\mathfrak{p}\cdot L^{q-1}(\mu)\subset L^q(\mu)$, we have the induced map

$$\mathfrak{p}\otimes V^q_\mu\to V^{q+1}_\mu \tag{3.2}$$

for each q. That is, the diagram

$$\begin{array}{ccc}\mathfrak{p}\otimes L^q(\mu) & \longrightarrow & L^{q+1}(\mu)\\ \downarrow & & \downarrow\\ \mathfrak{p}\otimes V^q_\mu & \longrightarrow & V^{q+1}_\mu\end{array} \tag{3.3}$$

commutes, where the upper horizontal map is the Clifford multiplication. For each q, we define a first order differential operator

$$\mathscr{D}\colon\ C^\infty(E_{V^q_\mu})\to C^\infty(E_{V^{q+1}_\mu})$$

as the operator associated to the symbol (3.2). The Dirac operator D maps $C^\infty(E_{L^q(\mu)})$ into $C^\infty(E_{L^{q+1}(\mu)})$, and by (3.3) the diagram

$$\begin{array}{ccc}C^\infty(E_{L^q(\mu)}) & \xrightarrow{D} & C^\infty(E_{L^{q+1}(\mu)})\\ \downarrow & & \downarrow\\ C^\infty(E_{V^q_\mu}) & \xrightarrow{\mathscr{D}} & C^\infty(E_{V^{q+1}_\mu})\end{array}$$

commutes for each q. By Lemma 3.1, one can see that D^2 maps $C^\infty(E_{L^q(\mu)})$ into itself for each q. Hence, in the sequence

$$C^\infty(E_{V^q_\mu})\xrightarrow{\mathscr{D}}C^\infty(E_{V^{q+1}_\mu})\xrightarrow{\mathscr{D}}C^\infty(E_{V^{q+2}_\mu})$$

we have $\mathscr{D}^2=0$. We thus have the following lemma.

Lemma 3.3. *For $\mu\in\mathscr{F}_0$ such that $\mu-\rho_n\in\mathscr{F}^k_0$, we have a differential complex $\mathbb{E}_\mu$:*

$$0\to C^\infty(E_{V_\mu})\xrightarrow{\mathscr{D}}C^\infty(E_{V^1_\mu})\xrightarrow{\mathscr{D}}\cdots\xrightarrow{\mathscr{D}}C^\infty(E_{V^m_\mu})\to 0$$

defined as above. The complex $\mathbb{E}_\mu$ *is then an elliptic complex with length less than or equal to* $m=\frac{1}{2}\dim X$. *Moreover, there exist isomorphisms*

$$\sum_{(-1)^q=\pm 1} V_\mu^q \xrightarrow{\sim} L^\pm \otimes V_{\mu-\rho_n}$$

such that, considering the formal adjoint operators $\mathscr{D}^*$ *of* $\mathscr{D}$ *under suitable metrics on the* V_μ^q, *we have* $D=\mathscr{D}+\mathscr{D}^*$ *under the above isomorphisms. Moreover, the laplacian* $\square=\mathscr{D}\mathscr{D}^*+\mathscr{D}^*\mathscr{D}$ *of* $\mathbb{E}_\mu$ *has the form*

$$\square=-\nu(\Omega)+(|\mu-\rho_n+\rho_k|^2-|\rho|^2)$$

where Ω *is the Casimir operator of* G.

Proof. The ellipticity of the $\mathbb{E}_\mu$ follows from that of D by the standard argument if we show $D=\mathscr{D}+\mathscr{D}^*$. Then, the formula for $\square$ also follows from Lemma 3.1. Introduce a Spin(p)-invariant metric $(\ ,\ )_L$ on L such that

$$(x\,u, v)_L+(u, x\,v)_L=0 \qquad \text{for } x\in\mathfrak{p},\ u, v\in L$$

([18, Lemma 4.1]). Consider the metric $(\ ,\ )_{L\otimes V_{\mu-\rho_n}}$ on $L\otimes V_{\mu-\rho_n}$ given in the usual way by the above metric on L and a K-invariant metric on $V_{\mu-\rho_n}$. It is then known that the Dirac operator D is formally self-adjoint with respect to the metric on $E_{L\otimes V_{\mu-\rho_n}}$ given by $(\ ,\ )_{L\otimes V_{\mu-\rho_n}}$ (X has an invariant volume element) ([18, Lemma 4.2]).

Define the embedding ι^q: $V_\mu^q \hookrightarrow L^q(\mu)$ by regarding V_μ^q as the orthogonal complement of $L^{q-1}(\mu)$ under the restriction of the metric $(\ ,\ )_{L\otimes V_{\mu-\rho_n}}$. Since $L^\pm\otimes V_{\mu-\rho_n}$ are mutually orthogonal, we have isomorphisms

$$\bigoplus_{(-1)^q=\pm 1}\iota^q\colon \bigoplus_{(-1)^q=\pm 1} V_\mu^q \xrightarrow{\sim} L^\pm\otimes V_{\mu-\rho_n}$$

in view of Lemma 3.2, (iii). We introduce a metric on each V_μ^q by the restriction of that of $L\otimes V_{\mu-\rho_n}$. If we denote by π^q: $L\otimes V_{\mu-\rho_n}\to V_\mu^q$ the orthogonal projection, then $(\iota^q)^*=\pi^q$ where $(\iota^q)^*$ denotes the adjoint of ι^q. Under those metrics, for $\mathscr{D}$: $C^\infty(E_{V_\mu^q})\to C^\infty(E_{V_\mu^{q+1}})$ and its formal adjoint $\mathscr{D}^*$: $C^\infty(E_{V_\mu^{q+1}})\to C^\infty(E_{V_\mu^q})$, we have $\mathscr{D}=\pi^{q+1}D\iota^q$ and $\mathscr{D}^*=\pi^q D\iota^{q+1}$ since D is formally self-adjoint. We shall show that the diagram

$$\begin{array}{ccc}
C^\infty(E_{V_\mu^q}) & \xrightarrow{\mathscr{D}+\mathscr{D}^*} & C^\infty(E_{V_\mu^{q+1}\oplus V_\mu^{q-1}}) \\
\Big\downarrow{\scriptstyle \iota^q} & & \Big\downarrow{\scriptstyle \iota^{q+1}\oplus\iota^{q-1}} \\
C^\infty(E_{L\otimes V_{\mu-\rho_n}}) & \xrightarrow{D} & C^\infty(E_{L\otimes V_{\mu-\rho_n}})
\end{array}$$

commutes. For $s\in C^\infty(E_{V_\mu^q})$, $(\mathscr{D}+\mathscr{D}^*)s=(\pi^{q+1}\oplus\pi^{q-1})Ds$. Therefore it suffices to show that Ds belongs to the image of $\iota^{q+1}\oplus\iota^{q-1}$.

For $s \in C^\infty(E_{L\otimes V_{\mu-\rho_n}})$, $s' \in C_0^\infty(E_{L\otimes V_{\mu-\rho_n}})$, we have a coupling

$$(s, s') = \int_G (s, s')_{L\otimes V_{\mu-\rho_n}} dg.$$

We note that $Ds \in C^\infty(E_{L^{q+1}(\mu)})$ for $s \in C^\infty(E_{V^q_\mu})$. For any $s' \in C_0^\infty(E_{L^{q-2}(\mu)})$, $(Ds, s') = (s, Ds') = 0$ since $Ds' \in C^\infty(E_{L^{q-1}(\mu)})$. On the other hand we have $(Ds, s'') = 0$ for any $s'' \in C_0^\infty(E_{V^q_\mu})$, since $Ds \in C^\infty(E_{L^\pm \otimes V_{\mu-\rho_n}})$ according to whether $(-1)^q = \mp 1$. Hence $Ds \in C^\infty(E_{V^{q-1}_\mu \oplus V^{q+1}_\mu})$, which implies our assertion. q.e.d.

We shall finally discuss the relationship of the elliptic complex $\mathbb{E}_\mu$ defined above with that obtained in [8]. We start with recalling the well-known Borel-Weil-Bott theorem about compact Lie groups.

For the compact group K, let $K^{\mathbb{C}}$ be the complexification of K and B the Borel subgroup of $K^{\mathbb{C}}$ corresponding to the Borel subalgebra $\mathfrak{b} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha\in P_k} \mathfrak{g}^\alpha$ given in Lemma 2.1. Consider the complex flag manifold $S = K^{\mathbb{C}}/B$ and put $s = \dim_C S = |P_k|$ as in §0. When a holomorphic B-module $\mathfrak{m}$ is given, one has the holomorphic vector bundle $\mathscr{V}_{\mathfrak{m}}$ over S associated to $\mathfrak{m}$. Then the cohomology space $H^i(S; \mathcal{O}(\mathscr{V}_{\mathfrak{m}}))$ with coefficients in the sheaf $\mathcal{O}(\mathscr{V}_{\mathfrak{m}})$ of germs of holomorphic sections of $\mathscr{V}_{\mathfrak{m}}$ has a $K^{\mathbb{C}}$- (or K-)module structure. As noted in §0, we denote by $H^i(\mathfrak{m})$ the above $K^{\mathbb{C}}$- (or K-)module given by the cohomology space $H^i(S; \mathcal{O}(\mathscr{V}_{\mathfrak{m}}))$.

For $\mu \in \mathscr{F}$, one has a unique holomorphic B-module l_μ of dimension one with weight μ extended trivially to the unipotent radical. For the K-module $H(l_\mu)$ we have the *Borel-Weil-Bott theorem:*

(i) *If* $(\mu - \rho_k, \alpha) = 0$ *for some* $\alpha \in P_k$, *then* $H^i(l_\mu) = 0$ *for every* i.

(ii) *If* $(\mu - \rho_k, \alpha) \neq 0$ $(\alpha \in P_k)$, *then* $H^i(l_\mu) = 0$ *for* $i \neq i_\mu$ *where* $i_\mu = |\{\alpha \in P_k; (\mu - \rho_k, \alpha) > 0\}|$. *In case* (ii) $H^{i_\mu}(l_\mu)$ *gives an irreducible* K*-module with highest weight* $w(\mu - \rho_k) - \rho_k \in \mathscr{F}_0$ *where* w *is the unique element in the Weyl group* W *for* (K, T) *such that* $w(\mu - \rho_k) \in \mathscr{F}_0$.

For the proof, see, for example, [2] or [12]. We notice that for the sign representation $\varepsilon(w)$ of $w \in W$, we have $\varepsilon(w) = (-1)^{i_\mu + s}$. As a special case, if $\mu \in \mathscr{F}_0$, then, among $H^i(l_{\mu+2\rho_k})$, the only nonvanishing $H^s(l_{\mu+2\rho_k})$ gives an irreducible K-module with highest weight μ.

We shall often use the following conventions. Let $\mathfrak{m}$ be a $K^{\mathbb{C}}$-module. By restriction to B, we have the B-module structure on $\mathfrak{m}$. Then the associated vector bundle $\mathscr{V}_{\mathfrak{m}}$ over S is holomorphically trivial and $\mathfrak{m} \simeq H^0(\mathfrak{m})$ as $K^{\mathbb{C}}$-modules. If another B-module $\mathfrak{n}$ is given, then, by the cup product, we have $K^{\mathbb{C}}$-module isomorphisms

$$\mathfrak{m} \otimes H^i(\mathfrak{n}) \simeq H^0(\mathfrak{m}) \otimes H^i(\mathfrak{n}) \xrightarrow{\sim} H^i(\mathfrak{m} \otimes \mathfrak{n}). \tag{3.4}$$

Even when $\mathfrak{m}$ is a real K-module, we have a K-module isomorphism

$$\mathfrak{m} \otimes H^i(\mathfrak{n}) \xrightarrow{\sim} H^i(\mathfrak{m} \otimes \mathfrak{n}) \tag{3.5}$$

in the sense that

$$\mathfrak{m}\otimes_{\mathbb{R}}H^i(\mathfrak{n})=\mathfrak{m}^{\mathbb{C}}\otimes_{\mathbb{C}}H^i(\mathfrak{n})\xrightarrow{\sim}H^i(\mathfrak{m}^{\mathbb{C}}\otimes_{\mathbb{C}}\mathfrak{n})=H^i(\mathfrak{m}\otimes_{\mathbb{R}}\mathfrak{n}).$$

For $\mu\in\mathscr{F}_0$, we now consider the following $\mathfrak{b}$-module exact sequence obtained by tensoring the one dimensional $\mathfrak{b}$-module $l_{\mu+2\rho_k-\rho_n}$ with the exact sequence in Lemma 2.2,

$$0\to L^{q-1}\otimes l_{\mu+2\rho_k-\rho_n}\to L^q\otimes l_{\mu+2\rho_k-\rho_n}\to \bigwedge^q\mathfrak{p}_-\otimes l_{\mu+2\rho_k}\to 0$$

for each q. It is easy to see that this turns out to be a B-module exact sequence. If $H^i(\bigwedge^q\mathfrak{p}_-\otimes l_{\mu+2\rho_k})=0$ for every $i<s$, then the associated long exact sequence of cohomology reduces to a short one,

$$\begin{aligned}0\to H^s(L^{q-1}\otimes l_{\mu+2\rho_k-\rho_n})\to H^s(L^q\otimes l_{\mu+2\rho_k-\rho_n})\\ \to H^s(\bigwedge^q\mathfrak{p}_-\otimes l_{\mu+2\rho_k})\to 0.\end{aligned}\tag{3.6}$$

The above vanishing condition on $H^i(\bigwedge^q\mathfrak{p}_-\otimes l_{\mu+2\rho_k})$ will dominate most of the remaining sections. Therefore, for the convenience of statement, we make the following definition.

Definition. We say that *$\mu\in\mathscr{F}$ satisfies the condition* (#), when $H^i(\bigwedge^q\mathfrak{p}_-\otimes l_{\mu+2\rho_k})=0$ for every $i<s$ and for every q, $0\leqq q\leqq m$.

We put, as K-modules,

$$\begin{aligned}L^q[\mu]=H^s(L^q\otimes l_{\mu+2\rho_k-\rho_n})\\ U^q_\mu=H^s(\bigwedge^q\mathfrak{p}_-\otimes l_{\mu+2\rho_k})\end{aligned}\tag{3.7}$$

for each q. In this terminology, under the condition (#) for $\mu\in\mathscr{F}_0$, we have an exact sequence by (3.6)

$$0\to L^{q-1}[\mu]\to L^q[\mu]\to U^q_\mu\to 0\tag{3.8}$$

for each q. By (3.4), we have an isomorphism

$$L\otimes V_{\mu-\rho_n}\xrightarrow{\sim}H^s(L\otimes l_{\mu+2\rho_k-\rho_n})=L^m[\mu].$$

Though L and $l_{\mu+2\rho_k-\rho_n}$ may not be B-modules, one can verify the above after taking a suitable covering of K (and so B). Setting $q=0$ in (3.8), we also have an isomorphism

$$L^0[\mu]\simeq V_\mu$$

by the Borel-Weil-Bott theorem. Hence under the condition (#) for μ, we have the filtration

$$V_\mu=L^0[\mu]\subset L^1[\mu]\subset\cdots\subset L^m[\mu]=L\otimes V_{\mu-\rho_n}.\tag{3.9}$$

Tensoring the Clifford multiplication $\mathfrak{p}\otimes L^{q-1}\to L^q$ with $l_{\mu+2\rho_k-\rho_n}$ gives rise to the map

$$\mathfrak{p}\otimes L^{q-1}[\mu]\to L^q[\mu]\tag{3.10}$$

which is the interpretation of the induced map

$$H^s(\mathfrak{p}\otimes L^{q-1}\otimes l_{\mu+2\rho_k-\rho_n})\to H^s(L^q\otimes l_{\mu+2\rho_k-\rho_n})$$

by (3.5) and (3.7). Under the embeddings (3.9), it is easy to see that the map (3.10) is nothing but the Clifford multiplication (or more precisely, the restriction to $L^{q-1}[\mu]$ of that of $L\otimes V_{\mu-\rho_n}$). By Lemma 2.3, we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{p}\otimes L^{q-1}[\mu] & \longrightarrow & L^q[\mu] \\ \Big\downarrow & & \Big\downarrow \\ \mathfrak{p}\otimes U^{q-1}_\mu & \longrightarrow & U^q_\mu \end{array} \tag{3.11}$$

where the lower horizontal map is given by the exterior multiplication

$$\mathfrak{p}\otimes\wedge^{q-1}\mathfrak{p}_-\otimes l_{\mu+2\rho_k}\to\wedge^q\mathfrak{p}_-\otimes l_{\mu+2\rho_k}.$$

We can consider the differential operator

$$\mathscr{D}'\colon\ C^\infty(E_{U^{q-1}_\mu})\to C^\infty(E_{U^q_\mu})$$

associated to the symbol $\mathfrak{p}\otimes U^{q-1}_\mu\to U^q_\mu$ in (3.11) composing the covariant differential operator ∇ like (3.1), and obtain the sequence $\mathbb{E}'_\mu$:

$$0\to C^\infty(E_{V_\mu})\to C^\infty(E_{U^1_\mu})\to\cdots\to C^\infty(E_{U^m_\mu})\to 0.$$

In [8], this $\mathbb{E}'_\mu$ has been shown to be an elliptic complex under the condition (#) for μ and further, under a specific choice of positive root system P. We will, however, see that $\mathbb{E}'_\mu$ is nothing but the elliptic complex $\mathbb{E}_\mu$ given in Lemma 3.3 only under the condition (#) for μ.

Lemma 3.4. *Under the condition* (#) *for* $\mu\in\mathscr{F}_0$, $L^q[\mu]=L^q(\mu)$ *for each* q, *hence* $U^q_\mu=V^q_\mu$. *Thus the elliptic complex* $\mathbb{E}_\mu$ *in Lemma* 3.3 *coincides with the sequence* $\mathbb{E}'_\mu$ *defined as above.*

Proof. The first assertion for $q=0$ has already been shown. Note that the map $\mathfrak{p}\otimes U^{q-1}_\mu\to U^q_\mu$ is surjective. Hence by (3.10) and (3.11)

$$L^q[\mu]\subset L^{q-1}[\mu]+\mathfrak{p}\cdot L^{q-1}[\mu],$$

and the inverse inclusion relation is clear. Hence

$$L^q[\mu]=L^{q-1}[\mu]+\mathfrak{p}\cdot L^{q-1}[\mu].$$

By the definition of $L^q(\mu)$ and $L^0(\mu)=L^0[\mu]$, we have $L^q(\mu)=L^q[\mu]$. Hence $U^q_\mu=V^q_\mu$. The second assertion follows from the first one and the diagram (3.11). q.e.d.

§ 4. Some Results of Schmid

Here we collect several results dealing with the compact flag manifold in Schmid's thesis [20] which will be used later. Since the results restated here (except Theorem 1) have simple proofs and Schmid's thesis [20] might not be available to everyone, we will reproduce the proofs for the sake of completeness. For Theorem 1, we give a more elementary proof in the next section. The notation and setting will be as in the previous sections.

We first start with a criterion for the condition (#) defined in § 3. In special cases, sharper results will be obtained (Lemma 9.1, § 9).

Lemma 4.1 ([20], Lemma 5.5). *Let $\mu\in\mathscr{F}$. If $(\mu+\rho_k-\langle Q\rangle,\alpha)\geqq 0$ for every $\alpha\in P_k$ and every $Q\subset P_n$, then μ satisfies the condition (#); i.e., $H^i(\bigwedge^q \mathfrak{p}_-\otimes l_{\mu+2\rho_k})=0$ for every $i<s$ and every q.*

Proof. One has a flag of B-modules

$$0=A_0\subset A_1\subset\cdots\subset A_r=\bigwedge^q \mathfrak{p}_-\otimes l_{\mu+2\rho_k}$$

such that

$$0\to A_{j-1}\to A_j\to l_{\mu+2\rho_k-\langle Q\rangle}\to 0$$

for some $Q\subset P_n$. By the Borel-Weil-Bott theorem, the assumption in the lemma implies that $H^i(l_{\mu+2\rho_k-\langle Q\rangle})=0$ for every $i<s$. The long cohomology exact sequence shows that if $H^i(A_{j-1})=0$ for every $i<s$, then $H^i(A_j)=0$. Here $H^i(A_0)=0$ for every i. Hence $H^i(A_r)=H^i(\bigwedge^q \mathfrak{p}_-\otimes l_{\mu+2\rho_k})=0$ for every $i<s$. q.e.d.

For $\mu\in\mathscr{F}$, $Q_l(\mu)$ denotes the number of distinct ways in which μ can be expressed as a sum of precisely l positive non-compact roots in P_n $(Q_0(0)=1)$. Put

$$Q(\mu)=\sum_{l\geqq 0} Q_l(\mu).$$

We define the *Blattner number* $b_\lambda(\mu)$ for $\lambda,\mu\in\mathscr{F}$ by

$$b_\lambda(\mu)=\sum_{w\in W}\varepsilon(w)\,Q\big(w(\mu+\rho_k)-(\lambda+2\rho_n+\rho_k)\big),$$

where W is the Weyl group for (K,T) as before. For this, one knows the following result.

Lemma 4.2 ([20], Lemma 6.10). *For $\mu\in\mathscr{F}$, $Q(\mu)$ is actually finite; hence $b_\lambda(\mu)$ is finite. Moreover, $b_\lambda(\lambda+2\rho_n)=1$ for $\lambda+2\rho_n\in\mathscr{F}_0$.*

Proof. If $Q_l(\mu)\neq 0$, then $\mu=\sum_{i=1}^{l}\alpha_i$ $(\alpha_i\in P_n)$. Hence

$$(\rho,\mu)=\sum_{i=1}^{l}(\rho,\alpha_i)\geqq l\min_{\alpha\in P_n}(\rho,\alpha).$$

Note that $\min_{\alpha\in P_n}(\rho,\alpha)>0$. Hence if l is sufficiently large, then $(\rho,\mu) < l\min_{\alpha\in P_n}(\rho,\alpha)$ which means $Q_l(\mu)=0$. Hence $Q(\mu)$ is finite. For $\lambda+2\rho_n\in\mathscr{F}_0$, $(\lambda+2\rho_n+\rho_k,\alpha)>0$ $(\alpha\in P_k)$. Hence for $w\neq 1$ in W, $\lambda+2\rho_n+\rho_k-w(\lambda+2\rho_n+\rho_k)$ is expressed as a sum of elements in P_k. Hence if

$$Q\big(w(\lambda+2\rho_n+\rho_k)-(\lambda+2\rho_n+\rho_k)\big)\neq 0$$

for $w\neq 1$, then a sum of elements in P turns out to be zero. This is a contradiction. Hence $b_\lambda(\lambda+2\rho_n)=Q(0)=Q_0(0)=1$. q.e.d.

Lemma 4.3 ([20], Lemma 5.3). *Assume the condition* (#) *for* $\lambda+2\rho_n\in\mathscr{F}_0$. *Let* $S^l(\mathfrak{p}_+)$ *be the B-module given by the l-th symmetric power of* $\mathfrak{p}_+$. *Then for the K-module given by the top-degree sheaf cohomology* $H^s(S^l(\mathfrak{p}_+)\otimes l_{\mu+2\rho})$ *and the irreducible K-module* V_μ *with highest weight* μ, *we have*

$$\big(H^s(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho}):V_\mu\big)=\sum_{w\in W}\varepsilon(w)\,Q_l\big(w(\mu+\rho_k)-(\lambda+2\rho_n+\rho_k)\big)$$

where (:) *denotes the intertwining number* (*see* § 0).

Proof. As in the proof of Lemma 4.1, let

$$0=A_0\subset A_1\subset\cdots\subset A_r=S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho}$$

be a flag of B-submodules such that

$$0\to A_{j-1}\to A_j\to l_{\lambda+2\rho+\alpha_1+\cdots+\alpha_l}\to 0$$

for some l-tuple $(\alpha_1,\ldots,\alpha_l)$ of P_n. For a B-module $\mathfrak{m}$, put

$$\chi(\mathfrak{m})=\sum_{i=0}^{s}(-1)^i[H^i(\mathfrak{m})]$$

in the character ring $\mathbb{Z}[\hat{K}]$ of K where $[H^i(\mathfrak{m})]$ denotes the elements in $\mathbb{Z}[\hat{K}]$ given by the K-module $H^i(\mathfrak{m})$. We then have

$$\chi\big(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho}\big)=\sum\chi(l_{\lambda+2\rho+\alpha_1+\cdots+\alpha_l})$$

where $(\alpha_1,\ldots,\alpha_l)$ runs over all unordered l-tuples in P_n. In Lemma 5.3 of the next section, we shall show that $H^i\big(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho}\big)=0$ $(i<s)$ if $\lambda+2\rho_n$ satisfies the condition (#). Hence in this case

$$\big[H^s(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho})\big]=(-1)^s\sum\chi(l_{\lambda+2\rho+\alpha_1+\cdots+\alpha_l}).$$

Applying the Borel-Weil-Bott theorem for $\chi(l_{\lambda+2\rho+\alpha_1+\cdots+\alpha_l})$, we have the lemma. q.e.d.

Remark 1. Schmid shows the formula under the less restrictive condition $(\lambda+2\rho_k,\alpha)>0$ $(\alpha\in P_k)$ using Griffiths' vanishing theorem for

$H^i(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho})$ which is sharper than the above one. For our later use, this sharper result, however, plays no roles, hence we have stated the weaker version.

We are now going back to the elliptic complex given in § 3. By Lemma 3.3, for $\lambda\in\mathscr{F}$ such that $\lambda+\rho_n\in\mathscr{F}_0^k$ (hence $\lambda+2\rho_n\in\mathscr{F}_0$), we have the elliptic complex $\mathbb{E}_{\lambda+2\rho_n}$:

$$0\to C^\infty(E_{V_{\lambda+2\rho_n}})\to C^\infty(E_{V^1_{\lambda+2\rho_n}})\to\cdots\to C^\infty(E_{V^m_{\lambda+2\rho_n}})\to 0$$

over the symmetric space $X=G/K$. Consider the 0-th cohomology $H^0(\mathbb{E}_{\lambda+2\rho_n})$ of this complex $\mathbb{E}_{\lambda+2\rho_n}$ in the C^∞-category, i.e., $H^0(\mathbb{E}_{\lambda+2\rho_n})$ is the kernel of the first $\mathscr{D}$: $C^\infty(E_{V_{\lambda+2\rho_n}})\to C^\infty(E_{V^1_{\lambda+2\rho_n}})$. Then $H^0(\mathbb{E}_{\lambda+2\rho_n})$ has a natural G-module structure. Denote by $H^0(\mathbb{E}_{\lambda+2\rho_n})^0$ the space of all K-finite vectors in $H^0(\mathbb{E}_{\lambda+2\rho_n})$, and by $(H^0(\mathbb{E}_{\lambda+2\rho_n})^0:V_\mu)$ the multiplicity of the V_μ in $H^0(\mathbb{E}_{\lambda+2\rho_n})^0$. Then we shall prove the following result in the next section.

Theorem 1. *Under the condition* (#) *for* $\lambda+2\rho_n\in\mathscr{F}_0$,

$$(H^0(\mathbb{E}_{\lambda+2\rho_n})^0:V_\mu)\leqq b_\lambda(\mu)\qquad \textit{for } \mu\in\mathscr{F}_0.$$

In particular, $(H^0(\mathbb{E}_{\lambda+2\rho_n})^0:V_{\lambda+2\rho_n})\leqq 1$.

Remark 2. By Lemma 3.4, our $\mathbb{E}_{\lambda+2\rho_n}$ is the same as the elliptic complex in [8], where the first term $\mathscr{D}$: $C^\infty(E_{V_{\lambda+2\rho_n}})\to C^\infty(E_{V^1_{\lambda+2\rho_n}})$ is shown to be the same as Schmid's operator $\mathscr{D}$ in [20]. In [20, Theorem 7.4], Schmid obtained a stronger result which gives equality in Theorem 1 under a more restrictive condition on λ than above. This result has been improved by him in [22] (also see [8, § 6]). For our final results on discrete series, only the inequalities stated in Theorem 1 will be used. Schmid's proof of this theorem uses complex analysis on the complex flag manifold G/T extensively, while our method in the subsequent section will rely on elementary differential calculus on the symmetric space $X=G/K$.

§ 5. K-Types of $H^0(\mathbb{E}_{\lambda+2\rho_n})$

The purpose of this section is to give the proof of Theorem 1 stated in § 4, based upon the framework given in § 1. Hence the situation will continue as in the final part of § 4.

We shall use the following adaptation of the Dolbeault lemma on differential forms with polynomial coefficients. Let S^l be the space of polynomials of homogeneous degree l in $2m$ variables

$$z_1,\bar z_1,z_2,\bar z_2,\ldots,z_m,\bar z_m,$$

and $\bigwedge^{0,q}$ the space of $(0,q)$ forms on $\mathbb{C}^m$ with constant coefficients, i.e., the vector space spanned by $d\bar z_{i_1}\wedge\cdots\wedge d\bar z_{i_q}$ for all $1\leqq i_1<\cdots<i_q\leqq m$.

The operator

$$\bar{\partial} = \sum_{i=1}^{m} \frac{\partial}{\partial \bar{z}_i} \otimes \varepsilon(d\bar{z}_i),$$

where $\varepsilon(d\bar{z}_i)$: $\bigwedge^{0,q} \to \bigwedge^{0,q+1}$ denotes the exterior multiplication, maps $S^{l-q} \otimes \bigwedge^{0,q}$ into $S^{l-q-1} \otimes \bigwedge^{0,q+1}$ for each l and q.

Lemma 5.1. *The sequence*

$$0 \to S^l_{\text{hol}} \to S^l \xrightarrow{\bar{\partial}} S^{l-1} \otimes \textstyle\bigwedge^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} S^{l-m} \otimes \textstyle\bigwedge^{0,m} \to 0$$

is exact, where S^l_{hol} is the space of holomorphic polynomials of homogeneous degree l on $\mathbb{C}^m$, i.e., polynomials in m-variables $z_1, \ldots, z_m$.

Proof. A standard proof of the ordinary Dolbeault lemma depends upon the solvability of the equation

$$\frac{\partial f}{\partial \bar{z}_i} = u$$

and the other part is quite formal by induction (see [5, Proof of 3, Theorem, p. 27]). It is clear that for any $p \in S^l$ there exists $q \in S^{l+1}$ such that

$$\frac{\partial q}{\partial \bar{z}_i} = p$$

and if p is holomorphic with respect to $z_{i+1}, \ldots, z_m$ then q is so chosen that q has the same property. With this fact, the proof can be carried out quite similarly to that in the above book. q.e.d.

We are now going back to the situation of the final part of the previous section. Let $\lambda \in \mathscr{F}$ satisfy $\lambda + \rho_n \in \mathscr{F}_0^k$. We then have the elliptic complex $\mathbb{E}_{\lambda+2\rho_n}$:

$$0 \to C^\infty(E_{V_{\lambda+2\rho_n}}) \to C^\infty(E_{V^1_{\lambda+2\rho_n}}) \to \cdots \to C^\infty(E_{V^m_{\lambda+2\rho_n}}) \to 0.$$

Hereafter throughout this section we assume the condition (#) for $\lambda + 2\rho_n$, i.e.,

$$H^i(\textstyle\bigwedge^q \mathfrak{p}_- \otimes l_{\lambda+2\rho}) = 0 \qquad \text{for every } i < s \text{ and every } q.$$

We can then identify $\mathbb{E}_{\lambda+2\rho_n}$ with $\mathbb{E}'_{\lambda+2\rho_n}$ defined cohomologically in § 3, by Lemma 3.4. More precisely, we may identify

$$V^q_{\lambda+2\rho_n} = H^s(\textstyle\bigwedge^q \mathfrak{p}_- \otimes l_{\lambda+2\rho}) \tag{5.1}$$

and the symbol of $\mathscr{D}$: $C^\infty(E_{V^q_{\lambda+2\rho_n}}) \to C^\infty(E_{V^{q+1}_{\lambda+2\rho_n}})$ is given by the map

$$\begin{aligned} \mathfrak{p} \otimes H^s(\textstyle\bigwedge^q \mathfrak{p}_- \otimes l_{\lambda+2\rho}) &= H^s(\mathfrak{p} \otimes \textstyle\bigwedge^q \mathfrak{p}_- \otimes l_{\lambda+2\rho}) \\ &\to H^s(\textstyle\bigwedge^{q+1} \mathfrak{p}_- \otimes l_{\lambda+2\rho}) \end{aligned} \tag{5.2}$$

where the first identity is an application of (3.5) and the map is induced by the exterior multiplication $\mathfrak{p} \otimes \bigwedge^q \mathfrak{p}_- \to \bigwedge^{q+1} \mathfrak{p}_-$.

We now fix the origin $o=\{K\}$ of the symmetric space $X=G/K$ and consider the "polynomialization" of each $\mathscr{D}$ at o in the sense of § 1. If we identify the cotangent space at o with $\mathfrak{p}$ and the fibre of $E_{V^q_{\lambda+2\rho_n}}$ at o with $V^q_{\lambda+2\rho_n}$, then the polynomialization

$$\mathscr{D}_q^{(l-q)}: S^{l-q}(\mathfrak{p})\otimes V^q_{\lambda+2\rho_n}\to S^{l-q-1}(\mathfrak{p})\otimes V^{q+1}_{\lambda+2\rho_n}$$

at the origin o is obtained by composing the map

$$S^{l-q}(\mathfrak{p})\otimes V^q_{\lambda+2\rho_n}\xrightarrow{d\otimes 1} S^{l-q-1}(\mathfrak{p})\otimes \mathfrak{p}\otimes V^q_{\lambda+2\rho_n}$$
$$\xrightarrow{1\otimes(\text{symbol})} S^{l-q-1}(\mathfrak{p})\otimes V^{q+1}_{\lambda+2\rho_n}$$

for each l, q, where $d=\sum_{i=1}^{2m}\frac{\partial}{\partial x_i}\otimes x_i$ for an orthonormal basis $\{x_i\}_{i=1}^{2m}$ of $\mathfrak{p}$. It is clear that those polynomializations $\mathscr{D}^{(\cdot)}$ are K-module maps.

Lemma 5.2. *Under the condition* (#) *for* $\lambda+2\rho_n\in\mathscr{F}_0$, *we have an exact sequence as K-modules*

$$0\to H^s(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho})\to S^l(\mathfrak{p})\otimes V_{\lambda+2\rho_n}$$
$$\to S^{l-1}(\mathfrak{p})\otimes V^1_{\lambda+2\rho_n}\to\cdots\to S^{l-m}(\mathfrak{p})\otimes V^m_{\lambda+2\rho_n}\to 0,$$

where the maps $S^{l-q}(\mathfrak{p})\otimes V^q_{\lambda+2\rho_n}\to S^{l-q-1}(\mathfrak{p})\otimes V^{q+1}_{\lambda+2\rho_n}$ *are the polynomializations* $\mathscr{D}_q^{(l-q)}$ *at the origin* $o\in X$ *for each* l *and* $0\leqq q\leqq m$. *The first inclusion map is the map*

$$H^s(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho})\to H^s(S^l(\mathfrak{p})\otimes l_{\lambda+2\rho})$$

induced from the natural inclusion

$$S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho}\hookrightarrow S^l(\mathfrak{p}^{\mathbb{C}})\otimes l_{\lambda+2\rho}=S^l(\mathfrak{p})\otimes l_{\lambda+2\rho}$$

where $H^s(S^l(\mathfrak{p})\otimes l_{\lambda+2\rho})=S^l(\mathfrak{p})\otimes V_{\lambda+2\rho_n}$ *by* (3.5), *and* $S^l(\mathfrak{p}_+)$ *is as in Lemma* 4.3.

For the proof, we first consider the exact sequence obtained from Lemma 5.1. Equip $\mathfrak{p}$ with a complex structure so that $\mathfrak{p}_-$ is identified with the antiholomorphic cotangent space. Then $\bar\partial: S^\cdot(\mathfrak{p})\otimes\bigwedge^\cdot\mathfrak{p}_-\to S^\cdot(\mathfrak{p})\otimes\bigwedge^\cdot\mathfrak{p}_-$ can be written as $\bar\partial=\sum_{\alpha\in P_n}\frac{\partial}{\partial x_{-\alpha}}\otimes\varepsilon(x_{-\alpha})$ where $x_{-\alpha}\in\mathfrak{p}_-$ is a root vector for $-\alpha\in -P_n$. The maps $\bar\partial$ are B-module maps when we consider $S^\cdot(\mathfrak{p})\otimes\bigwedge^\cdot\mathfrak{p}_-$ as B-modules, since $\bar\partial$ is the composite of

$$S^\cdot(\mathfrak{p})\otimes\bigwedge^\cdot\mathfrak{p}_-\xrightarrow{d\otimes 1} S^\cdot(\mathfrak{p})\otimes\mathfrak{p}\otimes\bigwedge^\cdot\mathfrak{p}_-\xrightarrow{1\otimes(\text{ext})} S^\cdot(\mathfrak{p})\otimes\bigwedge^\cdot\mathfrak{p}_-$$

where both maps are B-module maps. Since the space of holomorphic polynomials $S^\cdot_{\text{hol}}$ is $S^\cdot(\mathfrak{p}_+)$ in our setting, we have a B-module exact sequence

$$0\to S^l(\mathfrak{p}_+)\to S^l(\mathfrak{p})\otimes\mathbb{C}\to S^{l-1}(\mathfrak{p})\otimes\mathfrak{p}_-\to\cdots\to S^{l-m}(\mathfrak{p})\otimes\bigwedge^m\mathfrak{p}_-\to 0. \quad (5.3)$$

We consider the sequence (5.3) tensored with the one dimensional B-module $l_{\lambda+2\rho}$ and their top-degree cohomologies.

Lemma 5.3. *Under the condition* (#) *for* $\lambda+2\rho_n$, *the sequence induced from* (5.3) *tensored with* $l_{\lambda+2\rho}$,

$$0\to H^s(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho})\to H^s(S^l(\mathfrak{p})\otimes l_{\lambda+2\rho})\to\cdots$$
$$\to H^s(S^{l-m}(\mathfrak{p})\otimes\wedge^m\mathfrak{p}_-\otimes l_{\lambda+2\rho})\to 0$$

is exact. Moreover, $H^i(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho})=0$ *for every* $i<s$.

Proof. In (5.3), define the subspace

$$W^{l,q}\subset S^{l-q}(\mathfrak{p})\otimes\wedge^q\mathfrak{p}_-$$

as the image of $\bar\partial$. We then have the short exact sequence

$$0\to W^{l,q}\to S^{l-q}(\mathfrak{p})\otimes\wedge^q\mathfrak{p}_-\xrightarrow{\bar\partial} W^{l,q+1}\to 0. \tag{5.4}$$

Note that $W^{l,m}=S^{l-m}(\mathfrak{p})\otimes\wedge^m\mathfrak{p}_-$. We then claim that, if

$$H^i(W^{l,q+1}\otimes l_{\lambda+2\rho})=0 \qquad (i<s),$$

then $H^i(W^{l,q}\otimes l_{\lambda+2\rho})=0$ $(i<s)$. In fact, considering the long cohomology exact sequence associated to (5.4) tensored with $l_{\lambda+2\rho}$, we have

$$H^i(S^{l-q}(\mathfrak{p})\otimes\wedge^q\mathfrak{p}_-\otimes l_{\lambda+2\rho})\simeq H^i(W^{l,q}\otimes l_{\lambda+2\rho})$$

for $i<s$, and

$$\begin{aligned}0\to H^s(W^{l,q}\otimes l_{\lambda+2\rho})&\to H^s(S^{l-q}(\mathfrak{p})\otimes\wedge^q\mathfrak{p}_-\otimes l_{\lambda+2\rho})\\ &\to H^s(W^{l,q+1}\otimes l_{\lambda+2\rho})\to 0\end{aligned} \tag{5.5}$$

by the assumption. On the other hand,

$$H^i(S^{l-q}(\mathfrak{p})\otimes\wedge^q\mathfrak{p}_-\otimes l_{\lambda+2\rho})=S^{l-q}(\mathfrak{p})\otimes H^i(\wedge^q\mathfrak{p}_-\otimes l_{\lambda+2\rho})=0$$

under the condition (#) for $\lambda+2\rho_n$ by (3.5). Hence

$$H^i(W^{l,q}\otimes l_{\lambda+2\rho})=0 \qquad \text{for every } i<s. \tag{5.6}$$

Now for $q=m$,

$$H^i(W^{l,m}\otimes l_{\lambda+2\rho})\simeq S^{l-m}(\mathfrak{p})\otimes H^i(\wedge^m\mathfrak{p}_-\otimes l_{\lambda+2\rho})$$

which vanishes for $i<s$ under the condition (#) for $\lambda+2\rho_n$. Hence (5.6) holds for every q and so (5.5) holds under the condition (#) for $\lambda+2\rho_n$. All statements in the lemma immediately follow from this. q.e.d.

We now prove Lemma 5.2. According to (3.5), the map in Lemma 5.3

$$S^{l-q}(\mathfrak{p})\otimes H^s(\wedge^q\mathfrak{p}_-\otimes l_{\lambda+2\rho})\to S^{l-q-1}(\mathfrak{p})\otimes H^s(\wedge^{q+1}\mathfrak{p}_-\otimes l_{\lambda+2\rho})$$

turns out to be the composite

$$S^{l-q}(\mathfrak{p})\otimes H^s(\wedge^q \mathfrak{p}_-\otimes l_{\lambda+2\rho}) \xrightarrow{d\otimes 1} S^{l-q-1}(\mathfrak{p})\otimes \mathfrak{p}\otimes H^s(\wedge^q \mathfrak{p}_-\otimes l_{\lambda+2\rho})$$
$$= S^{l-q-1}(\mathfrak{p})\otimes H^s(\mathfrak{p}\otimes \wedge^q \mathfrak{p}_-\otimes l_{\lambda+2\rho}) \to S^{l-q-1}(\mathfrak{p})\otimes H^s(\wedge^{q+1} \mathfrak{p}_-\otimes l_{\lambda+2\rho}),$$

where the last map is induced by the exterior multiplication $\mathfrak{p}\otimes \wedge^q \mathfrak{p}_- \to \wedge^{q+1} \mathfrak{p}_-$, which is nothing but the symbol map (5.2). Hence the maps in Lemma 5.3 coincide with the polynomializations $\mathscr{D}_q^{(l-q)}$ at the origin $o\in X$ in Lemma 5.2 under our identifications (5.1). Thus we have completed the proof of Lemma 5.2.

The proof of Theorem 1 in § 4 goes as follows. For a moment, put $F=H^0(\mathbb{E}_{\lambda+2\rho_n})^0$, the K-module consisting of K-finite vectors in the kernel of $\mathscr{D}$: $C^\infty(E_{V_{\lambda+2\rho_n}}) \to C^\infty(E_{V^1_{\lambda+2\rho_n}})$. Considering the filtration of $C^\infty(E_{V_{\lambda+2\rho_n}})$ by $I^l = I^l_o(E_{V_{\lambda+2\rho_n}})$ at the origin $o\in X$ as in § 1, we make the filtration of F by $F^l = F\cap I^l$ (l is a non-negative integer). By Lemma 1.1, we have a natural inclusion

$$F^l/F^{l+1} \hookrightarrow \operatorname{Ker} \mathscr{D}_0^{(l)} \quad \text{for each } l$$

where the right hand side is the kernel of the polynomialization

$$\mathscr{D}_0^{(l)}:\ S^l(\mathfrak{p})\otimes V_{\lambda+2\rho_n} \to S^{l-1}(\mathfrak{p})\otimes V^1_{\lambda+2\rho_n}$$

of the first $\mathscr{D}$ in $\mathbb{E}_{\lambda+2\rho_n}$ at the origin o. The inclusion is clearly that of K-modules. Since $\mathscr{D}$ is elliptic, F consists of real analytic sections; hence $\bigcap_{l=0}^{\infty} F^l = 0$. Therefore

$$F \simeq \operatorname{Gr} F^\cdot = \bigoplus_{l=0}^{\infty} F^l/F^{l+1} \hookrightarrow \bigoplus_{l=0}^{\infty} \operatorname{Ker} \mathscr{D}_0^{(l)}$$

as K-modules. By Lemma 5.2,

$$\bigoplus_{l=0}^{\infty} \operatorname{Ker} \mathscr{D}_0^{(l)} \simeq \bigoplus_{l=0}^{\infty} H^s(S^l(\mathfrak{p}_+)\otimes l_{\lambda+2\rho}).$$

In view of Lemmas 4.2 and 4.3, we see

$$(F:V_\mu) \leqq b_\lambda(\mu)$$

for $\mu\in\mathscr{F}_0$. We thus have completed the proof of Theorem 1.

§ 6. Vanishing for L^2-Cohomologies of $\mathbb{E}_{\lambda+2\rho_n}$

For $\lambda\in\mathscr{F}$ such that $\lambda+\rho_n\in\mathscr{F}_0^k$ we consider the elliptic complex $\mathbb{E}_{\lambda+2\rho_n}$

$$0 \to C^\infty(E_{V_{\lambda+2\rho_n}}) \to C^\infty(E_{V^1_{\lambda+2\rho_n}}) \to \cdots \to C^\infty(E_{V^m_{\lambda+2\rho_n}}) \to 0$$

given in Lemma 3.3. Equip each $V^q_{\lambda+2\rho_n}$ with the K-invariant inner product $(\ ,\ )_{V^q_{\lambda+2\rho_n}}$ considered in Lemma 3.3. Let Γ be a finitely generated discrete subgroup of G. Denote by $C^\infty(\Gamma; E_{V^q_{\lambda+2\rho_n}})$ the subspace consisting of Γ-invariant sections in $C^\infty(E_{V^q_{\lambda+2\rho_n}})$. Fixing a Haar measure dg on G, one can define, in the usual way, the inner product

$$(s, s')_\Gamma = \int_{\Gamma \backslash G} (s, s')_{V^q_{\lambda+2\rho_n}} dg$$

whenever this has meaning for $s, s' \in C^\infty(\Gamma; E_{V^q_{\lambda+2\rho_n}})$ as in [10, § 1]. Denote by $L^2(\Gamma; E_{V^q_{\lambda+2\rho_n}})$ the Hilbert space consisting of Γ-invariant L^2 sections of $E_{V^q_{\lambda+2\rho_n}}$ under the inner product $(\ ,\ )_\Gamma$, i.e., the completion of the pre-Hilbert space

$$\{s \in C^\infty(\Gamma; E_{V^q_{\lambda+2\rho_n}});\ \|S\|_\Gamma < \infty\},$$

where $\|s\|^2_\Gamma = (s, s)_\Gamma$. Letting the differential operators $\mathscr{D}$ act on $L^2(\Gamma; E_{V^q_{\lambda+2\rho_n}})$ extending maximally, we have the L^2-complex

$$0 \to L^2(\Gamma; E_{V_{\lambda+2\rho_n}}) \to L^2(\Gamma; E_{V^1_{\lambda+2\rho_n}}) \to \cdots \to L^2(\Gamma; E_{V^m_{\lambda+2\rho_n}}) \to 0. \quad (6.1)$$

As noticed in [10, § 1, (1.3)], due to the completeness of the Riemann metric on $\Gamma \backslash X$ (when we assume that Γ is torsion-free), the L^2-complex (6.1) enjoys a nice property. That is, similar to [18, Lemma 4.3], if

$$\|s\|_\Gamma < \infty \quad \text{and} \quad \|\Box s\|_\Gamma < \infty \qquad \text{for } s \in C^\infty(\Gamma; E_{V^q_{\lambda+2\rho_n}}),$$

then

$$\|\mathscr{D} s\|^2_\Gamma + \|\mathscr{D}^* s\|^2_\Gamma < \infty \quad \text{and} \quad (\Box s, s)_\Gamma = (s, \Box s)_\Gamma = \|\mathscr{D} s\|^2_\Gamma + \|\mathscr{D}^* s\|^2_\Gamma \quad (6.2)$$

and the laplacian $\Box$ is essentially self-adjoint, where $\mathscr{D}^*$, $\Box$ are as in Lemma 3.3.

Defining the L^2-cohomologies of (6.1) by

$$H^q_{(2)}(\Gamma; \mathbb{E}_{\lambda+2\rho_n}) = \{s \in L^2(\Gamma; E_{V^q_{\lambda+2\rho_n}});\ \Box s = 0\}$$

where $\Box$ acts on $L^2(\Gamma; E_{V^q_{\lambda+2\rho_n}})$ in the sense of distributions, one has

$$H^q_{(2)}(\Gamma; \mathbb{E}_{\lambda+2\rho_n}) = \{s \in C^\infty(\Gamma; E_{V^q_{\lambda+2\rho_n}});\ \|s\|_\Gamma < \infty, \mathscr{D} s = \mathscr{D}^* s = 0\},$$

by the above property (6.2).

We have the following vanishing theorem.

Theorem 2. *Let* $\lambda \in \mathscr{F}_0$. *Assume*

(1) $(\lambda, \alpha) > 0$ *for every* $\alpha \in P_n$, *and*

(2) *for every* $Q \subset P_n$ *such that* $\lambda + 2\rho_n - \langle Q \rangle \in \mathscr{F}_0$, *we also have* $\lambda + \rho_n - \langle Q \rangle \in \mathscr{F}_0^k$.

Then $H^q_{(2)}(\Gamma; \mathbb{E}_{\lambda + 2\rho_n}) = 0$ *for every* $q > 0$.

Proof. Applying the results of Kostant [12], we first show that

$$|\lambda + \rho - \langle Q \rangle| < |\lambda + \rho| \qquad \text{for every } Q \neq \emptyset \text{ in } P_n \tag{6.3}$$

under the assumption (1). By [12, Lemma 5.9], $\rho - \langle Q \rangle \in \mathscr{F}$ is a weight of the irreducible $\mathfrak{g}^{\mathbb{C}}$-module with highest weight ρ. Hence

$$|\rho|^2 \geqq |\rho - \langle Q \rangle|^2$$

by [12, Lemma 5.8]. This gives

$$-2(\rho, \langle Q \rangle) + |\langle Q \rangle|^2 \leqq 0.$$

On the other hand, by the assumption (1),

$$-2(\lambda, \langle Q \rangle) < 0 \qquad \text{for } Q \neq \emptyset \text{ in } P_n.$$

Hence $-2(\lambda + \rho, \langle Q \rangle) + |\langle Q \rangle|^2 < 0$ for $Q \neq \emptyset$ in P_n, which implies (6.3) immediately.

To prove the vanishing, we now assume $H^q_{(2)}(\Gamma; \mathbb{E}_{\lambda + 2\rho_n}) \neq 0$ for some $q > 0$. Then there exists non-zero $s \in L^2(\Gamma; E_{V^q_{\lambda + 2\rho_n}})$ such that

$$\nu(\Omega)\, s = (|\lambda + \rho|^2 - |\rho|^2)\, s,$$

by the formula of $\square$ given in Lemma 3.3. Therefore we may assume that there exists an irreducible component $V_\mu \subset V^q_{\lambda + 2\rho_n}$ with highest weight $\mu \in \mathscr{F}_0$ for which there exists non-zero

$$s \in L^2(\Gamma; E_{V_\mu}) \tag{6.4}$$

such that $\nu(\Omega)\, s = (|\lambda + \rho|^2 - |\rho|^2)\, s$, where $L^2(\Gamma; E_{V_\mu}) \subset L^2(\Gamma; E_{V^q_{\lambda + 2\rho_n}})$ is the space of Γ-invariant L^2 sections of the vector bundle E_{V_μ} associated to V_μ under the metric restricted to V_μ of $(\ ,\ )_{V^q_{\lambda + 2\rho_n}}$. By the construction of $\mathbb{E}_{\lambda + 2\rho_n}$ (in Lemma 3.3), $V_\mu \subset V_{\lambda + \rho_n} \otimes L$ and $\mu \neq \lambda + 2\rho_n$. Since the highest weight of an irreducible component of a tensor product can be expressed as a sum of the highest weight of one and a weight of another, there exists $Q \neq \emptyset$ in P_n such that

$$\mu = \lambda + 2\rho_n - \langle Q \rangle \in \mathscr{F}_0$$

by the observation in § 2. By the assumption (2), $\mu - \rho_n \in \mathscr{F}_0^k$. Hence one can consider the elliptic complex $\mathbb{E}_\mu$ by Lemma 3.3, whose laplacian $\square_\mu$ has a form

$$\square_\mu = -\nu(\Omega) + (|\mu - \rho_n + \rho_k|^2 - |\rho|^2).$$

The corresponding L^2-complex (6.1) of $\mathbb{E}_\mu$ has the first term $L^2(\Gamma;\, E_{V_\mu})$. Letting $\square_\mu$ act on $s\in L^2(\Gamma;\, E_{V_\mu})$ as chosen in (6.4), one has

$$\begin{aligned}\square_\mu s &= -\nu(\Omega)\, s + (|\mu-\rho_n+\rho_k|^2-|\rho|^2)\, s\\ &= (|\rho|^2-|\lambda+\rho|^2)\, s + (|\mu-\rho_n+\rho_k|^2-|\rho|^2)\, s\\ &= (|\mu-\rho_n+\rho_k|^2-|\lambda+\rho|^2)\, s\\ &= (|\lambda+\rho-\langle Q\rangle|^2-|\lambda+\rho|^2)\, s,\end{aligned}$$

where one should note that $s\in C^\infty(\Gamma;\, E_{V_\mu})$. Since $\square_\mu s=\mathscr{D}^*\mathscr{D} s$ in $\mathbb{E}_\mu$, this leads to

$$\|\mathscr{D} s\|_\Gamma^2 = (|\lambda+\rho-\langle Q\rangle|^2-|\lambda+\rho|^2)\|s\|_\Gamma^2$$

in view of (6.2). By (6.3), the right hand side is strictly negative, while the left hand side is non-negative. This is a contradiction; hence the theorem. q.e.d.

Remark 1. As is seen in the above proof and in the proof of [18, Theorem 2], the assumption (2) can be replaced by (2)′ for every irreducible K-submodule V_μ in $V_{\lambda+\rho_n}\otimes V_{\rho_n}$, $\mu-\rho_n\in\mathscr{F}_0^k$ where V_{ρ_n} is the irreducible $\mathfrak{k}$-module with highest weight $\rho_n\in\mathscr{F}_0^k$.

§ 7. *K*-Types of Discrete Classes

In this section we shall prove (I), (II) and (III) of the theorem stated in the introduction, applying Theorems 1 and 2 of the previous sections.

We first recall the basic facts on the discrete series $\mathscr{E}_2$ for G according to Harish-Chandra [7]. Since G has been assumed to have a compact Cartan subgroup T, the discrete series $\mathscr{E}_2$ of G is non-empty. Denote by $\mathscr{F}'$ the set of regular integral linear forms in $\mathscr{F}$, i.e.,

$$\mathscr{F}'=\{\Lambda\in\mathscr{F};\, (\Lambda,\alpha)\neq 0\ (\alpha\in\Sigma)\}.$$

Harish-Chandra's surjection ω: $\mathscr{F}'\to\mathscr{E}_2$, which has been referred to in the introduction, has the following properties. First $\omega(\Lambda)=\omega(\Lambda')$ for $\Lambda,\Lambda'\in\mathscr{F}'$ if and only if there exists a $w\in W$ such that $w\Lambda=\Lambda'$. Here, as before, W denotes the Weyl group for the pair (G, T) which is identified with that for (K, T). Secondly when $\Lambda\in\mathscr{F}'$ is given, a positive root system P in Σ is chosen such that

$$P=\{\alpha\in\Sigma;\, (\Lambda,\alpha)>0\}. \tag{7.1}$$

Then, the restriction of the distribution character $\Theta_{\omega(\Lambda)}$ of $\omega(\Lambda)$ to the set of regular elements of T has the form

$$(-1)^m\Big(\sum_{w\in W}\varepsilon(w)\, e^{w\Lambda}\Big)\Big/\prod_{\alpha\in P}\big(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}\big)$$

where $m=|P_n|$.

For $\Lambda \in \mathscr{F}'$, choose P as in (7.1) and put

$$\lambda = \Lambda - \rho. \tag{7.2}$$

Then λ is dominant with respect to P and $\lambda + \rho_n \in \mathscr{F}_0^k$. Hence we can consider the elliptic complex $\mathbb{E}_{\lambda+2\rho_n}$ and its L^2-cohomologies as in § 6. Specifically for $\Gamma = \{1\}$ the identity group, $H^q_{(2)}(\{1\}; \mathbb{E}_{\lambda+2\rho_n})$ gives a unitary representation π^q_λ for each $0 \leqq q \leqq m$. As in [18, § 6], one can see that π^q_λ is a finite sum of discrete series representations since the representation space is an eigenspace of the Casimir operator; hence it has the distribution character Trace π^q_λ. Based on the method of Narasinhan-Okamoto [16], one then has the alternating sum formula

$$\sum_{q=0}^{m} (-1)^q \operatorname{Trace} \pi^q_\lambda = \Theta_{\omega(\lambda+\rho)}$$

in view of Lemma 3.3 and [18, Theorem 1]. Combining this with Theorem 2 in § 6, we have the following realization theorem.

Theorem 3. *For $\Lambda \in \mathscr{F}'$, define λ by (7.2). Assume moreover that λ satisfies assumptions (1), (2) of Theorem 2 in § 6. Then the representation π^0_λ on $H^0_{(2)}(\{1\}; \mathbb{E}_{\lambda+2\rho_n})$ belongs to the discrete class $\omega(\lambda+\rho) = \omega(\Lambda)$.*

We are now ready to show (I) and (III) of the theorem in the introduction. Since the assumption (ii)^b in the introduction is the same as (2) in Theorem 2, (I) now follows from Theorem 3 and the formula for the laplacian in Lemma 3.3. For (III), the space of K-finite vectors of $H^0_{(2)}(\{1\}; \mathbb{E}_{\lambda+2\rho_n})$ is clearly embedded in $H^0(\mathbb{E}_{\lambda+2\rho_n})^0$ of §§ 4, 5 as K-modules. By Lemma 4.1, the condition (#) for $\lambda + 2\rho_n$ is satisfied under the assumption (ii)^a in the introduction. Hence by Theorem 1 in § 4, the multiplicity of V_μ in π^0_λ is dominated by the Blattner number $b_\lambda(\mu)$ for every $\mu \in \mathscr{F}_0$. We thus have shown one part of Blattner's conjecture (III).

The proof of the lowest K-type Theorem (II) can be carried out quite similarly to that of [21, Lemma 9], by combining Theorem 1 with Theorem 3. For the sake of completeness and later use, we reproduce the arguments, some of which have been unpublished. We start with a lemma which is the last proposition in [20, Announcement]. Let $C^\infty(\mathbb{1}_G) \otimes \operatorname{End} V_{\lambda+2\rho_n}$ (resp. $C^\infty(\mathbb{1}_G) \otimes \operatorname{Hom}(V_{\lambda+2\rho_n}, V^1_{\lambda+2\rho_n})$) be the space of C^∞-functions on G with values in $\operatorname{End} V_{\lambda+2\rho_n}$ (resp. $\operatorname{Hom}(V_{\lambda+2\rho_n}, V^1_{\lambda+2\rho_n})$). Define the first-order differential operator

$$\tilde{\mathscr{D}}\colon C^\infty(\mathbb{1}_G) \otimes \operatorname{End} V_{\lambda+2\rho_n} \to C^\infty(\mathbb{1}_G) \otimes \operatorname{Hom}(V_{\lambda+2\rho_n}, V^1_{\lambda+2\rho_n}) \tag{7.3}$$

as the composite of

$$\nabla \otimes 1_{V^*_{\lambda+2\rho_n}}\colon C^\infty(\mathbb{1}_G) \otimes V_{\lambda+2\rho_n} \otimes V^*_{\lambda+2\rho_n} \to C^\infty(\mathbb{1}_G) \otimes \mathfrak{p} \otimes V_{\lambda+2\rho_n} \otimes V^*_{\lambda+2\rho_n}$$

and

$$1_{C^\infty(\mathbb{1}_G)} \otimes \sigma \otimes 1_{V^*_{\lambda+2\rho_n}}: \ C^\infty(\mathbb{1}_G) \otimes \mathfrak{p} \otimes V_{\lambda+2\rho_n} \otimes V^*_{\lambda+2\rho_n} \to C^\infty(\mathbb{1}_G) \otimes V^1_{\lambda+2\rho_n} \otimes V^*_{\lambda+2\rho_n}$$

where V is as in (3.1) and $\sigma: \mathfrak{p} \otimes V_{\lambda+2\rho_n} \to V^1_{\lambda+2\rho_n}$ is the symbol as in (3.2). In the above, we have adopted the natural identifications

$$\operatorname{End} V_{\lambda+2\rho_n} = V_{\lambda+2\rho_n} \otimes V^*_{\lambda+2\rho_n}, \quad \operatorname{Hom}(V_{\lambda+2\rho_n}, V^1_{\lambda+2\rho_n}) = V^1_{\lambda+2\rho_n} \otimes V^*_{\lambda+2\rho_n}$$

where $V^*_{\lambda+2\rho_n}$ is the dual space of $V_{\lambda+2\rho_n}$. Then Schmid obtains the following result.

Lemma 7.1. *Assume $(H^0(\mathbb{E}_{\lambda+2\rho_n})^0 : V_{\lambda+2\rho_n}) \leqq 1$. There then exists at most one $F \in C^\infty(\mathbb{1}_G) \otimes \operatorname{End} V_{\lambda+2\rho_n}$ such that*

(1) *$F(1)$ is the identity map of $V_{\lambda+2\rho_n}$,*

(2) *$F(k_1 g k_2) = k_2^{-1} F(g) k_1^{-1}$ for $k_1, k_2 \in K$ and $g \in G$,*

(3) *$\tilde{\mathscr{D}} F = 0$.*

In this statement, (3) *can be replaced by*

(3)′ *$\int_K \chi(k)(v(x) F)(gk)\, dk = 0$ for every character χ of an irreducible component of $V^1_{\lambda+2\rho_n}$. Here for every $x \in \mathfrak{p}$ $v(x)$ denotes the action of x on F as a left invariant vector field.*

Proof. Let $F \in C^\infty(\mathbb{1}_G) \otimes \operatorname{End} V_{\lambda+2\rho_n}$ satisfy (1), (2), (3). For $v \in V_{\lambda+2\rho_n}$, define $F_v \in C^\infty(\mathbb{1}_G) \otimes V_{\lambda+2\rho_n}$ by $F_v(g) = F(g)(v) \in V_{\lambda+2\rho_n}$ $(g \in G)$. By (2), F_v can be regarded as an element of $C^\infty(E_{V_{\lambda+2\rho_n}})$, and by (3), F_v is annihilated by $\mathscr{D}: C^\infty(E_{V_{\lambda+2\rho_n}}) \to C^\infty(E_{V^1_{\lambda+2\rho_n}})$ in $\mathbb{E}_{\lambda+2\rho_n}$. Define the map

$$\varphi: \ V_{\lambda+2\rho_n} \to H^0(\mathbb{E}_{\lambda+2\rho_n})$$

by $\varphi(v) = F_v$ for $v \in V_{\lambda+2\rho_n}$. By (2), φ is then a K-module map and φ is non-zero by (1). Since $(H^0(\mathbb{E}_{\lambda+2\rho_n})^0 : V_{\lambda+2\rho_n}) \leqq 1$, φ is unique by (1) and $(H^0(\mathbb{E}_{\lambda+2\rho_n})^0 : V_{\lambda+2\rho_n}) = 1$. Since $F(g)(v) = \varphi(v)(g)$, such F must be unique.

We next show that (3)′ implies (3). It is easily seen that every component of the K-module $V_{\lambda+2\rho_n} \otimes \mathfrak{p}$ has multiplicity one. Hence by the definition (7.3) of $\tilde{\mathscr{D}}$, in order that $\tilde{\mathscr{D}} F = 0$, it is sufficient that

$$\int_K \bar{\chi}(k)\, k_i\big(\sum_i x_i \otimes (v(x_i) F)\big)\, dk = 0$$

for every character χ of an irreducible component of $V^1_{\lambda+2\rho_n}$, where $k_i(\sum_i x_i \otimes (v(x_i) F))$ denotes the action of k on the part $\mathfrak{p} \otimes V_{\lambda+2\rho_n}$ in $\mathfrak{p} \otimes V_{\lambda+2\rho_n} \otimes V^*_{\lambda+2\rho_n}$. In view of the K-invariance of $\sum_i x_i \otimes v(x_i)$, the above means

$$\sum_i x_i \otimes \int_K \bar{\chi}(k)(v(x_i) F)(g k^{-1})\, dk = 0$$

for $g\in G$. If (3)' holds for F, then the above holds for F; hence (3) holds. q.e.d.

We notice that if $V_\mu \subset V^1_{\lambda+2\rho_n}$ is an irreducible component, then $\mu=\lambda+2\rho_n-\alpha$ for some $\alpha\in P_n$ by the definition of $V^1_{\lambda+2\rho_n}$ in § 3. Using Lemma 7.1, the following lemma can then be proved in quite the same way as in the proof of [21, Lemma 9], so we omit the proof.

Lemma 7.2. *Assume* $(H^0(\mathbb{E}_{\lambda+2\rho_n})^0 : V_{\lambda+2\rho_n}) \leq 1$ *for a* $\lambda\in\mathscr{F}$ such that $\lambda+\rho_n\in\mathscr{F}_0^k$. *Let* π *be an irreducible unitary representation such that the restriction* $\pi|K$ *contains* $V_{\lambda+2\rho_n}$ *but does not contain* $V_{\lambda+2\rho_n-\alpha}$ *for every* $\alpha\in P_n$ *such that* $\lambda+2\rho_n-\alpha\in\mathscr{F}_0$. *Then* π *is unique up to equivalence.*

Now Theorem (II) in the introduction immediately follows from this. Under the assumption (ii)[a], the assumption in Lemma 7.2 is satisfied in view of Theorem 1 and Lemma 4.1. It is easy to see that $b_\lambda(\lambda+2\rho_n-\alpha)=0$ for $\alpha\in P_n$. Hence by Theorem (III)

$$(\omega(\lambda+\rho)|K : V_{\lambda+2\rho_n-\alpha})=0 \qquad (\alpha\in P_n).$$

As observed before, we actually know that

$$(\omega(\lambda+\rho)|K : V_{\lambda+2\rho_n})=1.$$

Hence Lemma 7.2 leads to the Lowest K-type Theorem (II).

§ 8. Multiplicity of Discrete Classes in $L^2(\Gamma\backslash G)$

In this section we shall concentrate on Theorem (IV) in the introduction. Let Γ be a discrete subgroup of G with compact quotient $\Gamma\backslash G$. Let $\hat{G}$ be the unitary dual of G, i.e., the set of equivalence classes of irreducible unitary representations, and denote by $N_\omega(\Gamma)$ the multiplicity of a class $\omega\in\hat{G}$ in the right regular representation on $L^2(\Gamma\backslash G)$. It is then well-known that $N_\omega(\Gamma)$ is finite for any $\omega\in\hat{G}$, and $L^2(\Gamma\backslash G)$ breaks up into a direct sum of irreducible components; i.e.,

$$L^2(\Gamma\backslash G)=\bigoplus_{\omega\in\hat{G}} N_\omega(\Gamma)\, H_\omega, \tag{8.1}$$

where $N_\omega(\Gamma)\, H_\omega$ is the sum of $N_\omega(\Gamma)$ copies of a representation space H_ω belonging to $\omega\in\hat{G}$.

We now consider the L^2-cohomologies of the complex (6.1) for our Γ and write

$$H^q(\Gamma;\mathbb{E}_{\lambda+2\rho_n})=H^q_{(2)}(\Gamma;\mathbb{E}_{\lambda+2\rho_n})$$

for $\lambda\in\mathscr{F}$ such that $\lambda+\rho_n\in\mathscr{F}_0^k$.

Lemma 8.1. *Assume* $\lambda+2\rho_n$ *satisfies the condition* (#) *in the sense of* § 3. *Suppose that there exists* $\omega_\lambda\in\hat{G}$ *for which the K-module* $H^0_{\omega_\lambda}$ *consisting of K-finite vectors in the representation space* H_{ω_λ} *of* ω_λ *can be embedded* *in* $H^0(\mathbb{E}_{\lambda+2\rho_n})^0$ *as* K*-modules and actually contains the* K*-type* $V_{\lambda+2\rho_n}$. *Then such* ω_λ *is unique in* $\hat{G}$ *and* $N_{\omega_\lambda}(\Gamma)=\dim H^0(\Gamma;\mathbb{E}_{\lambda+2\rho_n})$.

Proof. The uniqueness of ω_λ is a consequence of Lemma 7.2 since the condition (#) for $\lambda+2\rho_n$ leads to the assumption of Lemma 7.2 and a representation belonging to $\omega_\lambda\in\hat{G}$ satisfies the condition of that lemma by Theorem 1. Note that $(\omega_\lambda|K:V_{\lambda+2\rho_n})=1$ by our assumption.

Let π be an irreducible unitary representation of G on a Hilbert space H_π. We define the linear operator

$$\pi(\mathscr{D})\colon\ \mathrm{Hom}_K(H_\pi, V_{\lambda+2\rho_n})\to \mathrm{Hom}_K(H_\pi, V^1_{\lambda+2\rho_n}) \tag{8.2}$$

following [21]. We first notice that the space $\mathrm{Hom}_K(H_\pi, V)$ of K-maps from H_π into a finite dimensional K-module V can be identified with $\mathrm{Hom}_K(H_\pi^0, V)$ by restricting maps to the space H_π^0 of K-finite vectors in H_π. Note that $\mathrm{Hom}_K(H_\pi, V)$ is finite dimensional. Since elements in H_π^0 are differentiable vectors and the Lie algebra $\mathfrak{g}$ of G has a skew-hermitian representation on H_π^0, we can define the map

$$\pi(\nabla)\colon \mathrm{Hom}_K(H_\pi^0, V_{\lambda+2\rho_n})\to \mathrm{Hom}_K(H_\pi^0, \mathfrak{p}\otimes V_{\lambda+2\rho_n})$$

by

$$\pi(\nabla)\varphi=\sum_{i=1}^{2m} x_i\otimes\varphi\circ\pi(x_i)\qquad \text{for }\ \varphi\in\mathrm{Hom}_K(H_\pi^0, V_{\lambda+2\rho_n})$$

where $\{x_i\}_{i=1}^{2m}$ is an orthonormal basis of $\mathfrak{p}$. Composing $\pi(\nabla)$ with the symbol map

$$\sigma\colon\ \mathfrak{p}\otimes V_{\lambda+2\rho_n}\to V^1_{\lambda+2\rho_n}$$

in (3.2), we define

$$\pi(\mathscr{D})\,\varphi=\sigma\circ\pi(\nabla)\,\varphi$$

which is easily shown to be in $\mathrm{Hom}_K(H_\pi^0, V^1_{\lambda+2\rho_n})$. Extending $\pi(\mathscr{D})\varphi$ uniquely to $\mathrm{Hom}_K(H_\pi, V^1_{\lambda+2\rho_n})$, we have the map (8.2). Defining the map

$$\sigma(x)\colon\ V_{\lambda+2\rho_n}\to V^1_{\lambda+2\rho_n}\qquad \text{for }\ x\in\mathfrak{p}$$

by $\sigma(x)\,v=\sigma(x\otimes v)$ for $v\in V_{\lambda+2\rho_n}$, we can write $\pi(\mathscr{D})$ as

$$\pi(\mathscr{D})\,\varphi=\sum_i\sigma(x_i)\circ\varphi\circ\pi(x_i) \tag{8.3}$$

for $\varphi\in\mathrm{Hom}_K(H_\pi^0, V_{\lambda+2\rho_n})$.

Let π_1,π_2 be two irreducible unitary representations of G and denote by $\mathrm{Ker}\,\pi_i(\mathscr{D})$ $(i=1,2)$ the kernel of the operator $\pi_i(\mathscr{D})$ in (8.2). We will show now that if $\mathrm{Ker}\,\pi_i(\mathscr{D})\neq 0$ for both $i=1$ and 2, then π_1 is equivalent to π_2, of course under the conditions stated in the lemma; i.e., $\mathrm{Ker}\,\pi(\mathscr{D})\neq 0$ assures the uniqueness of π up to equivalence. Write for a moment $\pi=\pi_i$ $(i=1$ or $2)$. Let $\varphi\neq 0$ in $\mathrm{Ker}\,\pi(\mathscr{D})$. Then $\pi|K$ contains $V_{\lambda+2\rho_n}$ and one can choose $\psi\in\mathrm{Hom}_K(V_{\lambda+2\rho_n}, H_\pi)$ such that $\varphi\circ\psi=$ identity. Define an End $V_{\lambda+2\rho_n}$-valued function $F\in C^\infty(\mathbb{1}_G)\otimes \operatorname{End} V_{\lambda+2\rho_n}$ by

$$F(g)=\varphi\circ\pi(g^{-1})\circ\psi \quad \text{for } g\in G.$$

Then F clearly satisfies the conditions (1), (2) in Lemma 7.1. We will show that F also satisfies (3), $\tilde{\mathscr{D}}F=0$. Equip the spaces $V_{\lambda+2\rho_n}$, $V^1_{\lambda+2\rho_n}$ and H_π with the inner products $(\ ,\ )_0$, $(\ ,\ )_1$ and $(\ ,\ )_\pi$ respectively. Denote by $\sigma(x)^*$ and φ^* the adjoint operators of $\sigma(x)$ and φ under the metrics. Note that $\pi(x)^*=-\pi(x)$ on H^0_π for $x\in\mathfrak{g}$. For $v\in V_{\lambda+2\rho_n}$, $u\in V^1_{\lambda+2\rho_n}$ and $g\in G$, we have

$$\begin{aligned}(\tilde{\mathscr{D}}F(g)(v),u)_1&=\sum_i(\sigma(x_i)(v(x_i)F)(g)(v),u)_1\\&=\sum_i((v(x_i)F)(g)(v),\sigma(x_i)^*u)_0\end{aligned}$$

by the definitions. Denoting by $e^{tx}\in G$ $(t\in\mathbb{R})$ the one-parameter subgroup generated by $x\in\mathfrak{g}$, we see the last term equals the value of

$$\sum_i\frac{d}{dt}(\varphi\circ\pi(e^{-tx_i})\,\pi(g^{-1})\circ\psi(v),\sigma(x_i)^*u)_0$$

at $t=0$. Since π is unitary, it equals

$$\sum_i(\pi(g^{-1})\circ\psi(v),\pi(x_i)\circ\varphi^*\circ\sigma(x_i)^*u)_\pi.$$

Since $\varphi\in\operatorname{Ker}\pi(\mathscr{D})$,

$$\sum_i\pi(x_i)\circ\varphi^*\circ\sigma(x_i)^*=0$$

by (8.3). Hence $\tilde{\mathscr{D}}F=0$. By Lemma 7.1, F is unique. Noting that F is, by definition, a spherical function of type $V_{\lambda+2\rho_n}$ of π, the uniqueness of F leads to the uniqueness of the equivalence class of π in $\hat{G}$.

Now the proof of Lemma 8.1 is immediate. Let

$$L^2(\Gamma\backslash G)=\bigoplus_{\omega\in\hat{G}}N_\omega(\Gamma)\,H_\omega$$

be the decomposition of (8.1). Note that $N_\omega(\Gamma)=N_{\omega^*}(\Gamma)$ for the contragredient class ω^* of ω. Then a standard reasoning [for example [21], Lemma 6′], we have

$$H^0(\Gamma;\mathbb{E}_{\lambda+2\rho_n})=\bigoplus_{\omega\in\hat{G}}N_\omega(\Gamma)\operatorname{Ker}\omega(\mathscr{D}) \tag{8.4}$$

where $\operatorname{Ker}\omega(\mathscr{D})=\operatorname{Ker}\pi(\mathscr{D})$ and π is a representation belonging to the class ω. From what has been proved just before, under the condition (#) for $\lambda+2\rho_n\in\mathscr{F}_0$, there exists at most one $\omega_0\in\hat{G}$ such that $\operatorname{Ker}\omega_0(\mathscr{D})\neq 0$. By the assumption of Lemma 8.1, about ω_λ we have $(\omega_\lambda|K:V_{\lambda+2\rho_n})=1$ and $(\omega_\lambda|K:V_{\lambda+2\rho_n-\alpha})=0$ for $\alpha\in P_n$ by Theorem 1. Hence

$$\operatorname{Hom}_K(H_{\omega_\lambda}:V^1_{\lambda+2\rho_n})=0$$

and by the definition (8.2), we actually have $\dim \operatorname{Ker} \omega_\lambda(\mathscr{D})=1$. Since $\operatorname{Ker} \omega(\mathscr{D})=0$ for $\omega \neq \omega_\lambda$ in $\hat{G}$, by (8.4)

$$\dim H^0(\Gamma; \mathbb{E}_{\lambda+2\rho_n})=N_{\omega_\lambda}(\Gamma).$$

This completes the proof of Lemma 8.1. q.e.d.

We now consider $\dim H^0(\Gamma; \mathbb{E}_{\lambda+2\rho_n})$. In view of the relationship of the elliptic omplex $\mathbb{E}_{\lambda+2\rho_n}$ with the Dirac operator observed in Lemma 3.3, the computation at the index level has already been given in our previous paper [10, § 2] using the Atiyah-Bott-Singer Lefschetz fixed point formula. We recall it here.

Following [10], we first introduce the function Ψ_λ on the elliptic elements of G (elements which are conjugate to those in T). Fixing an element $\gamma \in T$ for a moment, let G_γ be the centralizer of γ in G, and G_γ^0 the identity component of G_γ. Then G_γ^0 contains T as its Cartan subgroup. Denote by W_γ the Weyl group for the pair (G_γ^0, T). Then W_γ can be identified with a subgroup of W, the Weyl group for (G, T). We denote by $[G_\gamma : G_\gamma^0]$ the number of connected components of G_γ. Denote by P_γ the set of positive roots $\alpha \in P$ such that $e^\alpha(\gamma)=1$ and put $\rho_\gamma=\frac{1}{2}\langle P_\gamma\rangle$. Then P_γ is a positive root system for $(\mathfrak{g}_\gamma^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ where $\mathfrak{g}_\gamma^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{g}_\gamma$ of G_γ^0. We denote by $W_\gamma^{\mathbb{C}}$ the Weyl group for the pair $(\mathfrak{g}_\gamma^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and by $|W_\gamma^{\mathbb{C}}|$ its order. We put $m_\gamma=\frac{1}{2}\dim_R G_\gamma/K_\gamma=|P_\gamma \cap P_n|$ where $K_\gamma=G_\gamma \cap K$.

We now define the function Ψ_λ on T by

$$\Psi_\lambda(\gamma)=(-1)^{m+m_\gamma}|W_\gamma^{\mathbb{C}}|^{-1}[G_\gamma : G_\gamma^0]^{-1} \prod_{\alpha \in P_\gamma} (\rho_\gamma, \alpha)^{-1} \Psi_\lambda'(\gamma)$$

for $\gamma \in T$, where

$$\Psi_\lambda'(\gamma)=\frac{\sum_{w \in W_\gamma \setminus W} \varepsilon(w)\, e^{w(\lambda+\rho)-\rho}(\gamma) \prod_{\alpha \in P_\gamma} (w(\lambda+\rho), \alpha)}{\prod_{\alpha \in P-P_\gamma} (1-e^{-\alpha}(\gamma))}.$$

(Notice that this Ψ_λ differs from that in [10] up to the sign $j(\lambda)$ and our W was denoted by W_G there.) It is then seen that Ψ_λ is invariant under the W-action on T; hence Ψ_λ can be extended to a function, which will be denoted also by Ψ_λ, on the set of all elliptic elements on G invariant on the conjugacy classes.

We normalize the Haar measure on G_γ so that, for every discrete subgroup Γ_0 of G_γ such that $\Gamma_0 \setminus G_\gamma/T$ is a compact manifold, the volume $v(\Gamma_0 \setminus G_\gamma)$ of $\Gamma_0 \setminus G_\gamma$ under this measure equals the Euler number of $\Gamma_0 \setminus G_\gamma/T$ up to the sign $(-1)^{m_\gamma}$.

Lemma 8.2. *For $\lambda \in \mathscr{F}$ such that $\lambda+\rho_n \in \mathscr{F}_0^k$,*

$$\sum_{q=0}^{m} (-1)^q \dim H^q(\Gamma; \mathbb{E}_{\lambda+2\rho_n})=\sum_{\gamma} v(\Gamma_\gamma \setminus G_\gamma)\, \Psi_\lambda(\gamma)$$

where $\Gamma_\gamma = \Gamma \cap G_\gamma$ and in the summation γ runs over representatives of all the Γ-conjugacy classes of elliptic elements in Γ.

Proof. Note that by Lemma 3.3 $\sum_{q=0}^{m} (-1)^q \dim H^q(\Gamma; \mathbb{E}_{\lambda+2\rho_n})$ equals the index of the Dirac operator

$$D\colon\ C^\infty(E_{L^+\otimes V_{\lambda+\rho_n}}) \to C^\infty(E_{L^-\otimes V_{\lambda+\rho_n}})$$

over $\Gamma\backslash X$, which is denoted by $\chi(\Gamma; \lambda)$ in [10]. Lemma 8.2 is hence an interpretation of [10, Theorem 3] in view of the difference of the definitions of Ψ_λ. q.e.d.

We are now ready to prove Theorem (IV) in the introduction. Let $\lambda \in \mathscr{F}$ satisfy $(\lambda+\rho, \alpha) > 0$ for all positive roots $\alpha \in P$. If λ moreover satisfies the assumptions (1) and (2) of Theorem 2 in § 6, the representation π_λ^0 on $H_{(2)}^0(\{1\}; \mathbb{E}_{\lambda+2\rho_n})$ belongs to the discrete class $\omega(\lambda+\rho) \in \mathscr{E}_2$. Let $\omega(\lambda+\rho)$ be taken as the $\omega_\lambda \in \hat{G}$ in Lemma 8.1. Hence we have

$$N_{\omega(\lambda+\rho)}(\Gamma) = \dim H^0(\Gamma; \mathbb{E}_{\lambda+2\rho_n})$$

under the condition $(\#)$ for $\lambda+2\rho_n$.

On the other hand, in this case we also know

$$H^q(\Gamma; \mathbb{E}_{\lambda+2\rho_n}) = 0 \qquad \text{for } q > 0$$

by Theorem 2. Therefore we have the formula for $\dim H^0(\Gamma; \mathbb{E}_{\lambda+2\rho_n})$ and hence $N_{\omega(\lambda+\rho)}(\Gamma)$.

Theorem 4. *Assume that $\lambda \in \mathscr{F}_0$ satisfies*

(1) $(\lambda, \alpha) > 0$ $(\alpha \in P_n)$,

(2)′ *the condition $(\#)$ for $\lambda+2\rho_n$ and the assumption* (2) *of Theorem* 2 *in* § 6. *Then for the discrete class $\omega(\lambda+\rho) \in \mathscr{E}_2$,*

$$N_{\omega(\lambda+\rho)}(\Gamma) = \dim H^0(\Gamma; \mathbb{E}_{\lambda+2\rho_n}) = \sum_{\gamma} v(\Gamma_\gamma \backslash G_\gamma)\, \Psi_\lambda(\gamma)$$

where the last term is as in Lemma 8.2.

To show Theorem (IV) in the introduction, it suffices to note that the assumptions $(\mathrm{ii})^a$ and $(\mathrm{ii})^b$ in the introduction imply (2)′ (by Lemma 4.1) and that, when Γ has no elliptic elements other than the identity, the last term in the formula reduces to $v(\Gamma\backslash G)\,\Psi_\lambda(1)$ which has been shown to equal $d_{\omega(\lambda+\rho)} v(\Gamma\backslash G)$ in [10, § 3].

Remark 2. A discrete class $\omega \in \mathscr{E}_2$ is called "integrable" when the K-finite matrix coefficients of ω are integrable functions on G. Let $\mathscr{E}_1$ be the set of all integrable discrete classes of G. Trombi and Varadarajan showed in [23, Theorem 8.2] that, if $\omega(\Lambda)\in\mathscr{E}_1$ for $\Lambda\in\mathscr{F}'$, then

$$|(\Lambda,\alpha)|>\tfrac{1}{2}\sum_{\beta\in P}|(\beta,\alpha)| \qquad \text{for every } \alpha\in P_n. \tag{8.5}$$

Now let P be chosen as in (7.1) for $\Lambda\in\mathscr{F}'$, and let π be the set of all simple roots in P. Set $\pi_k=\pi\cap P_k$ and $\pi_n=\pi\cap P_n$. Then one knows $\pi_n\neq\emptyset$. For $\alpha\in P_n$, put

$$k(\alpha)=\sum_{\beta\in P}|(\beta,\alpha)|/|\alpha|^2,$$

which is a positive integer. In the Weyl chamber defined by

$$\{\mu\in\mathrm{Hom}(\sqrt{-1}\,\mathfrak{t},\mathbb{R});\ (\mu,\alpha)\geqq 0\ (\alpha\in\pi)\},$$

if $\omega(\Lambda)\in\mathscr{E}_1$, then Λ must be away from every wall defined by $\alpha\in\pi_n$ at least by "distance" $k(\alpha)+1$ by the above Trombi-Varadarajan criterion. On the other hand, though the assumption (2)′ in Theorem 4 means that $\Lambda=\lambda+\rho$ is a "little" away from the walls defined by π_k, the assumption (1) means only that Λ is not one of the points "closest" to the walls defined by π_n.

Theorem 4 may not cover all integrable classes $\omega(\Lambda)=\omega(\lambda+\rho)\in\mathscr{E}_1$, in general, because of the assumption (2)′. However, for simple groups G, since the integers $k(\alpha)$ seem to become larger as $\dim G$ become larger, it seems, in general, that Theorem 4 assures the validity of a simple formula for $N_{\omega(\Lambda)}(\Gamma)$ for infinitely many $\omega(\Lambda)\in\mathscr{E}_2-\mathscr{E}_1$. As mentioned in the introduction, though this is not true for $SL(2,\mathbb{R})$, in the following section, we shall see that the above situation actually occurs for $G=SU(m,1)$ and $\mathrm{Spin}(2m,1)$ $(m\neq 1)$.

§ 9. Examples; $SU(m,1)$, $\mathrm{Spin}(2m,1)$ and Holomorphic Discrete Series

Finally, we illustrate our results in some cases, in which somewhat sharper results will be obtained. The condition (#) is automatically satisfied if λ lies in a specific Weyl chamber. Next in the cases $E1$, $E2$ and $E3$ illustrated later, we shall show that every statement of the theorem in the introduction is true without imposing the assumption (ii).

We call a positive root system P *admissible* when $[[\mathfrak{p}_+,\mathfrak{p}_+],\mathfrak{p}_+]=0$ for $\mathfrak{p}_+=\sum_{\alpha\in P_n}\mathfrak{g}^\alpha$, or equivalently, $\alpha+\beta+\gamma$ is never a root for every $\alpha,\beta,\gamma\in P_n$. We know that there always exist admissible positive root systems ([8], Lemma 3.2). (There, actually, admissibility is stronger condition.) We prove the following lemma.

Lemma 9.1. *Let a positive root system P be admissible. Assume that $\lambda\in\mathscr{F}_0$ under this choice of a positive root system. Then the condition* (#)

for $\lambda+2\rho_n$ is satisfied, i.e.,

$$H^i(\wedge^q \mathfrak{p}_- \otimes l_{\lambda+2\rho})=0$$

for every $i<s$ and every q.

For the proof, it suffices to prove $H^i(\wedge^{\cdot} \mathfrak{p}_- \otimes l_{-\lambda})=0$ for every $i>0$. In fact, by the Serre duality, the dual of $H^i(\wedge^q \mathfrak{p}_- \otimes l_{\lambda+2\rho})$ is $H^{s-i}(\wedge^q \mathfrak{p}_+ \otimes l_{-\lambda-2\rho_n})$ since the canonical line bundle over $K^{\mathbb{C}}/B$ is the line bundle associated to the B-module $l_{2\rho_k}$. One easily sees that the dual of $\wedge^{m-q} \mathfrak{p}_+$ equals $\wedge^q \mathfrak{p}_+ \otimes l_{-2\rho_n}$ while it also equals $\wedge^{m-q} \mathfrak{p}_-$. Hence the dual of $H^i(\wedge^q \mathfrak{p}_- \otimes l_{\lambda+2\rho_n})$ equals $H^{s-i}(\wedge^{m-q} \mathfrak{p}_- \otimes l_{-\lambda})$, which shows our assertion.

Consider the parabolic subgroup Q of $K^{\mathbb{C}}$ consisting of elements $k\in K^{\mathbb{C}}$ such that $\mathrm{Ad}(k)\mathfrak{p}_+=\mathfrak{p}_+$, which clearly contains our Borel subgroup B. Let $Q=MU$ be the Levi decomposition where U is the unipotent radical of Q and M is the reductive supplement containing the maximal torus T. When a holomorphic Q-module W is given, we denote by $H^i(K^{\mathbb{C}}/Q;\, W)$ the $K^{\mathbb{C}}$-module given by the i-th cohomology space with coefficients in the sheaf of germs of holomorphic sections of the holomorphic vector bundle over $K^{\mathbb{C}}/Q$ associated to W. Let $\lambda\in\mathscr{F}_0$ and $W_{-\lambda}$ the irreducible M-module with lowest weight $-\lambda$, extended to the Q-module trivially on U. The $\mathfrak{p}_-=\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_+$ has the Q-module structure by quotient. One can then see that

$$H^i(\wedge^q \mathfrak{p}_- \otimes l_{-\lambda})\simeq H^i(K^{\mathbb{C}}/Q;\, \wedge^q \mathfrak{p}_- \otimes W_{-\lambda})$$

for every i and q. For this, consider the fibration $K^{\mathbb{C}}/B\to K^{\mathbb{C}}/Q$ with fibres isomorphic to $Q/B\simeq M/M\cap B$. By the standard argument of the Leray spectral sequence (see [2, § 11]), it suffices to see that under the above fibration the zero-th direct image of the sheaf associated to $\wedge^q \mathfrak{p}_- \otimes l_{-\lambda}$ is isomorphic to that associated to $\wedge^q \mathfrak{p}_- \otimes W_{-\lambda}$ and that all the other direct images vanish. This can be easily seen by means of the Borel-Weil-Bott theorem [2].

Now the following can be also easily seen by Bott's observations in [2] for the Borel-Weil-Bott theorem for the parabolic Q in $K^{\mathbb{C}}$. Let $\mathfrak{u}$ denote the Lie algebra of the unipotent radical U of Q. Suppose V is a $K^{\mathbb{C}}$-module and put

$$W=V^{\mathfrak{u}}=\{v\in V;\ xv=0 \text{ for any } x\in\mathfrak{u}\},$$

so that W is a Q-module. Then

$$H^i(K^{\mathbb{C}}/Q;\, W^*)=0 \qquad (i>0)$$

and $H^0(K^{\mathbb{C}}/Q;\, W^*)$ gives a $K^{\mathbb{C}}$-module isomorphic to V^*, where V^*, W^* are the contragredient modules of V, W respectively. Furthermore, if W_1

is a Q-module which is isomorphic to W as M-modules, then again, the same conclusion holds for the vanishing of $H^i(K^{\mathbb{C}}/Q; W_1^*)$. This follows by using induction on a length of the Jordan-Hölder filtration of W_1^*. Denoting by W_λ the irreducible M-module with highest weight λ extended to the Q-module trivially on U, one sees that the dual of $\wedge \mathfrak{p}_+ \otimes W_\lambda$ is isomorphic to $\wedge \mathfrak{p}_- \otimes W_{-\lambda}$. Hence in order to show $H^i(K^{\mathbb{C}}/Q; \wedge \mathfrak{p}_- \otimes W_{-\lambda}) = 0$ $(i > 0)$, from which Lemma 9.1 follows, it suffices to prove the next lemma.

Lemma 9.2. *When the positive root system P is admissible, $\wedge \mathfrak{p}_+ \otimes W_\lambda$ is isomorphic to $(L \otimes V_{\lambda+\rho_n})^{\mathfrak{u}}$ as M-modules, where $L \otimes V_{\lambda+\rho_n}$ is as in the previous sections.*

Proof. Let $\mathfrak{m}$ denote the Lie algebra corresponding to M. We first note that $W_{\lambda+\rho_n} = V_{\lambda+\rho_n}^{\mathfrak{u}}$ is an irreducible $\mathfrak{m}$-module with highest weight $\lambda+\rho_n$ which occurs with multiplicity one in $V_{\lambda+\rho_n}$. Moreover, since the irreducible $\mathfrak{m}$-module W_{ρ_n} with highest weight ρ_n is one-dimensional. $W_{\lambda+\rho_n} \simeq W_{\rho_n} \otimes W_\lambda$ and $L \otimes W_{\rho_n} \simeq \wedge \mathfrak{p}_+$ as M-modules by a similar observation to the one in § 2. Hence $L \otimes W_{\lambda+\rho_n} \simeq \wedge \mathfrak{p}_+ \otimes W_\lambda$ as M-modules. Define the M-module projection $\mathrm{id} \otimes p$: $L \otimes V_{\lambda+\rho_n} \to L \otimes W_{\lambda+\rho_n}$ where p: $V_{\lambda+\rho_n} \to W_{\lambda+\rho_n}$ is the unique $\mathfrak{m}$-module projection. Then it is easily seen that $\mathrm{id} \otimes p$ maps the subspace $(L \otimes V_{\lambda+\rho_n})^{\mathfrak{u}}$ injectively into $L \otimes W_{\lambda+\rho_n}$. It therefore suffices to see the surjectivity of this restriction of $\mathrm{id} \otimes p$ to $(L \otimes V_{\lambda+\rho_n})^{\mathfrak{u}}$.

We first consider the special case that $\lambda = 0$ and the K is mapped onto the full special orthogonal group $SO(\mathfrak{p})$ via the adjoint action (the case of $G = \mathrm{Spin}(2m, 1)$). That is, let $Q_1 = \{g \in SO(\mathfrak{p}^{\mathbb{C}}); g \cdot \mathfrak{p}_+ = \mathfrak{p}_+\}$ be the parabolic subgroup of $SO(\mathfrak{p}^{\mathbb{C}}) \simeq SO(2m, \mathbb{C})$ and let $Q_1 = M_1 U_1$ be the Levi decomposition as before. Note that $U_1 = \{g \in Q_1; g x = x \, (x \in \mathfrak{p}_+)\}$. Let $\mathfrak{u}_1$ be the Lie algebra of U_1. We will then show that $\wedge \mathfrak{p}_+$ is isomorphic to $(L \otimes L^+)^{\mathfrak{u}_1}$ as M_1-modules where L^+ is as in (2.3). Noting that L^+ is identical to V_{ρ_n} in case $\mathfrak{k}^{\mathbb{C}} = \mathfrak{so}(\mathfrak{p}^{\mathbb{C}})$ and that $L \otimes (L^+)^{\mathfrak{u}_1} \simeq \wedge \mathfrak{p}_+$ as M_1-modules where $(L^+)^{\mathfrak{u}_1}$ is one-dimensional, by the above observation we have to show that the injection $(L \otimes L^+)^{\mathfrak{u}_1} \to L \otimes (L^+)^{\mathfrak{u}_1}$ is surjective. For this, it suffices to see that the numbers of M_1-irreducible components in both sides coincide. First, since M_1 is isomorphic to $GL(m, \mathbb{C})$ embedded in $SO(2m, \mathbb{C})$, every $\wedge^q \mathfrak{p}_+$ is irreducible as an M_1-module. Hence $L \otimes (L^+)^{\mathfrak{u}_1} \simeq \wedge \mathfrak{p}_+$ has $m+1$ M_1-irreducible components. On the other hand, the number of M_1-irreducible components of $(L \otimes L^+)^{\mathfrak{u}_1}$ equals that of $SO(\mathfrak{p})$-irreducible components of $L \otimes L^+$, which will be shown to be also $m+1$ as follows. In fact, it is known that $L \otimes L^+$ and $L \otimes L^-$ have the same number of $SO(\mathfrak{p})$-irreducible components and that $L \otimes L \simeq \wedge \mathfrak{p}^{\mathbb{C}}$ as $SO(\mathfrak{p})$-modules. Now $\wedge^q \mathfrak{p}^{\mathbb{C}}$ is $SO(\mathfrak{p})$-irreducible if $q \neq m$ and $\wedge^m \mathfrak{p}^{\mathbb{C}}$ breaks up into two irreducible $SO(\mathfrak{p})$-modules. Thus the number of

$SO(\mathfrak{p})$-irreducible components of $\wedge \mathfrak{p}^{\mathbb{C}} \simeq L \otimes L$ is $2m+2$, which implies our assertion.

Before we go into the general case, we shall secondly see that if P is admissible, then $\mathfrak{u} \subset \mathfrak{u}_1$ where $\mathfrak{u}_1$ is the above defined nilpotent Lie algebra. For this, it suffices to see that $\mathfrak{u}$ acts trivially on $\mathfrak{p}_+$. We note that $\mathfrak{u} = [\mathfrak{p}_+, \mathfrak{p}_+]$. In fact, since $\mathfrak{u}$ is spanned by root vectors, $\mathfrak{g}^\alpha \subset \mathfrak{u}$ $(\alpha \in P_k)$ if and only if $\mathfrak{g}^{-\alpha} \not\subset \mathfrak{m}$, i.e., if and only if there exists $\beta \in P_n$ such that $[\mathfrak{g}^{-\alpha}, \mathfrak{g}^\beta] \neq 0$ and $[\mathfrak{g}^{-\alpha}, \mathfrak{g}^\beta] \not\subset \mathfrak{p}_+$, which means $\alpha - \beta \in P_n$. Then $\alpha = (\alpha - \beta) + \beta$; hence $\mathfrak{g}^\alpha = [\mathfrak{g}^{\alpha-\beta}, \mathfrak{g}^\beta] \subset [\mathfrak{p}_+, \mathfrak{p}_+]$. Hence $\mathfrak{u} = [\mathfrak{p}_+, \mathfrak{p}_+]$. Now if we assume that P is admissible, then $[[\mathfrak{p}_+, \mathfrak{p}_+], \mathfrak{p}_+] = 0$, which means $[\mathfrak{u}, \mathfrak{p}_+] = 0$. Hence our assertion.

Finally, we shall show the surjectivity of the injection

$$\mathrm{id} \otimes p\colon (L \otimes V_{\lambda+\rho_n})^{\mathfrak{u}} \to L \otimes W_{\lambda+\rho_n}.$$

We have to prove: given any element $w_0 \otimes w \in W_{\rho_n} \otimes W_\lambda \simeq W_{\lambda+\rho_n}$ and any element $l \in L$, there exists an element $v \in (L \otimes V_{\lambda+\rho_n})^{\mathfrak{u}}$ such that $(\mathrm{id} \otimes p)(v) = l \otimes w_0 \otimes w$. Noticing that $w_0 \in W_{\rho_n} = (L^+)^{\mathfrak{u}_1}$, let $\bar{z} \in (L \otimes L^+)^{\mathfrak{u}_1}$ be the inverse image of $l \otimes w_0 \in L \otimes (L^+)^{\mathfrak{u}_1}$ under the established isomorphism $(L \otimes L^+)^{\mathfrak{u}_1} \xrightarrow{\sim} L \otimes (L^+)^{\mathfrak{u}_1}$. Next let $z \in L \otimes V_{\rho_n}$ be the image of the $\bar{z}$ under the K-module projection $L \otimes L^+ \to L \otimes V_{\rho_n}$. Since $\mathfrak{u} \subset \mathfrak{u}_1$ under our assumption, we then have $z \in (L \otimes V_{\rho_n})^{\mathfrak{u}}$. Considering the unique projection $V_{\rho_n} \otimes V_\lambda \to V_{\lambda+\rho_n}$, we have the composed map

$$L \otimes V_{\rho_n} \otimes W_\lambda \hookrightarrow L \otimes V_{\rho_n} \otimes V_\lambda \to L \otimes V_{\lambda+\rho_n}.$$

Let $v \in L \otimes V_{\lambda+\rho_n}$ be the image of $z \otimes w \in L \otimes V_{\rho_n} \otimes W_\lambda$ under the above map. It is clear that $v \in (L \otimes V_{\lambda+\rho_n})^{\mathfrak{u}}$. Noting that there is only one M-module projection $L^+ \otimes V_\lambda \to W_{\lambda+\rho_n}$, it can be seen that $(\mathrm{id} \otimes p)(v) = l \otimes w_0 \otimes w$. Thus we have completed the proof of Lemma 9.2, hence that of Lemma 9.1. q.e.d.

By Lemma 9.1, we have the following proposition.

Proposition 9.1. *If the positive root system P is admissible, then every statement of the theorem in the introduction is true under the assumption* (i) *and* (ii)$^{\mathrm{b}}$.

In all the following cases E 1, E 2 and E 3, we notice that every positive root system will be admissible in our sense.

E 1. $G = SU(m, 1)$.

Let $\{e_i\}_{i=1}^{m+1}$ be an orthogonal basis of the euclidean space $\mathbb{R}^{m+1}$ such that

$$(e_i, e_j) = (2(m+1))^{-1}\, \delta_{ij} \qquad (1 \leqq i, j \leqq m+1)$$

where δ_{ij} is the Kronecker symbol. Then, as a root system for $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(m+1, \mathbb{C})$, one may consider

$$\Sigma = \{\pm(e_i - e_j);\ 1 \leqq i < j \leqq m+1\}$$

with the above inner product. Since $\mathfrak{k}^{\mathbb{C}} = \mathfrak{sl}(m, \mathbb{C}) \oplus \mathbb{C}$ embedded in $\mathfrak{sl}(m+1, \mathbb{C})$, one may assume that the set of compact roots is

$$\Sigma_k = \{\pm(e_i - e_j);\ 1 \leqq i < j \leqq m\}.$$

We fix a positive root system P_k for Σ_k as

$$P_k = \{e_i - e_j;\ 1 \leqq i < j \leqq m\}.$$

In the set of non-compact roots

$$\Sigma_n = \{\pm(e_i - e_{m+1});\ 1 \leqq i \leqq m\},$$

we define for $l = 0, 1, \ldots, m$,

$$P_n^{(l)} = \left\{\varepsilon_i(e_i - e_{m+1});\ \begin{array}{ll} \varepsilon_i = 1 & \text{for} \quad 1 \leqq i \leqq l \\ \varepsilon_i = -1 & \text{for}\ l+1 \leqq i \leqq m \end{array}\right\}.$$

Since the Weyl group W for $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ may be regarded as the symmetric group acting on $\{e_i;\ 1 \leqq i \leqq m\}$, $P^{(l)} = P_k \cup P_n^{(l)}$ $(0 \leqq l \leqq m)$ exhaust all W-inequivalent positive root systems in Σ.

Since $\mathrm{Hom}(\sqrt{-1}\,\mathfrak{t}, \mathbb{R})$ can be identified with the hyperplane in $\mathbb{R}^{m+1}$ defined by $\sum_{i=1}^{m+1} m_i e_i$ such that $\sum_{i=1}^{m+1} m_i = 0$, $\Lambda = \sum_{i=1}^{m+1} m_i e_i \in \mathscr{F}$ if and only if $\sum_{i=1}^{m+1} m_i = 0$ and $2(\Lambda, \alpha_{ij})/|\alpha_{ij}|^2 = m_i - m_j \in \mathbb{Z}$ where $\alpha_{ij} = e_i - e_j \in \Sigma$ $(i \neq j)$. It is It is seen that $\Lambda = \sum_{i=1}^{m+1} m_i e_i$ is regular dominant with respect to $P^{(l)}$ if and only if

$$m_1 > m_2 > \cdots > m_l > m_{m+1} > m_{l+1} > \cdots > m_m.$$

For $\rho^{(l)} = \frac{1}{2}\langle P^{(l)}\rangle$, $\rho_n^{(l)} = \frac{1}{2}\langle P_n^{(l)}\rangle$, we have

$$\rho^{(l)} = \tfrac{1}{2}\left\{\sum_{i=1}^{l}(m-2i+2)\,e_i + \sum_{j=l+1}^{m}(m-2j)\,e_j + (m-2l)\,e_{m+1}\right\},$$

$$\rho_n^{(l)} = \tfrac{1}{2}\left\{\sum_{i=1}^{l} e_i - \sum_{j=l+1}^{m} e_j + (m+2l)\,e_{m+1}\right\}$$

Now it is easy to see that every $P^{(l)}$ is admissible in the sense of the beginning part of this section. Hence by Proposition 9.1, every statement of the theorem in the introduction is true without (ii)[a].

It is seen that $\lambda = \Lambda - \rho^{(l)}$ satisfies the assumption (i) of the theorem in the introduction if and only if

$$m_1 > \cdots > m_m, \quad m_l - m_{m+1} \geqq 2 \quad \text{and} \quad m_{l+1} - m_{m+1} \leqq -2.$$

In this case, we see that the assumption (ii)[b] *is automatically satisfied.* In fact, suppose $\lambda + 2\rho_n^{(l)} - \langle Q \rangle \in \mathscr{F}_0$ for $Q \subset P_n^{(l)}$. Then $(\lambda + \rho_n^{(l)} - \langle Q \rangle, \alpha) \geqq -(\rho_n^{(l)}, \alpha)$ $(\alpha \in P_k)$. It then suffices to show that $(\lambda + \rho_n^{(l)} - \langle Q \rangle, \alpha) \geqq 0$ $(\alpha \in P_k)$. When $l = 0$ or m, then $(\rho_n^{(l)}, \alpha) = 0$ $(\alpha \in P_k)$.

Hence one may assume that $l \neq 0$ nor m.

Putting $\alpha_i = e_i - e_{i+1}$ $(1 \leqq i \leqq m-1)$, it suffices to check the above inequality for α_i. First $(\rho_n^{(l)}, \alpha_i) = 0$ for $i \neq l$, hence for α_i $(i \neq l)$ the inequality holds. Secondly, $2(\rho_n^{(l)}, \alpha_l)/|\alpha_l|^2 = 1$. Since $m_l - m_{m+1} \geqq 2$ and $m_{l+1} - m_{m+1} \leqq -2$, we have $m_l - m_{l+1} \geqq 4$. Hence $2(\lambda, \alpha_l)/|\alpha_l|^2 = m_l - m_{l+1} - 2 \geqq 2$. For $Q \subset P_n^{(l)}$, $2(\langle Q \rangle, \alpha_l)/|\alpha_l|^2$ is possibly 0, 1 or 2. Hence the inequality holds also for α_l.

Moreover, one can see that $k(\alpha) = m$ for $\alpha \in \Sigma_n$ where $k(\alpha)$ is the number defined in Remark 2 in § 8. Hence, if $\omega(\Lambda) \in \mathscr{E}_1$ $\left(\Lambda = \sum_{i=1}^{m+1} m_i e_i\right)$, then $m_l - m_{m+1} \geqq m+1$ and $m_{l+1} - m_{m+1} \leqq -(m+1)$. By this, in case $m \neq 1$, for every Weyl chamber positive for $P^{(l)}$ there exist infinitely many non-integrable $\omega(\Lambda) \in \mathscr{E}_2$ which satisfy the assumption (i).

E 2. $G = \mathrm{Spin}(2m, 1)$.

Let $\{e_i\}_{i=1}^m$ be an orthogonal basis of $\mathbb{R}^m$ such that $(e_i, e_j) = (2(2m-1))^{-1} \delta_{ij}$. The root system of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(2m+1, \mathbb{C})$ is then $\Sigma = \Sigma_k \cup \Sigma_n$, where $\Sigma_k = \{\pm(e_i - e_j), \pm(e_i + e_j);\ 1 \leqq i < j \leqq m\}$ is the set of compact roots, and $\Sigma_n = \{\pm e_i;\ 1 \leqq i \leqq m\}$ is that of non-compact roots. Fix

$$P_k = \{e_i + e_j, e_i - e_j;\ 1 \leqq j < i \leqq m\},$$

and put $P_n^{(\pm)} = \{\pm e_1, e_2, \ldots, e_m\}$. Then these two $P^{(\pm)} = P_k \cup P_n^{(\pm)}$ exhaust all W-inequivalent positive root systems. $\Lambda = \sum_{i=1}^m m_i e_i \in \mathscr{F}$ if and only if $2m_1 \in \mathbb{Z}$ and $m_i - m_j \in \mathbb{Z}$ $(\forall i, j)$. One sees that Λ is regular dominant with respect to $P^{(+)}$ if and only if $0 < m_1 < m_2 < \cdots < m_m$; with respect to $P^{(-)}$ if and only if $-m_2 < m_1 < 0 < m_2 < m_3 < \cdots < m_m$. Putting $\lambda = \Lambda - \rho^{(\pm)}$ $(\rho^{(\pm)} = \frac{1}{2}\langle P^{(\pm)} \rangle)$, the condition $(\lambda, \alpha) > 0$ $(\alpha \in P_n^{(\pm)})$ means additionally that $m_1 \geqq 1$ for $P^{(+)}$; $m_1 \leqq -1$ for $P^{(-)}$. It is easily seen that $P^{(\pm)}$ is admissible; hence we do not care about the assumption (ii)[a] in the introduction. *It is also easy to see that if* $(\lambda, \alpha) > 0$ $(\alpha \in P_n^{(\pm)})$, *then the assumption* (ii)[b] *is automatically satisfied.*

Moreover, $k(\alpha) = 2(m - \frac{1}{2})$ $(\alpha \in \Sigma_n)$. Hence, if $\omega(\Lambda) \in \mathscr{E}_1$, then $m_1 \geqq m$ for $P^{(+)}$, or $m_1 \leqq -m$ for $P^{(-)}$. Incidentally, for this group, the sufficient condition given in [23, Theorem 8.2] in order that $\omega(\Lambda) \in \mathscr{E}_1$ coincides with the above necessary condition, which also coincides with the condition given by Schmid [20]. Hence

$$\mathscr{E}_1=\left\{\omega(\Lambda)\in\mathscr{E}_2;\ \begin{array}{l} m\leqq m_1<m_2<\cdots<m_m, \text{ or} \\ -m_2<m_1\leqq -m \text{ and } m_2<\cdots<m_m \end{array}\right\}.$$

We have thus seen the following fact.

Proposition 9.2. *For* $G=SU(m,1)$ *or* $\mathrm{Spin}(2m,1)$, *every statement of the theorem in the introduction is true under the assumption* (i). *Moreover the set of discrete classes satisfying* (i) *covers all integrable classes, and, when* $m\neq 1$, *contains infinitely many non-integrable classes.*

Remark. If one uses Schmid's work for $\mathrm{Spin}(2m,1)$ in the last half of [20], it can be verified that the statements (I), (II) and (III) in the theorem are true for all discrete classes for $\mathrm{Spin}(2m,1)$ (i.e., without imposing (i)). Also the equality (Blattner's conjecture) holds in (III). (See [8, § 6].)

E 3. Holomorphic discrete series.

In this case, everything has been beautifully worked out mainly by Harish-Chandra [6]. However, to see how the matters can be simplified in this case, we illustrate it here. Nothing new is thus contained in this paragraph.

The discrete class $\omega(\Lambda)$ $(\Lambda\in\mathscr{F}')$ is called "holomorphic" when, for the positive root system P such that $P=\{\alpha\in\Sigma;(\Lambda,\alpha)>0\}$, $\alpha+\beta\notin P_k$ for any $\alpha,\beta\in P_n$. Then the subspaces $\mathfrak{p}_\pm=\sum_{\alpha\in P_n}\mathfrak{g}^{\pm\alpha}$ must be abelian subalgebras of $\mathfrak{g}^{\mathbb{C}}$ and stable under the K-action.

We notice that the condition (#) for $\lambda+2\rho_n\in\mathscr{F}_0$ is always satisfied, i.e.,

$$H^i(\wedge^q\mathfrak{p}_-\otimes l_{\lambda+2\rho})=H^0(\wedge^q\mathfrak{p}_-)\otimes H^i(l_{\lambda+2\rho})=0 \qquad (i<s)$$

since $\mathfrak{p}_-$ is itself a $K^{\mathbb{C}}$-module. (Of course, our P is admissible.) Also the assumption (2) in Theorem 2 in § 6 is automatically satisfied since $(\rho_n,\alpha)=0$ for $\alpha\in P_k$. The vanishing thus obtained by Theorem 2 is due to Matsushima-Murakami [15; Theorem 2]. We have thus seen that for a holomorphic $\omega(\Lambda)\in\mathscr{E}_2$, the theorem in the introduction holds under the assumption (i). Further by the results of [6], (I), (II) and (III) (together with the equalities) have been known to be true.

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