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Spectral asymmetry and Riemannian geometry. III

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1. *Introduction.* In Parts I and II of this paper ((4), (5)) we studied the 'spectral asymmetry' of certain elliptic self-adjoint operators arising in Riemannian geometry. More precisely, for any elliptic self-adjoint operator A on a compact manifold we defined

$$\eta_A(s) = \sum_{\lambda \neq 0} \operatorname{sign} \lambda \, |\lambda|^{-s}, \tag{1·1}$$

where λ runs over the eigenvalues of A. For the particular operators of interest in Riemannian geometry we showed that $\eta_A(s)$ had an analytic continuation to the whole complex s-plane, with simple poles, and that $s = 0$ was not a pole. The real number $\eta_A(0)$, which is a measure of 'spectral asymmetry', was studied in detail particularly in relation to representations of the fundamental group.

In this part of the paper we shall study the function $\eta_A(s)$ for arbitrary elliptic self-adjoint operators. Whereas Part II was, in a sense, concerned with odd-dimensional analogues of the Riemann–Roch theorem, Part III will be analogous to the general index theorem. In particular topological arguments based on K-theory will be used in an essential way, just as in the proof of the index theorem in (7).

We begin in section 2 by establishing the basic analytical properties of the function $\eta_A(s)$, showing in particular that it has an analytic continuation to the whole s-plane with only simple poles. However, as explained in the introduction to Part I, $s = 0$ is a possible pole. In the particular case of the Riemannian operators of Part I the finiteness of $\eta_A(0)$ was established as a consequence of the main theorem of Part I. An alternative proof, based on invariance theory, will also be given in section 4. For general operators however there seems to be no direct analytical argument to eliminate the pole at $s = 0$. The example at the end of section 4 illustrates the difficulty. What the analysis does show is that the residue at $s = 0$ is unchanged under continuous variation of A: in other words the residue $R(A)$ is a homotopy invariant. Note also that if A is positive (or negative) its η-function coincides with $\pm \zeta_A(s)$, the ζ-function of Seeley (14), and this is known to be finite at $s = 0$. Thus in studying $R(A)$ we may disregard positive (or negative) operators.

In section 3 therefore we undertake a topological study of self-adjoint elliptic operators. Factoring out by positive (or negative) operators we show (Proposition (3·1)) that the homotopy classes form an abelian group naturally isomorphic to $K^1(TX)$ where TX is the cotangent bundle of X and K denotes K-theory with compact supports. The residue $R(A)$ then induces a homomorphism $K^1(TX) \to R$. To show that this is zero it is therefore enough, for each X, to check that $R(A) = 0$ for sufficiently many operators A corresponding to a set of rational generators of $K^1(TX)$. In section 4 we show that the basic Riemannian operators provide such a set of generators provided $\dim X$ is *odd*. Since for these operators, as we have observed, $R(A) = 0$ it follows that $R(A) = 0$ for all operators A on *odd-dimensional* manifolds (Theorem 4·5). In (3) we stated that $\eta_A(s)$ was always finite but at present we do not know how to prove this for even-dimensional manifolds. The difficulty in extending the proof to even-dimensions is somewhat technical and is discussed in section 4.

The second half of the paper is concerned with the effect of 'twisting' the η-function of A by a unitary representation α of the fundamental group as in Part II. We obtain in this way an invariant $\tilde{\eta}_\alpha(0, A) \in R/Z$ (see section 5) which is a homotopy invariant of A. This is the generalization of the invariants introduced in Part II, section 3, and our aim is to derive a topological formula for this invariant in terms of α and of the class of A in $K^1(TX)$. This formula, which we refer to as the *index theorem for flat bundles*, is stated in section 5. It generalizes the Riemann–Roch theorem for flat bundles of Part I, section 3, and the proof for *odd-dimensional manifolds* occupies the last three sections of the paper.

In formulating the theorem we have first to show how to associate to the representation α of $\pi_1(X)$ an element in $K^{-1}(X; R/Z)$, generalizing the case of finite groups treated in Part I, section 5. This involves giving an appropriate definition of K-theory with coefficients in R/Z and the definition we adopt (in section 5) was suggested to us by G. B. Segal. Essentially this definition reduces us to the case of Q/Z-coefficients (treated in Part I, section 5) and of R-coefficients, where we can use differential forms and real cohomology.

The proof of (5·3) – the index theorem for flat bundles – breaks up correspondingly into two parts, one dealing with the R-component and the other with the Q/Z-component. The R-component is treated in section 6 by topological methods as in section 5, that is we use the results of Part I for the Riemannian operators and then apply K-theory to deduce the general case.

The Q/Z-component is treated in section 8 by converting the problem into one concerning the index of a family of elliptic operators, and then applying the index theorem for families as given in (9). This conversion depends on interpreting the Q/Z-component in terms of 'spectral flow', a notion which is explained in section 7 and is of some independent interest.

We hope elsewhere to give another proof of Theorem (5·3) in which the results of section 6 are obtained in an entirely different manner quite independent of Part I of this paper. In particular this will apply equally to even and odd dimensional manifolds. The idea of this alternative proof is to interpret the R-component of section 6 as an index in a type II von Neumann algebra situation, using the basic results of M. Breuer.

Just as in the treatment of the index theorem in (7) it is convenient to consider *pseudo-differential* operators. There are various minor variations in the notion of pseudo-differential operators, depending on how large a class of operators one wants to include. Since we shall be relying heavily on the analysis in (14), we shall employ the term in the same sense as there (where they are referred to as Calderon–Zygmund operators). In particular a pseudo-differential operator P of order m has a well-defined *leading symbol* $\sigma_m(P)$ which is a homogeneous function of degree m on the non-zero cotangent vectors. By contrast the *complete symbol* of P means the class of P modulo operators with C^∞ kernel, and is represented in local coordinates by a formal series

$$\sum_{j=0}^{\infty} \sigma_{m-j}(x, \xi).$$

In our use of K-theory we shall sometimes write K^1 and sometimes K^{-1}. By the periodicity theorem these are the same and the exponent should be viewed as an integer mod 2. Our reason for using both ± 1 is to try to see clearly where the periodicity theorem enters explicitly. In particular if one generalized everything to the real case (as in (10)), then the distinction would become important because the periodicity then is mod 8.

One final comment concerns the question of signs. There are many places where sign conventions enter and although we have endeavoured to be consistent we have not belaboured the point. Any error in sign in the final results will more readily be found by computing examples than by checking all the intermediate stages.

2. *Analytical properties of* $\eta_A(s)$. We begin by recalling the general results of Seeley (14) on zeta-functions of elliptic operators. Let B be an elliptic pseudo-differential operator of positive order m on a compact n-dimensional manifold. We assume that B is self-adjoint and positive so that it has eigenvalues $\mu > 0$ and we define its zeta-function by

$$\zeta_B(s) = \operatorname{Tr} B^{-s} = \sum_{\mu>0} \mu^{-s}.$$

This converges for $\operatorname{Re}(s) > n/m$ giving a holomorphic function of the complex variable s in this half-plane. Moreover $\zeta_B(s)$ can be analytically continued to the whole s-plane as a meromorphic function with simple poles. More precisely, for every integer $N \geqslant -n$, we have

$$\zeta_B(s) = \sum_{\substack{k=-n \\ k\neq 0}}^{N} \frac{a_k}{s+k/m} + \phi_N(s) \tag{2·1}$$

where $\phi_N(s)$ is holomorphic for $\operatorname{Re}(s) > -N/m$. Finally the coefficients a_k can be computed by an explicit integral formula

$$a_k = \int \alpha_k, \tag{2·2}$$

where α_k is constructed from the complete symbol of B. Note that $\zeta_B(s)$ is finite at $s = 0$; its value there, denoted by a_0, is also given by a formula of type (2·2).

If $B = B_u$ depends in a C^∞ manner on u then Seeley's analysis allows us to consider everything as a C^∞ function of u. More precisely $\zeta_B(s, u) = \operatorname{Tr} B_u^{-s}$ has an expansion of the form (2·1) in which the coefficients $a_k(u)$ are now C^∞ functions of u (as follows from (2·2)), and

$$u \mapsto \phi_N(s, u)$$

is a C^∞ map into the space of holomorphic functions in the half-plane $\operatorname{Re}(s) > -N/m$.

Finally if we relax the positivity condition on B and allow B to be non-negative, so that $\mu = 0$ may be an eigenvalue, all the above results continue to hold with minor modifications. We now define

$$\zeta_B(s) = \sum_{\mu>0} \mu^{-s}$$

and its properties can be deduced from those of the zeta-function of $B+H$ where H denotes the projection onto the null-space of B (note that H is an operator with C^∞ kernel). In particular

$$\zeta_B(0) + \operatorname{tr} H = \int \alpha_0, \tag{2·3}$$

where α_0 is the expression occurring in (2·2) and is constructed from the complete symbol of $B+H$, which is the same as that of B. Of course, when we come to consider a variation B_u in which the dimension of the null-space varies, the corresponding functions $\zeta(s,u)$ will have simple discontinuities (as functions of u) which we can easily allow for.

We come now to our η-functions, so let A be a self-adjoint elliptic pseudo-differential operator of positive order m on a compact n-dimensional manifold and define its η-function as before by

$$\eta_A(s) = \sum_{\lambda \neq 0} \operatorname{sign} \lambda \, |\lambda|^{-s}, \tag{2·4}$$

where λ runs over the eigenvalues of A. Since A^2 is non-negative of order $2m$ its zeta-function $\sum_{\lambda\neq 0} (\lambda^2)^{-s}$ converges for $\operatorname{Re} s > n/2m$. Hence (2·4) converges absolutely for $\operatorname{Re}(s) > n/m$ showing that $\eta_A(s)$ is holomorphic in this half-plane.

According to Seeley, $|A| = (A^2)^{\frac{1}{2}}$ (taking the positive square-root) is again a pseudo-differential operator. Hence

$$B_1 = \tfrac{3}{2}|A| + \tfrac{1}{2}A, \quad B_2 = \tfrac{3}{2}|A| - \tfrac{1}{2}A$$

are pseudo-differential. If μ, $-v$ denote the positive and negative eigenvalues of A, the eigenvalues of B_1 are 2μ, v while those of B_2 are μ, $2v$. Thus both are elliptic self-adjoint and non-negative, so their zeta functions $\zeta_1(s)$, $\zeta_2(s)$ are defined. Moreover

$$\begin{aligned}\zeta_1(s) - \zeta_2(s) &= \{\Sigma(2\mu)^{-s} + \Sigma v^{-s}\} - \{\Sigma\mu^{-s} + \Sigma(2v)^{-s}\},\\ &= (2^{-s}-1)\{\Sigma\mu^{-s} - \Sigma v^{-s}\},\\ &= (2^{-s}-1)\,\eta_A(s).\end{aligned}$$

Thus

$$\eta_A(s) = \frac{1}{2^{-s}-1}\{\zeta_1(s) - \zeta_2(s)\}. \tag{2·5}$$

Since $\zeta_1(s)$ and $\zeta_2(s)$ are both meromorphic in the whole s-plane, (2·5) provides an analytic continuation for $\eta_A(s)$ to the whole s-plane. At $s = 0$ the zeta-functions are finite but, because of the factor $(2^{-s}-1)^{-1}$ in (2·5), $\eta_A(s)$ has a priori a simple pole. Its residue there is given by

$$R(A) = \operatorname{Res}_{s=0} \eta_A(s) = -\frac{1}{\log 2}(\zeta_1(0) - \zeta_2(0)). \tag{2·6}$$

Using (2·3) we obtain an explicit integral formula

$$R(A) = \int \omega, \tag{2·7}$$

where $\omega = -(\log 2)^{-1}(\alpha_0^1 - \alpha_0^2)$, α_0^i arising from the operator B_i. Since B_i is a linear combination of A and $|A|$, and since the complete symbol of $|A| = (A^2)^{\frac{1}{2}}$ is expressible by (14) in terms of that of A, it follows that ω is constructed out of the symbol of A.

Summarizing our observations so far we have

PROPOSITION (2·8). *$\eta_A(s)$ extends to a meromorphic function in the whole s-plane with simple poles. Its residue at $s = 0$ is given by the integral formula* (2·7) *where ω is constructed out of the complete symbol of A.*

Our aim is to show that $\eta_A(s)$ is actually *finite* at $s = 0$ or equivalently that its residue $R(A)$ vanishes. As a first step in this direction we shall prove that $R(A)$ is a *constant* under smooth variation of A. For this purpose we shall consider a smooth 1-parameter family A_u and we shall show that

$$\frac{d}{du}R(A_u) = 0.$$

To allow for the discontinuities produced by the zero-eigenvalues it is convenient to decompose the series in (2·4) into two parts, corresponding to values λ with $|\lambda| < C$ or $|\lambda| > C$, where C is any number which is not $\pm$ an eigenvalue. To study $\eta(s,u) = \eta_{A_u}(s)$ near $u = 0$ we pick C not an eigenvalue of $\pm A_0$, then by continuity it is not an eigenvalue of any $\pm A_u$ for small u. Thus we may write

$$\eta(s,u) = \eta'(s,u) + \eta''(s,u),$$

where η' is the *finite* sum over eigenvalues λ with $|\lambda| < C$. As a function of s (with u fixed) $\eta'(s,u)$ is holomorphic and

$$\eta'(0,u) = \sum_{0<|\lambda(u)|<C} \operatorname{sign}\lambda(u)$$

is an *integer*. In particular η and η'' have the same residue $R(u) = R(A_u)$ at $s = 0$, confirming the fact that $R(A)$ depends only on the complete symbol of A. Moreover, if we define $\bar{\eta}$ to be the function with values in the C/Z obtained from η by reducing modulo Z, we see that $\bar{\eta}(0,u) = \bar{\eta}''(0,u)$. In other words *in studying $\bar{\eta}(0,u)$ there is no loss of generality in putting all eigenvalues in $(-C,C)$ equal to zero, or, better still, equal to one.* Thus we may assume that A_u is always invertible.

In order to compute $\dfrac{d}{du}R(u)$ we shall first need to differentiate zeta-functions with respect to a parameter:

PROPOSITION (2·9). *Let B_u be a C^∞ one-parameter family of positive self-adjoint elliptic operators of positive order m. Put $\zeta(s,u) = \zeta_{B_u}(s)$ and $\dot{B}_u = \dfrac{d}{du}(B_u)$, then for large* Re (s)

$$\frac{d}{du}\zeta(s,u) = -s\operatorname{tr}(\dot{B}B^{-s-1}).$$

Proof. $\zeta(s,u) = \dfrac{1}{2\pi i}\operatorname{tr}\displaystyle\int_\Gamma \lambda^{-s}(B_u-\lambda)^{-1}d\lambda$, where Γ is a suitable contour enclosing the positive real axis. For large Re (s) the convergence of this expression is ensured by the estimates in (14). To differentiate this we note that

$$(B_u-\lambda)^{-1}(B_u-\lambda) = 1$$

and so

$$\left\{\frac{d}{du}(B_u-\lambda)^{-1}\right\}(B_u-\lambda)+(B_u-\lambda)^{-1}\dot{B}_u = 0$$

or

$$\frac{d}{du}(B_u-\lambda)^{-1} = -(B_u-\lambda)^{-1}\dot{B}_u(B_u-\lambda)^{-1}.$$

Since trace is linear it commutes with d/du, hence

$$\frac{d}{du}\zeta(s,u) = -\frac{1}{2\pi i}\operatorname{tr}\int_\Gamma \lambda^{-s}(B_u-\lambda)^{-1}\dot{B}_u(B_u-\lambda)^{-1}d\lambda.$$

If the order m of B is sufficiently large $(B_u-\lambda)^{-1}$ is of trace class and has a continuous Schwartz kernel which is bounded as a function of λ. This follows from the estimate on Sobolev norms ((14), Cor. 1). Hence for large m we may interchange integration over Γ with taking traces since this amounts to changing the order of integration in an absolutely convergent double integral (over $X\times\Gamma$). Since $\operatorname{tr}PQ = \operatorname{tr}QP$ whenever P is of trace class and Q is bounded it follows that

$$\frac{d}{du}\zeta(s,u) = -\frac{1}{2\pi i}\int_\Gamma \lambda^{-s}\operatorname{tr}\dot{B}_u(B_u-\lambda)^{-2}d\lambda.$$

The Schwartz kernel of $\dot{B}_u(B_u-\lambda)^{-2}$ has the same good properties as that of $(B_u-\lambda)^{-1}$, hence we can again interchange tr and $\int_\Gamma$, giving

$$\begin{aligned}\frac{d}{du}\zeta(s,u) &= -\frac{1}{2\pi i}\operatorname{tr}\int_\Gamma \lambda^{-s}\dot{B}_u(B_u-\lambda)^{-2}d\lambda,\\ &= -\frac{1}{2\pi i}\operatorname{tr}\dot{B}_u\int_\Gamma s\lambda^{-s-1}(B_u-\lambda)^{-1}d\lambda \quad \text{(integrating by parts)},\\ &= -s\operatorname{tr}(\dot{B}_u B_u^{-s-1}),\end{aligned}$$

proving the Proposition for the case of large m. The general case follows easily from this by taking powers. In fact, replacing B by a large integral power B^N we obtain

$$\frac{d}{du}\zeta(Ns,u) = -s\operatorname{tr}\frac{d}{du}(B_u^N)\,B_u^{-N(s+1)}$$

and since

$$\frac{d}{du}(B_u^N) = \sum_{i=1}^{N} B_u^i\dot{B}_u B_u^{N-i-1},$$

we have

$$\frac{d}{du}\zeta(Ns,u) = -s\sum_{i=1}^{N}\operatorname{tr}B_u^i\dot{B}_u B_u^{-Ns-i-1}.$$

The Lemma will then follow (with Ns for s) provided we can justify permuting factors under the trace so that

$$\operatorname{tr}B_u^i(\dot{B}_u B_u^{-Ns-i-1}) = \operatorname{tr}(\dot{B}_u B_u^{-Ns-i-1})B_u^i = \operatorname{tr}\dot{B}_u B_u^{-Ns-1}.$$

Since B_u^i is unbounded this requires a little care, but we can proceed as follows. For $\mathrm{Re}(s)$ large we have (omitting the subscript u)

$$\begin{aligned}\operatorname{tr} B^i\dot{B}B^{-Ns-i-1} &= \operatorname{tr}(B^i\dot{B}B^{-\frac{1}{2}Ns})(B^{-\frac{1}{2}Ns-i-1}),\\ &= \operatorname{tr}(B^{-\frac{1}{2}Ns-i-1})(B^i\dot{B}B^{-\frac{1}{2}Ns}),\\ &= \operatorname{tr}(B^{-\frac{1}{2}Ns-1})(\dot{B}B^{-\frac{1}{2}Ns}) = \operatorname{tr}\dot{B}B^{-Ns-1}.\end{aligned}$$

All operators in parentheses ( ) are of trace class (for $\mathrm{Re}(s)$ large) and this justifies the interchange of order.

The corresponding result for the η-function is

PROPOSITION (2·10). *Let A_u be a C^∞ one-parameter family of self-adjoint invertible operators of positive order m. Put $\eta(s,u) = \eta_{A_u}(s)$ and $\dot{A}_u = \frac{d}{du}(A_u)$, then for large* $\mathrm{Re}(s)$

$$\frac{d}{du}\eta(s,u) = -s\operatorname{tr}\dot{A}_u(A_u^2)^{-\frac{1}{2}(s+1)}.$$

Proof. The argument is essentially the same as in (2·9) except we use two contours Γ^+ and Γ^- instead of Γ where Γ^+ is the line $\mathrm{Re}(\lambda) = \epsilon > 0$ (oriented in the sense of $\mathrm{Im}(\lambda)$ increasing), and $\Gamma^- = -(\Gamma^+)$. Then

$$\eta(s,u) = \frac{\operatorname{tr}}{2\pi i}\left\{\int_{\Gamma^+}\lambda^{-s}(A_u-\lambda)^{-1}d\lambda - (-1)^s\int_{\Gamma^-}\lambda^{-s}(A_u-\lambda)^{-1}d\lambda\right\}.$$

Here λ^{-s} denotes the branch $|\lambda|^{-s}\exp(-is\arg\lambda)$ where $-\frac{3}{2}\pi < \arg\lambda < \frac{1}{2}\pi$ and in particular $(-1)^s = \exp(-\pi is)$. We now proceed as in (2·9) computing the u-derivatives of each contour integral separately. Seeley's estimates (14) still apply in this situation† and we obtain

$$\frac{d}{du}\eta(s,u) = -s\operatorname{tr}\{\dot{A}_u(A_u^{-s-1}P_+) + (-1)^{s+1}\dot{A}_u(A_u^{-s-1}P_-)\},$$

where P_+ and P_- are the spectral projections of A_u corresponding to $\lambda > 0$ and $\lambda < 0$ respectively. Hence

$$\begin{aligned}\frac{d}{du}\eta(s,u) &= -s\operatorname{tr}\dot{A}_u\{A_u^{-s-1}P_+ + (-A_u)^{-s-1}P_-\}\\ &= -s\operatorname{tr}\dot{A}_u|A|^{-s-1} = -s\operatorname{tr}\dot{A}_u(A_u^2)^{-\frac{1}{2}(s+1)}\end{aligned}$$

as required.

Remark. An alternative proof can be given by differentiating (2·5) with respect to a parameter u and applying (2·9). This gives

$$\frac{d}{du}\eta(s,u) = s(1-2^{-s})^{-2}\operatorname{tr}\{\tfrac{3}{2}|A|^{\cdot}(B_1^{-s-1}-B_2^{-s-1}) + \tfrac{1}{2}\dot{A}(B_1^{-s-1}+B_2^{-s-1})\}.$$

A little manipulation, using $\operatorname{tr}PQ = \operatorname{tr}QP$ and the fact that $|A|, B_1, B_2$ all commute with A, yields the formula of (2·10).

We can now proceed and prove the promised assertion concerning the residue of $\eta_A(s)$:

† Seeley only requires that the top-order symbol of A takes values in $C-\tau R^+$ for some non-zero $\tau\in C$. For positive A we usually take $\tau = 1$ but here we can take $\tau = i$.

Proposition (2·11). *Let A be a self-adjoint elliptic operator and let $R(A) = \mathrm{Res}_{s=0}\,\eta_A(s)$. Then $R(A)$ is constant for continuous variation of A.*

Proof. By remarks made earlier we may suppose A invertible and it is enough to prove that, for any C^∞ one-parameter family A_u, $d/du\,(R(u))$ vanishes for $u = 0$, where $R(u) = R(A_u)$. Now the self-adjoint operator $B_u = |A_0| + u\dot{A}_0$ will be elliptic and positive for small u. If $\zeta(s,u)$ denotes its zeta-function, (2·9) and (2·10) show that for large $\mathrm{Re}\,(s)$ $\zeta(s,u)$ and $\eta(s,u) = \eta_{A_u}(s)$ have the same u-derivative at $u = 0$. Now, as pointed out earlier, (2·1) extended to include dependence on a smooth parameter u shows that $d/du\,(\zeta(s,u))$ has an analytic continuation in s which is finite at $s = 0$. The same is therefore true for $d/du\,(\eta(s,u))$ at $u = 0$. But (2·1) and (2·6) show that $\eta(s,u)$ is of the form

$$\eta(s,u) = \frac{R(u)}{s} + \sum_{0 \neq k = -n}^{N} \frac{b_k(u)}{s + k/m} + \phi_N(s,u)$$

where $u \to \phi_N(s,u)$ is a C^∞ map into the space of holomorphic functions in the half-plane $\mathrm{Re}\,(s) > -N/m$. Differentiating with respect to u and then putting $u = 0$ we see that the resulting function of s has residue $[d/du\,(R(u))]_{u=0}$ at $s = 0$. But we have already proved that this function of s is finite at $s = 0$, so that its residue has to vanish. Thus $d/du\,(R(u))$ vanishes at $u = 0$ as required.

In the proof of (2·11) we saw that

$$\frac{d}{du}\eta(s,u) = \frac{d}{du}\zeta(s,u) \quad \text{for} \quad u = 0, s = 0$$

(where the value at $s = 0$ is obtained as usual by analytic continuation).

But the values of ζ-functions at $s = 0$ and of their u-derivatives are given by explicit integral formulae. In view of our earlier observations about the function $\bar{\eta}$ with vaues in C/Z this gives

Proposition (2·12). *Let A_u be a C^∞ one-parameter family of elliptic self-adjoint operators of positive order and let $\bar{\eta}(s,u)$ be the corresponding η-function reduced modulo Z. Then $(d\bar{\eta}/du)_{u=0}$ is holomorphic at $s = 0$ and its value there is given by an explicit integral formula constructed out of the complete symbols of A_0 and $\dot{A}_0$.*

We shall now develop some important consequences of (2·12) in connexion with flat bundles. Let $\alpha: \pi_1(X) \to U(N)$ be a unitary representation of the fundamental group of X. This defines a flat vector bundle V_α over X with hermitian metric. If $A: C^\infty(X,E) \to C^\infty(X,E)$ is a differential operator acting on the sections of the vector bundle E then A extends naturally to give a differential operator

$$A_\alpha: C^\infty(X, E \otimes V_\alpha) \to C^\infty(X, E \otimes V_\alpha).$$

The reason is that the transition matrices of V_α (with respect to its natural local bases) are *constants*. Moreover if A is self-adjoint then, because α is unitary, A_α will be self-adjoint. Hence we can consider its η-function $\eta_\alpha(s,A) = \eta_{A_\alpha}(s)$. We shall be interested in comparing this with the corresponding η-function for the trivial representation and reducing modulo Z, so we put

$$\tilde{\eta}_\alpha(s,A) = \bar{\eta}_\alpha(s,A) - N\bar{\eta}(s,A). \tag{2·13}$$

The point about the two operators A_α and $A_N = A \oplus A \oplus \ldots \oplus A$ (N times) is that they are locally isomorphic, and hence any invariant which is given by an explicit integral formula, computed locally, will coincide for A_α and A_N. We apply this observation twice. In the first place (2·8) implies that $R(A_\alpha) = R(A_N) = NR(A)$ and hence $\tilde{\eta}_\alpha(0,A)$ is *finite*. Next we apply (2·12) to a one-parameter family $A(u)$ and we deduce that $[(d/du)\,\tilde{\eta}_\alpha(s,A(u))]_{s=0} = 0$. As in the proof of Proposition (2·11) we can interchange the order of d/du and put $s = 0$. Hence $(d/du)\ \tilde{\eta}_\alpha(0,\ A(u)) = 0$ which shows that $\tilde{\eta}_\alpha(0,A)$ is a *homotopy invariant* of A.

If A is only a pseudo-differential operator the preceding discussion applies with one minor modification, namely that there is no unique natural way of defining the operator A_α. However, using a partition of unity, we can construct operators A_α which act on $C^\infty(X, E \otimes V_\alpha)$ and whose *complete symbol* is $\sigma(A) \otimes 1_\alpha$ where $\sigma(A)$ is the complete symbol of A.

Summarizing all this we have proved

PROPOSITION (2·14). *Let A be a self-adjoint pseudo-differential elliptic operator of positive order acting on $C^\infty(X,E)$ and let $\alpha:\pi_1(X) \to U(N)$ be a unitary representation. Let A_α be any self-adjoint extension of A to $C^\infty(X, E\otimes V_\alpha)$, so that the complete symbols are related by*

$$\sigma(A_\alpha) = \sigma(A) \otimes 1_\alpha,$$

where V_α is the flat bundle associated to α and 1_α is the identity endomorphism of V_α. Define $\tilde{\eta}_\alpha(s,A)$ by

$$\tilde{\eta}_\alpha(s,A) = \bar{\eta}_{A_\alpha}(s) - N\bar{\eta}_A(s).$$

Then $\tilde{\eta}_\alpha(0,A)$ is finite and is a homotopy invariant of A. It takes values in R/Z.

This Proposition generalizes the particular cases studied in Part II, section 3, where we considered particular operators A associated to a Riemannian metric, and the homotopy invariance of (2·14) reduced to saying that $\tilde{\eta}_\alpha(0,A)$ was independent of the metric. Just as in Part II, section 3, we can refine our η-invariant by using the function

$$\xi(s) = \frac{h+\eta(s)}{2},$$

where h is the dimension of the null-space. Propositions (2·11), (2·13) and (2·14) all continue to hold with η replaced by ξ.

3. *K-theory of self-adjoint symbols.* In section 2 we ended up with certain homotopy invariants of self-adjoint elliptic operators. In this section we shall set up the appropriate algebraic machinery to deal with such homotopy invariants.

If A is a self-adjoint elliptic operator of order m acting on $C^\infty(X,E)$ its homotopy class depends only on the homotopy class of its leading symbol $\sigma_m(A)$. For each $x\in X$ and each non-zero cotangent vector ξ at x, $\sigma_m(A)\,(x,\xi)$ is a self-adjoint invertible operator on the (finite-dimensional) vector space E_x. As a function of ξ it is homogeneous of degree m and so is determined by its restriction to the cotangent sphere bundle $S(X)$. Thus $\sigma_m(A)$ may be viewed as a C^∞ self-adjoint automorphism of the vector π^*E, where $\pi: S(X) \to X$ is the projection.

We shall therefore consider pairs (E, σ) where E is a hermitian vector bundle over X and σ is a self-adjoint automorphism of π^*E. For brevity such a σ will be referred to as a *self-adjoint symbol* on X. We shall consider homotopy classes of such symbols and write $\sigma \sim \sigma'$ for homotopy equivalence. Up to homotopy a self-adjoint symbol σ can always be replaced by its unitary part $U\sigma = \sigma(\sigma^2)^{-\frac{1}{2}}$. This satisfies $(U\sigma)^2 = 1$ and so gives an orthogonal decomposition $\pi^*E = E_\sigma^+ + E_\sigma^-$ where E_σ^+ and E_σ^- are the ± 1-eigenspaces of $U\sigma$. A self-adjoint symbol for which $E_\sigma^- = 0$ is called *positive* and one for which $E_\sigma^+ = 0$ is called *negative*. We now define two self-adjoint symbols σ, σ' to be *stably equivalent*, written $\sigma \approx \sigma'$, if

$$\sigma \oplus \alpha \oplus \beta \sim \sigma' \oplus \alpha' \oplus \beta'$$

where α, α' are positive and β, β' are negative. The important thing to note in this definition is that the decomposition of the bundle on which $\alpha \oplus \beta$ acts is *already defined over* X.

PROPOSITION (3·1). *The stable classes of self-adjoint symbols on X are in bijective correspondence with the elements of $K^1(TX)$, where TX is the cotangent bundle. The correspondence is obtained by assigning to a self-adjoint symbol σ the element $\delta[E_\sigma^+]$, where δ is the coboundary homomorphism.*

$$\delta: K(SX) \to K^1(TX).$$

Proof. It is enough to consider symbols σ for which $\dim E_\sigma^+$ and $\dim E_\sigma^-$ are both larger than $\dim X$ (since we can always add $\alpha \oplus \beta$ to ensure this). Then the isomorphism classes of E_σ^+, E are determined by the elements $[E_\sigma^+] \in K(SX)$ and $[E] \in K(X)$. Moreover† the embedding $E_\sigma^+ \to E$ (hence the decomposition of E) is also unique up to homotopy. Thus the homotopy classes of such σ correspond bijectively to pairs $[E], [E_\sigma^+]$ of the corresponding dimension. The equivalence $\sigma \approx \sigma + \alpha + \beta$ corresponds to an equivalence

$$[E], [E^+] \approx [E] \oplus [F], \quad [E^+] \oplus [\pi^*G]$$

where F, G are bundles on X. The equivalence classes generated by this relation are clearly just the elements of the cokernel of

$$1 \oplus \pi^*: K(X) \oplus K(X) \to K(X) \oplus K(SX).$$

Hence the stable classes of self-adjoint symbols correspond bijectively to the cokernel of

$$K(X) \to K(SX)$$

† This is a standard 'stability' argument: the space Hom* (E_y^+, E_y) of embeddings is $GL(N, C)/GL(N^-, C)$ where $N = \dim E$, $N^- = \dim E^-$ and so is $2N^-$-connected; since

$$2N^- > \dim SX$$

there is (up to homotopy) a unique section of the fibre bundle Hom* (E^+, E), i.e. a unique global embedding $E^+ \to E$.

or equivalently, from the exact sequence, to the image of the coboundary

$$\delta: K(SX) \to K^1(BX, SX) = K^1(TX)$$

where BX is the unit ball bundle of X. It remains to observe that δ is *surjective*. If $\dim X$ is odd this is clear because we have a section $X \to SX$ (a nowhere zero tangent vector field). If $\dim X = 2n$ we note that the image of

$$j: K^1(TX) \to K^1(X)$$

has filtration $\geqslant 2n$ because BX/SX is $(2n-1)$-connected. On the other hand the filtration groups of K^1 satisfy $K^1_{2q} = K^1_{2q+1}$ (6) so that

$$K^1_{2n}(X) = K^1_{2n+1}(X) = 0 \quad \text{since} \quad \dim X < 2n+1.$$

Thus $\operatorname{Im}\delta = \operatorname{Ker} j$ is the whole of $K^1(TX)$ as required.

Proposition (3·1) shows that the stable class $[\sigma]$ of a self-adjoint symbol σ on X may be viewed as an element of $K^1(TX)$. This should be contrasted with the usual theory of elliptic symbols in which the symbol class is an element of $K^0(TX)$. The relation between the two cases can be clarified by using the ideas in (11), where to a self-adjoint Fredholm operator A one associates the one-parameter family

$$A_t = I\cos t + iA\sin t \quad 0 \leqslant t \leqslant \pi. \tag{3·2}$$

On the symbol level if σ is a self-adjoint symbol we consider the family

$$\left.\begin{aligned} \sigma_t &= I\cos t + i\sigma\sin t, & 0 \leqslant t \leqslant \pi, \\ &= (\cos t + i\sin t)I, & \pi \leqslant t \leqslant 2\pi, \end{aligned}\right\} \tag{3·3}$$

of elliptic symbols on X parametrized by a point t on the circle S^1. This defines an element $\alpha \in K^0(S^1 \times TX)$ whose restriction to $\{0\} \times TX$ is trivial and so can be regarded as an element of $K^{-1}(TX)$: it is essentially the same as the element $[\sigma]$, or rather α corresponds to $[\sigma]$ under the periodicity isomorphism

$$K^1(TX) \to K^{-1}(TX).$$

This is a direct consequence of their definitions and the fact that equations (3·3) are just Bott's original definition (12) for the periodicity map

$$G_N \to \Omega U(N),$$

where G_N is the Grassmannian of all subspaces of $\mathbf{C}^N$ (identified with the self-adjoint unitary matrices) and Ω denotes the loop space.

For an elliptic operator on X its index is computed, (7), from its symbol by the 'topological index' homomorphism $K(TX) \to Z$. More generally, (9), a family of such operators parametrized by a compact space P has an index in $K(P)$ which is computed from σ by a topological index $K(P \times TX) \to K(P)$. Using the equations (3·2), extended to 2π as in (3·3), we can derive corresponding results for families of self-adjoint elliptic operators.

If $A = \{A_p\}$ is a family of self-adjoint elliptic operators of order 0 parametrized by $p \in P$ we obtain a mapping $P \to \hat{\mathfrak{F}}$ where $\hat{\mathfrak{F}}$ is the space of self-adjoint Fredholm operators on a Hilbert space. The (analytical) index of the family A can be considered to be the homotopy class of this map. Now in (11) it is shown that $\hat{\mathfrak{F}}$ (or equivalently $i\hat{\mathfrak{F}}$) consists of two contractible components $\hat{\mathfrak{F}}_+$ (essentially positive) and $\hat{\mathfrak{F}}_-$ (essentially negative) together with a third component $\hat{\mathfrak{F}}_*$ which is a classifying space for K^{-1}. Thus discarding essentially positive (or negative) families we get an index in $K^{-1}(P)$. In fact if $B = \{A_{t,p}\}$ is the corresponding family of elliptic operators, parametrized by $S^1 \times P$, given by (3·2), we have

$$K^{-1}(P) \ni \operatorname{index} A = \operatorname{index} B \in K^0(S^1 \times P).$$

This is because the map $\hat{\mathfrak{F}}_* \to \Omega\mathfrak{F}$, identifying $\hat{\mathfrak{F}}_*$ as a classifying space for K^{-1}, is also given by (3·2).

The symbol of the family B is given by (3·3). Hence applying the index theorem for B we get an index formula for A which may be stated as follows:

Theorem (3·4). *Let $A = \{A_p\}$ be a family of self-adjoint elliptic operators parametrized by $p \in P$ (a compact space). Then the index of A can be computed by applying the topological index homomorphism $K^1(P \times TX) \to K^1(P)$ to the symbol class of A.*

Remark. As usual operators of order m can be reduced to operators of order 0 so we have omitted any reference to order in (3·4).

Theorem (3·4) is not much more than a retranslation of the index theorem of (11). We shall however consider some special cases later which are interesting in themselves and are also a necessary step towards our 'index theorem for flat bundles'.

4. *Finiteness of $\eta(0)$.* In section 2 we showed that $R(A) = \operatorname{Res}_{s=0} \eta_A(s)$ is a *homotopy invariant* of the self-adjoint elliptic operator A. It therefore depends only on the homotopy class of the leading symbol $\sigma_m(A)$ where m is the order of A. For a positive operator A we have $\eta_A(s) = \zeta_A(s)$ and this is finite at $s = 0$ by the results of Seeley (14). Similarly for a negative operator. Since $R(A \oplus B) = R(A) + R(B)$ it follows that $R(A)$ depends only on the stable class of $\sigma_m(A)$ as defined in section 3. By Proposition (3·1) the stable class $[\sigma_m(A)]$ can be identified with an element of $K^1(TX)$. The additivity of $R(A)$ then implies that it is given by a homomorphism $\rho_m: K^1(TX) \to R$, that is

$$R(A) = \rho_m[\sigma_m(A)].$$

The order m of A is not of any importance. In fact putting $B = A\,|A|^{k-1}$ we see that

$$\eta_B(s) = \eta_A(ks)$$

hence $R(B) = R(A)/k$. Thus if we normalize the residue by multiplying by the order, putting

$$\tilde{R}(A) = mR(A),$$

we have $\tilde{R}(A) = \tilde{R}(B)$. On the other hand $\sigma_{mk}(B)$ and $\sigma_m(A)$ have the same unitary part on $S(X)$ and hence define the same element of $K^1(TX)$. Thus putting $\tilde{\rho} = m\rho_m$ we see that, independently of the order,

$$\tilde{R}(A) = \tilde{\rho}[\sigma_m(A)].$$

To prove that $\eta(0)$ is finite, i.e. that $\tilde{R} = 0$, it is therefore sufficient to verify this for a set of A whose classes $[\sigma_m(A)]$ generate $K^1(TX)$ over Q. Now the first order Riemannian examples studied in Parts I and II do in fact provide such a set of operators if X is *odd-dimensional* and orientable, as we shall now show.

If X is an oriented Riemannian manifold of dimension $2l-1$ we defined in Part II, section 2, a self-adjoint elliptic operator acting on forms of even degree by

$$B^{ev}\phi = i^l(-1)^{p+1}(*d-d*)\phi, \quad \deg\phi = 2p$$

(in Part I this operator was only introduced for l even). The importance of this operator is due to the following

LEMMA (4·2). *The element* $[\sigma_1(B^{ev})] \in K^1(TX)$, *when restricted to a fixed fibre* T_x *of* TX, *gives* 2^{l-1} *times a generator of*

$$K^1(T_x) \cong K^1(R^{2l-1}) \cong Z.$$

Proof. If we construct the signature operator A on $X \times R$ then, as pointed out in Part I, section 4, A is of the form

$$A = J\left(\frac{\partial}{\partial u}+B\right), \tag{4·3}$$

where $u \in R$, J is a constant unitary matrix with $J^2 = -1$, and $B = B^{ev} \oplus B^{odd}$ with B^{odd} (acting on odd forms) isomorphic to B^{ev} by $\phi \mapsto (-1)^p *\phi$ (for $\deg\phi = 2p$). Now it is well-known (see, for example, (8), section 6) that the symbol of A restricted to a fibre of $T(X\times R)$ gives 2^l times a generator of $K^0(R^{2l}) \cong Z$. On the other hand formula (4·3) implies that the symbol class of A is the suspension of the self-adjoint symbol class of B, i.e. they correspond via the isomorphism

$$K^0(R^{2l}) \cong K^1(R^{2l-1}).$$

This follows from formula (3·3) and the observations made there on the relation between elliptic symbols and self-adjoint elliptic symbols. Hence

$$[\sigma_1(B)] = 2[\sigma_1(B^{ev})] = 2^l g$$

where g is a generator of $K^1(R^{2l-1})$, which proves the lemma.

Now $K^1(TX)$ is a module over $K(X)$ and a standard argument (cf. (1)) shows that an element $\sigma \in K^1(TX)$ is a module generator provided its restriction to a fibre T_x is a generator g of $K^1(T_x)$. More generally if σ restricts to a non-zero multiple of g then σ is a module generator after tensoring with the rationals. The module product $\sigma . v$ corresponds on the operator level to extending a self-adjoint operator A, acting on $C^\infty(X,E)$, to a self-adjoint operator A_V, acting on $C^\infty(X, E\otimes V)$. Such an extension involves choosing a metric and connexion on V but its leading symbol is just $\sigma_m(A)\otimes 1_V$. For the operator B^{ev} above such extensions have been explicitly defined in Parts I and II. From Lemma (4·2) and these remarks we deduce:

PROPOSITION (4·4). *Let X be an oriented $(2l-1)$-dimensional Riemannian manifold. Then the symbols of the self-adjoint operators B^{ev}_V, as V varies over the vector bundles on X, generate a subgroup of $K^1(TX)$ of finite index.*

The next step is to establish the finiteness of $\eta(0)$ for the special operators B_V^{ev}. This can be done in one of two ways.

Method 1. By general cobordism theory some multiple of (X, V) is a boundary, that is there is a $2l$-dimensional oriented manifold X' with $\partial X'$ consisting of N copies of X and a vector bundle V' which restricts on each boundary component to a copy of V. Now we apply our main boundary value problem of Part I which tells us that $\eta_A(s)$ is finite at $s = 0$, where A is N copies of B_V^{ev}: in fact Part I gives a formula for the value of $\eta_A(0)$.

Method 2. By Proposition (2·8) we know that the residue $R(A)$ of $\eta_A(s)$ at $s = 0$ is given by an explicit integral formula $\int \omega$. For our particular operator $A = B_V^{ev}$ the integrand can be expressed in terms of the metric tensor g of X and the metric and connexion of V. Moreover replacing g by k^2g where k is a positive constant replaces A by $k^{-1}\epsilon A\epsilon^{-1}$, where ϵ is the automorphism on forms given by $\epsilon(\phi) = k^p(\phi)$ for a p-form ϕ (cf. (2), p. 306, for a similar calculation): hence $\zeta(A, s) = \operatorname{Tr} A^{-s}$ gets replaced by $\zeta(k^{-1}\epsilon A\epsilon^{-1}, s) = \zeta(k^{-1}A, s)$. Formula (2·6), together with the homogeneity properties of ζ-functions (see (2)), then shows that the form $\omega = \omega(g)$ satisfies

$$\omega(k^2g) = \omega(g).$$

The generalized Gilkey Theorem ((2), Theorem II) then asserts that ω is a Pontrjagin–Chern form and is therefore identically zero since $\dim X$ is *odd*.

Remark. Method 1 uses cobordism theory while Method 2 uses the invariant theory of (2). Note that the invariant theory is not needed in Method 1 just to prove finiteness: it is only required for the explicit evaluation of $\eta(0)$.

For a non-orientable manifold X we can take the oriented double cover $\tilde{X} \to X$. If A is any self-adjoint elliptic operator on X we can lift it up to a self-adjoint elliptic operator $\tilde{A}$ on $\tilde{X}$. This is clear if A is differential. For a pseudo-differential A the choice of $\tilde{A}$ is not unique but its complete symbol is unique – see section 2 for essentially the same construction in the language of flat bundles. Since $R(A)$ is given by a local integral formula (Proposition (2·8)) it follows that $R(\tilde{A}) = 2R(A)$. Hence the finiteness of $\eta(0)$ on $\tilde{X}$ implies the finiteness on X.

Thus we have finally proved our objective when $\dim X$ is odd, namely

THEOREM (4·5). *Let X be an odd-dimensional manifold, A a self-adjoint elliptic pseudo-differential operator of positive order on X. Then $\eta_A(s)$ is holomorphic at $s = 0$.*

In (3) this Theorem was asserted for all X. Unfortunately we do not yet know how to deal with even-dimensional X. The usual device, to change parity, is to replace X by $X \times S^1$. Now it is true that if A is self-adjoint and elliptic on X and B is elliptic on Y, with order B = order A, then

$$P = \begin{pmatrix} A \otimes 1 & 1 \otimes B^* \\ 1 \otimes B & -A \otimes 1 \end{pmatrix}$$

is self-adjoint and elliptic on $X \times Y$ and

$$\eta_P(s) = (\operatorname{index} B)\,\eta_A(s).$$

If A, B are differential operators then so is P but unfortunately if A or B is pseudo-differential P is *not* pseudo-differential. On the other hand we cannot use only differential operators because index $B = 0$ if B is differential and dim Y is odd – for example if Y is the circle. In (7), in studying the index, this problem is not serious because P can be approximated (with respect to suitable Sobolev-norms) by pseudo-differential operators and the index is norm-continuous. The residue $R(A)$ is a more sophisticated invariant, depending (in its explicit integral formula) on much of the complete symbol of A, and so is not obviously continuous under a crude norm approximation. For the present therefore the case of even-dimensional manifolds must be left open.

The vanishing of the index for a differential operator on an odd-dimensional manifold is a consequence of the symmetry (or skew-symmetry) of its leading symbol and can be proved analytically as in (8) or by topological methods (showing that the symbol gives an element of finite order in $K(TX)$). There is a similar result for the residue $R(A)$, namely if A is differential and order $A + \dim X$ is odd then $R(A) = 0$. Again this may be proved analytically or topologically. Thus for even-dimensional Y, $\eta_A(0)$ is finite for differential A of *odd* order: the unsettled case is for differential A of even order. The reason why the order of A plays a role for this question (and not for the index) is that $R(-A) = -R(A)$ whereas index$(-B) =$ index$B$.

We conclude this section with an example which shows why the finiteness of $\eta_A(s)$ at $s = 0$ is not something which comes out trivially from the analysis. By definition $\eta_A(s)$ is, for Re(s) large, the trace of the operator

$$T_s = A\,|A|^{-s-1}.$$

The kernel $T_s(x, y)$ is (for large Res) a continuous function of $(x, y) \in X \times X$: it takes its value in the vector space Hom(E_y, E_x) once we have picked a C^∞ measure on X. In more concrete terms $T_s(x, y)$ is a square matrix and

$$\eta_A(s) = \int_X \operatorname{tr} T_s(x, x)\,dx$$

depends only on the diagonal entries of this matrix. Now general analysis of the Seeley-type gives only that the entries of $T_s(x, x)$ have at most a simple pole at $s = 0$. In fact the example we shall give will show that such a simple pole really occurs but it disappears when we take the trace. It requires a fairly subtle argument therefore to prove $\eta(0)$ is finite while allowing the kernel $T_s(x, y)$ to admit a pole.

Our example is in fact the simplest case of our basic Riemannian operator of Part I, namely the operator B^{ev} acting on even forms for a 3-dimensional manifold X. Explicit computation in this case shows that, near $s = 0$,

$$T_s(x, x) = \frac{1}{s}\begin{pmatrix} 0 & T_{02} \\ T_{20} & T_{22} \end{pmatrix} + R(s),$$

where $R(s)$ is analytic in s. If we put $V = T^*_x(X)$, then the components of the above matrix act as follows:

(1) 0 is the zero endomorphism of $\Lambda^0 V$.

(2) $T_{02}: \Lambda^2 V \to \Lambda^0 V$ is defined by

$$T_{02}(\phi) = c_1 \langle *d\tau, \phi \rangle, \quad \text{for} \quad \phi \in \Lambda^2 V,$$

where τ is the scalar curvature of the manifold X, $\tau = \Sigma_{i,j} R^i_{jij}$, R being the curvature tensor, and c_1 is a non-zero numerical constant.

(3) $T_{02}: \Lambda^0 V \to \Lambda^2 V$ is defined by

$$T_{20} f = c_1 f * d\tau, \quad f \in \Lambda^0 V.$$

(4) $T_{22} \in \operatorname{End} \Lambda^2 V \cong \Lambda^2 V \otimes \Lambda^2 V$ (using the metric) is given by $T_{22} = c_2 \chi(\nabla \tilde{R})$ where $\nabla \tilde{R}$ is the covariant derivative of the Ricci tensor $\tilde{R}$, $\chi: \otimes^3 V \to \Lambda^2 V \otimes \Lambda^2 V$ is given by

$$\chi(v_1 \otimes v_2 \otimes v_3) = (*v_3) \otimes v_1 \wedge v_2$$

and c_2 is a numerical constant.

Now, in general for a Riemannian manifold, the endomorphisms T_{02}, T_{20} and T_{22} are *non-zero* so that $T_s(x,x)$ has a pole at $s = 0$. However Trace $T_s(x,x)$ = Trace T_{22} is zero because (Trace T_{22}) $*1$, up to a numerical constant, equals the image of $\nabla \tilde{R}$ under the skew-symmetrizing map $\tilde{\chi}: \otimes^3 V \to \Lambda^3 V$, and the Bianchi identities imply that $\tilde{\chi}(\nabla \tilde{R}) = 0$. Thus Trace $T_s(x,x)$ is finite at $s = 0$, confirming the finiteness of $\eta_A(0)$.

5. *The index theorem for flat bundles.* In section 2 we defined a reduced η-function associated to a self-adjoint elliptic operator A and a unitary representation α by

$$\tilde{\eta}_\alpha(s, A) = \eta_{A_\alpha}(s) - N\eta_A(s), \quad \operatorname{mod} Z,$$

where A_α is A 'twisted by α' and N is the dimension of α. We proved in Proposition (2·14) that $\tilde{\eta}_\alpha(0, A)$ is finite and is a homotopy invariant of A. It therefore only depends on the leading symbol of A (and on α). In this section we shall formulate an 'index theorem' which gives a topological formula for $\tilde{\eta}_\alpha(0, A)$ in terms of α and of the leading symbol of A. In fact we shall work with the slightly more precise invariant $\tilde{\xi}_\alpha(0, A)$ based on

$$\xi(s) = \frac{h + \eta(s)}{2},$$

where h is the dimension of the null-space.

We shall first show that $\tilde{\xi}_\alpha(0, A)$ depends only on the class $[\sigma_m(A)] \in K^1(TX)$ as defined in section 3. For this we must show the following

(i) $\tilde{\xi}_\alpha(0, A)$ is additive under direct sums $(A = A_1 \oplus A_2)$,

(iii) $\tilde{\xi}_\alpha(0, A)$ is independent of the order m of A $(m > 0)$,

(iii) $\tilde{\xi}_\alpha(0, A)$ vanishes if $\pm A$ is non-negative.

Now (i) is clear. To prove (ii) we replace A and A_α by B and B_α where

$$B = |A|^k A, \quad B_\alpha = |A_\alpha|^k A_\alpha, \quad k = \frac{1-m}{m}.$$

Here $|A|$ and $|A_\alpha|$ have the same null-spaces as A and A_α, and their inverses (which we need for $m > 1$) are defined on the orthogonal complement of these null-spaces.

Then B has order 1 and B_α is an extension or twisting of B by α in the sense of (2·14), namely the complete symbol of B_α is $\sigma(B) \otimes 1_\alpha$ (because $\sigma(A_\alpha) = \sigma(A) \otimes 1_\alpha$ and $\sigma(|A|)$ depends only on $\sigma(A)$). Moreover

$$\eta_A(s) = \eta_B(ms), \quad \eta_\alpha(s, A) = \eta_\alpha(ms, B),$$
$$h(A) = h(B), \qquad h(A_\alpha) = h(B_\alpha),$$

and so $\tilde{\xi}_\alpha(0, A) = \tilde{\xi}_\alpha(0, B)$ as required.

For (iii) we note that if $A \geqslant 0$ we have

$$h + \eta_A(s) = h + \zeta_A(s)$$

and so, by Seeley (14), $h + \eta_A(0) = h + \zeta_A(0)$ is given by an integral formula (see section 2). Since A_α and NA are locally isomorphic the integral formulae for $h_\alpha + \eta_\alpha(0, A)$ and $N(h + \eta(0, A))$ coincide and so their difference, and hence $\tilde{\xi}_\alpha(0, A)$, vanishes.

Thus for fixed α the mapping $A \mapsto \tilde{\xi}_\alpha(0, A)$ induces a homomorphism

$$\mathrm{ind}_\alpha : K^1(TX) \to R/Z \tag{5·1}$$

which we shall refer to as the analytical index of A (or of the symbol class of A) with respect to the flat bundle α. Our aim is to identify ind_α topologically.

In Part II, section 5, we introduced K-theory with Q/Z-coefficients and showed that a representation α of $\pi_1(X)$ factoring through a *finite* group defined an element $[\alpha] \in K^{-1}(X, Q/Z)$. Let us assume for the moment that we have extended this to R/Z-coefficients, so that any unitary representation α of $\pi_1(X)$ defines an element $[\alpha] \in K^{-1}(X, R/Z)$. Assuming $K^1(\ , R/Z)$ has the expected formal properties we have a pairing

$$K^{-1}(X, R/Z) \otimes K^1(TX) \to K^0(TX, R/Z)$$

and a homomorphism

$$\mathrm{Ind}: K^0(TX, R/Z) \to R/Z$$

which extends the usual topological index of (7). For each α we therefore obtain a topologically defined homomorphism

$$\mathrm{Ind}_\alpha : K^1(TX) \to R/Z \tag{5·2}$$

given by $\sigma \mapsto -\mathrm{Ind}([\alpha].\sigma)$ (or $\mathrm{Ind}(\sigma.[\alpha])$ since products in K^1 are skew-symmetric) and we shall prove the following:

(5·3) *Index Theorem for flat bundles.* If X is odd-dimensional the analytical index ind_α of (5·1) coincides with the topological index Ind_α of (5·2).

Remarks. (1) It seems likely that the theorem should also hold when $\dim X$ is even. The difficulties here are essentially technical and similar to those discussed in section 4 in connexion with the finiteness of $\eta(0)$. As mentioned in the Introduction we hope to give elsewhere an altogether different proof of (5·3) which will work in all cases. This alternative proof will depend on the use of von Neumann algebras (see Remark 4 at the end of this section).

(2) If X is a Spin^c-manifold, $\sigma \in K^1(TX)$ the symbol class of its Dirac operator then we get a homomorphism

$$K^{-1}(X, R/Z) \to R/Z$$

given by $u \mapsto \operatorname{Ind}(u\sigma)$ which coincides with the direct image homomorphism. When R/Z is replaced by Q/Z this was used in Part II, section 3, definition II. We see therefore that Theorem (3·6) of Part II is indeed a special case of Theorem (5·3).

Remark. Perhaps we should point out, at this stage, that there are two natural direct image maps in K-theory depending on which Thom isomorphism one uses. For a complex vector bundle V over X we can construct a generator for $K(V)$ over $K(X)$ by using either the exterior algebra of V or of its dual V^*. We use V^* and, as a consequence, the direct image $K(X) \to Z$ for an even-dimensional almost complex or Spin^c-manifold X is given cohomologically by the Riemannian–Roch formula

$$V \mapsto \{\operatorname{ch} V . \mathscr{T}(X)\}[X]$$

where $\mathscr{T} = e^{c\frac{1}{2}}\,\hat{a}$ is the total Todd polynomial.

We return now to the *problem of defining* K-theory *with* R/Z-*coefficients*. The method we shall follow is due to G. B. Segal and rests on the fact that we already know how to define $K^*(\ ,Q/Z)$, which is the 'torsion part'. The 'free part', namely K-theory with Q- or R-coefficients may just be defined by tensoring with Q or R:

$$K^*(X;Q) = K^*(X) \otimes Q, \quad K^*(X,R) = K^*(X) \otimes R.$$

To put the two parts together we now *define* $K^*(X, R/Z)$ *as the cokernel of the natural homomorphism*

$$(p, -j): K^*(X,Q) \to K^*(X,Q/Z) \oplus K^*(X,R).$$

The first component p of this is induced by passing to the limit in the diagram

$$\begin{array}{ccc} K^*(X) & \to & K^*(X, Z/nZ) \\ \downarrow m & & \downarrow \\ K^*(X) & \to & K^*(X, Z/mnZ). \end{array}$$

The second component, induced by the inclusion $j: Q \to R$, is *injective* for all X. It is this fact which makes the cokernel satisfy the exactness axiom of a cohomology theory. If $Y \subset X$ we have long exact sequences for the two cohomology functors $K^*(\ ,Q)$ and $K^*(\ ,Q/Z) \oplus K^*(\ ,R)$: regarding the first as a sub chain-complex of the second we see that the quotient complex is also acyclic (exact).

All the formal properties of ordinary K-theory now extend to $K^*(\ ,R/Z)$ including the appropriate cup-products and direct images. In view of the Chern character isomorphism

$$\operatorname{ch}: K^*(X,R) \to H^*(X,R)$$

we may identify $K^*(X,R)$ with real cohomology, K^0 corresponding to even dimensions and K^1 to odd dimensions.

We shall now show how† to associate to a unitary representation α of $\pi_1(X)$ an element of $K^{-1}(X, R/Z)$. We shall make some choices in order to construct an element $a \in K^{-1}(X, Q/Z)$ and an element $b \in K^{-1}(X,R)$. Altering our choices will alter (a,b) to $(a+p(c), b-j(c))$ for some $c \in K^{-1}(X,Q)$ so that we get a unique element in the cokernel of $(p, -j)$. The details are as follows.

† This construction is also due to G. B. Segal.

(i) *The choices.* The vector bundle V_α defined by α is flat so its real Chern classes vanish, hence some multiple kV_α is (unitarily) trivial: we choose k and the (unitary) trivialization $\phi: kV_\alpha \to kV_N$, where $V_N = X \times C^N$.

(ii) *The element a.* As explained in Part II, section 5, the triple $\{V_\alpha, V_N, \phi\}$ defines an element a_k of $K^{-1}(X, Z/kZ)$, hence an element a of $K^{-1}(X, Q/Z)$.

(iii) *The element b.* The trivial bundle $V_{kN} = kV_N$ has, besides its own trivial connexion γ_0, another flat connexion γ_1, coming from the flat connexion of kV_α by the isomorphism ϕ. We can therefore construct odd-dimensional characteristic classes in the manner of Chern–Simons (13). More precisely let γ_t be a one-parameter family of connexions joining γ_0 to γ_1 (e.g. $\gamma_t = t\gamma_1 + (1-t)\gamma_0$) interpreted as connexion for the trivial bundle on $I \times X$, where I is the unit interval $[0, 1]$. We form the Chern forms c_i of this connexion and then integrate with respect to $t \in I$ to get $(2i-1)$-forms ω_i on X. The cohomology class (ω_i) of ω_i is independent of the choice of path. Applying this to the Chern character ch, rather than the c_i, and then dividing by k we obtain our (mixed) odd-dimensional cohomology class β on X. We put $b = \mathrm{ch}^{-1}(\beta)$.

(iv) *Effect of the choices.* For fixed k we can alter ϕ by any automorphism of $X \times C^{Nk}$. Such an automorphism defines an element u_k of $K^{-1}(X)$. It is routine to check that a_k gets altered to $a_k + p_k(u_k)$ where

$$p_k: K^{-1}(X) \to K^{-1}(X, Z/kZ)$$

is 'reduction mod k', and that b alters to

$$b - \frac{J(\mathrm{ch}\, u_k)}{k},$$

where $J: K^{-1}(X) \to K^{-1}(X, R)$ is the natural map. Putting $u = k^{-1}u_k \in K^{-1}(X, Q)$ it follows that $(a, b) \mapsto (a + p(u), b - j(u))$. If we alter k to a multiple kl and ϕ to $\phi \otimes 1_l$ then a and b are both unaltered.

Remarks. (1) If our representation α is not unitary then the above construction works in the same way to give an element of $K^{-1}(X, C/Z)$. The point is that for unitary connexions the Chern forms are real but for a general linear connexion they are complex.

(2) We have assumed that X is a manifold but this is not really necessary. If X is a finite complex we can embed it in Euclidean space and use differential forms on a neighbourhood to represent cohomology. Thus for any discrete group Γ a unitary representation $\alpha: \Gamma \to U(N)$ defines an element of $K^{-1}(B_\Gamma, R/Z)$, this group being defined as the limit over finite subcomplexes of the classifying space B_Γ.

(3) For a finite group G the construction of Part II, section 5, is clearly compatible with the more general construction given above. The point is that for finite G there is a preferred class of trivializations ϕ for which the element b is always zero and the corresponding element a is unique.

(4) It is possible to give a more direct bundle-theoretic definition of

$$[\alpha] \in K^{-1}(X, R/Z)$$

without any reference to cohomology. This requires the use of von Neumann algebras and will be described elsewhere.

The proof of Theorem (5·3) will be carried out in the remaining sections. It will be divided into parts corresponding to the elements a and b in the construction above. In section 6 we shall deal with the R-component b using the main results of Part I: this proceeds very similarly to the finiteness proof in section 4. In section 7 we shall digress a little to discuss the notion of spectral flow which is of independent interest but is also used in section 8 to convert the Q/Z-component a into the index of a suitable family of elliptic operators. We then appeal to the index theorem for families of (9) to complete the treatment of the torsion part of Theorem (5·3). Note that the odd-dimensionality of X is used in section 6 but not in section 8.

Theorem (5·3) can also be extended to non-unitary representations α. Although the twisted operator A_α is then not self-adjoint it has a self-adjoint symbol and this is enough to define $\eta_\alpha(s, A)$. We use a suitable contour integral as in the proof of (2·10). In passing from $\eta(s)$ to $\xi(s)$ we now take h to be the dimension arising from the spectrum on the imaginary axis, each λ contributing $\dim \operatorname{Ker}(A_\alpha - \lambda)^N$ for large N. $\tilde{\xi}_\alpha(0, A)$ is now a *complex* number modulo integers and, as noted in Remark (1) above, α defines an element $[\alpha] \in K^{-1}(X, C/Z)$. Theorem (5·3) continues to hold in this situation as an equation in C/Z. The proof proceeds as for the unitary case by separating out the C-component and the torsion component. The latter is covered by the proof of section 8 while the C-component can be treated by analytic continuation from the R-component proof of section 6. We shall make brief remarks on this at the appropriate place.

6. *Trivialized flat bundles.* In this section we shall assume that $\dim X$ is odd. It is then convenient, though not essential, to use the finiteness of $\eta_A(0)$ established in section 4.

In section 5 we studied $\tilde{\eta}_\alpha(0, A)$ which is a homotopy invariant of A depending on α and takes values in R/Z. In this section we shall refine this to a *real-valued* invariant on the assumption that the bundle V_α defined by α is trivial and that a fixed (unitary) trivialization $\phi: V_\alpha \to V_N = X \times C^N$ has been chosen. The main point is that A_α and A_N can then be regarded as acting on the *same* bundle: if A operates on $C^\infty(X, E)$ then A_N and A_α (via ϕ) operate on $C^\infty(X, NE)$.

We have seen that $f(B) = \xi_B(0)$ is a real-valued function of B whose reduction modulo $Z, \bar{f}(B)$ is C^∞; that is f has integer discontinuities but is otherwise smooth. The differential $\omega = d\bar{f}$ is therefore a well-defined closed 1-form on the space of all B: we shall not worry about the precise definition of 1-forms on infinite-dimensional spaces since we shall only use the restriction to one or two parameter families. On any component of the space of all B (for a fixed manifold, bundle and order) we can therefore introduce the indefinite integral g of ω:

$$g(B) = \int_{B_0}^{B} \omega$$

which depends on B_0 and on the homotopy class of the path from B_0 to B. Clearly g is f made continuous by eliminating the integer jumps. If we further fix the leading symbol of B then we get a contractible space and so the homotopy class from B_0 to B is unique.

We apply these considerations to the operators A_α, A_N above. They are both obtained by twisting A and so their leading symbols coincide (via ϕ). Hence we have a uniquely defined integral

$$\int_{A_N}^{A_\alpha} \omega. \tag{6·1}$$

On the one hand this is clearly continuous in all the data. On the other hand we already know that its reduction modulo Z is (for fixed α) a homotopy invariant of A. Hence (6·1) defines a homotopy invariant of A depending on α, ϕ which takes *real values*: we denote it by $\operatorname{ind}(\alpha, \phi, A)$.

Our aim is to give an explicit formula for $\operatorname{ind}(\alpha, \phi, A)$ in terms of the class

$$[\sigma_m(A)] \in K^1(TX)$$

and of the pair (α, ϕ). As explained in section 5 the two flat connexions on $X \times \mathrm{C}^N$ enable us to define a cohomology class

$$\beta(\alpha, \phi) \in H^{\mathrm{odd}}(X, R)$$

and hence an element

$$b(\alpha, \phi) \in K^{-1}(X, R)$$

with $\operatorname{ch} b = \beta$. Our explicit formula is then as follows:

PROPOSITION (6·2). $\operatorname{ind}(\alpha, \phi, A) = -\operatorname{Ind}_R\{b(\alpha,\phi)\,[\sigma_m(A)]\}$ *where* $\operatorname{Ind}_R: K(TX, R) \to \mathrm{R}$ *is induced from the topological index*: $K(TX) \to Z$ *by tensoring with* R.

Remarks. (1) It is clear that, after reducing modulo Z, (6·2) yields a special case of Theorem (5·3) – in which the torsion component vanishes. The extra precision in having (6·2) *as an equation in* R is however necessary for our proof. (2) Since the topological index is given by

$$u \mapsto \{\operatorname{ch} u \,.\, \mathscr{I}(X)\}\,[TX],$$

where $\mathscr{I}(X)$ is the index class of X[8] we can rewrite (6·2) cohomologically as

$$\operatorname{ind}(\alpha, \phi, A) = -\{\beta(\alpha, \phi)\operatorname{ch}[\sigma_m(A)]\,\mathscr{I}(X)\}\,[TX]. \tag{6·3}$$

We have already noted that $\operatorname{ind}(\alpha, \phi, A)$ is a homotopy invariant of A. It is also additive for direct sums and independent of the order m of A, because $\xi_A(0)$ is (see section 2). Hence it depends only on the symbol class $[\sigma_m(A)] \in K^1(TX)$. Thus, for fixed (α, ϕ), $A \mapsto \operatorname{ind}(\alpha, \phi, A)$ induces a homomorphism

$$K^1(TX) \to R.$$

To establish (6·2) for a given X it is therefore sufficient to check it for a set of operators A whose symbol classes generate $K^1(TX)$ over Q. *Assume now that X is orientable*; then, as proved in Proposition (4·4), we get such a generating set by taking $A = B_W$ where W runs over the vector bundles on X and B is the basic self-adjoint operator acting on differential forms which was studied in Parts I and II.

To prove (6·2), or equivalently (6·3), for $A = B_W$ we shall apply the main theorem of Part I to the manifold $I \times X$ with the product metric and a suitable connexion on the bundle $N\tilde{W}$, where $\tilde{W}$ is W lifted to $I \times X$. For any unitary connexion c on NW we have a corresponding extension of B, acting on forms with coefficients in NW, which we denote by A_c. Given two connexions c_0 and c_1 we join them by any C^∞ path of connexions c_t, $0 \leqslant t \leqslant 1$, which is constant near $t = 0$ and $t = 1$. This gives a connexion on $N\tilde{W}$ and we now apply the main theorem of Part I to the signature operator of $I \times X$ with coefficients in $N\tilde{W}$. This gives the formula

$$\xi_1(0) - \xi_0(0) = \int_{I \times X} (\operatorname{ch} N\tilde{W})\, \mathscr{L}' - \operatorname{index} D, \tag{6·4}$$

where ξ_i is the ξ-function of A_{c_i}, $\operatorname{ch} N\tilde{W}$ denotes the Chern character form of $N\tilde{W}$ constructed from its connexion, $\mathscr{L}' = \mathscr{L}'(p)$ is the universal polynomial in the Pontrjagin form of $I \times X$ associated to the signature operator† and index D is a certain integer. If we now vary c_1 the integral expression in (6·4) varies continuously while the term index D may jump. Hence the integral expression must coincide with the 'continuous part' of the left-hand side namely with the integral (6·1) where A_α, A_N are replaced by A_{c_0}, A_{c_1}. In particular, taking c_0, c_1 to be the two connexions on NW induced by the two flat connexions on $X \times \mathbb{C}^N$ and any fixed connexion on W, we see that

$$\operatorname{ind}(\alpha, \phi, A) = \int_{I \times X} (\operatorname{ch} N\tilde{W})\, \mathscr{L}'. \tag{6·5}$$

Recalling the definition of $\beta(\alpha, \phi)$ we see that (6·5) reduces to

$$\operatorname{ind}(\alpha, \phi, A) = \{\beta(\alpha, \phi)\, \mathscr{L}'(X)\}[X]. \tag{6·6}$$

It remains to verify that this cohomological formula coincides with (6·3) via the Thom isomorphism

$$H^*(X) \cong H^*(TX).$$

Now on an even-dimensional manifold Y, $\mathscr{L}'$ corresponds via the Thom isomorphism to $\operatorname{ch}[\sigma(D)] . \mathscr{I}(Y)$, where D is the generalized Signature operator – this is essentially the way it arises in (8), section 6. Taking $Y = I \times X$ we have $D = \sigma_0(\partial/\partial u + A)$ where u is the inward normal coordinate (see Part II, section 4, where the Signature operator is denoted by A and our A is denoted by B) and σ_0 is a fixed matrix. As explained in the proof of Lemma (4·2) this formula for D shows that $[\sigma(D)]$ restricted to $1 \times X$ is the suspension of the class of the self-adjoint symbol $\sigma(A)$ in $K^1(TX)$. Actually, because the inward normal is the negative direction in the standard orientation conventions, we get $-[\sigma(A)]$. Hence $\mathscr{L}'$ corresponds to $-\operatorname{ch}[\sigma(A)]\, \mathscr{I}(X)$ and so (6·6) coincides with (6·3) as asserted.

We have now proved (6·2) when X is *orientable* (and $\dim X$ is odd). For non-orientable X we introduce its oriented double covering $\tilde{X} \to X$. Since the right-hand

† Explicitly

$$\mathscr{L} = \prod_{i=1}^{l} \frac{x_i}{\tanh \frac{1}{2} x^i}$$

where the p_j are interpreted as the elementary symmetric functions of $x_1^2, \ldots, x_l^2$ ((8), (6·4)).

side in (6·2) or (6·3) is clearly multiplicative for finite coverings it will be enough to prove the corresponding result for $\operatorname{ind}(\alpha, \phi, A)$. But by definition this is given by the integral (6·1). Moreover by Proposition (2·12), the 1-form $\omega = \overline{df}$ in (6·1) is itself given by an integral over X, in which the integrand is computed locally from the complete symbols. Hence $\operatorname{ind}(\alpha, \phi, A)$ is multiplicative for finite coverings.

Note that this argument, as well as the argument in the orientable case, works because we now have a real-valued invariant. If we had kept our invariant in R/Z then our proof would not have dealt with the 2-torsion because it involves taking multiples (by a power of 2).

Everything in this section extends in a fairly straightforward way to the non-unitary case, R being replaced by C. The only point that requires comment concerns (6·4) which came from applying the main theorem of Part I. Working modulo Z we ignore the term index D and we then regard both sides of (6·4) as functions of the connexion c_1. They are both defined for any connexion, not necessarily unitary, and (6·4) asserts equality for unitary connexions. But the space of all connexions is a complex affine space and the unitary connexions can be viewed as the real points of this space.

We may therefore extend (6·4) by analytic continuation provided both sides are analytic. The Chern forms appearing on the right are polynomial functions of the connexion and so certainly analytic. To see that $\tilde{\xi}_{A_c}(0)$ is analytic in the connexion c we note that the operator A_c depends affine linearly on c and that $\tilde{\xi}_{A_c}(0)$ can be defined by a suitably Cauchy integral in the λ-plane, using the resolvent $(A_c - \lambda)^{-1}$. Thus (6·4) holds for all connexions and in particular for the flat connexion α.

7. *Spectral flow.* In this section we make a slight digression to discuss the notion of 'spectral flow' and to relate it to the main theorem of Part I. As mentioned in the introduction to Part I spectral flow provided one of the clues leading to the introduction of the function $\eta(s)$ – this was work carried out in collaboration with G. Lusztig. Moreover in section 8 the connexion with spectral flow will be applied to complete the proof of our general index theorem for flat bundles.

For a family A_t of elliptic self-adjoint operators of positive order we have in section 6 already replaced the function $f(t) = \xi_{A_t}(0)$ by its continuous part $g(t)$, so that

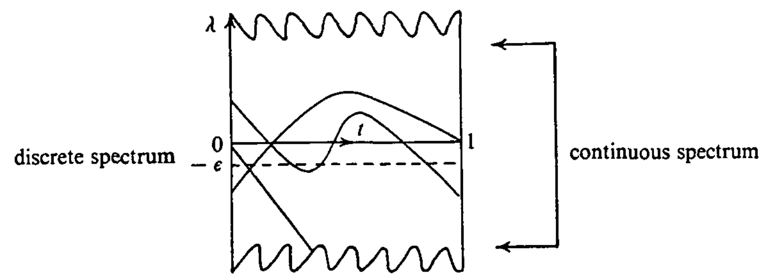
$$f(t) = g(t) + j(t), \quad j(0) = 0, \tag{7·1}$$

where $j(t)$ is integer-valued. From its definition we see that $j(t)$ can be interpreted as the net number of eigenvalues of A_u that cross the origin as u runs from 0 to t. More precisely $j(t)$ increases by 1 every time an eigenvalue $\lambda < 0$ changes to one $\geqslant 0$ and decreases by 1 when the reverse happens.

Whereas $f(t)$ and $g(t)$ depend on the asymptotic properties of the eigenvalues, and require that A_t be a pseudo-differential operator of positive order, their difference $j(t)$ can be defined whenever we have a family F_t of *self-adjoint Fredholm operators.* Assuming that F_t is continuous in t we can still count eigenvalues crossing zero: even though F_t may have a continuous spectrum, 0 is always in the discrete spectrum. To give a precise definition of $j(t)$ (for $t = 1$ say) we consider the graph of Spec F_t:

$$\mathscr{S} = \cup \operatorname{Spec} F_t.$$

It is a closed subset of the (t, λ)-plane. We define $j(1)$ to be the *intersection number* of $\mathscr{S}$ with the line $\lambda = -\epsilon$ where ϵ is any sufficiently small positive number (we can take $\epsilon = 0$ if F_0 and F_1 are both invertible). This intersection number may be defined as in elementary topology using any convenient approximation of the family F_t, e.g. by an analytic or piece-wise linear family.



We shall refer to $j(1)$ as the *spectral flow* from F_0 to F_1.

A case of particular importance is when $F_1 = F_0$ so that we have a family parametrized by the circle S^1. The graph above can then be viewed as on the cylinder $S^1 \times R$ (identifying $t = 0$ with $t = 1$), and the spectral flow is the intersection number of $\mathscr{S}$ with *any* circle $S^1 \times \lambda$. The spectral flow is now a homotopy-invariant of the family and it defines a homomorphism

$$sf: \pi_1(\hat{\mathfrak{F}}_*) \rightarrow Z \tag{7·2}$$

where $\hat{\mathfrak{F}}_*$ is the non-trivial component of the self-adjoint Fredholm operators as in section 3.

Now we have the homotopy equivalence $\hat{\mathfrak{F}}_* \rightarrow \Omega\mathfrak{F}$ established in (11) giving an isomorphism

$$\left.\begin{aligned} \pi_1(\hat{\mathfrak{F}}_*) &\cong \pi_2(\mathfrak{F}), \\ &\cong Z \quad \text{by Bott periodicity.} \end{aligned}\right\} \tag{7·3}$$

It is easy to construct a family with spectral flow equal to 1 (take A_t with eigenvalues $n + t, n \in Z$) and so (7·2) is an isomorphism and coincides up to sign with (7·3). The sign will be clarified shortly.

Thus Theorem (3·4) can be viewed as giving an explicit formula for the spectral flow of a family A_t of elliptic self-adjoint operators parametrized by $t \in S^1$, namely

Theorem (7·4). *Let A_t be a family of self-adjoint elliptic operators on X parametrized by $t \in S^1$. Then the spectral flow of the family is given by Ind $[\sigma\{A_t\}]$, where*

$$[\sigma\{A_t\}] \in K^1(S^1 \times TX)$$

is the symbol of the family and

$$\mathrm{Ind}: K^1(S^1 \times TX) \rightarrow K^1(S^1) \cong Z$$

denotes the topological index.

Remark. There is a slightly more general version of (7·4) in which the product $S^1 \times X$ is replaced by a fibre bundle with fibre X and base S^1. Only minor changes in notation and proof are involved.

For a first order family A_t we can also compute the spectral flow by using the main theorem of Part I. We take the operator $D = \partial/\partial t + A_t$ on $X \times I$ and we assume (using a deformation) that A_t is constant near $t = 0$. There are then two index problems we can consider:

(i) the index of D with the boundary condition (2·3) of Part I;

(ii) the index of D with periodic boundary condition, i.e. the index of the operator on $X \times S^1$.

The boundary contributions from $X \times 0$ and $X \times 1$ cancel when we apply the index theorem (3·10) of Part I to compute (i), and the integral term coincides with that for problem (ii). Hence (i) and (ii) have the same index. On the other hand, index D in (i) is the same as the spectral flow of the family A_t: this point was essentially made in discussing (6·4), except that the interest there was in the continuous part $g(t)$ whereas we now want the residual jump function $j(t)$. Thus the spectral flow of the family A_t is equal to the index of $\partial/\partial t + A_t$ on $X \times S^1$.

It is now a routine matter to compare, on the K-theory level, the index of $\partial/\partial t + A_t$ on $X \times S^1$ with the answer given by applying Theorem (3·4). We know they differ at most in sign and a careful check of all sign conventions shows that the sign is $+$.

Remark. The two approaches to the family A_t, converting spectral flow into an index, can be thought of as 'suspension' and 'desuspension'. In one case we go up from a 1-parameter family to a 2-parameter family, whereas in the other case we go down to a single operator. The results are compatible via the periodicity theorem. Note that desuspension is technically more difficult and requires that A_t be a family of first-order elliptic operators.

8. *Mod k index theorem.* In this section we shall prove a 'mod k index theorem' which will deal with the torsion part of Theorem (5·3). The proof will be quite independent of Part I but will use the ideas of section 7.

By a mod k family of self-adjoint elliptic operators we shall mean a family

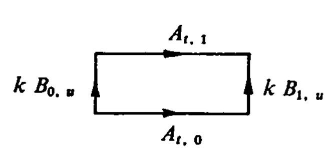
$$A_t \quad 0 \leqslant t \leqslant 1$$

with given isomorphisms

$$A_0 \cong B_0 \oplus \dots \oplus B_0 \quad k \text{ times}$$

$$A_1 \cong B_1 \oplus \dots \oplus B_1 \quad k \text{ times}$$

for some B_0, B_1. If $A_{t,u}$ $0 \leqslant u \leqslant 1$ is a homotopy of such a mod k family then we have a family of self-adjoint operators parametrized by the (t, u)-square $0 \leqslant t \leqslant 1$, $0 \leqslant u \leqslant 1$



The total spectral flow around the boundary is therefore zero and this gives

$$sf\{A_{t,1}\} - sf\{A_{t,0}\} = k(sf\{B_{1,u}\} - sf\{B_{0,u}\}).$$

Thus the spectral flow of A_t reduced modulo k is a homotopy invariant: we denote it by $sf_k\{A_t\}$.

If $B_0 \cong B_1$ then our family $\{A_t\}$ $0 \leqslant t \leqslant 1$ can be closed up into a family $\{\bar{A}_y\}$ parametrized by $y \in S^1$, and $sf_k\{A_t\}$ is then the mod k reduction of $sf\{\bar{A}_y\}$, the integer spectral flow invariant of section 7. More generally this holds whenever B_0 is homotopic to B_1 through self-adjoint elliptic operators.

If it were always possible to find such a homotopy connecting B_0 and B_1 there would really be no point in introducing this mod k spectral flow. In general however B_0 and B_1 need not be homotopic even though kB_0 is homotopic to kB_1 (by our family $\{A_t\}$). This is because the group $K^1(TX)$ of self-adjoint symbols may have elements of order k. On the other hand if we consider, not elliptic operators on X, but abstract self-adjoint Fredholm operators, the situation simplifies because the space $\hat{\mathfrak{F}}_*$ (see section 3) is connected while the trivial components of $\hat{\mathfrak{F}}$ can be ignored. This means that our mod k spectral flow can always be viewed as the mod k reduction of an integer spectral flow for a suitable family $S^1 \to \hat{\mathfrak{F}}_*$. However this in itself is no help in effectively computing our mod k index because we only have index theorems (e.g. Theorem (3·4)) for families of elliptic operators.

Our next task therefore is to show how to compute our mod k spectral flow by converting it, in a somewhat more sophisticated way, into the index of a certain family of elliptic operators. Our parameter space will be, not S^1, but $S^1 \times M_k$ where M_k is the Moore space of Part II, section 5, namely M_k is the 'cofibre' of the degree k map $S^1 \to S^1$.

Let H be the line-bundle on M_k induced from the Hopf bundle on S^2 by the natural map $M_k \to S^2$: then $[H]-1$ generates $\tilde{K}(M_k)$ which is cyclic of order k. Moreover kH is trivial so let $\theta: kH \to k$ be a trivialization.† Now form the family $A_{(t,m)}$ of self-adjoint operators parametrized by $(t,m) \in I \times M_k$ where $A_{(t,m)} = A_t \otimes I_{H_m}$, and I_{H_m} denotes the identity automorphism of H_m (fibre of H over m).

Using the isomorphism θ we have

$$\left.\begin{aligned} A_{(0,m)} &\cong B_0 \otimes I_{kH_m} \cong B_0 \otimes I_{km}, \\ A_{(1,m)} &\cong B_1 \otimes I_{kH_m} \cong B_1 \otimes I_{km}. \end{aligned}\right\} \tag{8·1}$$

Now take a second family over $I \times M_k$ in which H is replaced by the trivial line-bundle. The isomorphisms (8·1) enable us to glue our two families together over $(\{0\} \cup \{1\}) \times M_k$ to obtain finally a family $\{\tilde{A}_y\}$ of self-adjoint operators parametrized by $y \in S^1 \times M_k$. The index of this family, in the sense of Theorem (3·4), is then an element of

$$K^1(S^1 \times M_k) \cong Z \oplus \tilde{K}(M_k) \cong Z \oplus Z/kZ. \tag{8·2}$$

The interesting component of this is equal to our mod k spectral flow as we shall now prove.

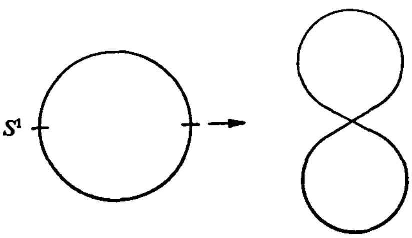
PROPOSITION (8·3). *Using the decomposition* (8·2) *we have*

$$\operatorname{index}\{\tilde{A}_y\} = (0, sf_k\{A_t\}).$$

Proof. The *interest* of (8·3) lies precisely in the fact that it re-interprets $sf_k\{A_t\}$ in terms of the index of the family $\{\tilde{A}_y\}$ of *elliptic* operators. However for the *proof* we

† Here k denotes the trivial k-dimensional bundle $M_k \times C^k$.

may pass to the abstract situation in which all operators are self-adjoint Fredholm operators. The construction of $\{\tilde{A}_y\}$ still makes sense in this context, and we now have the added simplification noted above that $sf_k\{A_t\}$ can be viewed as the mod k reduction of the integer spectral flow $sf\{\bar{A}_u\}$. Here $\bar{A}_u$ $(u \in S^1)$ denotes the extension of $\{A_t\}$ to a family over S^1 given by a homotopy† $B_0 \sim B_1$ in $\hat{\mathfrak{F}}_*$. In terms of $\bar{A}$ the family $\tilde{A}$ may be described more simply as follows. Let $f: S^1 \to S^1 \vee S^1$ be the map which pinches a circle to a figure eight



and put $F = f \times 1: S^1 \times M_k \to (S^1 \vee S^1) \times M_k = (S^1 \times M_k) \cup_{M_k} (S^1 \times M_k)$. Then

$$\tilde{A} = F^*(\bar{A} \otimes I_H \cup \bar{A}' \otimes I_1)$$

where $\bar{A}'$ denotes the family $\bar{A}$ in the opposite sense (u replaced by $-u$). Hence, taking the index of both families, we get

$$\begin{aligned} \operatorname{index} \tilde{A} &= (\operatorname{index} \bar{A}) \otimes [H] + (\operatorname{index} \bar{A}') \otimes [1] \\ &= (\operatorname{index} \bar{A}) \otimes ([H] - 1). \end{aligned}$$

Here index $\tilde{A} \in K^1(S^1 \times M_k)$ while index $\bar{A} \in K^1(S^1)$. Using (8·2), and recalling that $[H]-1$ is the natural generator of $\tilde{K}(M)$, we can rewrite this as

$$\operatorname{index} \tilde{A} = (0; \operatorname{index} \bar{A} \bmod k).$$

On the other hand we saw, in section 7, that for an S^1-family of self-adjoint Fredholm operators the index coincided with the spectral flow. Applying this to $\bar{A}$ we have

$$\begin{aligned} \operatorname{index} \tilde{A} &= (0, sf(\bar{A}) \bmod k) \\ &= (0, sf_k(A)) \end{aligned}$$

completing the proof.

Proposition (8·3) together with Theorem (3·4) therefore gives an explicit formula for our mod k spectral flow. We now apply this to the following special situation.

Let E, F be vector bundles on X and $\phi: kE \to kF$ an isomorphism. Then as explained in Part II, section 5, we get an element

$$[E, F, \phi] \in K^{-1}(X, Z/kZ) = K^{-1}(X \times M_k, X \times \text{point}).$$

If A is a self-adjoint elliptic operator on X we take extensions A_E, A_F, that is self-adjoint operators on X whose leading symbols are obtained by tensoring the leading symbol of A by the identity endomorphisms of E, F respectively. Then, using ϕ, we

† If B_0 or B_1 are essentially positive (or negative) so is the whole family $\{A_t\}$ and both sides of (8·3) are easily seen to be zero. We therefore ignore these cases.

can regard

$$kA_E = A_E \oplus \ldots \oplus A_E \quad k \text{ times}$$

and

$$kA_F = A_F \oplus \ldots \oplus A_F \quad k \text{ times}$$

as acting on the same bundle. They have the same leading symbol hence can be connected in a homotopically unique way by a family A_t with $A_0 = kA_E, A_1 = kA_F$. This mod k family then has a mod k spectral flow which is a homotopy invariant of all our data, and which we shall denote by $\operatorname{ind}_k(E, F, \phi; A)$. The symbol of the corresponding family $\tilde{A}$, parametrized by $S^1 \times M_k$, is given by

$$[\sigma(\tilde{A})] = [\sigma(A)].[E, F, \phi]$$

using the multiplication

$$K^1(TX) \otimes K^{-1}(X \times M_k) \to K^0(TX \times M_k).$$

Applying (8·3) and Theorem (3·4) (with $P = S^1 \times M_k$) and rewriting things in terms of K-theory with coefficients in Z/kZ we obtain the following:

(8·4). *Mod k Index Theorem.*

$$\operatorname{Ind}_k(E, F, \phi; A) = \operatorname{Ind}_k([\sigma_m(A)][E, F, \phi])$$

where $\operatorname{Ind}_k: K^0(TX, Z/kZ) \to Z/kZ$ *is the topological index for* Z/kZ *coefficients.*

Theorem (5·3) – our general index theorem for flat bundles formulated in section 5 – now follows by combining (8·4) with (6·2). The proof is just a matter of collecting the notation. We take the triple (E, F, ϕ) in (8·4) to be the triple (kV_α, kV_N, ϕ) of section 5. For the family A_t used in (8·4) we then compute the difference $\xi(0, A_1) - \xi(0, A_0)$ in terms of its continuous part and the spectral flow as in (7·1). With the notation of section 6 we have

$$\xi(0, A_1) - \xi(0, A_0) = \operatorname{ind}(k\alpha, k\phi, A) + sf\{A_t\}.$$

Dividing by k and observing that $A_1 = kA_\alpha, A_0 = kA_N$ and

$$\frac{1}{k} sf\{A_t\} = \operatorname{ind}_k(V_\alpha, V_N, \phi; A) \mod Z$$

we obtain

$$\tilde{\xi}_\alpha(0, A) = \xi(0, A) - \xi(0, A_N) = \frac{1}{k}\operatorname{ind}(k\alpha, k\phi, A) + \operatorname{ind}_k(V_\alpha, V_N, \phi; A) \mod Z.$$

Applying (6·2) and (8·4) this becomes

$$\tilde{\xi}_\alpha(0, A) = -\operatorname{Ind}_R(b\sigma) + \operatorname{Ind}_k[\sigma a]$$

where $\sigma = [\sigma_m(A)]$, $b = b(\alpha, \phi)$, $a = [V_\alpha, V_N, \phi]$ and Ind denotes topological index. But from the way we defined the element $[\alpha] \in K^{-1}(X, R/Z)$ in section 5 this equation is essentially the same as

$$\tilde{\xi}_\alpha(0, A) = -\operatorname{Ind}([\alpha]\sigma)$$

which is the assertion of Theorem (5·3). The proof is therefore complete.

An incidental consequence of our mod k index theorem is that it provides yet another definition of the Q/Z invariants of Part II, section 3, for Spin^c-manifolds with *finite* fundamental group G. This is:

V. *The spectral flow definition.* If V_α is the flat vector bundle defined by the representation $\alpha:\pi_1(Y)\to G\to U(N)$ we choose a trivialization $kV_\alpha\to kN$ coming from (a finite skeleton of) B_G. This makes kD_α and kND operators on the same bundle (D denotes the Dirac operator of Y) and we can join them (say linearly) by a one-parameter family of self-adjoint elliptic operators. The spectral flow of this family divided by $-k$ is the value of our invariant in Q/Z.

The equivalence of definitions V and II (of Part II, section 3) is given by (8·4).

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